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A New Seminorm for d-Tuples of A-Bounded Operators and Their Applications

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Abstract: The aim of this paper was to introduce and investigate a new seminorm of operator tuples on a complex Hilbert space $\mathcal{H}$ when an additional semi-inner product structure defined by a positive (semi-definite) operator A on $\mathcal{H}$ is considered. We prove the equality between this new seminorm and the well-known A-joint seminorm in the case of A-doubly-commuting tuples of A-hyponormal operators. This study is an extension of a well-known result in [Results Math 75, 93(2020)] and allows us to show that the following equalities $r_A(\mathbf{T}) = \omega_A(\mathbf{T}) = \|\mathbf{T}\|_A$ hold for every A-doubly-commuting d-tuple of A-hyponormal operators $\mathbf{T} = (T_1, \ldots, T_d)$. Here, $r_A(\mathbf{T})$, $\|\mathbf{T}\|_A$, and $\omega_A(\mathbf{T})$ denote the A-joint spectral radius, the A-joint operator seminorm, and the A-joint numerical radius of $\mathbf{T}$, respectively.

Keywords: positive operator; A-adjoint operator; A-joint operator seminorm; A-hyponormal operator; A-joint spectral radius; A-joint numerical radius

MSC: 47B65; 47A05; 47A12; 46C05; 47B20; 47A10

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1. Introduction

In functional analyses, many authors have studied the tuples of operators. For example, we refer to [1–5] and the references therein.

Consider a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, where the norm induced by $\langle\cdot,\cdot\rangle$ is denoted by $\|\cdot\|$. The set $\mathbb{B}(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$ with identity $I_{\mathcal{H}}$ (or shortly I). If $\mathcal{H}$ is n-dimensional, we identify $\mathbb{B}(\mathcal{H})$ with the space $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field and denote its identity by I_n. In what follows, by an operator, we mean a bounded linear operator. We will mention some specific notions of an operator, i.e., the null space of every operator T is denoted by $\mathcal{N}(T)$, its range by $\mathcal{R}(T)$, and T^* is the adjoint of T. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. We write $T \geq 0$ if T is positive. If $T \geq 0$, then $T^{1/2}$ means the square root of T. The commutator of two operators $T, S \in \mathbb{B}(\mathcal{H})$ is defined as $[T, S] := TS - ST$. It is easy to see that $[T - \lambda I, S - \mu I] = [T, S]$, for every $\lambda, \mu \in \mathbb{C}$ and $T, S \in \mathbb{B}(\mathcal{H})$. Recall that $T \in \mathbb{B}(\mathcal{H})$ is called normal (respectively hyponormal) if $[T^*, T] = 0$ (respectively, $[T^*, T] \geq 0$).

Next, we present some inequalities related to operators that we need in the future. First, we give the classical Schwarz inequality for a positive operator $T \in \mathbb{B}(\mathcal{H})$:

$$|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle, \tag{1}$$

where $x, y \in \mathcal{H}$.

In [6], Halmos obtains a result similar to the inequality above

$$|\langle Tx, x\rangle| \leq \langle |T|x, x\rangle^{1/2}\langle |T^*|x, x\rangle^{1/2}$$

for every $T \in \mathbb{B}(\mathcal{H})$ and for any $x, y \in \mathcal{H}$. In [7], Kato proves a Schwarz-type inequality (1), which generalizes the inequality of Halmos:

$$|\langle Tx, y\rangle|^2 \leq \langle |T|^{2\theta}x, x\rangle\langle |T^*|^{2(1-\theta)}y, y\rangle \tag{2}$$

for all operators $T \in \mathbb{B}(\mathcal{H})$, for every vector $x, y \in \mathcal{H}$, and $\theta \in [0, 1]$. McCarthy [8] gives an important inequality in the theory of operators as follows:

Lemma 1 (Theorem 1.4 in [8])**.** *Let $T \in \mathbb{B}(\mathcal{H})$ be a positive operator and $x \in \mathcal{H}$ satisfy $\|x\| = 1$. Then, for $r \geq 1$,*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle.$$

For $0 \leq r \leq 1$, the above inequality is reversed.

In what follows, we assume that A is a positive nonzero operator that defines the following positive semi-definite sesquilinear form:

$$\langle\cdot,\cdot\rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$
$$(x, y) \mapsto \langle x, y\rangle_A := \langle Ax, y\rangle = \langle A^{1/2}x, A^{1/2}y\rangle.$$

The seminorm induced by $\langle\cdot,\cdot\rangle_A$ is given by $\|x\|_A = \sqrt{\langle x, x\rangle_A}$, $\forall\, x \in \mathcal{H}$. It can be seen that $\|\cdot\|_A$ is a norm on $\mathcal{H}$ if and only if A is injective, and the semi-Hilbert space $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$.

Definition 1 ([9])**.** *An operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint of $T \in \mathbb{B}(\mathcal{H})$, if we have $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ ($AS = T^*A$) for every $x, y \in \mathcal{H}$.*

The existence of an A-adjoint operator is not guaranteed. Thus, we denote by $\mathbb{B}_A(\mathcal{H})$ the set of all operators that admit A-adjoints. Using Douglas' theorem [10], we obtain the following:

$$\mathbb{B}_A(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H})\,;\ \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}$$

and

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H})\,;\ \exists\, c > 0;\ \|Tx\|_A \leq c\|x\|_A,\ \forall\, x \in \mathcal{H}\}.$$

When $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we say that T is A-bounded. The sets $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are subalgebras of $\mathbb{B}(\mathcal{H})$, which are neither closed nor dense in $\mathbb{B}(\mathcal{H})$. Moreover, the inclusions

$$\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$$

hold. We have equality if A is injective and $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$, where $\overline{\mathcal{R}(A)}$ means the closure of $\mathcal{R}(A)$ in the norm topology of $\mathcal{H}$ (see [11]). Further, $\langle\cdot,\cdot\rangle_A$ gives the following seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$

$$\|T\|_A := \sup_{\substack{x \in \overline{\mathcal{R}(A)},\\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x \in \mathcal{H},\\ \|x\|_A = 1}} \|Tx\|_A = \sup_{\substack{x, y \in \mathcal{H},\\ \|x\|_A = \|y\|_A = 1}} |\langle Tx, y\rangle_A| < \infty \tag{3}$$

(see [12] and the references therein). It is useful to note that if $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$. Further, $\|TS\|_A \leq \|T\|_A\|S\|_A$ for any $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Let $X^\dagger$ denote the Moore–Penrose pseudo-inverse of an operator X (for more details concerning this operator, see [11]). Following [11], we have: $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ implies that $A^{1/2}T(A^{1/2})^{\dagger} \in \mathbb{B}(\mathcal{H})$ and

$$\|T\|_A = \left\|A^{1/2}T(A^{1/2})^{\dagger}\right\|. \tag{4}$$

In 2012, Saddi [13] introduced the A-numerical radius of an operator $T \in \mathbb{B}(\mathcal{H})$ by

$$\omega_A(T) := \sup\{|\langle Tx, x\rangle_A| \,;\, x \in \mathcal{H}, \|x\|_A = 1\}.$$

In 2020, the concept of the A-spectral radius of A-bounded operators was defined in [14] as follows:

$$r_A(T) := \inf_{n\geq 1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|_A^{\frac{1}{n}}. \tag{5}$$

Note that $\|T\|_A$ and $\omega_A(T)$ may equal $+\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [14]). However, the following relation shows that $\|\cdot\|_A$ and $\omega_A(\cdot)$ are equivalent seminorms on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$:

$$r_A(T) \leq \max\left\{\frac{1}{2}\|T\|_A, r_A(T)\right\} \leq \omega_A(T) \leq \|T\|_A. \tag{6}$$

For the proof of (6), we refer to the following references [12,14]. If $A = I$, then the classical definitions of the operator norm, numerical radius, and spectral radius for Hilbert space operators are obtained and are simply denoted by $\|\cdot\|$, $\omega(\cdot)$ and $r(\cdot)$.

If $T \in \mathbb{B}_A(\mathcal{H})$, then by Douglas's theorem [10] there exists a unique solution, given by $T^{\sharp_A}$, of the following problem

$$AX = T^*A, \;\; \mathcal{R}(X) \subseteq \overline{\mathcal{R}(A)}.$$

Note that $T^{\sharp_A} = A^{\dagger}T^*A$, where $A^{\dagger}$ is the Moore–Penrose pseudo-inverse of A (see [11]). If $T, S \in \mathbb{B}_A(\mathcal{H})$, then $TS, \alpha T + \beta S \in \mathbb{B}_A(\mathcal{H})$ for every $\alpha, \beta \in \mathbb{R}$ and we have $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$ and $(\alpha T + \beta S)^{\sharp_A} = \overline{\alpha}T^{\sharp_A} + \overline{\beta}S^{\sharp_A}$. Moreover, if $P_{\overline{\mathcal{R}(A)}}$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$, then for a given $T \in \mathbb{B}_A(\mathcal{H})$, we have $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ and $\left((T^{\sharp_A})^{\sharp_A}\right)^{\sharp_A} = T^{\sharp_A}$. For more details about the operator $T^{\sharp_A}$, one can see [9,11,15]. Furthermore, we recall that an operator T is said to be A-positive if AT is a positive operator and we write $T \geq_A 0$. It can be observed that A-positive operators are in $\mathbb{B}_A(\mathcal{H})$. For $T, S \in \mathbb{B}(\mathcal{H})$, the notation $T \geq_A S$ means $T - S \geq_A 0$. When $A = I$, then $T \geq_I S$ will simply be denoted by $T \geq S$.

The structure of this paper is organized as follows: in Section 2, we give some notions that characterize a d-tuple of operators $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$. In Section 3, we introduce a new joint norm of tuples of operators that generalizes the joint norm given in (12) and define the class of doubly-commuting tuples of hyponormal operators acting on an A-weighted Hilbert space, where A is a positive operator that is not assumed to be invertible. We proved a generalization of the well-known result due to G. Popescu [16]. We also present an inequality that characterizes the Euclidean norm of an operator tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$. In Section 4, we give several characterizations related to the operators from $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and the operators from $\mathbb{B}(\mathbf{R}(A^{1/2}))$. For an A-doubly-commuting d-tuple of hyponormal operators, we prove the equalities $\|\mathbf{T}\|_{e,A} = \|\mathbf{T}\|_A$ and $r_A(\mathbf{T}) = \|\mathbf{T}\|_A = \omega_A(\mathbf{T})$. The motivation for our investigation comes from a recent paper [17].

2. Preliminaries

To prepare the framework in which we will work, we present in this section some notions and notations that will be useful in this paper.

Let $\mathbb{N}$ and $\mathbb{N}^*$ denote the set of nonnegative and positive integers, respectively. Let $d \in \mathbb{N}^*$ and $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$ be a d-tuple of operators. If $[T_i, T_j] = 0$ for all $i, j \in \{1, \ldots, d\}$, then $\mathbf{T}$ is said to be a commuting tuple. Moreover, if $\mathbf{T}$ is a commuting d-tuple of operators and $[T_i^*, T_j] = 0$ for every $1 \leq i \neq j \leq d$, then it is called a doubly-commuting operator tuple.

In the next definition, we recall two important classes of operators in semi-Hilbert spaces.

Definition 2 ([14])**.** *An operator $T \in \mathbb{B}_A(\mathcal{H})$ is called*

(i) *A-normal if $[T^{\sharp_A}, T] = 0$;*

(ii) *A-hyponormal if $[T^{\sharp_A}, T] \geq_A 0$.*

For some results concerning the above two classes of operators, see [14] and the references therein. For $T \in \mathbb{B}_A(\mathcal{H})$, the equalities

$$r_A(T) = \omega_A(T) = \|T\|_A \tag{7}$$

hold for the class of A-normal, A-hyponormal, and A-positive operators (see [14]). Since $T^{\sharp_A}T \geq_A 0$ and $TT^{\sharp_A} \geq_A 0$, then an application of the second equality in (7) together with the last equality in (3) shows that

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2. \tag{8}$$

Now, associated with a d-tuple of operators, $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$ (not necessarily commuting), the following quantities

$$\|\mathbf{T}\|_A := \sup\left\{ \sqrt{\sum_{j=1}^{d} \|T_j x\|_A^2}\,;\ x \in \mathcal{H},\ \|x\|_A = 1 \right\},$$

and

$$\omega_A(\mathbf{T}) := \sup\left\{ \sqrt{\sum_{j=1}^{d} |\langle T_j x, x\rangle_A|^2}\,;\ x \in \mathcal{H},\ \|x\|_A = 1 \right\}$$

are defined in [12]. If $T_j \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ for all $j \in \{1, \ldots, d\}$, then one can verify that $\|\cdot\|_A$ and $\omega_A(\cdot)$ two seminorms on $\mathbb{B}_{A^{1/2}}(\mathcal{H})^d$. Notice that $\omega_A(\mathbf{T})$ and $\|\mathbf{T}\|_A$ are called the A-joint numerical radius and the A-joint operator seminorm of $\mathbf{T}$, respectively.

In [18], H. Baklouti et al. introduced the concept of the A-joint spectral radius associated with a d-tuple of commuting operators $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ as follows

$$r_A(\mathbf{T}) := \inf_{n\in\mathbb{N}^*} \left\| \sum_{\substack{|\alpha|=n,\\ \alpha\in\mathbb{N}^d}} \frac{n!}{\alpha!} \left(\mathbf{T}^{\sharp_A}\right)^{\alpha} \mathbf{T}^{\alpha} \right\|_A^{\frac{1}{2n}} = \lim_{n\to\infty} \left\| \sum_{\substack{|\alpha|=n,\\ \alpha\in\mathbb{N}^d}} \frac{n!}{\alpha!} \left(\mathbf{T}^{\sharp_A}\right)^{\alpha} \mathbf{T}^{\alpha} \right\|_A^{\frac{1}{2n}}, \tag{9}$$

where $\mathbf{T}^{\sharp_A} = (T_1^{\sharp_A}, \ldots, T_d^{\sharp_A})$. Moreover, for the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we will use the following notations:

$$\mathbf{T}^{\alpha} := \prod_{k=1}^{d} T_k^{\alpha_k},\ |\alpha| := \sum_{j=1}^{d} |\alpha_j| \quad \text{and} \quad \alpha! := \prod_{k=1}^{d} \alpha_k!.$$

We mention here that the second equality in (9) has also been proved by Baklouti et al. in [18]. Notice that for every commuting operator tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$, we have

$$r_A(\mathbf{T}) \leq \max\left\{ \frac{1}{2\sqrt{d}} \|\mathbf{T}\|_A, r_A(\mathbf{T}) \right\} \leq \omega_A(\mathbf{T}) \leq \|\mathbf{T}\|_A \tag{10}$$

(see Theorem 2.4 in [12] and Theorem 2.2 in [19]). In [19], it is stated that if $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ is any d-tuple of commuting A-normal operators, then

$$r_A(\mathbf{T}) = \omega_A(\mathbf{T}) = \|\mathbf{T}\|_A. \tag{11}$$

One of the main targets of this work is to establish the equalities in (11) for a new class of multivariable operators.

Next, for $A = I$, we define $r_I(\mathbf{T})$, $\omega_I(\mathbf{T})$, and $\|\mathbf{T}\|_I$ which will simply be denoted by $r(\mathbf{T})$, $\omega(\mathbf{T})$ and $\|\mathbf{T}\|$, respectively. Thus, we obtain

$$\|\mathbf{T}\| := \sup\left\{\sqrt{\sum_{j=1}^{d}\|T_jx\|^2}\,;\ x\in\mathcal{H},\ \|x\|=1\right\},$$

and

$$\omega(\mathbf{T}) := \sup\left\{\sqrt{\sum_{j=1}^{d}|\langle T_jx,x\rangle|^2}\,;\ x\in\mathcal{H},\ \|x\|=1\right\}.$$

The last equality is given in [20] by M. Chō and M. Takaguchi and is the Euclidean operator radius of an operator tuple $\mathbf{T} = (T_1,\ldots,T_d) \in \mathbb{B}(\mathcal{H})^d$, see also [16].

For $\mathbf{T} = (T_1,\ldots,T_d) \in \mathbb{B}(\mathcal{H})^d$, G. Popescu defined in [16] the following quantity

$$\|\mathbf{T}\|_e := \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d}\|\lambda_1T_1+\ldots+\lambda_dT_d\|, \tag{12}$$

where $\mathbb{B}_d$ denotes the open unit ball of $\mathbb{C}^d$ with respect to the Euclidean norm, i.e.,

$$\mathbb{B}_d := \left\{\lambda = (\lambda_1,\ldots,\lambda_d)\in\mathbb{C}^d\,;\ \|\lambda\|_2^2 := \sum_{j=1}^{d}|\lambda_j|^2 < 1\right\}.$$

It is clear that we can change $\mathbb{B}_d$ with its closure in (12) without changing the value of $\|\mathbf{T}\|_e$. Note that $\|\cdot\|_e$ defines a norm on $\mathbb{B}(\mathcal{H})^d$. Moreover, in [17], the following equality is established:

$$\|\mathbf{T}\| = \|\mathbf{T}\|_e \tag{13}$$

for every doubly-commuting d-tuple of hyponormal operators $\mathbf{T}$. It is important to mention that G. Popescu proved in [16] that the following inequalities hold

$$\frac{1}{\sqrt{d}}\sqrt{\left\|\sum_{j=1}^{d}T_jT_j^*\right\|} \le \|\mathbf{T}\|_e \le \sqrt{\left\|\sum_{j=1}^{d}T_jT_j^*\right\|} \tag{14}$$

for any d-tuple $\mathbf{T} = (T_1,\ldots,T_d) \in \mathbb{B}(\mathcal{H})^d$. Furthermore, it has been shown in [16] that the constants $\frac{1}{\sqrt{d}}$ and 1 are the best choices possible.

3. New Joint Seminorm for Operator Tuples

In this section, we aim to introduce and investigate a new joint seminorm for d-tuples of A-bounded operators. An alternative and easy proof of a well-known result due to G. Popescu [16] is established.

First, we introduce the following definition, which is a natural generalization of (12).

Definition 3. *Let $\mathbf{T} = (T_1,\ldots,T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$. The A-Euclidean seminorm of the d-tuple of A-bounded operators $\mathbf{T}$ is given by*

$$\|\mathbf{T}\|_{e,A} := \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d}\|\lambda_1T_1+\ldots+\lambda_dT_d\|_A.$$

In the next proposition, we state some connections between the seminorms $\|\cdot\|_{e,A}$ and $\|\cdot\|_A$.

Proposition 1. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ *be the d-tuple of the operators. Then, the following assertions hold:*

(1) $\|\mathbf{T}\|_{e,A} \leq \|\mathbf{T}\|_A$;

(2) *If* $T_k \in \mathbb{B}_A(\mathcal{H})$ *for all* $k \in \{1, \ldots, d\}$, *then* $\|\mathbf{T}^{\sharp_A}\|_{e,A} = \|\mathbf{T}\|_{e,A}$ *and*

$$\frac{1}{\sqrt{d}} \max\left\{\|\mathbf{T}\|_A, \|\mathbf{T}^{\sharp_A}\|_A\right\} \leq \|\mathbf{T}\|_{e,A}, \tag{15}$$

where $\mathbf{T}^{\sharp_A} = (T_1^{\sharp_A}, \ldots, T_d^{\sharp_A})$.

Proof. (1) Let $x \in \mathcal{H}$ and $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{B}_d$. Then, by applying the Cauchy–Schwarz inequality (in short (C–S)) and making several calculations, we deduce that

$$\begin{aligned}
\left\|\sum_{j=1}^d \lambda_j T_j x\right\|_A^2 &= \langle \left(\sum_{j=1}^d \lambda_j T_j\right)x, \left(\sum_{k=1}^d \lambda_k T_k\right)x\rangle_A \\
&= \sum_{j=1}^d \sum_{k=1}^d \lambda_j \overline{\lambda_k} \langle T_j x, T_k x\rangle_A \\
&\leq \sum_{j=1}^d \sum_{k=1}^d |\lambda_j| \times |\lambda_k| \times \|T_j x\|_A \|T_k x\|_A \\
&= \left(\sum_{j=1}^d |\lambda_j| \times \|T_j x\|_A\right)^2.
\end{aligned}$$

By applying the inequality (C–S) again, we obtain the following inequality

$$\left\|\sum_{k=1}^d \lambda_k T_k x\right\|_A^2 \leq \|\lambda\|_2^2 \left(\sum_{j=1}^d \|T_j x\|_A^2\right).$$

Then, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$, we find

$$\left\|\sum_{k=1}^d \lambda_k T_k\right\|_A \leq \|\lambda\|_2 \|\mathbf{T}\|_A.$$

So, the desired inequality is proved by taking the supremum over all $\lambda \in \mathbb{B}_d$.

(2) The fact that $\|\mathbf{T}^{\sharp_A}\|_{e,A} = \|\mathbf{T}\|_{e,A}$ follows trivially since $\|X^{\sharp_A}\|_A = \|X\|_A$ for all $X \in \mathbb{B}_A(\mathcal{H})$. Now, in order to prove (15), we need to recall from [16] the following facts: if we denote by $\mathbb{S}_d$ the unit sphere of $\mathbb{C}^d$ and σ the rotation-invariant positive Borel measure on $\mathbb{S}_d$ for which $\sigma(\mathbb{S}_d) = 1$, then for all $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{C}^d$, we have

$$\int_{\mathbb{S}_d} |\mu_k|^2 d\sigma(\mu) = \frac{1}{d},\ \forall\, k \in \{1, \ldots, d\} \ \text{ and } \ \int_{\mathbb{S}_d} \mu_i \overline{\mu_j}\, d\sigma(\mu) = 0,\ \ \forall\, 1 \leq i \neq j \leq d. \tag{16}$$

Now, let $\overline{\mathbb{B}}_d$ denote the closed unit ball of $\mathbb{C}^d$. It is clear that

$$\begin{aligned}
\|\mathbf{T}\|_{e,A}^2 &= \sup_{(\lambda_1,\ldots,\lambda_d) \in \mathbb{B}_d} \left\|\sum_{j=1}^d \lambda_j T_j\right\|_A^2 \\
&= \sup_{(\lambda_1,\ldots,\lambda_d) \in \overline{\mathbb{B}}_d} \left\|\sum_{j=1}^d \lambda_j T_j\right\|_A^2.
\end{aligned}$$

Further, by using (8), we see that

$$\|\mathbf{T}\|_{e,A}^2 = \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \left\| \left(\sum_{j=1}^d \lambda_j T_j\right)^{\sharp_A} \left(\sum_{j=1}^d \lambda_j T_j\right) \right\|_A$$
$$= \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \sup_{\|x\|_A=1} \langle \left(\sum_{j=1}^d \overline{\lambda_j} T_j^{\sharp_A}\right)\left(\sum_{j=1}^d \lambda_j T_j\right) x, x\rangle_A$$
$$= \sup_{\|x\|_A=1} \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \sum_{i,j=1}^d \overline{\lambda_i}\lambda_j \langle T_i^{\sharp_A} T_j x, x\rangle_A.$$

On the other hand, since $\sigma(\mathbb{S}_d) = 1$, then it follows that

$$\sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \sum_{i,j=1}^d \overline{\lambda_i}\lambda_j \langle T_i^{\sharp_A} T_j x, x\rangle_A = \int_{\mathbb{S}_d} \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \sum_{i,j=1}^d \overline{\lambda_i}\lambda_j \langle T_i^{\sharp_A} T_j x, x\rangle_A \, d\sigma(\mu)$$
$$\geq \int_{\mathbb{S}_d} \sum_{i,j=1}^d \overline{\mu_i}\mu_j \langle T_i^{\sharp_A} T_j x, x\rangle_A \, d\sigma(\mu),$$

for all $x \in \mathcal{H}$. This implies that, through (16),

$$\|\mathbf{T}\|_{e,A}^2 \geq \sup_{\|x\|_A=1} \int_{\mathbb{S}_d} \sum_{i,j=1}^d \mu_i\overline{\mu_j} \langle T_i^{\sharp_A} T_j x, x\rangle_A \, d\sigma(\mu)$$
$$= \frac{1}{d} \sup_{\|x\|_A=1} \sum_{i=1}^d \langle T_i^{\sharp_A} T_i x, x\rangle_A = \frac{1}{d}\|\mathbf{T}\|_A^2.$$

This proves that

$$\frac{1}{d}\|\mathbf{T}\|_A \leq \|\mathbf{T}\|_{e,A}. \tag{17}$$

By replacing T_k by $T_k^{\sharp_A}$ in (17) and then using the fact that $\|\mathbf{T}^{\sharp_A}\|_{e,A} = \|\mathbf{T}\|_{e,A}$, we have

$$\frac{1}{d}\|\mathbf{T}^{\sharp_A}\|_A \leq \|\mathbf{T}\|_{e,A}. \tag{18}$$

Combining (17) together with (18) yields (15) as desired. □

Remark 1. *(1) If* $\mathbf{T} = (T_1,\ldots,T_d) \in \mathbb{B}_A(\mathcal{H})^d$*, then clearly* $\sum_{k=1}^d T_k^{\sharp_A} T_k \geq_A 0$*. Hence, a direct application of* (7) *shows that*

$$\|\mathbf{T}\|_A = \sqrt{\left\|\sum_{j=1}^d T_j^{\sharp_A} T_j\right\|_A}. \tag{19}$$

It should be mentioned here that the equality $\|\mathbf{T}^{\sharp_A}\|_A = \|\mathbf{T}\|_A$ *may not be correct even if* $\mathbf{T}$ *is a commuting operator tuple. Indeed, let us consider the following matrices in* $\mathcal{M}_3$*:* $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$*,* $T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*. We remark that* $[T_1, T_2] = 0$*. Furthermore, by using* *the fact that* $T_k^{\sharp_A} = A^{\dagger}T_k^*A$ *with* $k \in \{1,2\}$*, it can be seen that* $T_1^{\sharp_A} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $T_2^{\sharp_A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$*. Further, by applying (19) and (4), we can show that*

$$\|(T_1, T_2)\|_A = \sqrt{\left\|T_1^{\sharp_A}T_1 + T_2^{\sharp_A}T_2\right\|_A} = 1$$

and

$$\|(T_1^{\sharp_A}, T_2^{\sharp_A})\|_A = \sqrt{\left\|(T_1^{\sharp_A})^{\sharp_A}T_1^{\sharp_A} + (T_2^{\sharp_A})^{\sharp_A}T_2^{\sharp_A}\right\|_A} = \frac{\sqrt{5}}{2}.$$

(2) In virtue of proposition 1, we infer that $\|\cdot\|_A$ *and* $\|\cdot\|_{e,A}$ *are equivalent seminorms on* $\mathbb{B}_A(\mathcal{H})^d$*.*

The following corollary provides a generalization and improvement of the well-known result due to G. Popescu [16].

Corollary 1. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ *be a d-tuple of operators. Then, the inequality*

$$\frac{1}{\sqrt{d}}\max\{\alpha, \beta\} \leq \|\mathbf{T}\|_{e,A} \leq \min\{\alpha, \beta\} \tag{20}$$

holds, where $\alpha = \sqrt{\left\|\sum_{k=1}^{d} T_kT_k^{\sharp_A}\right\|_A}$ *and* $\beta = \sqrt{\left\|\sum_{k=1}^{d} T_k^{\sharp_A}T_k\right\|_A}$.

Proof. By applying Proposition 1 together with (19), we deduce that

$$\frac{1}{\sqrt{d}}\sqrt{\left\|\sum_{k=1}^{d} T_k^{\sharp_A}T_k\right\|_A} \leq \|\mathbf{T}\|_{e,A} \leq \sqrt{\left\|\sum_{k=1}^{d} T_k^{\sharp_A}T_k\right\|_A}. \tag{21}$$

By replacing T_k by $T_k^{\sharp_A}$ in (21), we can see that

$$\frac{1}{\sqrt{d}}\sqrt{\left\|\left(\sum_{k=1}^{d} T_kT_k^{\sharp_A}\right)^{\sharp_A}\right\|_A} \leq \|\mathbf{T}^{\sharp_A}\|_{e,A} \leq \sqrt{\left\|\left(\sum_{k=1}^{d} T_kT_k^{\sharp_A}\right)^{\sharp_A}\right\|_A},$$

from which we have

$$\frac{1}{\sqrt{d}}\sqrt{\left\|\sum_{k=1}^{d} T_kT_k^{\sharp_A}\right\|_A} \leq \|\mathbf{T}\|_{e,A} \leq \sqrt{\left\|\sum_{k=1}^{d} T_kT_k^{\sharp_A}\right\|_A}. \tag{22}$$

A combination of (21) together with (22) yields (20) as desired. □

Remark 2. *Note that the following equality*

$$\left\|\sum_{j=1}^{d} T_j^{\sharp_A}T_j\right\|_A = \left\|\sum_{j=1}^{d} T_jT_j^{\sharp_A}\right\|_A$$

may not be correct for some d-tuple of operators $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ *even if* $A = I$. *Indeed, we consider the following matrices in* $\mathcal{M}_2$: $A = I_2$, $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *It is not difficult to check that*

$$\left\| \sum_{j=1}^{2} T_j T_j^* \right\| = 1 \neq 2 = \left\| \sum_{j=1}^{2} T_j^* T_j \right\|.$$

In the next theorem, we give a new formula of $\|\mathbf{T}\|_{A,e}$ for $\mathbf{T} \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$, which allows us to prove that $\|\cdot\|_{A,e}$ and $\|\cdot\|_A$ are two equivalent seminorms on $\mathbb{B}_{A^{1/2}}(\mathcal{H})^d$. Notice that our new techniques provide an alternative and easy proof of the inequalities (14), which were first proved in [16].

Theorem 1. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$. *Then, the equality*

$$\|\mathbf{T}\|_{A,e} = \sup\left\{ \sqrt{\sum_{j=1}^{d} |\langle T_j x, y\rangle_A|^2};\ x, y \in \mathcal{H},\ \|x\|_A = \|y\|_A = 1 \right\} \tag{23}$$

holds.

Proof. By using (3), we see that

$$\begin{aligned} \|\mathbf{T}\|_{e,A} &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \|\lambda_1 T_1 + \ldots + \lambda_d T_d\|_A \\ &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \sup\left\{ \left\| \sum_{k=1}^{d} \lambda_k T_k x \right\|_A;\ x \in \mathcal{H},\ \|x\|_A = 1 \right\} \\ &= \sup\left\{ \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \left\| \sum_{k=1}^{d} \lambda_k T_k x \right\|_A;\ x \in \mathcal{H},\ \|x\|_A = 1 \right\}. \end{aligned} \tag{24}$$

Moreover, recall from [19] that for complex numbers $z_1, \ldots, z_d$, we have

$$\sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \left| \sum_{k=1}^{d} \lambda_k z_k \right| = \sqrt{\sum_{k=1}^{d} |z_k|^2}. \tag{25}$$

Now, let $x, y \in \mathcal{H}$. By using (25), we have

$$\begin{aligned} \sqrt{\sum_{j=1}^{d} |\langle T_j x, y\rangle_A|^2} &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \left| \sum_{j=1}^{d} \lambda_j \langle T_j x, y\rangle_A \right| \\ &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \left| \langle \left(\sum_{j=1}^{d} \lambda_j T_j x \right), y\rangle_A \right| \end{aligned}$$

Hence, by taking the supremum over all $y \in \mathcal{H}$ with $\|y\|_A = 1$ in the last equality we have

$$\sup_{\substack{y\in\mathcal{H},\\ \|y\|_A=1}} \sqrt{\sum_{k=1}^{d} |\langle T_k x, y\rangle_A|^2} = \sup_{\substack{y\in\mathcal{H},\\ \|y\|_A=1}} \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \left| \langle \left(\sum_{k=1}^{d} \lambda_k T_k x \right), y\rangle_A \right|.$$

This yields that

$$\sup_{\substack{y\in\mathcal{H},\\ \|y\|_A=1}} \sqrt{\sum_{k=1}^{d} |\langle T_k x, y\rangle_A|^2} = \sup_{(\lambda_1,\dots,\lambda_d)\in\mathbb{B}_d} \left[\sup_{\substack{y\in\mathcal{H},\\ \|y\|_A=1}} \left| \left\langle \left(\sum_{k=1}^{d} \lambda_k T_k x \right), y \right\rangle_A \right| \right]. \tag{26}$$

On the other hand, it is not difficult to check that

$$\sup\{|\langle u, v\rangle_A|\,;\; v\in\mathcal{H},\; \|v\|_A = 1\} = \|u\|_A, \quad \forall u\in\mathcal{H}. \tag{27}$$

Thus, by using (26) and (27), we obtain

$$\sup_{\substack{y\in\mathcal{H},\\ \|y\|_A=1}} \sqrt{\sum_{k=1}^{d} |\langle T_k x, y\rangle_A|^2} = \sup_{(\lambda_1,\dots,\lambda_d)\in\mathbb{B}_d} \left\| \sum_{k=1}^{d} \lambda_k T_k x \right\|_A. \tag{28}$$

Combining (28) together with (24) yields (23) as required, and, hence, the proof is complete. □

Remark 3. *By letting $A = I$ in (23), we obtain a well-known result established by Dragomir in Theorem 9 in [21], and when the 2-tuple is $(T, T^{\sharp_A})$, where $T \in \mathbb{B}_A(\mathcal{H})$, we obtain a recent result in [22].*

The following corollary is an application of Theorem 1 and provides an improvement of the results given in Proposition 1 since $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Moreover, the new Formula (23) enables us to derive an alternative and easy proof of the inequalities (14).

Corollary 2. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ be a d-tuple of operators. Then,*

$$\frac{1}{\sqrt{d}}\|\mathbf{T}\|_A \le \|\mathbf{T}\|_{e,A} \le \|\mathbf{T}\|_A. \tag{29}$$

Proof. By using (23) and then applying the inequality (C–S), we easily prove the second inequality in (29). Now, let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Assume that $T_k x \notin \mathcal{N}(A)$ for all $k \in \{1, \dots, d\}$ and let

$$y_k = \frac{T_k x}{\|T_k x\|_A}, \quad \forall\, k \in \{1, \dots, d\}.$$

(If $T_{k_0} x \in \mathcal{N}(A)$ for some $k_0 \in \{1, \dots, d\}$, we choose $y_{k_0} = x$). We clearly have

$$\|y_k\|_A = 1 \quad \text{and} \quad |\langle T_k x, y_k\rangle_A|^2 = \|T_k x\|_A^2, \quad \forall\, k \in \{1, \dots, d\}.$$

Thus, by applying (23), we have

$$\|\mathbf{T}\|_{A,e}^2 \ge \sum_{k=1}^{d} |\langle T_k x, y_k\rangle_A|^2 \ge |\langle T_1 x, y_1\rangle_A|^2 = \|T_1 x\|_A^2.$$

Similarly, we prove that $\|\mathbf{T}\|_{A,e}^2 \ge \|T_i x\|_A^2$ for all $i \in \{1, \dots, d\}$. This yields

$$d\|\mathbf{T}\|_{A,e}^2 \ge \sum_{k=1}^{d} \|T_k x\|_A^2.$$

Therefore, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the last inequality, we have

$$\|\mathbf{T}\|_{A,e} \ge \frac{1}{\sqrt{d}}\|\mathbf{T}\|_A.$$

Hence, the proof is complete. □

Remark 4. *By letting $A = I$ in (29) and then replacing T_k with T_k^* for all $k \in \{1, \ldots, d\}$ we easily obtain the inequalities (14) that have been already established by G. Popescu in [16] by using a different argument.*

To establish our next result, we require the following lemma.

Lemma 2. *For any vectors $x_1, x_2, \ldots, x_d$ in $\mathcal{H}$ and for arbitrary complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_d$, with $\lambda_i \neq 0$, $i = \overline{1,d}$, we have*

$$\sum_{i=1}^{d} \|x_i\|_A^2 \geq \frac{\left\| \sum_{i=1}^{d} \lambda_i x_i \right\|_A^2}{\sum_{i=1}^{d} |\lambda_i|^2} + \max_{i,j \in \{1,\ldots,d\}} \frac{\|\overline{\lambda_i} x_j - \overline{\lambda_j} x_i\|_A^2}{|\lambda_i|^2 + |\lambda_j|^2}$$

for any $d \geq 2$.

Proof. We use, as in [23] or [24], the technique of the monotony of a sequence. Consider the sequence

$$S_d = \sum_{i=1}^{d} \|x_i\|_A^2 - \frac{\left\| \sum_{i=1}^{d} \lambda_i x_i \right\|_A^2}{\sum_{i=1}^{d} |\lambda_i|^2}, d \geq 1.$$

By studying the monotony of sequence $S_k, k \leq d$, we have

$$S_{k+1} - S_k = \|x_{k+1}\|_A^2 + \frac{\left\| \sum_{i=1}^{k} \lambda_i x_i \right\|_A^2}{\sum_{i=1}^{k} |\lambda_i|^2} - \frac{\left\| \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1} \right\|_A^2}{\sum_{i=1}^{k} |\lambda_i|^2 + |\lambda_{k+1}|^2}.$$

For two vectors $x, y \in \mathcal{H}$ and for complex numbers $\lambda, \mu \neq 0$, the following equality holds:

$$\frac{\|x\|_A^2}{|\lambda|^2} + \frac{\|y\|_A^2}{|\mu|^2} - \frac{\|x+y\|_A^2}{|\lambda|^2 + |\mu|^2} = \frac{\| |\mu|^2 x - |\lambda|^2 y \|_A^2}{|\lambda|^2 |\mu|^2 (|\lambda|^2 + |\mu|^2)}. \tag{30}$$

Since the term on the right side of equality (30) is positive, then we have

$$\frac{\|x\|_A^2}{|\lambda|^2} + \frac{\|y\|_A^2}{|\mu|^2} \geq \frac{\|x+y\|_A^2}{|\lambda|^2 + |\mu|^2}. \tag{31}$$

Now, using the inequality from (31), we have:

$$\|x_{k+1}\|_A^2 + \frac{\left\| \sum_{i=1}^{k} \lambda_i x_i \right\|_A^2}{\sum_{i=1}^{k} |\lambda_i|^2} = \frac{\left\| \lambda_{k+1} x_{k+1} \right\|_A^2}{|\lambda_{k+1}|^2} + \frac{\left\| \sum_{i=1}^{k} \lambda_i x_i \right\|_A^2}{\sum_{i=1}^{k} |\lambda_i|^2} \geq \frac{\left\| \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1} \right\|_A^2}{\sum_{i=1}^{k} |\lambda_i|^2 + |\lambda_{k+1}|^2}.$$

It is easy to see that $S_{k+1} - S_k \geq 0$, that is, the sequence S_k is increasing. Therefore, we deduce that

$$S_d \geq S_{d-1} \geq \ldots \geq S_2 \geq S_1 = 0.$$

However, by applying relation (30) for $x = \lambda_1 x_1$, $y = \lambda_2 x_2$, $\lambda = \lambda_1$ and $\mu = \lambda_2$, we obtain

$$\begin{aligned} S_2 &= \|x_1\|_A^2 + \|x_2\|_A^2 - \frac{\|\lambda_1 x_1 + \lambda_2 x_2\|_A^2}{|\lambda_1|^2 + |\lambda_2|^2} \\ &= \frac{\|\overline{\lambda_1} x_2 - \overline{\lambda_2} x_1\|_A^2}{|\lambda_1|^2 + |\lambda_2|^2}. \end{aligned}$$

Taking into account that we can rearrange the terms of the two sequences, we obtain the inequality:

$$S_d = \sum_{i=1}^{d} \|x_i\|_A^2 - \frac{\left\| \sum_{i=1}^{d} \lambda_i x_i \right\|_A^2}{\sum_{i=1}^{d} |\lambda_i|^2} \geq S_2 = \frac{\|\overline{\lambda_1} x_2 - \overline{\lambda_2} x_1\|_A^2}{|\lambda_1|^2 + |\lambda_2|^2}.$$

Consequently, we deduce the inequality of the statement. □

We are now able to establish the following result.

Theorem 2. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ be a d-tuple of operators. Then, the inequality*

$$\|\mathbf{T}\|_A \geq \|\mathbf{T}\|_{e,A} + \max_{i,j \in \{1,\ldots,d\}} \sup_{(\lambda_1,\ldots,\lambda_d) \in \mathbb{B}_d} \inf_{\|x\|_A = 1} \|(\overline{\lambda_i} T_j - \overline{\lambda_j} T_i)x\|_A^2 \tag{32}$$

holds for any $d \geq 2$.

Proof. In Lemma 2, set $x_i = T_i x$ for all $i \in \{1, \ldots, d\}$, then

$$\sum_{i=1}^{d} \|T_i x\|_A^2 \geq \frac{\left\| \sum_{i=1}^{d} \lambda_i T_i x \right\|_A^2}{\sum_{i=1}^{d} |\lambda_i|^2} + \max_{i,j \in \{1,\ldots,d\}} \frac{\|\overline{\lambda_i} T_j x - \overline{\lambda_j} T_i x\|_A^2}{|\lambda_i|^2 + |\lambda_j|^2}. \tag{33}$$

First, we take the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in relation (33), we deduce

$$\|\mathbf{T}\|_A \geq \frac{\left\| \sum_{i=1}^{d} \lambda_i T_i \right\|_A^2}{\sum_{i=1}^{d} |\lambda_i|^2} + \max_{i,j \in \{1,\ldots,d\}} \inf_{\|x\|_A = 1} \frac{\|(\overline{\lambda_i} T_j - \overline{\lambda_j} T_i)x\|_A^2}{|\lambda_i|^2 + |\lambda_j|^2}. \tag{34}$$

Therefore, it we take the supremum over all $(\lambda_1, \ldots, \lambda_d) \in \mathbb{B}_d$ in relation (34), then we find the inequality of the statement. □

Remark 5. *By letting $A = I$ in (32), we obtain*

$$\|\mathbf{T}\| \geq \|\mathbf{T}\|_e + \max_{i,j \in \{1,\ldots,d\}} \sup_{(\lambda_1,\ldots,\lambda_d) \in \mathbb{B}_d} \inf_{\|x\| = 1} \|(\overline{\lambda_i} T_j - \overline{\lambda_j} T_i)x\|^2$$

for any d-tuple of operators $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$ and $d \geq 2$.

Next, we will present a result that characterizes the Euclidean norm of an operator tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$.

Proposition 2. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$ be a d-tuple of operators. The following inequality holds:*

$$\|\mathbf{T}\|_e \leq \|\mathbf{T}\|^{\theta} \|\mathbf{T}^*\|^{1-\theta}, \tag{35}$$

where $\theta \in [0,1]$. *Here* $\mathbf{T}^* = (T_1^*, \ldots, T_d^*)$.

Proof. First, we will prove a radon-type inequality,

$$\sum_{k=1}^{d} a_k^{\theta} b_k^{1-\theta} \leq \left(\sum_{k=1}^{d} a_k\right)^{\theta} \left(\sum_{k=1}^{d} b_k\right)^{1-\theta}, \tag{36}$$

for every $a_k \geq 0$ and $b_k > 0$ with $k \in \{1, \ldots, d\}$. If we apply the Jensen inequality for the function $f(x) = x^{\theta}$, which is concave for $\theta \in [0,1]$, we deduce

$$\frac{\sum_{k=1}^{d} b_k \left(\frac{a_k}{b_k}\right)^{\theta}}{\sum_{k=1}^{d} b_k} \leq \left(\frac{\sum_{k=1}^{d} b_k \left(\frac{a_k}{b_k}\right)}{\sum_{k=1}^{d} b_k}\right)^{\theta},$$

which is equivalent to inequality (36). In [25], Dragomir applied Hölder's inequality for this. For $\theta \geq 1$, the function $f(x) = x^{\theta}$ is convex and the inequality sign in (36) is flipped, obtaining the classical Radon inequality

$$\sum_{k=1}^{d} \frac{a_k^{\theta}}{b_k^{\theta-1}} \geq \frac{\left(\sum_{k=1}^{d} a_k\right)^{\theta}}{\left(\sum_{k=1}^{d} b_k\right)^{\theta-1}}.$$

Thus, we have

$$\begin{aligned}
\sum_{k=1}^{d} |\langle T_k x, y\rangle|^2 &\overset{Kato}{\leq} \sum_{k=1}^{d} \langle |T_k|^{2\theta} x, x\rangle \langle |T_k^*|^{2(1-\theta)} y, y\rangle \\
&\overset{McCarthy}{\leq} \sum_{k=1}^{d} \langle |T_k|^2 x, x\rangle^{\theta} \langle |T_k^*|^2 y, y\rangle^{1-\theta} \\
&\overset{(36)}{\leq} \left(\sum_{k=1}^{d} \langle |T_k|^2 x, x\rangle\right)^{\theta} \left(\sum_{k=1}^{d} \langle |T_k^*|^2 y, y\rangle\right)^{1-\theta} \\
&= \left[\left(\sum_{k=1}^{d} \|T_k x\|^2\right)^{1/2}\right]^{2\theta} \left[\left(\sum_{k=1}^{d} \|T_k^* y\|^2\right)^{1/2}\right]^{2(1-\theta)}.
\end{aligned}$$

Therefore, we deduce

$$\left(\sum_{k=1}^{d} |\langle T_k x, y\rangle|^2\right)^{1/2} \leq \left[\left(\sum_{k=1}^{d} \|T_k x\|^2\right)^{1/2}\right]^{\theta} \left[\left(\sum_{k=1}^{d} \|T_k^* y\|^2\right)^{1/2}\right]^{1-\theta}. \tag{37}$$

Consequently, by taking the supremum over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ in inequality (37) and taking into account the equality from Theorem 1 for $A = I$, we obtain the desired result. □

Remark 6. *If we take* $\theta = 1$ *in relation* (35), *then* $\|\mathbf{T}\|_e \leq \|\mathbf{T}\|$ *and for* $\theta = 0$ *in the same relation, we obtain* $\|\mathbf{T}\|_e \leq \|\mathbf{T}^*\|$. *Using the Kittaneh–Manasrah inequality [26] and inequality* (35), *we found the following inequality:*

$$\|\mathbf{T}\|_e \leq \theta\|\mathbf{T}\| + (1-\theta)\|\mathbf{T}^*\| - \min\{\theta, 1-\theta\}\left(\sqrt{\|\mathbf{T}\|} - \sqrt{\|\mathbf{T}^*\|}\right)^2$$

for all $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$ *a d-tuple of positive operators.*

4. A-Doubly-Commuting Tuples of A-Hyponormal Operators

In this section, we give several characterizations related to the operators from $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and the operators from $\mathbb{B}(\mathbf{R}(A^{1/2}))$. For an A-doubly-commuting d-tuple of hyponormal operators, we proved the equalities $\|\mathbf{T}\|_{e,A} = \|\mathbf{T}\|_A$ and $r_A(\mathbf{T}) = \|\mathbf{T}\|_A = \omega_A(\mathbf{T})$.

Let us introduce the following definition.

Definition 4. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$. The d-tuple $\mathbf{T}$ is said to be A-doubly commuting if:*

(i) it is commuting, i.e., $[T_i, T_k] = 0$ for all $i, k \in \{1, \dots, d\}$,

(ii) $T_i^{\sharp_A} T_k = T_k T_i^{\sharp_A}$ for all $1 \le i \ne k \le d$.

In this section, we will study the connection between $\|\mathbf{T}\|_A$ and $\|\mathbf{T}\|_{e,A}$, when $\mathbf{T}$ is a d-tuple of A-doubly-commuting tuples of A-hyponormal operators. For this purpose, we need to recall some aspects: the semi-inner product $\langle \cdot, \cdot \rangle_A$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{N}(A)$ is given by

$$(\overline{x}, \overline{y}) = \langle Ax, y \rangle,$$

for any $\overline{x} = x + \mathcal{N}(A), \overline{y} = y + \mathcal{N}(A) \in \mathcal{H}/\mathcal{N}(A)$. We remark that $\big(\mathcal{H}/\mathcal{N}(A), (\cdot,\cdot)\big)$ is not complete unless $\mathcal{R}(A)$ is closed in $\mathcal{H}$. However, L. de Branges and J. Rovnyak [27] proved that the completion of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathbf{R}(A^{1/2}) := \Big(\mathcal{R}(A^{1/2}), \langle \cdot, \cdot \rangle_{\mathbf{R}(A^{1/2})}\Big)$, where $\langle \cdot, \cdot \rangle_{\mathbf{R}(A^{1/2})}$ is given by

$$\langle A^{1/2}\delta, A^{1/2}\xi \rangle_{\mathbf{R}(A^{1/2})} := \langle P_{\overline{\mathcal{R}(A)}}\delta, P_{\overline{\mathcal{R}(A)}}\xi \rangle, \ \forall\, \delta, \xi \in \mathcal{H}.$$

It is obvious that $\| \cdot \|_{\mathbf{R}(A^{1/2})}$ stands for the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\mathbf{R}(A^{1/2})}$. We mention here that, in view of Proposition 2.1. in [15], we have $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{1/2})$. Further, since $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2})$, then we observe that

$$\langle A\delta, A\xi \rangle_{\mathbf{R}(A^{1/2})} = \langle \delta, \xi \rangle_A, \quad \forall\, \delta, \xi \in \mathcal{H}, \tag{38}$$

which implies that

$$\|A\xi\|_{\mathbf{R}(A^{1/2})} = \|\xi\|_A, \ \forall\, x \in \mathcal{H}. \tag{39}$$

In the next proposition, we give an interesting connection between operators in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and operators in $\mathbb{B}(\mathbf{R}(A^{1/2}))$. Furthermore, we summarize in the same proposition some useful properties.

Proposition 3 ([14,15,19,28])**.** *Let $T \in \mathbb{B}(\mathcal{H})$. Then there exists $\widetilde{T} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $\widetilde{T} Z_A = Z_A T$ if and only if $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. In such cases $\widetilde{T}$ is unique. Here $Z_A : \mathcal{H} \to \mathbf{R}(A^{1/2})$ is defined by $Z_A x = Ax$. Furthermore, for every $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$, we have the following properties:*

(1) $\|T\|_A = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$ and $r_A(T) = r(\widetilde{T})$.

(2) If $T \in \mathbb{B}_A(\mathcal{H})$, then $\widetilde{T^{\sharp_A}} = (\widetilde{T})^*$ and $\widetilde{(T^{\sharp_A})^{\sharp_A}} = \widetilde{T}$.

(3) If $T_k \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ and $\eta_k \in \mathbb{C}$ for all $k \in \{1, \dots, d\}$, then we have

(i) $\displaystyle\sum_{k=1}^{d} \eta_k T_k \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ and $\displaystyle\widetilde{\sum_{k=1}^{d} \eta_k T_k} = \sum_{k=1}^{d} \eta_k \widetilde{T}_k$;

(ii) $\displaystyle\prod_{k=1}^{d} \eta_k T_k \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ and $\displaystyle\widetilde{\prod_{k=1}^{d} \eta_k T_k} = \prod_{k=1}^{d} \eta_k \widetilde{T}_k$.

(4) If $\mathbf{T} = (T_1, \dots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ such that $T_i T_j = T_j T_i$ for all $i, j \in \{1, \dots, d\}$, then $r_A(\mathbf{T}) = r(\widetilde{\mathbf{T}})$, where $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \dots, \widetilde{T}_d)$.

The following lemma is also useful in proving our results in this section.

Lemma 3. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then, the following assertions hold:*

(i) *If $T \geq_A S$, then $\widetilde{T} \geq \widetilde{S}$;*

(ii) *If $T \in \mathbb{B}_A(\mathcal{H})$ is A-hyponormal, then $\widetilde{T}$ is hyponormal on $\mathbf{R}(A^{1/2})$.*

Proof. (1) Let $x \in \mathcal{H}$. By applying (38) together with Proposition 3 (3), we have

$$\begin{aligned}\langle (T-S)x, x\rangle_A &= \langle A(T-S)x, Ax\rangle_{\mathbf{R}(A^{1/2})}\\ &= \langle \widetilde{(T-S)}Ax, Ax\rangle_{\mathbf{R}(A^{1/2})}\\ &= \langle (\widetilde{T}-\widetilde{S})Ax, Ax\rangle_{\mathbf{R}(A^{1/2})} \geq 0,\end{aligned}$$

where the last inequality follows by using the fact that $T \geq_A S$. Now, by using the fact that $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{1/2})$, we can check that

$$\langle (\widetilde{T}-\widetilde{S})A^{1/2}x, A^{1/2}x\rangle_{\mathbf{R}(A^{1/2})} \geq 0, \quad \forall\, x \in \mathcal{H}.$$

Hence, $\widetilde{T} - \widetilde{S}$ is a positive operator on the Hilbert space $\mathbf{R}(A^{1/2})$. Therefore, $\widetilde{T} \geq \widetilde{S}$ as required.

(2) Since T is A-hyponormal, then $T^{\sharp_A}T \geq_A TT^{\sharp_A}$. Hence, by the first assertion of this lemma, we deduce that $\widetilde{T^{\sharp_A}T} \geq \widetilde{TT^{\sharp_A}}$. By Proposition 3, this implies that $(\widetilde{T})^*\widetilde{T} \geq \widetilde{T}(\widetilde{T})^*$. Therefore, $\widetilde{T}$ is an hyponormal operator on $\mathbf{R}(A^{1/2})$. □

Now, we are able to prove the following result.

Theorem 3. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ be an A-doubly-commuting d-tuple of A-hyponormal operators. Then, the following equality*

$$\|\mathbf{T}\|_{e,A} = \|\mathbf{T}\|_A \tag{40}$$

holds.

Proof. Since $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$, then by applying Proposition 3 we deduce that for each $k \in \{1, \ldots, d\}$ there exists $\widetilde{T}_k \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T_k = \widetilde{T}_k Z_A$. Another application of Proposition 3 shows that

$$\begin{aligned}\|\mathbf{T}\|_{e,A} &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \|\lambda_1 T_1 + \ldots + \lambda_d T_d\|_A\\ &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \|\widetilde{\lambda_1 T_1 + \ldots + \lambda_d T_d}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \|\lambda_1 \widetilde{T}_1 + \ldots + \lambda_d \widetilde{T}_d\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))},\end{aligned}$$

and hence,

$$\|\mathbf{T}\|_{e,A} = \|\widetilde{\mathbf{T}}\|_e, \tag{41}$$

where $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \ldots, \widetilde{T}_d)$. On the other hand, we observe that the A-joint seminorm of $\mathbf{T}$ can be written as: $\|\mathbf{T}\|_A = \sup\{\|\boldsymbol{\lambda}\|_2\,;\ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \Omega_A(\mathbf{T})\}$ where

$$\Omega_A(\mathbf{T}) := \Big\{ \big(\|T_1 x\|_A, \ldots, \|T_d x\|_A\big)\,;\ x \in \mathcal{H},\ \|x\|_A = 1 \Big\}.$$

If $A = I$, we simply denote $\Omega_I(\cdot)$ by $\Omega(\cdot)$. In particular, we observe that

$$\Omega(\widetilde{\mathbf{T}}) = \Big\{ \big(\|\widetilde{T}_1 y\|_{\mathbf{R}(A^{1/2})}, \ldots, \|\widetilde{T}_d y\|_{\mathbf{R}(A^{1/2})} \big) \,;\, y \in \mathbf{R}(A^{1/2}),\ \|y\|_{\mathbf{R}(A^{1/2})} = 1 \Big\}.$$

Now, by using the decomposition $\mathcal{H} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A^{1/2})}$ together with (39), we obtain

$$\begin{aligned}\Omega_A(\mathbf{T}) &= \Big\{ \big(\|T_1 x\|_A, \ldots, \|T_d x\|_A \big) \,;\, x \in \overline{\mathcal{R}(A^{1/2})} \,;\, \|x\|_A = 1 \Big\} \\ &= \Big\{ \big(\|AT_1 x\|_{\mathbf{R}(A^{1/2})}, \ldots, \|AT_d x\|_{\mathbf{R}(A^{1/2})} \big) \,;\, x \in \overline{\mathcal{R}(A^{1/2})},\ \|Ax\|_{\mathbf{R}(A^{1/2})} = 1 \Big\},\end{aligned}$$

from which

$$\Omega_A(\mathbf{T}) = \Big\{ \big(\|\widetilde{T}_1 Ax\|_{\mathbf{R}(A^{1/2})}, \ldots, \|\widetilde{T}_d Ax\|_{\mathbf{R}(A^{1/2})} \big) \,;\, x \in \overline{\mathcal{R}(A^{1/2})},\ \|Ax\|_{\mathbf{R}(A^{1/2})} = 1 \Big\}. \quad (42)$$

This immediately implies that

$$\Omega_A(\mathbf{T}) \subseteq \Omega(\widetilde{\mathbf{T}}) \quad (43)$$

Furthermore, it can be seen that

$$\begin{aligned}&\Omega(\widetilde{\mathbf{T}}) \\ &= \Big\{ \big(\|\widetilde{T}_1 A^{1/2} x\|_{\mathbf{R}(A^{1/2})}, \ldots, \|\widetilde{T}_d A^{1/2} x\|_{\mathbf{R}(A^{1/2})} \big) \,;\, x \in \mathcal{H},\ \|A^{1/2} x\|_{\mathbf{R}(A^{1/2})} = 1 \Big\} \\ &= \Big\{ \big(\|\widetilde{T}_1 A^{1/2} x\|_{\mathbf{R}(A^{1/2})}, \ldots, \|\widetilde{T}_d A^{1/2} x\|_{\mathbf{R}(A^{1/2})} \big) \,;\, x \in \overline{\mathcal{R}(A^{1/2})},\ \|A^{1/2} x\|_{\mathbf{R}(A^{1/2})} = 1 \Big\}.\end{aligned}$$

Now, let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \Omega(\widetilde{\mathbf{T}})$. Then, there exists $x \in \overline{\mathcal{R}(A^{1/2})}$ satisfying

$$\|A^{1/2} x\|_{\mathbf{R}(A^{1/2})} = 1 \ \text{ and } \ \lambda_i = \|\widetilde{T}_i A^{1/2} x\|_{\mathbf{R}(A^{1/2})},\ \ \forall\, i \in \{1, \ldots, d\}. \quad (44)$$

Since the subspace $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{1/2})$, then there exists a sequence $\{\xi_n\}$, which may be assumed to be in $\overline{\mathcal{R}(A^{1/2})}$ (because of the fact that $\mathcal{H} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A^{1/2})}$) such that $A^{1/2} x = \lim_{n \to +\infty} A\xi_n$. Thus, by taking (44) into consideration, it holds that

$$\lim_{n \to +\infty} \|A\xi_n\|_{\mathbf{R}(A^{1/2})} = 1 \ \text{ and } \ \lambda_i = \lim_{n \to +\infty} \|\widetilde{T}_i A\xi_n\|_{\mathbf{R}(A^{1/2})}, \quad (45)$$

for all $i \in \{1, \ldots, d\}$. Now, set $\theta_n := \frac{\xi_n}{\|A\xi_n\|_{\mathbf{R}(A^{1/2})}}$. Clearly, we have $\|A\theta_n\|_{\mathbf{R}(A^{1/2})} = 1$. Further, by using (45), it can be checked that

$$\lim_{n \to +\infty} \|\widetilde{T}_i A\theta_n\|_{\mathbf{R}(A^{1/2})} = \lambda_i,\ \ \forall\, i \in \{1, \ldots, d\}.$$

This implies that, through (42), $\boldsymbol{\lambda} \in \overline{\Omega_A(\mathbf{T})}$ and, therefore,

$$\Omega(\widetilde{\mathbf{T}}) \subseteq \overline{\Omega_A(\mathbf{T})}. \quad (46)$$

From (43) and (46), we deduce that $\overline{\Omega(\widetilde{\mathbf{T}})} = \overline{\Omega_A(\mathbf{T})}$. Therefore, we infer that

$$\|\mathbf{T}\|_A = \|\widetilde{\mathbf{T}}\|. \quad (47)$$

On the other hand, since $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ is an A-doubly-commuting d-tuple of A-hyponormal operators, then

$$T_i T_j = T_j T_i,\ \forall\, i,j \in \{1, \ldots, d\}, \quad T_i^{\sharp_A} T_j = T_j T_i^{\sharp_A},\ \forall\, 1 \le i \ne j \le d.$$

$$\text{and} \quad T_k^{\sharp_A} T_k \ge_A T_k T_k^{\sharp_A},\ \forall\, k \in \{1, \ldots, d\}.$$

Therefore, by applying Proposition 3 together with Lemma 3 (ii), we have

$$\widetilde{T}_i\widetilde{T}_j = \widetilde{T}_j\widetilde{T}_i,\ \forall\, i,j \in \{1,\ldots,d\}\ \text{ and } (\widetilde{T}_i)^*\widetilde{T}_j = \widetilde{T}_j(\widetilde{T}_i)^*,\ \forall\, 1 \le i \ne j \le d.$$

$$\text{and} \quad (\widetilde{T}_k)^*\widetilde{T}_k \ge \widetilde{T}_k(\widetilde{T}_k)^*,\ \forall\, k \in \{1,\ldots,d\}.$$

Hence, $\widetilde{\mathbf{T}} = (\widetilde{T}_1,\ldots,\widetilde{T}_d)$ is a d-tuple of doubly-commuting hyponormal operators on the Hilbert space $\mathbf{R}(A^{1/2})$. Therefore, by (13), we have

$$\|\widetilde{\mathbf{T}}\| = \|\widetilde{\mathbf{T}}\|_e.$$

This completes the proof by taking (41) and (47) into consideration. □

Remark 7. *Note that the converse of Theorem 3 need not be correct as shown in the next example.*

Example 1. *Let us consider the same matrices in $\mathcal{M}_3$ given in Remark 1, i.e., $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. By Remark 1, we have $\|(T_1,T_2)\|_A = 1$. Further, we see that*

$$\|(T_1,T_2)\|_{e,A} = \sup_{(\lambda_1,\lambda_2)\in\mathbb{B}_2} \|\lambda_1 T_1 + \lambda_2 T_2\|_A = \sup_{(\lambda_1,\lambda_2)\in\overline{\mathbb{B}}_2} \|\lambda_1 T_1 + \lambda_2 T_2\|_A, \tag{48}$$

where $\overline{\mathbb{B}}_2$ means the closed unit ball of $\mathbb{C}^2$. So, by using (4) and making direct calculations, we show that

$$\|(T_1,T_2)\|_{e,A} = \sup_{|\lambda_1|^2+|\lambda_1|^2\le 1}\left(\sup_{|x|^2+|y|^2+|z|^2=1}\left|\frac{\lambda_1}{\sqrt{2}}y + \lambda_2 z\right|\right).$$

By making use of the Cauchy–Schwarz inequality, it can be easily checked that $\|(T_1,T_2)\|_{e,A} \le 1$. On the other hand, by using (48) and then (4), we see that

$$\|(T_1,T_2)\|_{e,A} \ge \|T_2\|_A = 1,$$

from which $\|(T_1,T_2)\|_{e,A} = 1$. So,

$$\|(T_1,T_2)\|_A = \|(T_1,T_2)\|_{e,A} = 1.$$

However, it can be verified that $T_1^{\sharp_A}T_2 \ne T_2T_1^{\sharp_A}$ and this (T_1,T_2) is not an A-doubly-commuting 2-tuple of A-hyponormal operators.

Remark 8. *According to our proof in Theorem 3, we remark that the equality*

$$\|(T_1,\ldots,T_d)\|_A = \|(\widetilde{T}_1,\ldots,\widetilde{T}_d)\|, \tag{49}$$

holds for every d-tuple of operators $(T_1,\ldots,T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$. Here, $\widetilde{T}_k \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ and verify $Z_AT_k = \widetilde{T}_kZ_A$ for all $k \in \{1,\ldots,d\}$. Note that (49) provides an improvement of a result of the second author in [19] since $\mathbb{B}_A(\mathcal{H})$ is in general a proper subspace of $\mathbb{B}_{A^{1/2}}(\mathcal{H})$.

In order to derive an important consequence from Theorem 3, we first introduce the following definition that is inspired by the work of G. Popescu [16].

Definition 5. *For $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$, we define a new A-joint numerical radius and a new A-joint spectral radius of $\mathbf{T}$ by setting*

$$r_{e,A}(\mathbf{T}) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} r_A(\lambda_1 T_1 + \ldots + \lambda_d T_d), \tag{50}$$

and

$$\omega_{e,A}(\mathbf{T}) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \omega_A(\lambda_1 T_1 + \ldots + \lambda_d T_d).$$

Remark 9. *If $A = I$, then $r_{e,I}(\cdot)$ will simply be denoted by $r_e(\cdot)$. Further, it is worth mentioning that the equality*

$$r(\mathbf{T}) = r_e(\mathbf{T}) \tag{51}$$

holds for every commuting operator tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$ (see Theorem 2.1 in [29] or [2]).

Now, as an application of Theorem 3, we state the following result.

Theorem 4. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ be an A-doubly-commuting d-tuple of A-hyponormal operators. Then*

$$r_A(\mathbf{T}) = \|\mathbf{T}\|_A = \omega_A(\mathbf{T}).$$

Proof. Since $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ is a d-tuple of A-doubly-commuting A-hyponormal operators, then in particular we have

$$T_k^{\sharp_A} T_l = T_l T_k^{\sharp_A},\ \forall\, 1 \le k \ne l \le d \ \text{ and } \ T_m^{\sharp_A} T_m \ge_A T_m T_m^{\sharp_A}, \tag{52}$$

for every $m \in \{1, \ldots, d\}$. Now, for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$, we let $S_{\boldsymbol{\lambda}} := \sum_{m=1}^{d} \lambda_m T_m$. Clearly, $S_{\boldsymbol{\lambda}} \in \mathbb{B}_A(\mathcal{H})$. By making simple calculations and using (52), we see that

$$\begin{aligned}
S_{\boldsymbol{\lambda}}^{\sharp_A} S_{\boldsymbol{\lambda}} &= \sum_{i=1}^{d}\sum_{j=1}^{d} \lambda_j \overline{\lambda_i} T_i^{\sharp_A} T_j \\
&= \sum_{i=1}^{d} |\lambda_i|^2 T_i^{\sharp_A} T_i + \sum_{i=1}^{d}\sum_{\substack{j=1,\\ j\ne i}}^{d} \lambda_j \overline{\lambda_i} T_i^{\sharp_A} T_j \\
&\ge_A \sum_{i=1}^{d} |\lambda_i|^2 T_i T_i^{\sharp_A} + \sum_{i=1}^{d}\sum_{\substack{j=1,\\ j\ne i}}^{d} \lambda_j \overline{\lambda_i} T_j T_i^{\sharp_A} = S_\lambda S_\lambda^{\sharp_A}.
\end{aligned}$$

Hence, $S_{\boldsymbol{\lambda}}$ is an A-hyponormal operator. This implies that, by (7),

$$r_A(S_{\boldsymbol{\lambda}}) = \omega_A(S_{\boldsymbol{\lambda}}) = \|S_{\boldsymbol{\lambda}}\|_A,$$

for all $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$. If we take the supremum over all $\lambda \in \mathbb{B}_d$ in the last equalities, then we obtain

$$r_{e,A}(\mathbf{T}) = \omega_{e,A}(\mathbf{T}) = \|\mathbf{T}\|_{e,A}. \tag{53}$$

Taking into account relation (25), we have

$$\omega_A(\mathbf{T}) = \sup_{\substack{x\in\mathcal{H},\\ \|x\|_A=1}} \sqrt{\sum_{j=1}^{d} |\langle T_j x, x\rangle_A|^2} = \sup_{\substack{x\in\mathcal{H},\\ \|x\|_A=1}} \left(\sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \left| \sum_{j=1}^{d} \lambda_j \langle T_j x, x\rangle_A \right| \right).$$

This implies that

$$\omega_A(\mathbf{T}) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \left[\sup_{\substack{x\in\mathcal{H},\\ \|x\|_A=1}} \left| \langle \left(\sum_{j=1}^{d} \lambda_j T_j \right) x, x \rangle_A \right| \right] = \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \omega_A \left(\sum_{j=1}^{d} \lambda_j T_j \right).$$

This proves

$$\omega_A(\mathbf{T}) = \omega_{e,A}(\mathbf{T}). \tag{54}$$

In view of Theorem 3, we have $\|\mathbf{T}\|_{e,A} = \|\mathbf{T}\|_A$. So, all that remains to be proven is that $r_A(\mathbf{T}) = r_{e,A}(\mathbf{T})$. Since we have $\mathbf{T} = (T_1,\ldots,T_d) \in \mathbb{B}_A(\mathcal{H})^d \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$, then by applying Proposition 3 we deduce that for each $j \in \{1,\ldots,d\}$ there exists $\widetilde{T}_j \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T_j = \widetilde{T}_j Z_A$. Hence, another application of Proposition 3 shows that

$$\begin{aligned} r_{e,A}(\mathbf{T}) &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} r_A(\lambda_1 T_1 + \ldots + \lambda_d T_d) \\ &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} r\big(\lambda_1 T_1 \widetilde{+ \ldots +} \lambda_d T_d\big) \\ &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} r\big(\lambda_1 \widetilde{T}_1 + \ldots + \lambda_d \widetilde{T}_d\big), \end{aligned}$$

and so,

$$r_{e,A}(\mathbf{T}) = r_e(\widetilde{\mathbf{T}}), \quad \text{where } \widetilde{\mathbf{T}} = (\widetilde{T}_1,\ldots,\widetilde{T}_d). \tag{55}$$

Since $\mathbf{T}$ is an A-doubly-commuting operator tuple, then it is commuting. So, similar to the proof of Theorem 3, we find that $\widetilde{\mathbf{T}}$ is a commuting d-tuple of operators in the Hilbert space $\mathbf{R}(A^{1/2})$. Therefore, by (51), we conclude that $r_e(\widetilde{\mathbf{T}}) = r(\widetilde{\mathbf{T}})$. Further, by Proposition 3 (4), we have $r_A(\mathbf{T}) = r(\widetilde{\mathbf{T}})$. Hence, by taking (55) into account, we deduce that

$$r_A(\mathbf{T}) = r_{e,A}(\mathbf{T}), \tag{56}$$

as desired. Thus, combining (53) with (54), (56) and (40) yields the desired result and the proof is complete. □

5. Conclusions

In this paper, we introduced a definition that is a generalization of (12). For $\mathbf{T} = (T_1,\ldots,T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$, the A-Euclidean seminorm of $\mathbf{T}$ is given by

$$\|\mathbf{T}\|_{e,A} := \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \|\lambda_1 T_1 + \ldots + \lambda_d T_d\|_A.$$

Consequently, our objective was to study a new joint norm of tuples of operators which generalizes the joint norm given in (12) and define the class of doubly-commuting tuples of hyponormal operators acting on an A-weighted Hilbert space, where A is a positive operator that is not assumed to be invertible. The motivation for our investigation comes from the recent paper [17].

This article was structured as follows: In Section 3, we investigated a new joint seminorm for d-tuples of A-bounded operators. An alternative and easy proof of a well-known result due to G. Popescu [16] was established. In Section 4, we give several characterizations related to the operators from $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and the operators from $\mathbb{B}(\mathbf{R}(A^{1/2}))$. For the A-doubly-commuting d-tuple of hyponormal operators, we proved the equalities $\|\mathbf{T}\|_{e,A} = \|\mathbf{T}\|_A$ and $r_A(\mathbf{T}) = \|\mathbf{T}\|_A = \omega_A(\mathbf{T})$.

In this paper, the ideas and methodologies used may serve as a starting point for future studies in this field. We will look for other connections of these seminorms for d-tuples of A-bounded operators by studying other possible characterizations. In future work, we will generalize the results given a countable collection of operators.

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References

1. Benhida, C.; Curto, R.E.; Lee, S.H.; Yoon, J. The spectral picture and joint spectral radius of the generalized spherical Aluthge transform. *Adv. Math.* **2022**, *408 Pt B*, 108602. [CrossRef]
2. Chavan, S.; Feki, K. Spherical symmetry of some unitary invariants for commuting tuples. *Oper. Matrices* **2021**, *15*, 1131–1139. [CrossRef]
3. Al Dohiman, A.A.; Ould Ahmed Mahmoud, S.A. Class of operators related to a (m, C)-isometric tuple of commuting operators. *J. Inequal. Appl.* **2022**, 105. [CrossRef]
4. Feki, K.; Yamazaki, T. Joint numerical radius of spherical Aluthge transforms of tuples of Hilbert space operators. *Math. Inequal. Appl.* **2021**, *24*, 405–420. [CrossRef]
5. Grover, P.; Singla, S. A distance formula for tuples of operators. *Linear Algebra Appl.* **2022**, *650*, 267–285. [CrossRef]
6. Halmos, P.R. *A Hilbert Space Problem Book*, 2nd ed.; Springer: New York, NY, USA, 1982.
7. Kato, T. Notes on some inequalities for linear operators. *Math. Ann.* **1952**, *125*, 208–212. [CrossRef]
8. Furuta, T.; Mićić, J.; Pečarić, J.; Seo, Y. *Mound–Pečarić Method in Operator Inequalities*; Element: Zagreb, Croatia, 2005.
9. Arias, M.L.; Corach, G.; Gonzalez, M.C. Partial isometries in semi-Hilbertian spaces. *Linear Algebra Appl.* **2008**, *428*, 1460–1475. [CrossRef]
10. Douglas, R.G. On majorization, factorization and range inclusion of operators in Hilbert space. *Proc. Amer. Math. Soc.* **1966**, *17*, 413–416. [CrossRef]
11. Arias, M.L.; Corach, G.; Gonzalez, M.C. Metric properties of projections in semi-Hilbertian spaces. *Integral Equ. Oper. Theory* **2008**, *62*, 11–28. [CrossRef]
12. Baklouti, H.; Feki, K.; Sid Ahmed, O.A.M. Joint numerical ranges of operators in semi-Hilbertian spaces. *Linear Algebra Appl.* **2018**, *555*, 266–284. [CrossRef]
13. Saddi, A. A-Normal operators in Semi-Hilbertian spaces. *Aust. J. Math. Anal. Appl.* **2012**, *9*, 1–12.
14. Feki, K. Spectral radius of semi-Hilbertian space operators and its applications. *Ann. Funct. Anal.* **2020**, *11*, 929–946. [CrossRef]
15. Arias, M.L.; Corach, G.; Gonzalez, M.C. Lifting properties in operator ranges. *Acta Sci. Math.* **2009**, *75*, 635–653.
16. Popescu, G. Unitary invariants in multivariable operator theory. *Mem. Am. Math. Soc.* **2009**, *200*, vi+91. [CrossRef]
17. Feki, K. A note on doubly-commuting tuples of hyponormal operators on Hilbert spaces. *Results Math.* **2020**, *75*, 93. [CrossRef]
18. Baklouti, H.; Feki, K.; Sid Ahmed, O.A.M. Joint normality of operators in semi-Hilbertian spaces. *Linear Multilinear Algebra* **2020**, *68*, 845–866. [CrossRef]
19. Feki, K. On tuples of commuting operators in positive semidefinite inner product spaces. *Linear Algebra Appl.* **2020**, *603*, 313–328. [CrossRef]
20. Chō, M.; Takaguchi, M. Boundary points of joint numerical ranges. *Pac. J. Math.* **1981**, *95*, 27–35. [CrossRef]
21. Dragomir, S.S. The hypo-Euclidean norm of an n-tuple of vectors in inner product spaces and applications. *Res. Rep. Collect.* **2006**, *9*. Available online: https://vuir.vu.edu.au/17507/1/EHNnTVIPSApp.pdf (accessed on 26 December 2022).
22. Bhunia, P.; Feki, K.; Paul, K. Generalized A-Numerical Radius of Operators and Related Inequalities. *Bull. Iran. Math. Soc.* **2022**, *48*, 3883–3907. [CrossRef]
23. Mărghidanu, D.; Díaz-Barrero, J.L.; Rădulescu, S. New refinements of some classical inequalities. *Math. Inequal. Appl.* **2009**, *12*, 513–518. [CrossRef]
24. Minculete, N. About the Cauchy–Bunyakovsky–Schwarz Inequality for Hilbert Space Operators. *Symmetry* **2021**, *13*, 305. [CrossRef]
25. Dragomir, S.S. Some Inequalities of Kato Type for Sequences of Operators in Hilbert Spaces. *Publ. RIMS Kyoto Univ.* **2012**, *46*, 937–955. [CrossRef]
26. Kittaneh, F.; Manasrah, Y. Improved Young and Heinz inequalities for matrices. *J. Math. Anal. Appl.* **2010**, *36*, 262–269. [CrossRef]
27. de Branges, L.; Rovnyak, J. *Square Summable Power Series*; Holt, Rinehert and Winston: New York, NY, USA, 1966.

28. Majdak, W.; Secelean, N.A.; Suciu, L. Ergodic properties of operators in some semi-Hilbertian spaces. *Linear Multilinear Algebra* **2013**, *61*, 139–159. [CrossRef]
29. Baklouti, H.; Feki, K. On joint spectral radius of commuting operators in Hilbert spaces. *Linear Algebra Appl.* **2018**, *557*, 455–463. [CrossRef]