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Common Fixed Point of Two L_2 Operators with Convergence Analysis and Application

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Abstract: This article introduces a new numerical algorithm for approximating the solution of the common fixed point problem for two operators defined on CAT(0) spaces, belonging to the class L_2, which was very recently introduced. The main results refer to Δ and strong convergence of the sequence generated by the new algorithm. A distinct feature of the adopted approach is the use of equivalent sequences.

Keywords: common fixed point; nonexpansive mappings; Δ-convergence; CAT(0) space; L_2 mappings

MSC: 47H10; 58A05

1. Introduction

Since its publication, Banach's Contraction Principle has been a permanent source of inspiration for many thousands of papers. The short statement: "any contraction on a complete metric space has a fixed point" encapsulates a tremendous amount of mathematics and has deep consequences which are still being discovered. A mapping $F : M \to M$, where (M, d) is a metric space, is called *contraction* if

$$d(F(x), F(y)) \leq Cd(x, y),$$

for any $x, y \in M$ and $0 \leq C < 1$. Computing successive compositions of a given contraction, known as the Picard iterative process, generates a sequence converging to the solution of the fixed point problem for F (find x such that $F(x) = x$). A natural extension is to consider the case $C = 1$, which leads to the class of *nonexpansive mappings*. However, as one can easily see, the Picard iteration for such mappings does not necessarily lead to a fixed point, as in the case of contractions. This fact has stimulated the search for new iterative schemes such as, for instance, [1–7]. At the same time, it is important to notice that the metric structure alone is not sufficient in order to apply more sophisticated iterative schemes. A richer structure is required, the most frequent one being that of normed spaces. Thus, we can identify three main aspects: a class of mappings, a space on which the given mappings act and an iterative scheme used to generate a sequence converging to the solution of the fixed point problem.

In 2011, Fuster and Gálvez [8] introduced a class of generalized nonexpansive mappings by the use of the so-called condition (L), in the context of Banach spaces. The emergence of this class of mappings can be seen as a natural development starting from the class of mappings satisfying Suzuki's condition (C) [9], via the class of those satisfying condition (E) [10]. For more details, we refer the reader to those original papers. In a more recent paper [11], in which a similar problem is discussed, the authors considered the class of operators which are closely related to those satisfying the condition (L), and which they called (L_2) class (for details, please see below). Thus, the class of mappings which we discuss in this paper is that of (L_2) operators.

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The framework for the results of this paper is that of CAT(0) spaces, i.e., metric spaces which satisfy additional axioms (details are provided in the sequel). Notable particular cases are Hilbert spaces, R-trees, etc. The structure of a CAT(0) space is such that it allows the introduction of a notion of convergence, called Δ-convergence, which is more general than the metric convergence [12]. Moreover, as it has been show by Kirk and Panyanak [12], many results on Banach spaces, involving weak convergence, have precise counterparts in the setting of CAT(0) spaces, where Δ-convergence is used instead.

A natural extension of the fixed point problem is the problem of common fixed point (find x such that $F(x) = x = G(x)$), thought in various frameworks for diverse classes of operators [13,14]. As an instrument for approximating the solution of the common fixed point problem, we introduce a new iterative scheme, inspired by [6], and whose strong and Δ-convergence is of interest. A distinct feature of this scheme is that using it involves nonstandard approaches based on equivalent sequences and which we believe could be useful to other authors as well.

We dedicate this paper to obtaining Δ and strong convergence results for the common fixed point problem for two operators F and G of the (L_2) class (shortly, L_2 operators), in CAT(0) spaces. For recent related results in this direction, we refer the reader to [13], where such a problem is studied for the class of (E) operators, or [15] for SKC mappings.

2. Preliminaries

Let (M, d) be a metric space. Given two distinct points x and y in M, a continuous mapping

$$\gamma\colon [0,a] \to M, \text{ with } \gamma(0) = x, \gamma(a) = y,$$

such that $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$, for any $t_1, t_2 \in [0,a]$, is called a *geodesic path* which joins x and y, while its image, denoted by $[x,y]$, is called the *geodesic segment* with endpoints the x and y. A metric space (M,d) is called *geodesic space* if any pair of distinct point can be joined by a geodesic. Moreover, if the geodesic is unique, then the space is called *uniquely geodesic*.

Three distinct points x, y, z in a uniquely geodesic metric space (M,d) constitute the vertices of a unique *geodesic triangle* denoted by $\Delta(x,y,z)$, whose sides are the geodesic segments $[x,y]$, $[y,z]$ and $[z,x]$. A triangle in the Euclidean plane $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$ such that

$$d(x,y) = d_E(\bar{x},\bar{y}), d(y,z) = d_E(\bar{y},\bar{z}), d(x,z) = d_E(\bar{x},\bar{z}),$$

where d_E is the Euclidean metric, is called a *comparison triangle* for $\Delta(x,y,z)$.

Definition 1 ([16,17])**.** *Let Δ be a geodesic triangle in a geodesic space (M,d) and let $\bar{\Delta}$ be a corresponding comparison triangle. We say that Δ satisfies the* CAT(0) *inequality if for all $x, y \in \Delta$ and the corresponding $\bar{x}, \bar{y} \in \bar{\Delta}$,*

$$d(x,y) \leq d_E(\bar{x},\bar{y}). \quad (1)$$

A geodesic space is said to be a CAT(0) *space if all its geodesic triangles satisfy the* CAT(0) *inequality.*

Other equivalent definitions can be encountered in the literature, for example [16–18].

Lemma 1 ([17])**.** *Let (M,d) be a* CAT(0) *space. Then*

(i) *(M,d) is uniquely geodesic.*
(ii) *For a given pair of distinct points x, y in M and a some $t \in [0,1]$, there exists a unique point $z \in [x,y]$, such that $d(x,z) = (1-t)d(x,y)$ and $d(y,z) = td(x,y)$. We denote this point by $z = tx \oplus (1-t)y$.*
(iii) *$[x,y] = \{tx \oplus (1-t)y \colon t \in [0,1]\}$.*
(iv) *$d(x,z) + d(z,y) = d(x,y)$ if and only if $z \in [x,y]$.*
(v) *The mapping $f\colon [0,1] \to [x,y]$, $f(t) = tx \oplus (1-t)ty$ is continuous and bijective.*

Lemma 2 ([17]). *Let (M,d) be a* CAT(0) *space. Then*

$$d(z, tx \oplus (1-t)y) \leq td(z,x) + (1-t)d(z,y), \tag{2}$$

$$d^2(z, tx \oplus (1-t)y) \leq td^2(z,x) + (1-t)d^2(z,y) - t(1-t)d^2(x,y), \tag{3}$$

for all $x, y, z \in M$ and $t \in [0,1]$.

Definition 2. *Let $\{x_n\}$ be a sequence in a complete* CAT(0) *space (M,d). The set*

$$\mathcal{C}(x_n) = \{x \in M \big| r(x, x_n) = r(x_n)\}$$

is called the asymptotic center of the sequence $\{x_n\}$, where

$$r(x_n) = \inf_{x \in M} r(x, x_n)$$

is the asymptotic radius, and

$$r(x, x_n) = \limsup_{n \to \infty} d(x, x_n).$$

In a complete CAT(0) space the asymptotic center associated to a given sequence consists of a single element (for details, please check [19]). This fact has allowed Kirk and Panyanak [12] to introduce a notion of convergence, called Δ-convergence, which is weaker than the metric convergence.

Definition 3 ([12]). *A sequence $\{x_n\}$ in a* CAT(0) *space (M,d) is said to be Δ-convergent to some point $x \in M$ and denote it by $x_n \overset{\Delta}{\to} x$, if x is the unique asymptotic center for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.*

The remarkable fact about this type of convergence is that many results involving weak convergence on Banach spaces have precise counterparts in the setting of CAT(0) spaces involving Δ-convergence. It is worth mentioning that CAT(0) spaces have the Opial property formulated with respect to Δ-convergence. We recall the below definition of Opial.

Definition 4 ([20]). *A Banach space X satisfies the Opial property if for any sequence $\{x_n\}$ in X, which converges weakly to x, the next inequality holds*

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|,$$

for any $y \neq x$.

Below are some results involving Δ-convergence which will be used in the sequel.

Lemma 3 ([12,17]). *Let (M,d) be a CAT(0) space.*

(i) Any bounded sequence in M has a Δ-convergent subsequence.
(ii) If C is a closed and convex subset in M, and $\{x_n\}$ is a bounded sequence in C, then $\mathcal{C}(x_n) \in C$.

The asymptotic center plays a key role in the following lemma, which will be used in the sequel to prove the coincidence of certain limits.

Lemma 4 ([17]). *Let $\{x_n\}$ be a bounded sequence in a*CAT(0) *space (M,d) with $\mathcal{C}(x_n) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ such that $\mathcal{C}(u_n) = \{u\}$ and the sequence $\{d(x_n,u)\}$ is convergent, then $x = u$.*

Fuster and Gálvez introduced, in 2011, the following generalized nonexpansive class of mappings, which extend the previously introduced classes of operators satisfying the conditions (*C*) [9] and (*E*) [10].

Definition 5 ([8])**.** *Let C be a nonempty subset of a* CAT(0) *space (M, d), and consider a mapping $T\colon C \to C$. One says that T fulfills the property (L) if the next two conditions are satisfied*

(i) *For any nonempty, closed, convex D of C, which is T-invariant (that is $TD \subseteq C$), there exists an almost fixed point sequence of T (shortly a.f.p.s, i.e., a sequence $\{x_n\}$ such that $\{d(x_n, Tx_n)\}$ is convergent to zero);*

(ii) *For any almost fixed point sequence $\{x_n\}$ of T in C, and $x \in C$, the following inequality holds true*

$$\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} d(x_n, x). \tag{4}$$

Henceforth, following [11], by L_2 mappings we shall mean those mappings which satisfy the condition (4) of the above definition.

Let C be a nonempty, and convex subset of a CAT(0) space (M, d). For two mappings $F, G\colon C \to C$, and $x_0 \in C$, we consider the next numerical scheme:

$$\begin{aligned} y_n &= (1 - a_n)x_n \oplus a_n F x_n, \\ z_n &= (1 - b_n)x_n \oplus b_n G y_n, \\ x_{n+1} &= (1 - c_n)F z_n \oplus c_n F y_n, \quad n \geq 0, \end{aligned} \tag{5}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences bounded away by 0 and 1.

Following is a standard and a very useful results which will be used in the sequel and which ends the section of preliminaries.

Lemma 5 ([21])**.** *In a complete* CAT(0) *space (M, d), consider a point x, and two sequences $\{x_n\}$, $\{y_n\}$. Let $\{t_n\}$ be a sequence of real numbers bounded away from 0 and 1. Suppose there exists $\ell \in \mathbb{R}$ such that*

$$\limsup_{n\to\infty} d(x_n, x) \leq \ell, \limsup_{n\to\infty} d(y_n, x) \leq \ell \text{ and } \lim_{n\to\infty} d(t_n x_n \oplus (1 - t_n)y_n, x) = \ell.$$

Then the sequence $\{d(x_n, y_n)\}$ converges to zero.

3. Main Results

The main goal in this section is to prove that the sequence $\{x_n\}$, generated by the algorithm (5) is Δ-convergent to a solution x of the common fixed point problem associated to two L_2 operators F and G. Additionally, we obtain a strong convergence result as well. We denote by $Fix(F, G)$ the set of common fixed points of two given operators F and G.

Firstly, let us show that a L_2 operator satisfies Browder's *demiclosed principle* (for details, please see [22]), where instead of weak convergence we assume Δ-convergence.

Lemma 6. *Let C be a subset in a complete* CAT(0) *space (M, d) and let $F\colon C \to M$ be a L_2 operator. If $\{x_n\} \subset C$ is an a.f.p.s. for F such that $x_n \xrightarrow{\Delta} x \in M$, then*

$$Fx = x.$$

Proof. As $x_n \xrightarrow{\Delta} x$, there exists a subsequence $\{x_{n_k}\}$ such that x is its unique asymptotic center. On the other hand, being a subsequence of $\{x_n\}$, $\{x_{n_k}\}$ is an a.f.p.s. for F as well. Since M has the Opial property and F satisfies the L_2 condition, we have

$$\limsup_{n\to\infty} d(x_{n_k}, Fx) \leq \limsup_{n_k\to\infty} d(x_{n_k}, x) \leq \limsup_{n\to\infty} d(x_{n_k}, Fx),$$

yielding

$$\limsup_{n\to\infty} d(x_{n_k}, Fx) = \limsup_{n_k\to\infty} d(x_{n_k}, x).$$

The conclusion follows from the uniqueness of the asymptotic center. □

Recall that a mapping F on a metric space M is called quasi-nonexpansive if

$$d(p, Fx) \leq d(p, x),$$

for all $x \in M$, where p is a fixed point of F. It can be esily seen that L_2 operators are quasi-nonexpansive. Thus, the following two results are valid for L_2 operators as well.

Lemma 7. *Let $F, G\colon C \to C$, where C is a subset of a* CAT(0) *space, be two quasi-nonexpansive operators. Then the set $Fix(F, G)$ is closed and convex.*

Proof. Let $\{y_n\}$ be a sequence in $Fix(F, G)$, convergent to some $y \in C$. It is, of course, an approximate fixed point sequence for both operators F and G. Since F and G are quasi-nonexpansive, we have that

$$\lim_{n\to\infty} d(y_n, Fy) \leq \lim_{n\to\infty} d(y_n, y) = 0,$$

implying $y = Fy$. Similarly, we obtain that $y = Gy$ and thus $Fix(F, G)$ is a closed set.

Let now $x, y \in Fix(F, G)$ and take z a point on the geodesic segment $[x, y]$. Suppose that $Fz \neq z$. Then

$$d(x, y) < d(Fz, x) + d(Fz, y) \leq d(z, x) + d(z, y) = d(x, y),$$

a contradiction which completes the proof. □

Lemma 8. *Let (M, d) be a* CAT(0) *space and C be a nonempty, closed and convex subset of M. Let $F, G : C \to C$ be two quasi-nonexpansive mappings such that $Fix(F, G) \neq \emptyset$. Then for the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, generated by the algorithm (5) and for any $q \in Fix(F, G)$, the following limits*

$$\lim_{n\to\infty} d(x_n, q), \lim_{n\to\infty} d(y_n, q), \lim_{n\to\infty} d(z_n, q)$$

exist and are equal.

Proof. Let $q \in Fix(F, G)$. Applying Lemma 2 and noticing that both F and G are quasi-nonexpansive mappings, it follows that

$$\begin{aligned} d(y_n, q) &= d((1 - a_n)x_n \oplus a_n Fx_n, q) \leq (1 - a_n)d(x_n, q) + a_n d(Fx_n, q) \\ &\leq (1 - a_n)d(x_n, q) + a_n d(x_n, q) = d(x_n, q), n \geq 0. \end{aligned} \tag{6}$$

A similar argument leads to

$$\begin{aligned} d(z_n, q) &= d((1 - b_n)x_n \oplus b_n Gy_n, q) \leq (1 - b_n)d(x_n, q) + b_n d(Gy_n, q) \\ &\leq (1 - b_n)d(x_n, q) + b_n d(y_n, q) \leq (1 - b_n)d(x_n, q) + b_n d(x_n, q) \\ &= d(x_n, q), n \geq 0. \end{aligned} \tag{7}$$

Finally, both inequalities (6) and (7), yield

$$\begin{aligned} d(x_{n+1}, q) &= d((1 - c_n)Fz_n \oplus c_n Fy_n, q) \leq (1 - c_n)d(Fz_n, q) + c_n d(Fy_n, q) \\ &\leq (1 - c_n)d(z_n, q) + c_n d(y_n, q) \leq (1 - c_n)d(x_n, q) + c_n d(x_n, q) \\ &= d(x_n, q), n \geq 0. \end{aligned} \tag{8}$$

The existence of $\lim_{n\to\infty} d(x_n, q)$ follows immediately from the fact the sequence of positive numbers $\{d(x_n, q)\}$ is decreasing.

According to (8), we have that

$$d(x_{n+1}, q) \leq (1 - c_n)d(z_n, q) + c_n d(y_n, q) \leq d(x_n, q), n \geq 0.$$

Taking $n \to \infty$ in the last inequalities, leads to

$$\lim_{n\to\infty}((1 - c_n)d(z_n, q) + c_n d(y_n, q)) = \ell,$$

where $\ell := \lim_{n\to\infty} d(x_n, q)$. On the other hand, from (6) and (7) we also have that

$$\limsup_{n\to\infty} d(y_n, q) \leq \limsup_{n\to\infty} d(x_n, q) \leq \ell, \text{and} \limsup_{n\to\infty} d(z_n, q) \leq \limsup_{n\to\infty} d(x_n, q) \leq \ell.$$

Applying now Lemma 5 for the sequences $\{y_n\}$ and $\{z_n\}$, yields $\lim_{n\to\infty} d(y_n, z_n) = 0$, which shows that

$$\lim_{n\to\infty} d(y_n, q) = \lim_{n\to\infty} d(z_n, q).$$

Letting $n \to \infty$ in (8), while keeping in mind that the sequence $\{c_n\}$ is bounded away by 0 and 1, we obtain that

$$\lim_{n\to\infty} d(x_n, q) = \lim_{n\to\infty} d(z_n, q),$$

which completes the proof. □

A closer look to our iterative scheme reveals the fact that it does not contain any term of the form Gx_n. This circumstance makes it rather difficult to establish whether $\{x_n\}$ is an approximate fixed point sequence for the mapping G, than for the mapping F. As we will see below, actually we do not need to show that x_n is an approximate fixed point sequence for both mappings. We will circumvent this obstacle by working with the sequence $\{y_n\}$.

Recall that two sequences $\{x_n\}$ and $\{y_n\}$ in a metric space are called equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Clearly, two equivalent sequences either converge to the same limit, or are both divergent. In fact, the same is true for the Δ-convergence.

Lemma 9. *If $\{x_n\}$ and $\{y_n\}$ are two equivalent sequences in a metric space (M, d), then $\mathcal{C}(x_n) = \mathcal{C}(y_n)$. Moreover, if $x_n \overset{\Delta}{\to} p \in M$, then $y_n \overset{\Delta}{\to} p$ as well.*

Proof. For any $x \in M$, taking lim sup in the inequalities $d(x_n, x) \leq d(x_n, y_n) + d(y_n, x)$ and $d(y_n, x) \leq d(x_n, y_n) + d(x_n, x)$, respectively, yields

$$r(x, x_n) = r(x, y_n),$$

implying $\mathcal{C}(x_n) = \mathcal{C}(y_n)$.

For the second part of the assertion, take an arbitrary subsequence $\{s'_{n_k}\}$ of $\{y_n\}$ and suppose that $x_n \overset{\Delta}{\to} p \in M$, i.e., p is the unique asymptotic center of any subsequence of $\{x_n\}$. There exists a corresponding subsequence $\{s_{n_k}\}$ of $\{x_n\}$, obtained by taking the elements with the exact same indexes, such that $\lim_{n\to\infty} d\left(s_{n_k}, s'_{n_k}\right) = 0$. Thus, $\mathcal{C}(s'_{n_k}) = \mathcal{C}(s_{n_k}) = \{p\}$. □

Lemma 10. *Let (M, d) be a complete* CAT(0) *space and C be a nonempty, closed and convex subset of M. Consider $F, G : C \to C$ be two quasi-nonexpansive mappings which have at least one common fixed point and let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated by the iterative scheme (5). Then*

(i) $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, z_n) = \lim_{n\to\infty} d(z_n, x_n) = 0;$
(ii) $\lim_{n\to\infty} d(x_n, Fx_n) = \lim_{n\to\infty} d(y_n, Gy_n) = 0.$

Proof. (i) Let $q \in Fix(F, G)$ and let $\ell = \lim_{n\to\infty} d(x_n, q)$, which exists according to Lemma 8. From the proof of Lemma 8 we already know that $\lim_{n\to\infty} d(y_n, z_n) = 0$.

Using the fact that G is a quasi-nonexpansive mapping and Lemma 2 in the second line of the algorithm (5), we obtain that

$$d(z_n, q) \le (1 - b_n)d(x_n, q) + b_n d(Gy_n, q) \le (1 - b_n)d(x_n, q) + b_n d(y_n, q) \le d(x_n, q).$$

Letting $n \to \infty$, it follows that

$$\ell = \limsup_{n\to\infty} d(z_n, q) \le \limsup_{n\to\infty} (1 - b_n)d(x_n, q) + b_n d(y_n, q) \le \limsup_{n\to\infty} d(x_n, q) = \ell.$$

As $\limsup_{n\to\infty} d(y_n, q) = \ell$ and the sequence $\{b_n\}$ is bounded away from 0 and 1, Lemma 5 can be applied for the sequences $\{x_n\}$ and $\{y_n\}$, leading to $\lim_{n\to\infty} d(x_n, y_n) = 0$.

The last limit from the statement is obtained by taking $n \to \infty$ in

$$d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n).$$

(ii) Using a similar argument as above, from the first line of the algorithm we have that

$$d(y_n, q) = d((1 - a_n)x_n \oplus a_n Fx_n, q) \le d(x_n, q).$$

Taking lim sup gives

$$\limsup_{n\to\infty} d((1 - a_n)x_n \oplus a_n Fx_n, q) = \ell,$$

where we have used the fact that F is quasi-nonexpansive implying that

$$\limsup_{n\to\infty} d(Fx_n, q) \le \limsup_{n\to\infty} d(x_n, q) \le \ell.$$

Thus, according to Lemma 5, applied for the sequences $\{x_n\}$ and $\{Fx_n\}$, it follows that

$$\lim_{n\to\infty} d(x_n, Fx_n) = 0,$$

i.e., $\{x_n\}$ is an approximate fixed point sequence for the mapping F. By an almost identical argument, from the second line of the algorithm (5) it follows that

$$\lim_{n\to\infty} d(x_n, Gy_n) = 0.$$

On the other hand, we have that $\lim_{n\to\infty} d(x_n, y_n) = 0$ which implies that

$$\lim_{n\to\infty} d(y_n, Gy_n) = 0,$$

i.e., $\{y_n\}$ is an approximate fixed point sequence for the mapping G. □

We now have everything prepared in order to show that the sequence $\{x_n\}$ generated by the iterative scheme (5) is Δ-convergent to a common fixed point of two mappings F and G which satisfies condition L_2, provided it exists.

Theorem 1. *Let (M,d) be a complete* CAT(0) *space and C be a nonempty, closed and convex subset of M. If $F,G\colon C\to C$ are two mappings satisfying the condition* (L_2) *such that $Fix(F,G)\neq\emptyset$, then the sequence $\{x_n\}$, generated by the algorithm* (5)*, is Δ-convergent to an element of $Fix(F,G)$.*

Proof. Let $\mathcal{W}_\Delta(x_n)$ be the reunion of all asymptotic centers associated to all subsequences of $\{x_n\}$ and take a subsequence $\{p_n\}$ whose asymptotic center is p. Let q be a common fixed point of F and G. Since $\lim\limits_{n\to\infty} d(x_n,q)$ exists, the sequence $\{x_n\}$ is bounded and thus $\{p_n\}$ is bounded too. According to Lemma 3, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ which is Δ-convergent to some $s\in C$, which is actually a fixed point for F. Indeed, $\{p_{n_k}\}$ being a subsequence of $\{x_n\}$, is an approximate fixed point sequence for F as well. Thus, according to Lemma 6, it follows that

$$Fs=s.$$

Let us show now that s is a fixed point for the mapping G as well. As before, take a subsequence of $\{y_n\}$, corresponding to $\{p_{n_k}\}$, by choosing the elements with exactly the same indexes. This gives us a subsequence $\{p'_{n_k}\}$ equivalent to $\{p_{n_k}\}$, which is also Δ-convergent to s. On the other hand, being a subsequence of $\{y_n\}$, $\{p'_{n_k}\}$ is an approximate fixed point sequence for the mapping G as well. Applying once again Lemma 6, yields

$$Gs=s.$$

Thus, we have proven that s is actually a common fixed point of F and G. This means, according to Lemma 8, that the sequence $\{d(x_n,s)\}$ is convergent and consequently, the same is true for the subsequence $\{d(p_n,s)\}$. Now, since $\mathcal{C}(p'_{n_k})=\{s\}$, according to Lemma 4, we have that $s=p=x$, where $\mathcal{C}(x_n)=\{x\}$. Since the subsequence $\{p_n\}$ was chosen arbitrarily, it follows that

$$\{x\}=\mathcal{C}(x_n)=\mathcal{W}(x_n)\subseteq Fix(F,G),$$

which completes the proof. □

Remark 1. *Taking G as the identity mapping, the scheme* (5) *reduces to the scheme of Sintunavarat and Pitea* [6]*, whereas the generated sequence is Δ-convergent to a fixed point of the mapping F.*

Turning now to the strong convergence, i.e., with respect to the metric topology, it is clear that if a sequence $\{x_n\}$ generated by the iterative scheme (5) converges to an element of $Fix(F,G)$, then necessarily $\liminf\limits_{n\to\infty} d(x_n,Fix(F,G))=0$, where

$$d(x_n,Fix(F,G))=\inf\{d(x_n,p)|p\in Fix(F,G)\}.$$

The next result shows that this condition is actually sufficient.

Proposition 1. *Let (M,d) be a* CAT(0) *space and C be a nonempty, closed, convex subset of M and let $F,G: C\to C$ be two operators having the property* (L_2)*. Then, the iterative sequence $\{x_n\}$ converges to a point in $Fix(F,G)$ if and only if $\liminf\limits_{n\to\infty} d(x_n,Fix(F,G))=0$.*

Proof. Suppose that $\liminf\limits_{n\to\infty} d(x_n,Fix(F,G))=0$. According to Lemma 8, the sequence $\{d(x_n,q)\}$ is decreasing, for any $q\in Fix(F,G)$. Thus,

$$d(x_{n+1},Fix(F,G))=\inf_{x\in Fix(F,G)} d(x_{n+1},x)\leq \inf_{x\in Fix(F,G)} d(x_n,x)=d(x_n,Fix(F,G)),$$

meaning that $\{d(x_n, Fix(F,G))\}$ is decreasing and therefore, convergent to 0. This means that, given $\varepsilon > 0$, there is a rank N and a point $p \in Fix(F,G)$ such that $d(x_n, p) < \varepsilon$, for $n \geq N$. In addition, from the inequalities

$$d(x_n, x_m) \leq d(x_n, p) + d(x_m, p) \leq 2\varepsilon, n, m \geq N,$$

it follows that $\{x_n\}$ is a Cauchy sequence and, since M is complete, it is convergent to some $x \in C$. Now, from the continuity of the metric, we have that $d(x, Fix(F,G)) = 0$. To complete the proof, it remains to show that $Fix(F,G)$ is closed. Indeed, let $\{y_n\}$ be a sequence in $Fix(F,G)$, convergent to some y. It is, of course, an approximate fixed point sequence for both mappings F and G. Applying the L_2 condition, say for F, we have that

$$\limsup_{n\to\infty} d(y_n, Fy) \leq \limsup_{n\to\infty} d(y_n, y) = 0,$$

which, by the uniqueness of the limit, it follows that $y = Fy$. Similarly, we obtain that $y = Gy$, and thus $y \in Fix(F,G)$, which completes the proof. □

4. Example

In this section we illustrate the results of the paper by means of an example. Let C be a point the Poincaré half-plane $\mathbb{H}^2 = \{(x,y) \in \mathbb{R} | y > 0\}$ in which the distance between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is given by the formula:

$$d_{\mathbb{H}^2}(p_1, p_2) = 2\ln\left(\frac{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} + \sqrt{(x_2-x_1)^2 + (y_2+y_1)^2}}{2\sqrt{y_1 y_2}}\right). \quad (9)$$

Let D be the disk centered at C with some fixed radius r and consider the mappings:

$$G(x,y) = (-x,y), (x,y) \in D,$$

and

$$FX = \begin{cases} \frac{1}{2}C \oplus \frac{1}{2}X, & X = (x,y) \in intD, \\ S_C\left(\frac{1}{2}C \oplus \frac{1}{2}X\right), & X = (x,y) \in \partial D, \end{cases}$$

where $S_C(Y)$ denotes the symmetric of the point Y w.r.t. the point C. In other words, $S_C(Y)$ is such that $C = \frac{1}{2}Y \oplus \frac{1}{2}S_C(Y)$.

From (9) it follows that the mapping G is nonexpansive and hence satisfies the condition L_2 as well.

Let us show that the mapping F is a L_2 mapping as well. Indeed, let $\{X_n\} \subset D$ be an a.f.p.s. for F, i.e., $\lim_{n\to\infty} d(X_n, FX_n) = 0$. It follows that there exists some $n_0 \geq 0$, such that $X_n \notin \partial D$, for $n \geq n_0$. On the other hand, for $X_n \notin \partial D$ one has $d(X_n, FX_n) = d(C, FX_n) = \frac{1}{2}d(C, X_n)$, for $n \geq n_0$. However, this means that, necessarily, $\lim_{n\to\infty} d(C, X_n) = 0$. Let now $x \in D$. Taking into account that $d(C, FX) = \frac{1}{2}d(C, X)$, while applying the triangle inequality, we have, for all $n \geq 0$,

$$\begin{aligned} d(X_n, FX) &\leq d(X_n, C) + d(C, FX) \leq d(X_n, C) + \frac{1}{2}d(C, X_n) + \frac{1}{2}d(X_n, X) \\ &= \frac{3}{2}d(C, X_n) + d(X_n, X). \end{aligned}$$

Taking lim sup of the first and the last term of the above relation, and keeping in mind that $\lim_{n\to\infty} d(X_n, FX_n) = 0$, leads to

$$\limsup_{n\to\infty} d(X_n, X) \leq \limsup_{n\to\infty} d(X_n, FX),$$

showing that F is a L_2 mapping.

Let us discuss now the convergence of the proposed algorithm. Since our further discussion does not involve the coordinates of the half-plane, we will use small letters to denote the points for simplicity (except the center). In addition, for simplicity, we will use the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$, with $a_n = b_n = c_n = \frac{1}{2}$, $n \geq 0$.

Denote by $d_n = d_{\mathbb{H}^2}(C, x_n)$, $n \geq 0$. For any $x_n \in intD$, we have the following distances estimates:

$$d(C, y_n) = d(C, \frac{1}{2}x_n \oplus \frac{1}{2}Fx_n) = \frac{3}{4}d_n$$

$$d(C, z_n) = d(C, \frac{1}{2}x_n \oplus \frac{1}{2}Gy_n) \leq \frac{1}{2}d(C, x_n) + \frac{1}{2}d(C, Gy_n) = \frac{7}{8}d_n$$

$$d(C, Fz_n) \leq \frac{7}{16}d_n$$

$$d(C, Fy_n) = \frac{1}{2}d(C, y_n) = \frac{3}{8}d_n$$

$$d(C, x_{n+1}) = d(C, \frac{1}{2}Fz_n \oplus \frac{1}{2}Fy_n) \leq \frac{1}{2}d(C, Fz_n) + \frac{1}{2}d(C, Fy_n) = \frac{12}{32}d_n,$$

where we have used the inequality (2).

It can be noticed that, starting with x_0 such that $d(C, x_0) < r$, all the points x_n,y_n,z_n, Fz_n, Fy_n, Gy_n, appearing at each iteration, belong to the interior of the disk D.

Thus, it follows that the sequence $\{x_n\}$ converges strongly to the point C. This fact implies the Δ-convergence as well.

5. Conclusions

In this article, we introduced a new iterative scheme for the approximation of the solution of the common fixed point problem associated to two L_2 operators acting on CAT(0) spaces. The class of L_2 operators has been recently introduced and extends many other well known classes such Suzuki type of operators. As the main result, we establish the Δ-convergence of the introduced algorithm. In addition, we provide a sufficient condition for strong convergence. Our study uses a new iterative algorithm and the technique based on the equivalent sequences represents a novel approach. As a natural development, we envision that the problem of best proximity point, or common best proximity point, could be studied in this setting [23].

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