mathematics

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The Algebraic Classification of Nilpotent Bicommutative Algebras

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Abstract: This paper is devoted to the complete algebraic classification of complex five-dimensional nilpotent bicommutative algebras.

Keywords: bicommutative algebras; nilpotent algebras; algebraic classification; central extension

MSC: 17A30

1. Introduction

The problem of classification of n-dimensional associative algebras was posed by Latyshev in Dniester Notebook, some achievements were made by Pikhtilkov, and it has also been discussed in the paper of Belov [1]. One of the classical problems in the theory of non-associative algebras is to classify (up to isomorphism) the algebras of dimension n from a certain variety defined by some family of polynomial identities. It is typical to focus on small dimensions, and there are two main directions for the classification: algebraic and geometric. Varieties such as Jordan, Lie, Leibniz or Zinbiel algebras have been studied from these two approaches ([2–8] and [5,7,9], respectively). In the present paper, we give the algebraic classification of five-dimensional nilpotent bicommutative algebras.

One-sided commutative algebras first appeared in the paper by Cayley [10] in 1857. The variety of bicommutative algebras is defined by the following identities of right- and left-commutativity:

$$(xy)z = (xz)y, \qquad x(yz) = y(xz).$$

It contains the commutative associative algebras as a subvariety; the square of each bicommutative algebra gives a structure of a commutative associative algebra [11]; and each bicommutative algebra is Lie admissible (in [11,12]. It was shown that any bicommutative algebra under commutator multiplication gives a metabelian Lie algebra). The variety of two-dimensional bicommutative algebras is described by Kaygorodov and Volkov; algebraic and geometric classification of four-dimensional nilpotent bicommutative algebras is given by Kaygorodov, Páez-Guillán and Voronin in [13]; algebraic classification of one-generated six-dimensional nilpotent bicommutative algebras is given by Kaygorodov, Páez-Guillán and Voronin in [14]. Bicommutative central extensions of n-dimensional restricted polynomial algebras are studied by Kaygorodov, Lopes and Páez-Guillán in [8]. The structure of the free bicommutative algebra of countable rank and its main numerical invariants were described by Dzhumadildaev, Ismailov and Tulenbaev [15]; see also the announcement [11]. They also proved that the bicommutative operad is not Koszul [15]. Shestakov and Zhang described automorphisms of finitely generated relatively free bicommutative algebras [16]. Drensky and Zhakhayev proved that every free bicommutative

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Citation: Abdurasulov, K.; Kaygorodov, I.; Khudoyberdiyev, A. The Algebraic Classification of Nilpotent Bicommutative Algebras. *Mathematics* **2023**, *11*, 777. https://doi.org/10.3390/math11030777

Academic Editor: Valery G. Romanovski

Received: 10 January 2023
Revised: 28 January 2023
Accepted: 30 January 2023
Published: 3 February 2023

algebra generated by one element is not noetherian, in the sense that it does not have finitely generated one-sided ideals, and they also obtained a positive solution of the Specht problem for any variety of bicommutative algebras over an arbitrary field of any characteristic [17]. Identities of two-dimensional bicommutative algebras and the invariant theory of free bicommutative algebras are studied by Drensky in [18,19]. Dzhumadildaev and Ismailov prove that every identity satisfied by the commutator multiplication in all bicommutative algebras is a consequence of anti-commutativity, the Jacobi and the metabelian identities [20]. They also proved that, in the anti-commutator case, every identity satisfied by the anti-commutator product in all bicommutative algebras is a consequence of commutativity and the two identities obtained in [20]. Bai, Chen and Zhang proved that the Gelfand–Kirillov dimension of a finitely generated bicommutative algebra is a nonnegative integer [21]. Bicommutative algebras are also known under the name of LR-algebras in a series of papers by Burde, Dekimpe and their co-authors [12,22,23]. The studied structures of LR-algebras on a certain Lie algebra. Burde, Dekimpe and Deschamps proved the existence of an LR-complete structure on a nilpotent Lie algebra of dimension n is equivalent to the existence of an n-dimensional abelian subgroup of the affine group $\mathrm{Aff}(N)$, which acts simply and transitively on N, where N is the connected and simply connected Lie group associated with n [12]. Burde, Dekimpe and Vercammen show that if a nilpotent Lie algebra admits an LR-structure, then it admits a complete LR-structure, i.e., the right multiplication for the LR-structure is always nilpotent. Extending this result, it is proven that a meta-solvable Lie algebra with two generators also admits a complete LR-structure [23].

Our method for classifying nilpotent bicommutative algebras is based on the calculation of central extensions of nilpotent algebras of smaller dimensions from the same variety. The algebraic study of central extensions of algebras has been an important topic for years [8,24,25]. First, Skjelbred and Sund used central extensions of Lie algebras to obtain a classification of nilpotent Lie algebras [25]. Note that the Skjelbred–Sund method of central extensions is an important tool in the classification of nilpotent algebras. Using the same method, small dimensional nilpotent (associative, terminal, Jordan, Lie, anticommutative) algebras, and some others have been described. Our main results related to the algebraic classification of the variety of bicommutative algebras are summarized below.

Theorem 1. *Up to isomorphism, there are infinitely many isomorphism classes of complex non-split non-one-generated five-dimensional nilpotent (non-two-step nilpotent) non-commutative bicommutative algebras, described explicitly in Section 3 in terms of* 77 *one-parameter families,* 20 *two-parameter families,* 3 *three-parameter families and* 107 *additional isomorphism classes.*

2. The Algebraic Classification of Nilpotent Bicommutative Algebras

2.1. Method of Classification of Nilpotent Algebras

The objective of this section is to give an analogue of the Skjelbred–Sund method for classifying nilpotent bicommutative algebras. As other analogues of this method were carefully explained in, for example, [13,24], we give only some important definitions, and refer the interested reader to the previous sources.

Let $(\mathbf{A}, \cdot)$ be a bicommutative algebra of dimension n over $\mathbb{C}$ and $\mathbb{V}$ a vector space of dimension s over $\mathbb{C}$. We define the $\mathbb{C}$-linear space $\mathrm{Z}^2(\mathbf{A},\mathbb{V})$ as the set of all bilinear maps $\theta\colon \mathbf{A}\times\mathbf{A} \longrightarrow \mathbb{V}$ such that

$$\theta(xy,z) = \theta(xz,y) \text{ and } \theta(x,yz) = \theta(y,xz).$$

These maps are called *cocycles*. Consider a linear map f from $\mathbf{A}$ to $\mathbb{V}$, and set $\delta f\colon \mathbf{A}\times\mathbf{A} \longrightarrow \mathbb{V}$ with $\delta f(x,y) = f(xy)$. Then, δf is a cocycle, and we define $\mathrm{B}^2(\mathbf{A},\mathbb{V}) = \{\theta = \delta f \ : f \in \mathrm{Hom}(\mathbf{A},\mathbb{V})\}$, which is a linear subspace of $\mathrm{Z}^2(\mathbf{A},\mathbb{V})$. Its elements are called *coboundaries*. The *second cohomology space* $\mathrm{H}^2(\mathbf{A},\mathbb{V})$ is defined to be the quotient space $\mathrm{Z}^2(\mathbf{A},\mathbb{V})/\mathrm{B}^2(\mathbf{A},\mathbb{V})$.

Let $\mathrm{Aut}(\mathbf{A})$ be the automorphism group of the bicommutative algebra $\mathbf{A}$ and let $\phi \in \mathrm{Aut}(\mathbf{A})$. Every $\theta \in \mathrm{Z}^2(\mathbf{A},\mathbb{V})$ defines $\phi\theta(x,y) = \theta(\phi(x),\phi(y))$, with $\phi\theta \in \mathrm{Z}^2(\mathbf{A},\mathbb{V})$. It is easily checked that $\mathrm{Aut}(\mathbf{A})$ acts on the right on $\mathrm{Z}^2(\mathbf{A},\mathbb{V})$, and that $\mathrm{B}^2(\mathbf{A},\mathbb{V})$ is invariant under the action of $\mathrm{Aut}(\mathbf{A})$. So, we have that $\mathrm{Aut}(\mathbf{A})$ acts on $\mathrm{H}^2(\mathbf{A},\mathbb{V})$.

Let θ be a cocycle, and consider the direct sum $\mathbf{A}_\theta = \mathbf{A} \oplus \mathbb{V}$ with the bilinear product "$[-,-]_{\mathbf{A}_\theta}$" defined by $[x+x', y+y']_{\mathbf{A}_\theta} = xy + \theta(x,y)$ for all $x,y \in \mathbf{A}, x',y' \in \mathbb{V}$. It is straightforward that $\mathbf{A}_\theta$ is a bicommutative algebra if and only if $\theta \in \mathrm{Z}^2(\mathbf{A},\mathbb{V})$; it is then a s-dimensional central extension of $\mathbf{A}$ by $\mathbb{V}$.

We also call the set $\mathrm{Ann}(\theta) = \{x \in \mathbf{A} : \theta(x,\mathbf{A}) + \theta(\mathbf{A},x) = 0\}$ the *annihilator* of θ. We recall that the *annihilator* of an algebra $\mathbf{A}$ is defined as the ideal $\mathrm{Ann}(\mathbf{A}) = \{x \in \mathbf{A} : x\mathbf{A} + \mathbf{A}x = 0\}$. Observe that $\mathrm{Ann}(\mathbf{A}_\theta) = \big(\mathrm{Ann}(\theta) \cap \mathrm{Ann}(\mathbf{A})\big) \oplus \mathbb{V}$.

Definition 1. *Let $\mathbf{A}$ be an algebra and I be a subspace of $\mathrm{Ann}(\mathbf{A})$. If $\mathbf{A} = \mathbf{A}_0 \oplus I$ as a direct sum of ideals, then I is called an annihilator component of $\mathbf{A}$.*

Definition 2. *A central extension of an algebra $\mathbf{A}$ without annihilator component is called a non-split central extension.*

The following result is fundamental for the classification method.

Lemma 1. *Let $\mathbf{A}$ be an n-dimensional bicommutative algebra such that $\dim(\mathrm{Ann}(\mathbf{A})) = s \neq 0$. Then there exists, up to isomorphism, a unique $(n-s)$-dimensional bicommutative algebra $\mathbf{A}'$ and a bilinear map $\theta \in \mathrm{Z}^2(\mathbf{A},\mathbb{V})$ with $\mathrm{Ann}(\mathbf{A}) \cap \mathrm{Ann}(\theta) = 0$, where $\mathbb{V}$ is a vector space of dimension s, such that $\mathbf{A} \cong {\mathbf{A}'}_\theta$ and $\mathbf{A}/\mathrm{Ann}(\mathbf{A}) \cong \mathbf{A}'$.*

For the proof, we refer the reader to Lemma 5 in [24].

Then, in order to decide when two bicommutative algebras with nonzero annihilator are isomorphic, it suffices to find conditions in terms of the cocycles.

Let us fix a basis $\{e_1,\ldots,e_s\}$ of $\mathbb{V}$, and $\theta \in \mathrm{Z}^2(\mathbf{A},\mathbb{V})$. Then, θ can be uniquely written as $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i$, where $\theta_i \in \mathrm{Z}^2(\mathbf{A},\mathbb{C})$. It holds that $\theta \in \mathrm{B}^2(\mathbf{A},\mathbb{V})$ if and only if all $\theta_i \in \mathrm{B}^2(\mathbf{A},\mathbb{C})$, and it also holds that $\mathrm{Ann}(\theta) = \mathrm{Ann}(\theta_1) \cap \ldots \cap \mathrm{Ann}(\theta_s)$. Furthermore, if $\mathrm{Ann}(\theta) \cap \mathrm{Ann}(\mathbf{A}) = 0$, then $\mathbf{A}_\theta$ has an annihilator component if and only if $[\theta_1],\ldots,[\theta_s]$ are linearly dependent in $\mathrm{H}^2(\mathbf{A},\mathbb{C})$ (see Lemma 13 in [24]).

Recall that, given a finite-dimensional vector space $\mathbb{V}$ over $\mathbb{C}$, the *Grassmannian* $G_k(\mathbb{V})$ is the set of all k-dimensional linear subspaces of $\mathbb{V}$. Let $G_s(\mathrm{H}^2(\mathbf{A},\mathbb{C}))$ be the Grassmannian of subspaces of dimension s in $\mathrm{H}^2(\mathbf{A},\mathbb{C})$. For $W = \langle[\theta_1],\ldots,[\theta_s]\rangle \in G_s(\mathrm{H}^2(\mathbf{A},\mathbb{C}))$ and $\phi \in \mathrm{Aut}(\mathbf{A})$, define $\phi W = \langle[\phi\theta_1],\ldots,[\phi\theta_s]\rangle$. It holds that $\phi W \in G_s(\mathrm{H}^2(\mathbf{A},\mathbb{C}))$, and this induces an action of $\mathrm{Aut}(\mathbf{A})$ on $G_s(\mathrm{H}^2(\mathbf{A},\mathbb{C}))$. We denote the orbit of $W \in G_s(\mathrm{H}^2(\mathbf{A},\mathbb{C}))$ under this action by $\mathrm{Orb}(W)$. Let

$$W_1 = \langle[\theta_1],\ldots,[\theta_s]\rangle, W_2 = \langle[\vartheta_1],\ldots,[\vartheta_s]\rangle \in G_s\Big(\mathrm{H}^2(\mathbf{A},\mathbb{C})\Big).$$

Similarly to Lemma 15 in [24], in case that $W_1 = W_2$, it holds that

$$\bigcap_{i=1}^{s} \mathrm{Ann}(\theta_i) \cap \mathrm{Ann}(\mathbf{A}) = \bigcap_{i=1}^{s} \mathrm{Ann}(\vartheta_i) \cap \mathrm{Ann}(\mathbf{A}),$$

and therefore the set

$$T_s(\mathbf{A}) = \left\{ W = \langle[\theta_1],\ldots,[\theta_s]\rangle \in G_s\Big(\mathrm{H}^2(\mathbf{A},\mathbb{C})\Big) : \bigcap_{i=1}^{s} \mathrm{Ann}(\theta_i) \cap \mathrm{Ann}(\mathbf{A}) = 0 \right\}$$

is well defined, and it is also stable under the action of $\mathrm{Aut}(\mathbf{A})$ (see Lemma 16 in [24]).

Now, let $\mathbb{V}$ be an s-dimensional linear space and let us denote by $E(\mathbf{A},\mathbb{V})$ the set of all non-split s-dimensional central extensions of $\mathbf{A}$ by $\mathbb{V}$. We can write

$$E(\mathbf{A},\mathbb{V}) = \left\{ \mathbf{A}_\theta : \theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i \text{ and } \langle [\theta_1], \ldots, [\theta_s] \rangle \in T_s(\mathbf{A}) \right\}.$$

Having established these results, we can determine whether two s-dimensional non-split central extensions $\mathbf{A}_\theta, \mathbf{A}_\vartheta$ are isomorphic or not. For the proof, see Lemma 17 in [24].

Lemma 2. *Let $\mathbf{A}_\theta, \mathbf{A}_\vartheta \in E(\mathbf{A},\mathbb{V})$. Suppose that $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i$ and $\vartheta(x,y) = \sum_{i=1}^{s} \vartheta_i(x,y)e_i$. Then the bicommutative algebras $\mathbf{A}_\theta$ and $\mathbf{A}_\vartheta$ are isomorphic if and only if*

$$\mathrm{Orb}\langle [\theta_1], \ldots, [\theta_s] \rangle = \mathrm{Orb}\langle [\vartheta_1], \ldots, [\vartheta_s] \rangle.$$

Then, it exists a bijective correspondence between the set of $\mathrm{Aut}(\mathbf{A})$-orbits on $T_s(\mathbf{A})$ and the set of isomorphism classes of $E(\mathbf{A},\mathbb{V})$. Consequently, we have a procedure that allows us, given a bicommutative algebra $\mathbf{A}'$ of dimension $n-s$, to construct all its non-split central extensions.

Procedure

Let $\mathbf{A}'$ be a bicommutative algebra of dimension $n-s$.

1. Determine $\mathrm{H}^2(\mathbf{A}',\mathbb{C})$, $\mathrm{Ann}(\mathbf{A}')$ and $\mathrm{Aut}(\mathbf{A}')$.
2. Determine the set of $\mathrm{Aut}(\mathbf{A}')$-orbits on $T_s(\mathbf{A}')$.
3. For each orbit, construct the bicommutative algebra associated with a representative of it.

It follows that, thanks to this procedure and to Lemma 1, we can classify all the nilpotent bicommutative algebras of dimension n, provided that the nilpotent bicommutative algebras of dimension $n-1$ are known.

2.2. Notations

Let $\mathbf{A}$ be a bicommutative algebra and fix a basis $\{e_1, \ldots, e_n\}$. We define the bilinear form $\Delta_{ij} \colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{C}$ by $\Delta_{ij}(e_l, e_m) = \delta_{il}\delta_{jm}$. Then, the set $\{\Delta_{ij} : 1 \le i,j \le n\}$ is a basis for the linear space of the bilinear forms on $\mathbf{A}$, and in particular, every $\theta \in \mathrm{Z}^2(\mathbf{A},\mathbb{V})$ can be uniquely written as $\theta = \sum_{1\le i,j\le n} c_{ij}\Delta_{ij}$, where $c_{ij} \in \mathbb{C}$. $\mathrm{H}^2_{com}(\mathfrak{N})$ is the subspace of commutative cocycles of $\mathrm{H}^2_{bicom}(\mathfrak{N})$, where $\mathrm{H}^2_{bicom}(\mathfrak{N})$ is the cohomology space for bicommutative cocycles of algebra $\mathfrak{N}$. Let us fix the following notations:

$\mathcal{B}^{i*}_j$	—	jth i-dimensional nilpotent bicommutative algebra with identity $xyz = 0$
$\mathcal{B}^{i}_j$	—	jth i-dimensional nilpotent "pure" bicommutative algebra (without identity $xyz = 0$)
$\mathfrak{N}_i$	—	ith $four$-dimensional two-step nilpotent algebra
B_i	—	ith non-split non-one-generated $five$-dimensional nilpotent (non-two-step nilpotent) non-commutative bicommutative algebra

2.3. One-Dimensional Central Extensions of Four-Dimensional Two-Step Nilpotent Bicommutative Algebras

2.3.1. The Description of Second Cohomology Spaces

In the following Table 1, we give the description of the second cohomology space of four-dimensional two-step nilpotent bicommutative algebras (see, [26]).

Table 1. The list of two-step nilpotent four-dimensional bicommutative algebras.

$\mathfrak{N}_{01}$: $e_1e_1 = e_2$ $\mathrm{H}^2_{com}(\mathfrak{N}_{01}) = \Big\langle [\Delta_{12}+\Delta_{21}], [\Delta_{13}+\Delta_{31}], [\Delta_{14}+\Delta_{41}], [\Delta_{33}], [\Delta_{34}+\Delta_{43}], [\Delta_{44}] \Big\rangle$ $\mathrm{H}^2_{bicom}(\mathfrak{N}_{01}) = \mathrm{H}^2_{com}(\mathfrak{N}_{01}) \oplus \Big\langle [\Delta_{21}], [\Delta_{31}], [\Delta_{41}], [\Delta_{43}] \Big\rangle$
$\mathfrak{N}_{02}$: $e_1e_1 = e_3 \quad e_2e_2 = e_4$ $\mathrm{H}^2_{com}(\mathfrak{N}_{02}) = \Big\langle [\Delta_{12}+\Delta_{21}], [\Delta_{13}+\Delta_{31}], [\Delta_{24}+\Delta_{42}] \Big\rangle$ $\mathrm{H}^2_{bicom}(\mathfrak{N}_{02}) = \mathrm{H}^2_{com}(\mathfrak{N}_{02}) \oplus \Big\langle [\Delta_{21}], [\Delta_{31}], [\Delta_{42}] \Big\rangle$
$\mathfrak{N}_{03}$: $e_1e_2 = e_3 \quad e_2e_1 = -e_3$ $\mathrm{H}^2(\mathfrak{N}_{03}) = \Big\langle [\Delta_{11}], [\Delta_{14}], [\Delta_{21}], [\Delta_{22}], [\Delta_{24}], [\Delta_{41}], [\Delta_{42}], [\Delta_{44}] \Big\rangle$
$\mathfrak{N}^{\alpha}_{04}$: $e_1e_1 = e_3 \quad e_1e_2 = e_3 \quad e_2e_2 = \alpha e_3$ $\mathrm{H}^2(\mathfrak{N}^{\alpha\neq 0}_{04}) = \Big\langle [\Delta_{12}], [\Delta_{14}], [\Delta_{21}], [\Delta_{22}], [\Delta_{24}], [\Delta_{41}], [\Delta_{42}], [\Delta_{44}] \Big\rangle = \Phi_\alpha$ $\mathrm{H}^2(\mathfrak{N}^{0}_{04}) = \Phi_0 \oplus \Big\langle [\Delta_{13}], [\Delta_{31}+\Delta_{32}] \Big\rangle$
$\mathfrak{N}_{05}$: $e_1e_1 = e_3 \quad e_1e_2 = e_3 \quad e_2e_1 = e_3$ $\mathrm{H}^2_{com}(\mathfrak{N}_{05}) = \Big\langle [\Delta_{11}], [\Delta_{14}], [\Delta_{21}], [\Delta_{22}], [\Delta_{24}], [\Delta_{41}], [\Delta_{42}], [\Delta_{44}] \Big\rangle$
$\mathfrak{N}_{06}$: $e_1e_2 = e_4 \quad e_3e_1 = e_4$ $\mathrm{H}^2(\mathfrak{N}_{06}) = \Big\langle [\Delta_{11}], [\Delta_{12}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23}], [\Delta_{32}], [\Delta_{33}] \Big\rangle$
$\mathfrak{N}_{07}$: $e_1e_2 = e_3 \quad e_2e_1 = e_4 \quad e_2e_2 = -e_3$ $\mathrm{H}^2(\mathfrak{N}_{07}) = \Big\langle [\Delta_{11}], [\Delta_{22}], [\Delta_{13}-\Delta_{23}], [\Delta_{24}], [\Delta_{32}], [\Delta_{41}] \Big\rangle$
$\mathfrak{N}^{\alpha}_{08}$: $e_1e_1 = e_3 \quad e_1e_2 = e_4 \quad e_2e_1 = -\alpha e_3 \quad e_2e_2 = -e_4$ $\mathrm{H}^2(\mathfrak{N}^{\alpha\neq 1}_{08}) = \Big\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{13}-\alpha\Delta_{23}], [\Delta_{14}-\Delta_{24}], [\Delta_{31}], [\Delta_{42}] \Big\rangle = \Phi_\alpha$ $\mathrm{H}^2(\mathfrak{N}^{1}_{08}) = \Phi_1 \oplus \Big\langle [\Delta_{32}+\Delta_{41}] \Big\rangle$
$\mathfrak{N}^{\alpha}_{09}$: $e_1e_1 = e_4 \quad e_1e_2 = \alpha e_4 \quad e_2e_1 = -\alpha e_4 \quad e_2e_2 = e_4 \quad e_3e_3 = e_4$ $\mathrm{H}^2(\mathfrak{N}^{\alpha}_{09}) = \Big\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23}], [\Delta_{31}], [\Delta_{32}], [\Delta_{33}] \Big\rangle$
$\mathfrak{N}_{10}$: $e_1e_2 = e_4 \quad e_1e_3 = e_4 \quad e_2e_1 = -e_4 \quad e_2e_2 = e_4 \quad e_3e_1 = e_4$ $\mathrm{H}^2(\mathfrak{N}_{10}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23}], [\Delta_{31}], [\Delta_{32}], [\Delta_{33}] \Big\rangle$
$\mathfrak{N}_{11}$: $e_1e_1 = e_4 \quad e_1e_2 = e_4 \quad e_2e_1 = -e_4 \quad e_3e_3 = e_4$ $\mathrm{H}^2(\mathfrak{N}_{11}) = \Big\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23}], [\Delta_{31}], [\Delta_{32}], [\Delta_{33}] \Big\rangle$
$\mathfrak{N}_{12}$: $e_1e_2 = e_3 \quad e_2e_1 = e_4$ $\mathrm{H}^2(\mathfrak{N}_{12}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{22}], [\Delta_{24}], [\Delta_{32}], [\Delta_{41}] \Big\rangle$
$\mathfrak{N}_{13}$: $e_1e_1 = e_4 \quad e_1e_2 = e_3 \quad e_2e_1 = -e_3 \quad e_2e_2 = 2e_3 + e_4$ $\mathrm{H}^2(\mathfrak{N}_{13}) = \Big\langle [\Delta_{21}], [\Delta_{22}], [\Delta_{14}+\Delta_{23}], [\Delta_{13}-2\Delta_{14}-\Delta_{24}], [\Delta_{32}-\Delta_{41}], [\Delta_{31}-2\Delta_{32}+\Delta_{42}] \Big\rangle$
$\mathfrak{N}^{\alpha}_{14}$: $e_1e_2 = e_4 \quad e_2e_1 = \alpha e_4 \quad e_2e_2 = e_3$ $\mathrm{H}^2(\mathfrak{N}^{\alpha}_{14}) = \Big\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{23}], [\Delta_{13}+\Delta_{24}], [\Delta_{32}], [\alpha\Delta_{31}+\Delta_{42}] \Big\rangle = \Phi_\alpha$
$\mathrm{H}^2(\mathfrak{N}^{0}_{14}) = \Phi_0 \oplus \Big\langle [\Delta_{14}] \Big\rangle$
$\mathfrak{N}_{15}$: $e_1e_2 = e_4 \quad e_2e_1 = -e_4 \quad e_3e_3 = e_4$ $\mathrm{H}^2(\mathfrak{N}_{15}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23}], [\Delta_{31}], [\Delta_{32}], [\Delta_{33}] \Big\rangle$

2.3.2. Central Extensions of $\mathfrak{N}_{01}$

Let us use the following notations:

$$\begin{array}{lllll}\nabla_1=[\Delta_{12}+\Delta_{21}], & \nabla_2=[\Delta_{13}+\Delta_{31}], & \nabla_3=[\Delta_{14}+\Delta_{41}], & \nabla_4=[\Delta_{33}], & \nabla_5=[\Delta_{34}+\Delta_{43}],\\ \nabla_6=[\Delta_{44}], & \nabla_7=[\Delta_{21}], & \nabla_8=[\Delta_{31}], & \nabla_9=[\Delta_{41}], & \nabla_{10}=[\Delta_{43}].\end{array}$$

Take $\theta=\sum\limits_{i=1}^{10}\alpha_i\nabla_i\in \mathrm{H}^2(\mathfrak{N}_{01})$. The automorphism group of $\mathfrak{N}_{01}$ consists of invertible matrices of the form

$$\phi=\begin{pmatrix} x & 0 & 0 & 0\\ q & x^2 & r & u\\ w & 0 & t & k\\ z & 0 & y & l\end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3\\ \alpha_1+\alpha_7 & 0 & 0 & 0\\ \alpha_2+\alpha_8 & 0 & \alpha_4 & \alpha_5\\ \alpha_3+\alpha_9 & 0 & \alpha_5+\alpha_{10} & \alpha_6\end{pmatrix}\phi=\begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* & \alpha_3^*\\ \alpha_1^*+\alpha_7^* & 0 & 0 & 0\\ \alpha_2^*+\alpha_8^* & 0 & \alpha_4^* & \alpha_5^*\\ \alpha_3^*+\alpha_9^* & 0 & \alpha_5^*+\alpha_{10}^* & \alpha_6^*\end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{01})$ on the subspace $\langle\sum\limits_{i=1}^{10}\alpha_i\nabla_i\rangle$ is given by $\langle\sum\limits_{i=1}^{10}\alpha_i^*\nabla_i\rangle$, where

$$\begin{array}{lll}
\alpha_1^* &=& x^3\alpha_1,\\
\alpha_2^* &=& rx\alpha_1+y(x\alpha_3+w\alpha_5+z\alpha_6)+t(x\alpha_2+w\alpha_4+z(\alpha_5+\alpha_{10})),\\
\alpha_3^* &=& ux\alpha_1+l(x\alpha_3+w\alpha_5+z\alpha_6)+k(x\alpha_2+w\alpha_4+z(\alpha_5+\alpha_{10})),\\
\alpha_4^* &=& t^2\alpha_4+y(2t\alpha_5+y\alpha_6+t\alpha_{10}),\\
\alpha_5^* &=& kt\alpha_4+(lt+ky)\alpha_5+y(l\alpha_6+k\alpha_{10})\\
\alpha_6^* &=& k^2\alpha_4+l(2k\alpha_5+l\alpha_6+k\alpha_{10}),\\
\alpha_7^* &=& x^3\alpha_7,\\
\alpha_8^* &=& rx\alpha_7+tx\alpha_8+xy\alpha_9+wy\alpha_{10}-tz\alpha_{10},\\
\alpha_9^* &=& ux\alpha_7+kx\alpha_8+lx\alpha_9+lw\alpha_{10}-kz\alpha_{10},\\
\alpha_{10}^* &=& (lt-ky)\alpha_{10}.
\end{array}$$

We are interested only in the cases with

$$(\alpha_1,\alpha_7)\neq(0,0),\ (\alpha_2,\alpha_4,\alpha_5,\alpha_8,\alpha_{10})\neq(0,0,0,0,0),$$
$$(\alpha_3,\alpha_5,\alpha_6,\alpha_9,\alpha_{10})\neq(0,0,0,0,0),\ (\alpha_7,\alpha_8,\alpha_9,\alpha_{10})\neq(0,0,0,0).$$

1. $\alpha_1=0,\ \alpha_7\neq 0$, then choosing $r=-\frac{tx\alpha_8+xy\alpha_9+(wy-tz)\alpha_{10}}{x\alpha_7}$, $u=-\frac{kx\alpha_8+lx\alpha_9+(lw-kz)\alpha_{10}}{x\alpha_7}$, we have $\alpha_8^*=\alpha_9^*=0$.
   The family of orbits $\langle\alpha_4\nabla_4+\alpha_5\nabla_5+\alpha_6\nabla_6+\alpha_{10}\nabla_{10}\rangle$ gives us a characterized structure of the three-dimensional ideal that has a one-dimensional extension of two dimensional subalgebra with basis $\{e_3,e_4\}$. Let us remember the classification of algebras of this type.

$$\begin{array}{lllll}
\mathcal{B}_{01}^{3*} & : & e_1e_1=e_2 & & \\
\mathcal{B}_{02}^{3*} & : & e_1e_1=e_3 & e_2e_2=e_3 & \\
\mathcal{B}_{03}^{3*} & : & e_1e_2=e_3 & e_2e_1=-e_3 & \\
\mathcal{B}_{04}^{3*}(\lambda) & : & e_1e_1=\lambda e_3 & e_2e_1=e_3 & e_2e_2=e_3
\end{array}$$

   Using the classification of three dimensional nilpotent algebras, we may consider the following cases.

   (a) $\alpha_4=\alpha_5=\alpha_6=\alpha_{10}=0$, i.e., three dimensional ideal is abelian. Then we may suppose $\alpha_2\neq 0$ and choosing $y=0$, $l=\alpha_2$, $k=-\alpha_3$, we obtain that $\alpha_3^*=0$, which implies $(\alpha_3^*, \alpha_5^*, \alpha_6^*, \alpha_9^*, \alpha_{10}^*) = (0,0,0,0,0)$. Thus, in this case we do not have new algebras.

(b) $\alpha_4 = 1, \alpha_5 = \alpha_6 = \alpha_{10} = 0$, i.e., three-dimensional ideal is isomorphic to $\mathcal{B}_{01}^{3*}$. Then, $\alpha_3 \neq 0$ and choosing $x = 1$, $t = 1$, $k = 0$, $y = 0$, $w = -\alpha_2$, $l = \frac{\alpha_7}{\alpha_3}$ and $t = \sqrt{\alpha_7}$, we have the representative $\langle \nabla_3 + \nabla_4 + \nabla_7 \rangle$.

(c) $\alpha_4 = \alpha_6 = 1, \alpha_5 = \alpha_{10} = 0$, i.e., three-dimensional ideal is isomorphic to $\mathcal{B}_{02}^{3*}$. Then, choosing $x = \frac{1}{\sqrt[3]{\alpha_7}}$, $k = y = 0$, $l = t = 1$, $w = -\frac{\alpha_2}{\sqrt[3]{\alpha_7}}$, $z = -\frac{\alpha_3}{\sqrt[3]{\alpha_7}}$, we have the representative $\langle \nabla_4 + \nabla_6 + \nabla_7 \rangle$.

(d) $\alpha_4 = \alpha_6 = 0, \alpha_5 = 1, \alpha_{10} = -2$, i.e., three-dimensional ideal is isomorphic to $\mathcal{B}_{03}^{3*}$. Then, choosing $x = \frac{1}{\sqrt[3]{\alpha_7}}$, $k = y = 0$, $l = t = 1$, $w = -\frac{\alpha_3}{\sqrt[3]{\alpha_7}}$, $z = \frac{\alpha_2}{\sqrt[3]{\alpha_7}}$, we have the representative $\langle \nabla_5 + \nabla_7 - 2\nabla_{10} \rangle$.

(e) $\alpha_4 = \lambda, \alpha_5 = 0, \alpha_6 = 1, \alpha_{10} = 1$, i.e., three-dimensional ideal is isomorphic to $\mathcal{B}_{04}^{3*}(\lambda)$.

i. If $\lambda \neq 0$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_7}}$, $k = 0$, $y = 0$, $l = t = 1$, $z = \frac{\alpha_3}{\sqrt[3]{\alpha_7}}$, and $w = \frac{\alpha_2 - \alpha_3}{\lambda \sqrt[3]{\alpha_7}}$, we have the family of representatives $\langle \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle_{\lambda \neq 0}$.

ii. If $\lambda = 0$ and $\alpha_2 = \alpha_3$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_7}}$, $k = 0$, $y = 0$, $l = t = 1$ and $z = \frac{\alpha_3}{\sqrt[3]{\alpha_7}}$, we have the representative $\langle \nabla_6 + \nabla_7 + \nabla_{10} \rangle$.

iii. If $\lambda = 0$ and $\alpha_2 \neq \alpha_3$, then choosing $x = \frac{(\alpha_2 - \alpha_3)^2}{\alpha_7}$, $k = 0$, $y = 0$, $l = t = \frac{(\alpha_2 - \alpha_3)^3}{\alpha_7}$ and $z = -\frac{\alpha_3(\alpha_2 - \alpha_3)^2}{\alpha_7}$, we have the representative $\langle \nabla_2 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle$.

2. $\alpha_1 \neq 0$, then choosing $\begin{array}{rcl} r & = & -\frac{tx\alpha_2 + xy\alpha_3 + tw\alpha_4 + wy\alpha_5 + tz\alpha_5 + yz\alpha_6 + tz\alpha_{10}}{x\alpha_1}, \\ u & = & -\frac{kx\alpha_2 + lx\alpha_3 + kw\alpha_4 + lw\alpha_5 + kz\alpha_5 + lz\alpha_6 + kz\alpha_{10}}{x\alpha_1}, \end{array}$ we have $\alpha_2^* = \alpha_3^* = 0$.

(a) $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_{10} = 0$, i.e., three-dimensional ideal is abelian. Then, we may suppose $\alpha_8 \neq 0$ and choosing $y = 0$, $l = \alpha_8$, $k = -\alpha_9$, we obtain that $\alpha_9^* = 0$, which implies $(\alpha_3^*, \alpha_5^*, \alpha_6^*, \alpha_9^*, \alpha_{10}^*) = (0,0,0,0,0)$. Thus, in this case we do not have new algebras.

(b) $\alpha_4 = 1, \alpha_5 = \alpha_6 = \alpha_{10} = 0$, i.e., three-dimensional ideal is isomorphic to $\mathcal{B}_{01}^{3*}$. Then, $\alpha_9 \neq 0$, and choosing $x = 1$, $k = 0$, $t = \sqrt{\alpha_1}$, $y = -\frac{\sqrt{\alpha_1}\alpha_8}{\alpha_9}$, $l = \frac{\alpha_1}{\alpha_9}$ and $w = 0$, we have the family of representatives $\langle \nabla_1 + \nabla_4 + \alpha\nabla_7 + \nabla_9 \rangle$.

(c) $\alpha_4 = \alpha_6 = 1, \alpha_5 = \alpha_{10} = 0$, i.e., three-dimensional ideal is isomorphic to $\mathcal{B}_{02}^{3*}$.

i. $\alpha_7 = 0$, then $\alpha_8^2 + \alpha_9^2 \neq 0$, then choosing $x = \frac{\alpha_8^2 + \alpha_9^2}{\alpha_1}$, $t = \frac{\alpha_8(\alpha_8^2 + \alpha_9^2)}{\alpha_1}$, $y = \frac{\alpha_9(\alpha_8^2 + \alpha_9^2)}{\alpha_1}$, $l = \frac{\alpha_8(\alpha_8^2 + \alpha_9^2)}{\alpha_1}$, $k = -\frac{\alpha_9(\alpha_8^2 + \alpha_9^2)}{\alpha_1}$, we have the representative $\langle \nabla_1 + \nabla_4 + \nabla_6 + \nabla_8 \rangle$.

ii. $\alpha_7 = 0$, $\alpha_8^2 + \alpha_9^2 = 0$, i.e., $\alpha_9 = \pm i\alpha_8 \neq 0$, then choosing $x = \sqrt{\alpha_8}$, $t = \frac{\alpha_1}{2}$, $y = \pm\frac{\alpha_1}{2i}$, $l = \pm ix\alpha_8$, $k = x\alpha_8$, have the representative $\langle \nabla_1 + \nabla_5 + \nabla_8 \rangle$.

iii. $\alpha_7 \neq 0$, then choosing $x = 1$, $y = k = 0$, $t = l = \sqrt{\alpha_7}$, $z = \frac{\alpha_1\alpha_9}{\alpha_7}$, $w = \frac{\alpha_1\alpha_8}{\alpha_7}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_4 + \nabla_6 + \nabla_7 \rangle_{\alpha \neq 0}$.

(d) $\alpha_4 = \alpha_6 = 0, \alpha_5 = 1, \alpha_{10} = -2$, i.e., three-dimensional ideal is isomorphic to $\mathcal{B}_{03}^{3*}$.

i. $2\alpha_1 + \alpha_7 \neq 0$, then choosing $x = 1$, $y = k = 0$, $t = \sqrt{\alpha_1}$, $l = 1$, $z - \frac{\alpha_1\alpha_8}{2\alpha_1 + \alpha_7}$ and $w = \frac{\alpha_1\alpha_9}{2\alpha_1 + \alpha_7}$, we have the family of representatives $\langle \nabla_1 + \nabla_5 + \alpha\nabla_7 - 2\nabla_{10} \rangle_{\alpha \neq -2}$.

ii. $2\alpha_1 + \alpha_7 = 0$, then in case of $(\alpha_8, \alpha_9) = (0,0)$, we have the representative $\langle \nabla_1 + \nabla_5 - 2\nabla_7 - 2\nabla_{10} \rangle$ and in case of $(\alpha_8, \alpha_9) \neq (0,0)$, without loss of generality we may assume $\alpha_8 \neq 0$ and choosing $x = 1$, $y = 0$,

$l = \alpha_8, k = -\alpha_9, t = \frac{\alpha_1}{\alpha_8}$, we have the representative $\langle \nabla_1 + \nabla_5 - 2\nabla_7 + \nabla_8 - 2\nabla_{10} \rangle$.

(e) $\alpha_4 = \lambda, \alpha_5 = 0, \alpha_6 = 1, \alpha_{10} = 1$, i.e., three-dimensional ideal is isomorphic to $\mathcal{B}_{04}^{3*}(\lambda)$.

i. $\alpha_1^2 + \alpha_1\alpha_7 + \lambda\alpha_7^2 \neq 0$, then choosing $x = 1, y = k = 0$ and
$$t = l = \sqrt{\alpha_7}, z = \frac{\alpha_1(\alpha_1\alpha_8 + \alpha_4\alpha_7\alpha_9)}{\alpha_1^2 + \alpha_1\alpha_7 + \alpha_4\alpha_7^2}, w = \frac{\alpha_1(\alpha_7(\alpha_8 - \alpha_9) + \alpha_1\alpha_9)}{\alpha_1^2 + \alpha_1\alpha_7 + \alpha_4\alpha_7^2},$$
we have the representative $\langle \alpha\nabla_1 + \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle$.

ii. $\alpha_1^2 + \alpha_1\alpha_7 + \lambda\alpha_7^2 = 0$, then choosing $y = k = 0, w = \frac{z\alpha_7}{\alpha_1} - x\alpha_9$, we have $\alpha_9^* = 0, \alpha_8^* = \frac{tx}{\alpha_1}(\alpha_1\alpha_8 - \lambda\alpha_7\alpha_9)$. Thus, in this case we have the representatives $\langle \frac{-1\pm\sqrt{1-4\lambda}}{2}\nabla_1 + \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle$ and $\langle \frac{-1\pm\sqrt{1-4\lambda}}{2}\nabla_1 + \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_8 + \nabla_{10} \rangle$ depending on $\alpha_1\alpha_8 = \lambda\alpha_7\alpha_9$ or not.

Summarizing all cases, we have the following distinct orbits

$$\begin{gathered}
\langle \nabla_3 + \nabla_4 + \nabla_7 \rangle, \langle \nabla_5 + \nabla_7 - 2\nabla_{10} \rangle, \langle \nabla_2 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle\ \langle \nabla_1 + \nabla_4 + \alpha\nabla_7 + \nabla_9 \rangle, \\
\langle \nabla_1 + \nabla_4 + \nabla_6 + \nabla_8 \rangle, \langle \nabla_1 + \nabla_5 + \nabla_8 \rangle, \langle \alpha\nabla_1 + \nabla_4 + \nabla_6 + \nabla_7 \rangle, \\
\langle \nabla_1 + \nabla_5 + \alpha\nabla_7 - 2\nabla_{10} \rangle, \\
\langle \nabla_1 + \nabla_5 - 2\nabla_7 + \nabla_8 - 2\nabla_{10} \rangle, \langle \alpha\nabla_1 + \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle, \\
\langle \tfrac{-1\pm\sqrt{1-4\lambda}}{2}\nabla_1 + \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_8 + \nabla_{10} \rangle,
\end{gathered}$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{01}, \mathrm{B}_{02}, \mathrm{B}_{03}, \mathrm{B}_{04}^{\alpha}, \mathrm{B}_{05}, \mathrm{B}_{06}, \mathrm{B}_{07}^{\alpha}, \mathrm{B}_{08}^{\alpha}, \mathrm{B}_{09}, \mathrm{B}_{10}^{\alpha,\lambda}, \mathrm{B}_{11}^{\lambda}, \mathrm{B}_{12}^{\lambda\neq\frac{1}{4}}.$$

2.3.3. Central Extensions of $\mathfrak{N}_{02}$

Let us use the following notations:

$$\begin{array}{lll}
\nabla_1 = [\Delta_{12} + \Delta_{21}], & \nabla_2 = [\Delta_{13} + \Delta_{31}], & \nabla_3 = [\Delta_{24} + \Delta_{42}], \\
\nabla_4 = [\Delta_{21}], & \nabla_5 = [\Delta_{31}], & \nabla_6 = [\Delta_{42}].
\end{array}$$

Take $\theta = \sum_{i=1}^{6} \alpha_i\nabla_i \in \mathrm{H}^2(\mathfrak{N}_{02})$. The automorphism group of $\mathfrak{N}_{02}$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ z & u & x^2 & 0 \\ t & v & 0 & y^2 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 & x & 0 & 0 \\ y & 0 & 0 & 0 \\ z & u & 0 & x^2 \\ t & v & y^2 & 0 \end{pmatrix}.$$

Since

$$\phi_1^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & 0 \\ \alpha_1 + \alpha_4 & 0 & 0 & \alpha_3 \\ \alpha_2 + \alpha_5 & 0 & 0 & 0 \\ 0 & \alpha_3 + \alpha_6 & 0 & 0 \end{pmatrix} \phi_1 = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* & 0 \\ \alpha_1^* + \alpha_4^* & \alpha^{**} & 0 & \alpha_3^* \\ \alpha_2^* + \alpha_5^* & 0 & 0 & 0 \\ 0 & \alpha_3^* + \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{02})$ on the subspace $\langle \sum_{i=1}^{10} \alpha_i\nabla_i \rangle$ is given by $\langle \sum_{i=1}^{10} \alpha_i^*\nabla_i \rangle$, where

$$\begin{array}{lllllllll}
\alpha_1^* & = & xy\alpha_1 + ux\alpha_2 + ty(\alpha_3 + \alpha_6), & \alpha_2^* & = & x^3\alpha_2, & \alpha_3^* & = & y^3\alpha_3, \\
\alpha_4^* & = & xy\alpha_4 + ux\alpha_5 - ty\alpha_6, & \alpha_5^* & = & x^3\alpha_5, & \alpha_6^* & = & y^3\alpha_6.
\end{array}$$

We are interested only in the cases with

$$(\alpha_3, \alpha_6) \neq (0,0), (\alpha_2, \alpha_5) \neq (0,0), (\alpha_4, \alpha_5, \alpha_6) \neq (0,0,0).$$

1. $(\alpha_5, \alpha_6) = (0,0)$, then $\alpha_2\alpha_3\alpha_4 \neq 0$ and by choosing $x = \frac{\alpha_4}{\sqrt[3]{\alpha_2^2\alpha_3}}$, $y = \frac{\alpha_4}{\sqrt[3]{\alpha_2\alpha_3^2}}$ and $t = -\frac{x(y\alpha_1+u\alpha_2)}{y\alpha_3}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle$;
2. $(\alpha_5, \alpha_6) \neq (0,0)$, then without loss of generality (maybe with an action of a suitable ϕ_2), we can suppose $\alpha_5 \neq 0$ and choosing $u = \frac{ty\alpha_6 - xy\alpha_4}{x\alpha_5}$, we have $\alpha_4^* = 0$.
   (a) $\alpha_3\alpha_5 + (\alpha_2+\alpha_5)\alpha_6 = 0$, then $\alpha_6 \neq 0$.
      i. if $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\sqrt[3]{\alpha_5^2\alpha_6}}$, $y = \frac{\alpha_1}{\sqrt[3]{\alpha_5\alpha_6^2}}$, we have the family of representatives $\langle \nabla_1 + \alpha\nabla_2 - (1+\alpha)\nabla_3 + \nabla_5 + \nabla_6 \rangle$;
      ii. if $\alpha_1 = 0$, then choosing $x = \sqrt[3]{\frac{\alpha_6}{\alpha_5}}$, $y = 1$, we have the family of representatives $\langle \alpha\nabla_2 - (1+\alpha)\nabla_3 + \nabla_5 + \nabla_6 \rangle$.
   (b) $\alpha_3\alpha_5 + (\alpha_2+\alpha_5)\alpha_6 \neq 0$, then choosing $t = -\frac{x\alpha_1\alpha_5}{\alpha_3\alpha_5+(\alpha_2+\alpha_5)\alpha_6}$, we have $\alpha_1^* = 0$.
      i. if $\alpha_6 = 0$, then choosing $x = \sqrt[3]{\frac{\alpha_3}{\alpha_5}}$, $y = 1$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_3 + \nabla_5 \rangle$;
      ii. if $\alpha_6 \neq 0$, then choosing $x = \sqrt[3]{\frac{\alpha_6}{\alpha_5}}$, $y = 1$, we have the family of representatives $\langle \alpha\nabla_2 + \beta\nabla_3 + \nabla_5 + \nabla_6 \rangle_{\beta \neq -(1+\alpha)}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle, \langle \nabla_1 + \alpha\nabla_2 - (1+\alpha)\nabla_3 + \nabla_5 + \nabla_6 \rangle,$$
$$\langle \alpha\nabla_2 + \nabla_3 + \nabla_5 \rangle, \langle \alpha\nabla_2 + \beta\nabla_3 + \nabla_5 + \nabla_6 \rangle^{O(\alpha,\beta)\simeq O(\beta,\alpha)},$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{13}, \mathrm{B}_{14}^{\alpha}, \mathrm{B}_{15}^{\alpha}, \mathrm{B}_{16}^{\alpha,\beta}.$$

2.3.4. Central Extensions of $\mathfrak{N}_{04}^0$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{13}], \quad \nabla_3 = [\Delta_{14}], \quad \nabla_4 = [\Delta_{21}], \quad \nabla_5 = [\Delta_{22}],$$
$$\nabla_6 = [\Delta_{24}], \quad \nabla_7 = [\Delta_{41}], \quad \nabla_8 = [\Delta_{42}], \quad \nabla_9 = [\Delta_{44}], \quad \nabla_{10} = [\Delta_{31} + \Delta_{32}].$$

Take $\theta = \sum_{i=1}^{10} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{04}^0)$. The automorphism group of $\mathfrak{N}_{04}^0$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x+y & 0 & 0 \\ z & t & x(x+y) & w \\ u & v & 0 & r \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & 0 & \alpha_6 \\ \alpha_{10} & \alpha_{10} & 0 & 0 \\ \alpha_7 & \alpha_8 & 0 & \alpha_9 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* + \alpha^* & \alpha_2^* & \alpha_3^* \\ \alpha_4^* & \alpha_5^* & 0 & \alpha_6^* \\ \alpha_{10}^* & \alpha_{10}^* & 0 & 0 \\ \alpha_7^* & \alpha_8^* & 0 & \alpha_9^* \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{01})$ on the subspace $\langle \sum_{i=1}^{10} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{10} \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned}
\alpha_1^* &= x^2\alpha_1 + x(t-z)\alpha_2 - x(u-v)\alpha_3 - xy\alpha_4 + xy\alpha_5 \\
&\qquad -y(u-v)\alpha_6 - ux\alpha_7 + ux\alpha_8 - u(u-v)\alpha_9, \\
\alpha_2^* &= x^2(x+y)\alpha_2, \\
\alpha_3^* &= wx\alpha_2 + r(x\alpha_3 + y\alpha_6 + u\alpha_9), \\
\alpha_4^* &= u((x+y)\alpha_6 + v\alpha_9) + x((x+y)\alpha_4 + v\alpha_7 + t\alpha_{10}) + y((x+y)\alpha_5 + v\alpha_8 + t\alpha_{10}), \\
\alpha_5^* &= v((x+y)\alpha_6 + v\alpha_9) + (x+y)((x+y)\alpha_5 + v\alpha_8 + t\alpha_{10}), \\
\alpha_6^* &= r((x+y)\alpha_6 + v\alpha_9), \\
\alpha_7^* &= r(x\alpha_7 + y\alpha_8 + u\alpha_9) + w(x+y)\alpha_{10}, \\
\alpha_8^* &= rv\alpha_9 + (x+y)(r\alpha_8 + w\alpha_{10}), \\
\alpha_9^* &= r^2\alpha_9, \\
\alpha_{10}^* &= x(x+y)^2\alpha_{10}.
\end{aligned}$$

Since we are interested only in the cases with

$$(\alpha_2, \alpha_{10}) \neq (0,0), \quad (\alpha_3, \alpha_6, \alpha_7, \alpha_8, \alpha_9) \neq (0,0,0,0,0),$$

consider the following subcases:

1. $\alpha_{10} = 0$, then $\alpha_2 \neq 0$ and choosing $w = -\frac{r(x\alpha_3+y\alpha_6+u\alpha_9)}{x\alpha_2}$ and
$$t = \frac{-x^2\alpha_1+xz\alpha_2+ux\alpha_3-vx\alpha_3+xy\alpha_4-xy\alpha_5+uy\alpha_6-vy\alpha_6+ux\alpha_7-ux\alpha_8+u^2\alpha_9-uv\alpha_9}{x\alpha_2},$$
we have $\alpha_1^* = \alpha_3^* = 0$.
   (a) $\alpha_9 \neq 0$, then choosing $u = -\frac{x\alpha_7+y\alpha_8}{\alpha_9},\ v = -\frac{(x+y)\alpha_8}{\alpha_9}$, we have $\alpha_7^* = \alpha_8^* = 0$.
      i. $\alpha_5 = \alpha_4 = \alpha_6 = 0$, then choosing $x = 1,\ y = 0,\ r = \sqrt{\frac{\alpha_2}{\alpha_9}}$, we have the representative $\langle \nabla_2 + \nabla_9 \rangle$;
      ii. $\alpha_5 = \alpha_4 = 0,\ \alpha_6 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_2\alpha_9-\alpha_6^2}{\alpha_6^2},\ r = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_6 + \nabla_9 \rangle$;
      iii. $\alpha_5 = 0,\ \alpha_4 \neq 0,\ \alpha_6 = 0$ then choosing $x = \frac{\alpha_4}{\alpha_2},\ y = 0,\ r = \frac{\sqrt{\alpha_4^3}}{\alpha_2\sqrt{\alpha_9}}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_9 \rangle$;
      iv. $\alpha_5 = 0,\ \alpha_4 \neq 0,\ \alpha_6 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_2},\ y = \frac{\alpha_4(\alpha_4\alpha_9-\alpha_6^2)}{\alpha_2\alpha_6^2}$, $r = \frac{\alpha_4^2}{\alpha_2\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_6 + \nabla_9 \rangle$;
      v. $\alpha_5 \neq 0,\ \alpha_4 = \alpha_5$, then choosing $x = 1,\ y = \frac{\alpha_2-\alpha_5}{\alpha_5},\ r = \frac{\alpha_2}{\sqrt{\alpha_5\alpha_9}}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_5 + \alpha\nabla_6 + \nabla_9 \rangle$;
      vi. $\alpha_5 \neq 0,\ \alpha_4 \neq \alpha_5$, then choosing $x = \frac{\alpha_5-\alpha_4}{\alpha_2},\ y = \frac{\alpha_4(\alpha_4-\alpha_5)}{\alpha_2\alpha_5},\ r = \frac{(\alpha_4-\alpha_5)^2}{\alpha_2\sqrt{\alpha_5\alpha_9}}$, we have the representative $\langle \nabla_2 + \nabla_5 + \alpha\nabla_6 + \nabla_9 \rangle$;
   (b) $\alpha_9 = 0,\ \alpha_8 \neq 0,\ \alpha_7 = \alpha_8$, then choosing $v = -\frac{x\alpha_4+y\alpha_5+u\alpha_6}{\alpha_8}$, we have $\alpha_4^* = 0$.
      i. $\alpha_6 = -\alpha_8, \alpha_5 = 0$, then choosing $x = 1,\ y = 0,\ r = \frac{\alpha_2}{\alpha_5}$, we have the representative $\langle \nabla_2 - \nabla_6 + \nabla_7 + \nabla_8 \rangle$;
      ii. $\alpha_6 = -\alpha_8, \alpha_5 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_2-\alpha_5}{\alpha_5},\ r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_5 - \nabla_6 + \nabla_7 + \nabla_8 \rangle$;
      iii. $\alpha_6 \neq -\alpha_8, \alpha_6 = 0,\ \alpha_5 = 0$, then choosing $x = 1,\ y = 0,\ r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_7 + \nabla_8 \rangle$;
      iv. $\alpha_6 \neq -\alpha_8, \alpha_6 = 0,\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_2},\ r = \frac{\alpha_5^2}{\alpha_2\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_5 + \nabla_7 + \nabla_8 \rangle$;
      v. $\alpha_6 \neq -\alpha_8, \alpha_6 \neq 0$, then choosing $x = 1,\ y = 0,\ r = \frac{\alpha_2}{\alpha_8}$, we have the family of representatives $\langle \nabla_2 + \alpha\nabla_6 + \nabla_7 + \nabla_8 \rangle_{\alpha \neq 0,-1}$.
   (c) $\alpha_9 = 0,\ \alpha_8 \neq 0,\ \alpha_7 \neq \alpha_8$, then choosing $y = -\frac{x\alpha_7}{\alpha_8}$, we have $\alpha_7^* = 0$. Hence,
      i. $\alpha_6 = -\alpha_8,\ \alpha_5 = 0$, then choosing $x = 1,\ u = \frac{\alpha_4}{\alpha_8},\ r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 - \nabla_6 + \nabla_8 \rangle$;

ii. $\alpha_6 = -\alpha_8$, $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_2}$, $u = \frac{\alpha_4\alpha_5}{\alpha_2\alpha_8}$, $r = \frac{\alpha_5^2}{\alpha_2\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_5 - \nabla_6 + \nabla_8 \rangle$;

iii. $\alpha_6 \neq -\alpha_8$, $\alpha_6 = \alpha_4 = 0$, then choosing $x = 1$, $v = -\frac{\alpha_5}{\alpha_8}$, $r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_8 \rangle$;

iv. $\alpha_6 \neq -\alpha_8$, $\alpha_6 = 0$, $\alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_2}$, $v = -\frac{\alpha_4\alpha_5}{\alpha_2\alpha_8}$, $r = \frac{\alpha_4^2}{\alpha_2\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_8 \rangle$;

v. $\alpha_6 \neq -\alpha_8$, $\alpha_6 \neq 0$, then choosing $x = 1$, $u = -\frac{\alpha_4}{\alpha_6}$, $v = -\frac{\alpha_5}{\alpha_6+\alpha_8}$, $r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 + \alpha\nabla_6 + \nabla_8 \rangle_{\alpha \neq 0,-1}$.

(d) $\alpha_9 = \alpha_8 = 0$, $\alpha_7 \neq 0$, then choosing $v = -\frac{(x+y)(x\alpha_4+y\alpha_5+u\alpha_6)}{x\alpha_7}$, we have $\alpha_4^* = 0$. Hence,

i. $\alpha_6 = \alpha_5 = 0$, then choosing $x = 1$, $y = 0$, $r = \frac{\alpha_2}{\alpha_7}$, we have the representative $\langle \nabla_2 + \nabla_7 \rangle$;

ii. $\alpha_6 = 0$, $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_2}$, $y = 0$, $r = \frac{\alpha_5^2}{\alpha_2\alpha_7}$, we have the representative $\langle \nabla_2 + \nabla_5 + \nabla_7 \rangle$;

iii. $\alpha_6 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_7-\alpha_6}{\alpha_6}$, $u = \frac{\alpha_5}{\alpha_6}$, $r = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_6 + \nabla_7 \rangle$.

(e) $\alpha_9 = \alpha_8 = \alpha_7 = 0$, $\alpha_6 \neq 0$, then choosing $x = 1$, $y = 0$, $u = -\frac{\alpha_4}{\alpha_6}$, $v = -\frac{\alpha_5}{\alpha_6}$, $r = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_6 \rangle$.

2. $\alpha_{10} \neq 0$, then choosing $w = -\frac{r((x+y)\alpha_8+v\alpha_9)}{(x+y)\alpha_{10}}$ and

$$t = -\frac{x(x+y)\alpha_4+y(x+y)\alpha_5+ux\alpha_6+uy\alpha_6+vx\alpha_7+vy\alpha_8+uv\alpha_9}{(x+y)\alpha_{10}},$$

we have $\alpha_4^* = \alpha_8^* = 0$. Now, we consider following subcases:

(a) $\alpha_9 \neq 0$, then choosing $u = -\frac{(x+y)\alpha_6+x\alpha_7}{\alpha_9}$, $v = -\frac{(x+y)\alpha_6}{\alpha_9}$, we have $\alpha_6^* = \alpha_7^* = 0$. Hence, we can suppose $\alpha_4 = \alpha_6 = \alpha_7 = \alpha_8 = 0$ and consider following cases:

i. $\alpha_2 = \alpha_5 = \alpha_1 = \alpha_3 = 0$, then choosing $x = 1$, $y = 0$, $r = \sqrt{\frac{\alpha_{10}}{\alpha_9}}$, we have the representative $\langle \nabla_9 + \nabla_{10} \rangle$;

ii. $\alpha_2 = \alpha_5 = \alpha_1 = 0$, $\alpha_3 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_3}{\sqrt{\alpha_9\alpha_{10}}} - 1$, $r = \frac{\alpha_3}{\alpha_9}$, we have the representative $\langle \nabla_3 + \nabla_9 + \nabla_{10} \rangle$;

iii. $\alpha_2 = \alpha_5 = 0$, $\alpha_1 \neq 0$, then choosing $x = 1$, $y = \sqrt{\frac{\alpha_1}{\alpha_{10}}} - 1$, $r = \sqrt{\frac{\alpha_1}{\alpha_9}}$, we have the family of representatives $\langle \nabla_1 + \alpha\nabla_3 + \nabla_9 + \nabla_{10} \rangle^{O(\alpha)\simeq O(-\alpha)}$;

iv. $\alpha_2 = 0$, $\alpha_5 \neq 0$, $\alpha_1 = \alpha_5$, $\alpha_3 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}}$, $y = 0$, $r = \frac{\alpha_5\sqrt{\alpha_5}}{\alpha_{10}\sqrt{\alpha_9}}$, we have the representative $\langle \nabla_1 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle$;

v. $\alpha_2 = 0$, $\alpha_5 \neq 0$, $\alpha_1 = \alpha_5$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_5^2\alpha_9}{\alpha_3^2\alpha_{10}}$, $y = \frac{\alpha_5(\alpha_3^2-\alpha_5\alpha_9)}{\alpha_3^2\alpha_{10}}$, $r = \frac{\alpha_5^2}{\alpha_3\alpha_{10}}$, we have the family of representatives $\langle \nabla_1 + \nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle$;

vi. $\alpha_2 = 0$, $\alpha_5 \neq 0$, $\alpha_1 \neq \alpha_5$, then choosing

$$x = -\frac{\alpha_5^2}{(\alpha_1-\alpha_5)\alpha_{10}}, y = \frac{\alpha_1\alpha_5}{(\alpha_1-\alpha_5)\alpha_{10}}, r = \frac{\alpha_5^2}{\sqrt{(\alpha_5-\alpha_1)\alpha_9\alpha_{10}}},$$

we have the family of representatives $\langle \alpha\nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle^{O(\alpha)\simeq O(-\alpha)}$;

vii. $\alpha_2 \neq 0$, $\alpha_5 = \alpha_3 = 0$, then choosing $x = 1$, $y = \frac{\alpha_2-\alpha_{10}}{\alpha_{10}}$, $z = \frac{\alpha_1}{\alpha_2}$, $r = \frac{\alpha_2}{\sqrt{\alpha_9\alpha_{10}}}$, we have the representative $\langle \nabla_2 + \nabla_9 + \nabla_{10} \rangle$;

viii. $\alpha_2 \neq 0$, $\alpha_5 = 0$, $\alpha_3 \neq 0$, then choosing

$$x = \frac{\alpha_3^2\alpha_{10}}{\alpha_2^2\alpha_9}, y = \frac{\alpha_3^2(\alpha_2-\alpha_{10})}{\alpha_2^2\alpha_9}, z = \frac{\alpha_1\alpha_3^2\alpha_{10}}{\alpha_2^3\alpha_9}, r = \frac{\alpha_3^3\alpha_{10}}{\alpha_2^2\alpha_9^2},$$

we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_9 + \nabla_{10} \rangle$;

ix. $\alpha_2 \neq 0$, $\alpha_5 \neq 0$, then choosing

$x = \frac{\alpha_5}{\alpha_2}, y = \frac{(\alpha_2-\alpha_{10})\alpha_5}{\alpha_2\alpha_{10}}, z = -\frac{\alpha_5\left(\alpha_2^2\alpha_5-2\alpha_2\alpha_5\alpha_{10}-(\alpha_1-\alpha_5)\alpha_{10}^2\right)}{\alpha_2^2\alpha_{10}^2}, r = \frac{\alpha_5\sqrt{\alpha_5}}{\sqrt{\alpha_2\alpha_9\alpha_{10}}},$
we have the family of representatives $\langle \nabla_2 + \alpha\nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10}\rangle^{O(\alpha)\simeq O(-\alpha)}$.

(b) $\alpha_9 = 0,\ \alpha_6 \neq 0$, then choosing $u = \frac{x(x+y)\alpha_5+v((x+y)\alpha_6-x\alpha_7)}{(x+y)\alpha_6}$, we have $\alpha_5^* = 0$. Hence, we have $\alpha_4 = \alpha_5 = \alpha_8 = \alpha_9 = 0$ and consider following cases:

i. $\alpha_2 = 0,\ \alpha_3 = \alpha_6,\ \alpha_7 = 0,\ \alpha_1 = 0$, then choosing $x = 1,\ y = 0,\ r = \frac{\alpha_{10}}{\alpha_6}$, we have the representative $\langle \nabla_3 + \nabla_6 + \nabla_{10}\rangle$;

ii. $\alpha_2 = 0,\ \alpha_3 = \alpha_6,\ \alpha_7 = 0,\ \alpha_1 \neq 0$, then choosing $x = 1, y = \sqrt{\frac{\alpha_1}{\alpha_{10}}} - 1$, $r = \frac{\sqrt{\alpha_1\alpha_{10}}}{\alpha_6}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_6 + \nabla_{10}\rangle$;

iii. $\alpha_2 = 0,\ \alpha_3 = \alpha_6,\ \alpha_7 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_7-\alpha_6}{\alpha_6},\ u = \frac{\alpha_1}{\alpha_7}$, $r = \frac{\alpha_7\alpha_{10}}{\alpha_6^2}$, we have the representative $\langle \nabla_3 + \nabla_6 + \nabla_7 + \nabla_{10}\rangle$;

iv. $\alpha_2 = 0,\ \alpha_3 \neq \alpha_6,\ \alpha_7 = \alpha_1 = 0$, then choosing $x = 1,\ y = -\frac{\alpha_3}{\alpha_6}$, $r = -\frac{(\alpha_3-\alpha_6)\alpha_{10}}{\alpha_6^2}$, we have the representative $\langle \nabla_6 + \nabla_{10}\rangle$;

v. $\alpha_2 = 0,\ \alpha_3 \neq \alpha_6,\ \alpha_7 = 0,\ \alpha_1 \neq 0$, then choosing
$$x = \frac{\alpha_1\alpha_6^2}{(\alpha_6-\alpha_3)^2\alpha_{10}},\ y = -\frac{\alpha_1\alpha_3\alpha_6}{(\alpha_6-\alpha_3)^2\alpha_{10}},\ r = \frac{\alpha_1^2\alpha_6^2}{(\alpha_6-\alpha_3)^3\alpha_{10}},$$
we have the representative $\langle \nabla_1 + \nabla_6 + \nabla_{10}\rangle$;

vi. $\alpha_2 = 0,\ \alpha_3 \neq \alpha_6,\ \alpha_7 \neq 0, \alpha_3 + \alpha_7 = \alpha_6,\ \alpha_1 = 0$, then choosing $x = 1$, $y = -\frac{\alpha_3}{\alpha_6}, r = \frac{(\alpha_6-\alpha_3)\alpha_{10}}{\alpha_6^2}$, we have the representative $\langle \nabla_6 + \nabla_7 + \nabla_{10}\rangle$;

vii. $\alpha_2 = 0,\ \alpha_3 \neq \alpha_6,\ \alpha_7 \neq 0,\ \alpha_3 + \alpha_7 = \alpha_6,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_6^2}{(\alpha_6-\alpha_3)^2\alpha_{10}}, y = -\frac{\alpha_1\alpha_3\alpha_6}{(\alpha_6-\alpha_3)^2\alpha_{10}}, r = \frac{\alpha_1^2\alpha_6^2}{(\alpha_6-\alpha_3)^3\alpha_{10}}$, we have the representative $\langle \nabla_1 + \nabla_6 + \nabla_7 + \nabla_{10}\rangle$;

viii. $\alpha_2 = 0,\ \alpha_3 \neq \alpha_6,\ \alpha_7 \neq 0, \alpha_3 + \alpha_7 \neq \alpha_6$, then choosing $x = 1, y = -\frac{\alpha_3}{\alpha_6}$, $r = \frac{(\alpha_6-\alpha_3)\alpha_{10}}{\alpha_6^2}$, we have the family of representatives $\langle \nabla_6 + \alpha\nabla_7 + \nabla_{10}\rangle_{\alpha\neq 0,1}$;

ix. $\alpha_2 \neq 0$, then choosing $z = \frac{x\alpha_1}{\alpha_2},\ v = 0$, we have $\alpha_1^* = 0$.

A. $\alpha_3 = \alpha_6$, then choosing $x = 1,\ y = \frac{\alpha_2-\alpha_{10}}{\alpha_{10}},\ r = \frac{\alpha_2}{\alpha_6}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \nabla_6 + \alpha\nabla_7 + \nabla_{10}\rangle$;

B. $\alpha_3 \neq \alpha_6$, then choosing $x = 1,\ y = -\frac{\alpha_3}{\alpha_6}, r = \frac{(\alpha_6-\alpha_3)\alpha_{10}}{\alpha_6^2}$, we have the family of representatives $\langle \beta\nabla_2 + \nabla_6 + \alpha\nabla_7 + \nabla_{10}\rangle_{\beta\neq 0}$.

(c) $\alpha_9 = \alpha_6 = 0,\ \alpha_7 \neq 0$, then choosing $v = \frac{(x+y)\alpha_5}{\alpha_7}$, we have $\alpha_5^* = 0$. Hence, we have $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_8 = \alpha_9 = 0$ and consider following cases:

i. $\alpha_2 = 0,\ \alpha_3 + \alpha_7 = 0,\ \alpha_1 = 0$, then choosing $x = 1,\ y = 0,\ r = \frac{\alpha_{10}}{\alpha_7}$, we have the representative $\langle -\nabla_3 + \nabla_7 + \nabla_{10}\rangle$;

ii. $\alpha_2 = 0,\ \alpha_3 + \alpha_7 = 0,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_{10}},\ y = 0,\ r = \frac{\alpha_1^2}{\alpha_7\alpha_{10}}$, we have the representative $\langle \nabla_1 - \nabla_3 + \nabla_7 + \nabla_{10}\rangle$;

iii. $\alpha_2 = 0,\ \alpha_3 + \alpha_7 \neq 0$, then choosing $x = 1,\ y = 0,\ u = \frac{\alpha_1}{\alpha_3+\alpha_7},\ r = \frac{\alpha_{10}}{\alpha_7}$, we have the family of representatives $\langle \alpha\nabla_3 + \nabla_7 + \nabla_{10}\rangle_{\alpha\neq -1}$;

iv. $\alpha_2 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_2-\alpha_{10}}{\alpha_{10}},\ z = \frac{\alpha_1}{\alpha_2},\ r = \frac{\alpha_2^2}{\alpha_7\alpha_{10}},\ u = 0$, we have the family of representatives $\langle \nabla_2 + \alpha\nabla_3 + \nabla_7 + \nabla_{10}\rangle$.

(d) $\alpha_9 = \alpha_7 = \alpha_6 = 0$, then $\alpha_3 \neq 0$, and choosing $u = \frac{(x\alpha_1+v\alpha_3+y\alpha_5)\alpha_{10}-\alpha_2(y\alpha_5+z\alpha_{10})}{\alpha_3\alpha_{10}}$, we obtain $\alpha_1^* = 0$. Hence, we have $\alpha_1 = \alpha_4 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = 0$ and consider following cases:

i. $\alpha_2 = \alpha_5 = 0$, then choosing $x = 1,\ y = 0,\ r = \frac{\alpha_{10}}{\alpha_3}$, we have the representative $\langle \nabla_3 + \nabla_{10}\rangle$;

ii. $\alpha_2 = 0,\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}},\ y = 0,\ r = \frac{\alpha_5^2}{\alpha_3\alpha_{10}}$, we have the representative $\langle \nabla_3 + \nabla_5 + \nabla_{10} \rangle$;

iii. $\alpha_2 \neq 0,\ \alpha_5 = 0$, then choosing $x = 1,\ y = \frac{\alpha_2 - \alpha_{10}}{\alpha_{10}},\ r = \frac{\alpha_2^2}{\alpha_3\alpha_{10}}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_{10} \rangle$;

iv. $\alpha_2 \neq 0,\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_2},\ y = \frac{(\alpha_2 - \alpha_{10})\alpha_5}{\alpha_2\alpha_{10}},\ r = \frac{\alpha_5^2}{\alpha_3\alpha_{10}}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5 + \nabla_{10} \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\begin{gathered}
\langle \nabla_2 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_6 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_6 + \nabla_9 \rangle,\\
\langle \nabla_2 + \nabla_4 + \nabla_5 + \alpha\nabla_6 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_5 + \alpha\nabla_6 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_5 + \nabla_7 + \nabla_8 \rangle,\\
\langle \nabla_2 + \nabla_5 - \nabla_6 + \nabla_7 + \nabla_8 \rangle, \langle \nabla_2 + \alpha\nabla_6 + \nabla_7 + \nabla_8 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_8 \rangle,\\
\langle \nabla_2 + \nabla_5 - \nabla_6 + \nabla_8 \rangle, \langle \nabla_2 + \alpha\nabla_6 + \nabla_8 \rangle, \langle \nabla_2 + \nabla_7 \rangle, \langle \nabla_2 + \nabla_5 + \nabla_7 \rangle, \langle \nabla_2 + \nabla_6 + \nabla_7 \rangle,\\
\langle \nabla_2 + \nabla_6 \rangle, \langle \nabla_9 + \nabla_{10} \rangle, \langle \nabla_3 + \nabla_9 + \nabla_{10} \rangle, \langle \nabla_1 + \alpha\nabla_3 + \nabla_9 + \nabla_{10} \rangle^{O(\alpha)\simeq O(-\alpha)},\\
\langle \nabla_1 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle, \langle \nabla_1 + \nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle, \langle \alpha\nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle^{O(\alpha)\simeq O(-\alpha)},\\
\langle \nabla_2 + \nabla_9 + \nabla_{10} \rangle, \langle \nabla_2 + \nabla_3 + \nabla_9 + \nabla_{10} \rangle, \langle \nabla_2 + \alpha\nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle^{O(\alpha)\simeq O(-\alpha)},\\
\langle \nabla_3 + \nabla_6 + \nabla_{10} \rangle, \langle \nabla_1 + \nabla_3 + \nabla_6 + \nabla_{10} \rangle, \langle \nabla_3 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle, \langle \nabla_1 + \nabla_6 + \nabla_{10} \rangle,\\
\langle \nabla_1 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle, \langle \nabla_2 + \nabla_3 + \nabla_6 + \alpha\nabla_7 + \nabla_{10} \rangle, \langle \beta\nabla_2 + \nabla_6 + \alpha\nabla_7 + \nabla_{10} \rangle,\\
\langle \nabla_1 - \nabla_3 + \nabla_7 + \nabla_{10} \rangle, \langle \alpha\nabla_3 + \nabla_7 + \nabla_{10} \rangle, \langle \nabla_2 + \alpha\nabla_3 + \nabla_7 + \nabla_{10} \rangle, \langle \nabla_3 + \nabla_{10} \rangle,\\
\langle \nabla_3 + \nabla_5 + \nabla_{10} \rangle, \langle \nabla_2 + \nabla_3 + \nabla_{10} \rangle, \langle \nabla_2 + \nabla_3 + \nabla_5 + \nabla_{10} \rangle,
\end{gathered}$$

which gives the following new algebras (see Section 3):

$$\begin{gathered}
\mathrm{B}_{17}, \mathrm{B}_{18}, \mathrm{B}_{19}, \mathrm{B}_{20}, \mathrm{B}_{21}^{\alpha}, \mathrm{B}_{22}^{\alpha}, \mathrm{B}_{23}, \mathrm{B}_{24}, \mathrm{B}_{25}^{\alpha}, \mathrm{B}_{26}, \mathrm{B}_{27}, \mathrm{B}_{28}^{\alpha}, \mathrm{B}_{29}, \mathrm{B}_{30}, \mathrm{B}_{31}, \mathrm{B}_{32}, \mathrm{B}_{33}, \mathrm{B}_{34}, \mathrm{B}_{35}^{\alpha}, \mathrm{B}_{36}, \mathrm{B}_{37},\\
\mathrm{B}_{38}^{\alpha}, \mathrm{B}_{39}, \mathrm{B}_{40}, \mathrm{B}_{41}^{\alpha}, \mathrm{B}_{42}, \mathrm{B}_{43}, \mathrm{B}_{44}, \mathrm{B}_{45}, \mathrm{B}_{46}, \mathrm{B}_{47}^{\alpha}, \mathrm{B}_{48}^{\alpha,\beta}, \mathrm{B}_{49}, \mathrm{B}_{50}^{\alpha}, \mathrm{B}_{51}^{\alpha}, \mathrm{B}_{52}, \mathrm{B}_{53}, \mathrm{B}_{54}, \mathrm{B}_{55}.
\end{gathered}$$

2.3.5. Central Extensions of $\mathfrak{N}_{07}$

Let us use the following notations:

$$\begin{array}{lll}
\nabla_1 = [\Delta_{11}], & \nabla_2 = [\Delta_{22}], & \nabla_3 = [\Delta_{13} - \Delta_{23}],\\
\nabla_4 = [\Delta_{24}], & \nabla_5 = [\Delta_{32}], & \nabla_6 = [\Delta_{41}].
\end{array}$$

Take $\theta = \sum\limits_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{07})$. The automorphism group of $\mathfrak{N}_{07}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0\\ 0 & x & 0 & 0\\ z & u & x^2 & 0\\ t & v & 0 & x^2 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_3 & 0\\ 0 & \alpha_2 & -\alpha_3 & \alpha_4\\ 0 & \alpha_5 & 0 & 0\\ \alpha_6 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_3^* & 0\\ \alpha^{**} & -\alpha^* + \alpha_2^* & -\alpha_3^* & \alpha_4^*\\ 0 & \alpha_5^* & 0 & 0\\ \alpha_6^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{07})$ on the subspace $\langle \sum\limits_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllllll}
\alpha_1^* & = & x(x\alpha_1 + z\alpha_3 + t\alpha_6), & \alpha_3^* & = & x^3\alpha_3, & \alpha_5^* & = & x^3\alpha_5,\\
\alpha_2^* & = & x(x\alpha_2 + v\alpha_4 + (u+z)\alpha_5), & \alpha_4^* & = & x^3\alpha_4, & \alpha_6^* & = & x^3\alpha_6.
\end{array}$$

We are interested only in the cases with $(\alpha_3, \alpha_5) \neq (0,0)$, $(\alpha_4, \alpha_6) \neq (0,0)$.

1. $\alpha_5 \neq 0$, then choosing $u = -\frac{x\alpha_2 + v\alpha_4 + z\alpha_5}{\alpha_5}$, we have $\alpha_2^* = 0$. Now we consider following subcases:
   (a) $\alpha_6 \neq 0$, then choosing $x = 1,\ z = 0,\ t = -\frac{\alpha_1}{\alpha_6}$, we have the family of representatives $\langle \beta\nabla_3 + \alpha\nabla_4 + \nabla_5 + \gamma\nabla_6 \rangle_{\gamma \neq 0}$;

(b) $\alpha_6 = 0, \alpha_3 \neq 0$, then choosing $x = 1,\ z = -\frac{\alpha_1}{\alpha_3}$, we have the family of representatives $\langle \beta\nabla_3 + \alpha\nabla_4 + \nabla_5 \rangle_{\alpha\beta\neq 0}$;

(c) $\alpha_6 = 0, \alpha_3 = 0,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_3}$, we have the family of representatives $\langle \nabla_1 + \alpha\nabla_4 + \nabla_5 \rangle_{\alpha\neq 0}$;

(d) $\alpha_6 = 0, \alpha_3 = 0,\ \alpha_1 = 0$, then we have the family of representatives $\langle \alpha\nabla_4 + \nabla_5 \rangle_{\alpha\neq 0}$.

2. $\alpha_5 = 0,\ \alpha_3 \neq 0$, then choosing $z = -\frac{x\alpha_1 + t\alpha_6}{\alpha_3}$, we have $\alpha_1^* = 0$.

   (a) $\alpha_4 \neq 0$, then choosing $x = 1,\ v = -\frac{\alpha_2}{\alpha_4}$, we have the family of representatives $\langle \nabla_3 + \beta\nabla_4 + \alpha\nabla_6 \rangle_{\beta\neq 0}$;

   (b) $\alpha_4 = 0, \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \alpha\nabla_6 \rangle_{\alpha\neq 0}$;

   (c) $\alpha_4 = 0, \alpha_2 = 0$, then we have the family of representatives $\langle \nabla_3 + \alpha\nabla_6 \rangle_{\alpha\neq 0}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_1 + \alpha\nabla_4 + \nabla_5 \rangle_{\alpha\neq 0},\ \langle \gamma\nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{(\alpha,\beta)\neq(0,0)},\ \langle \nabla_3 + \alpha\nabla_4 + \beta\nabla_6 \rangle_{(\alpha,\beta)\neq(0,0)},$$
$$\langle \nabla_2 + \nabla_3 + \alpha\nabla_6 \rangle_{\alpha\neq 0},$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{56}^{\alpha\neq 0},\ \mathrm{B}_{57}^{(\alpha,\beta,\gamma)\neq(0,0,\gamma)},\ \mathrm{B}_{58}^{(\alpha,\beta)\neq(0,0)},\ \mathrm{B}_{59}^{\alpha\neq 0}.$$

2.3.6. Central Extensions of $\mathfrak{N}_{08}^{\alpha\neq 1}$

Let us use the following notations:

$$\begin{array}{lll} \nabla_1 = [\Delta_{12}], & \nabla_2 = [\Delta_{21}], & \nabla_3 = [\Delta_{13} - \alpha\Delta_{23}], \\ \nabla_4 = [\Delta_{14} - \Delta_{24}], & \nabla_5 = [\Delta_{31}], & \nabla_6 = [\Delta_{42}]. \end{array}$$

Take $\theta = \sum\limits_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{08}^{\alpha\neq 1})$. The automorphism group of $\mathfrak{N}_{08}^{\alpha\neq 1}$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ t & v & x^2 & 0 \\ u & w & 0 & x^2 \end{pmatrix}, \quad \phi_2(\alpha\neq 0) = \begin{pmatrix} 0 & \alpha x & 0 & 0 \\ x & 0 & 0 & 0 \\ t & v & 0 & -\alpha^2 x^2 \\ u & w & -x^2 & 0 \end{pmatrix}.$$

Since

$$\phi_1^T \begin{pmatrix} 0 & \alpha_1 & \alpha_3 & \alpha_4 \\ \alpha_2 & 0 & -\alpha\alpha_3 & -\alpha_4 \\ \alpha_5 & 0 & 0 & 0 \\ 0 & \alpha_6 & 0 & 0 \end{pmatrix} \phi_1 = \begin{pmatrix} \alpha^* & \alpha_1^* + \alpha^{**} & \alpha_3^* & \alpha_4^* \\ \alpha_2^* - \alpha\alpha^* & -\alpha^{**} & -\alpha\alpha_3^* & -\alpha_4^* \\ \alpha_5^* & 0 & 0 & 0 \\ 0 & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{08}^{\alpha\neq 1})$ on the subspace $\langle \sum\limits_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{llll} \alpha_1^* = x(x\alpha_1 + (v - v\alpha)\alpha_3 + (u + w)\alpha_6), & \alpha_3^* = x^3\alpha_3, & \alpha_5^* = x^3\alpha_5, \\ \alpha_2^* = x(x\alpha_2 + u(\alpha - 1)\alpha_4 + (v + t\alpha)\alpha_5), & \alpha_4^* = x^3\alpha_4, & \alpha_6^* = x^3\alpha_6. \end{array}$$

We are interested only in the cases with $(\alpha_3, \alpha_5) \neq (0,0)$, $(\alpha_4, \alpha_6) \neq (0,0)$.

1. $\alpha_5 = \alpha_6 = 0$, then $\alpha_3\alpha_4 \neq 0$, and choosing $x = 1,\ u = \frac{\alpha_2}{(1-\alpha)\alpha_4},\ v = -\frac{\alpha_1}{(1-\alpha)\alpha_3}$, we have $\alpha_1^* = \alpha_2^* = 0$ and obtain the family of representatives $\langle \nabla_3 + \beta\nabla_4 \rangle_{\beta\neq 0}$.
2. $(\alpha_5, \alpha_6) \neq (0,0)$, $\alpha \neq 0$, then with an action of a suitable ϕ_2, we can suppose $\alpha_5 \neq 0$ and choosing $v = -\frac{x\alpha_2 + u(\alpha-1)\alpha_4 + t\alpha\alpha_5}{\alpha_5}$, we can suppose $\alpha_2^* = 0$. Now we consider following subcases:

(a) $\alpha_3 = \alpha_6 = \alpha_1 = 0$, then we have the family of representatives $\langle \beta\nabla_4 + \nabla_5 \rangle_{\beta \neq 0}$;

(b) $\alpha_3 = \alpha_6 = 0,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_5}$, we have the family of representatives $\langle \nabla_1 + \beta\nabla_4 + \nabla_5 \rangle_{\beta \neq 0}$;

(c) $\alpha_3 = 0,\ \alpha_6 \neq 0$, then choosing $x = 1,\ u = -\frac{\alpha_1}{\alpha_6},\ w = 0$, we have the family of representatives $\langle \gamma\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\beta \neq 0}$;

(d) $\alpha_3 \neq 0$, then choosing $x = 1,\ t = -\frac{(\alpha-1)\alpha_2\alpha_3 + \alpha_1\alpha_5}{(\alpha-1)\alpha\alpha_3\alpha_5},\ u = 0,\ w = 0$, we have the family of representatives $\langle \gamma\nabla_3 + \delta\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\gamma \neq 0,\ (\beta,\delta) \neq (0,0)}$.

3. $(\alpha_5, \alpha_6) \neq (0,0)$, $\alpha = 0$. If $\alpha_5 \neq 0$, then we obtain the previous cases. Thus, we consider the case of $\alpha_5 = 0$. Then, $\alpha_3\alpha_6 \neq 0$ and choosing $v = -\frac{x\alpha_1 + (u+w)\alpha_6}{\alpha_3}$, we can suppose $\alpha_1^* = 0$. Now, we consider following subcases:

(a) $\alpha_4 = 0,\ \alpha_2 = 0$, then we have the family of representatives $\langle \beta\nabla_3 + \nabla_6 \rangle_{\beta \neq 0}$,

(b) $\alpha_4 = 0, \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_6}$, we have the family of representatives $\langle \nabla_2 + \beta\nabla_3 + \nabla_6 \rangle_{\beta \neq 0}$,

(c) $\alpha_4 \neq 0$, then choosing $x = 1,\ u = \frac{\alpha_2}{\alpha_4}$, we obtain $\alpha_2^* = 0$ and obtain the family of representatives $\langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_6 \rangle_{\alpha=0, \beta \neq 0, \gamma \neq 0}$.

2.3.7. Central Extensions of $\mathfrak{N}_{08}^1$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{13} - \Delta_{23}], \quad \nabla_4 = [\Delta_{14} - \Delta_{24}],$$
$$\nabla_5 = [\Delta_{31}], \quad \nabla_6 = [\Delta_{42}], \quad \nabla_7 = [\Delta_{32} + \Delta_{41}].$$

Take $\theta = \sum\limits_{i=1}^{7} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{08}^1)$. The automorphism group of $\mathfrak{N}_{08}^1$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & y & 0 & 0 \\ x+y-z & z & 0 & 0 \\ t & v & x(z-y) & y(z-y) \\ u & w & (x+y-z)(z-y) & z(z-y) \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_3 & \alpha_4 \\ \alpha_2 & 0 & -\alpha_3 & -\alpha_4 \\ \alpha_5 & \alpha_7 & 0 & 0 \\ \alpha_7 & \alpha_6 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* + \alpha^{**} & \alpha_3^* & \alpha_4^* \\ \alpha_2^* - \alpha^* & -\alpha^{**} & -\alpha_3^* & -\alpha_4^* \\ \alpha_5^* & \alpha_7^* & 0 & 0 \\ \alpha_7^* & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{08}^1)$ on the subspace $\langle \sum\limits_{i=1}^{7} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{7} \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned}
\alpha_1^* &= (x+y)z\alpha_1 + y(x+y)\alpha_2 + y(t+v)\alpha_5 + z(u+w)\alpha_6 + (y(u+w) + z(t+v))\alpha_7, \\
\alpha_2^* &= (x+y)(x+y-z)\alpha_1 + x(x+y)\alpha_2 + x(t+v)\alpha_5 + (x+y-z)(u+w)\alpha_6 \\
&\qquad + ((x+y-z)(t+v) + x(u+w))\alpha_7, \\
\alpha_3^* &= (y-z)^2(x\alpha_3 + (x+y-z)\alpha_4), \\
\alpha_4^* &= (y-z)^2(y\alpha_3 + z\alpha_4), \\
\alpha_5^* &= (z-y)(x^2\alpha_5 + (x+y-z)((x+y-z)\alpha_6 + 2x\alpha_7)), \\
\alpha_6^* &= (z-y)(y^2\alpha_5 + z(z\alpha_6 + 2y\alpha_7)), \\
\alpha_7^* &= (z-y)(xy\alpha_5 + (x+y-z)z\alpha_6 + (y(y-z) + x(y+z))\alpha_7).
\end{aligned}$$

We are interested only in the cases with

$$(\alpha_3, \alpha_5, \alpha_7) \neq (0,0,0), (\alpha_4, \alpha_6, \alpha_7) \neq (0,0,0).$$

1. $(\alpha_5, \alpha_6, \alpha_7) = (0,0,0)$, then $\alpha_3 \neq 0$, $\alpha_4 \neq 0$. If $\alpha_3 \neq -\alpha_4$, then choosing $z = -\frac{y\alpha_3}{\alpha_4}$, we obtain that $\alpha_4^* = 0$, which implies $(\alpha_4^*, \alpha_6^*, \alpha_7^*) = (0,0,0)$. Thus, we have that $\alpha_3 = -\alpha_4$.
   (a) $(\alpha_1, \alpha_2) = (0,0)$, then we have the representative $\langle \nabla_3 - \nabla_4 \rangle$;
   (b) $(\alpha_1, \alpha_2) \neq (0,0)$, without loss of generality, we can suppose $\alpha_1 \neq 0$.
      i. $\alpha_1 = -\alpha_2$, then choosing $x = \frac{\alpha_3}{\alpha_1}$, $y = 0$, $z = 1$, we have the representative $\langle \nabla_1 - \nabla_2 + \nabla_3 - \nabla_4 \rangle$;
      ii. $\alpha_1 \neq -\alpha_2$, then choosing $x = \frac{\alpha_1^3}{(\alpha_1+\alpha_2)^2\alpha_3}$, $y = 0$, $z = \frac{\alpha_1^2}{(\alpha_1+\alpha_2)\alpha_3}$, we have the representative $\langle \nabla_1 + \nabla_3 - \nabla_4 \rangle$.
2. $(\alpha_5, \alpha_6, \alpha_7) \neq (0,0,0)$, then without loss of generality we can assume $\alpha_5 \neq 0$ and consider following subcases:
   (a) $\alpha_6\alpha_5 = \alpha_7^2, \alpha_7 = -\alpha_5$, $\alpha_4 = -\alpha_3$, $\alpha_2 = -\alpha_1$, then taking $x = 1$, $y = u = v = w = 0$, $t = \frac{\alpha_1}{\alpha_5}$, we have the family of representatives $\langle \beta\nabla_3 - \beta\nabla_4 + \nabla_5 + \nabla_6 - \nabla_7 \rangle$;
   (b) $\alpha_6\alpha_5 = \alpha_7^2, \alpha_7 = -\alpha_5$, $\alpha_4 = -\alpha_3$, $\alpha_1 \neq -\alpha_2$, then taking
   $$x = z = \frac{(\alpha_1+\alpha_2)}{\alpha_5}, y = u = v = w = 0, t = \frac{\alpha_1(\alpha_1+\alpha_2)}{\alpha_5^2},$$
   we have the family of representatives $\langle \nabla_2 + \beta\nabla_3 - \beta\nabla_4 + \nabla_5 + \nabla_6 - \nabla_7 \rangle$;
   (c) $\alpha_6\alpha_5 = \alpha_7^2$, $\alpha_7 = -\alpha_5$, $\alpha_3 \neq -\alpha_4$, then we can suppose $\alpha_3 \neq 0$ and choosing $u = v = w = 0$, $y = -\frac{z\alpha_4}{\alpha_3}$, $t = \frac{(x\alpha_3 - z\alpha_4)(\alpha_1\alpha_3 - \alpha_2\alpha_4)}{\alpha_3(\alpha_3+\alpha_4)\alpha_5}$, we can suppose $\alpha_1 = \alpha_4 = 0$.
      i. if $\alpha_2 = 0$, then choosing $x = 1$, $z = \frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5 + \nabla_6 - \nabla_7 \rangle$;
      ii. if $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2\alpha_5^2}{\alpha_3^3}$, $z = \frac{\alpha_2\alpha_5}{\alpha_3^2}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5 + \nabla_6 - \nabla_7 \rangle$.
   (d) $\alpha_6\alpha_5 = \alpha_7^2, \alpha_5 \neq -\alpha_7$, then choosing
   $$y = -\frac{z\alpha_7}{\alpha_5}, t = u = w = 0, v = \frac{\alpha_1((z-x)\alpha_5 + z\alpha_7) - x\alpha_2\alpha_5}{\alpha_5(\alpha_5+\alpha_7)},$$
   we can suppose $\alpha_2 = \alpha_6 = \alpha_7 = 0$. Since $(\alpha_4, \alpha_6, \alpha_7) \neq (0,0,0)$, we have that $\alpha_4 \neq 0$.
      i. $\alpha_1 = 0$, then choosing $x = \sqrt{\frac{\alpha_4}{\alpha_5}}$, $z = 1$, we have the family of representatives $\langle \beta\nabla_3 + \nabla_4 + \nabla_5 \rangle$;
      ii. $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_5}$, $z = \frac{\alpha_1}{\sqrt{\alpha_4\alpha_5}}$, we have the family of representatives $\langle \nabla_1 + \beta\nabla_3 + \nabla_4 + \nabla_5 \rangle$.
   (e) $\alpha_6\alpha_5 \neq \alpha_7^2$, then choosing suitable value of z and y such that $y \neq z$, we can suppose $\alpha_6 = 0$ and $\alpha_7^* \neq 0$. Then, choosing $t = -\frac{x\alpha_1}{\alpha_7}$, $y = u = v = 0$, $w = \frac{x(\alpha_1\alpha_5 - \alpha_2\alpha_7)}{\alpha_7^2}$, we have $\alpha_1^* = \alpha_2^* = 0$.
      i. if $\alpha_5 \neq -2\alpha_7$, then choosing $x = 1$, $z = \frac{\alpha_5+2\alpha_7}{3\alpha_7}$, we have the family of representatives $\langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_5 + \nabla_7 \rangle$;
      ii. if $\alpha_5 = -2\alpha_7$, then then we have the family of representatives $\langle \beta\nabla_3 - 2\nabla_5 + \nabla_7 \rangle$ and $\langle \beta\nabla_3 + \nabla_4 - 2\nabla_5 + \nabla_7 \rangle$ depending on whether $\alpha_4 = 0$ or not.

Summarizing all cases of the central extension of the algebra $\mathfrak{N}_{08}^{\alpha}$, we have the following distinct orbits,

in case of $\alpha \neq 1$:
$$\langle \nabla_3 + \beta\nabla_4 \rangle_{\beta \neq 0}, \langle \nabla_1 + \beta\nabla_4 + \nabla_5 \rangle_{\beta \neq 0}, \langle \delta\nabla_3 + \gamma\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{(\beta,\gamma) \neq (0,0)},$$
in case of $\alpha = 0$:
$$\langle \nabla_2 + \beta\nabla_3 + \nabla_6 \rangle_{\beta \neq 0}, \langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_6 \rangle_{\beta \neq 0},$$

in case of $\alpha = 1$:

$$\langle \nabla_3 - \nabla_4 \rangle, \langle \nabla_1 + \nabla_3 - \nabla_4 \rangle, \langle \nabla_1 - \nabla_2 + \nabla_3 - \nabla_4 \rangle, \langle \beta\nabla_3 - \beta\nabla_4 + \nabla_5 + \nabla_6 - \nabla_7 \rangle,$$
$$\langle \nabla_2 + \beta\nabla_3 - \beta\nabla_4 + \nabla_5 + \nabla_6 - \nabla_7 \rangle, \langle \nabla_3 + \nabla_5 + \nabla_6 - \nabla_7 \rangle, \langle \nabla_2 + \nabla_3 + \nabla_5 + \nabla_6 - \nabla_7 \rangle,$$
$$\langle \beta\nabla_3 + \nabla_4 + \nabla_5 \rangle, \langle \nabla_1 + \beta\nabla_3 + \nabla_4 + \nabla_5 \rangle, \langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_5 + \nabla_7 \rangle, \langle \beta\nabla_3 - 2\nabla_5 + \nabla_7 \rangle,$$
$$\langle \beta\nabla_3 + \nabla_4 - 2\nabla_5 + \nabla_7 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{60}^{\alpha\neq 1,\beta\neq 0}, \mathrm{B}_{61}^{\alpha\neq 1,\beta\neq 0}, \mathrm{B}_{62}^{\alpha\neq 1,(\beta,\gamma)\neq(0,0),\delta}, \mathrm{B}_{63}^{\beta\neq 0}, \mathrm{B}_{64}^{\beta\neq 0,\gamma}, \mathrm{B}_{65},$$
$$\mathrm{B}_{66}, \mathrm{B}_{67}, \mathrm{B}_{68}^{\beta}, \mathrm{B}_{69}^{\beta}, \mathrm{B}_{70}, \mathrm{B}_{71}, \mathrm{B}_{72}^{\beta}, \mathrm{B}_{73}^{\beta}, \mathrm{B}_{74}^{\beta,\gamma}, \mathrm{B}_{75}^{\beta}, \mathrm{B}_{76}^{\beta}.$$

2.3.8. Central Extensions of $\mathfrak{N}_{12}$

Let us use the following notations: $\begin{array}{lll} \nabla_1 = [\Delta_{11}], & \nabla_2 = [\Delta_{13}], & \nabla_3 = [\Delta_{22}], \\ \nabla_4 = [\Delta_{24}], & \nabla_5 = [\Delta_{32}], & \nabla_6 = [\Delta_{41}]. \end{array}$

Take $\theta = \sum\limits_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{12})$. The automorphism group of $\mathfrak{N}_{12}$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ z & v & xy & 0 \\ u & t & 0 & xy \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 & x & 0 & 0 \\ y & 0 & 0 & 0 \\ z & v & 0 & xy \\ u & t & xy & 0 \end{pmatrix}.$$

Since

$$\phi_1^T \begin{pmatrix} \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & \alpha_3 & 0 & \alpha_4 \\ 0 & \alpha_5 & 0 & 0 \\ \alpha_6 & 0 & 0 & 0 \end{pmatrix} \phi_1 = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_2^* & 0 \\ \alpha^{**} & \alpha_3^* & 0 & \alpha_4^* \\ 0 & \alpha_5^* & 0 & 0 \\ \alpha_6^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{12})$ on the subspace $\langle \sum\limits_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

for ϕ_1: $\begin{array}{lllllllll} \alpha_1^* & = & x(x\alpha_1 + z\alpha_2 + t\alpha_6) & \alpha_3^* & = & y(y\alpha_3 + v\alpha_4 + u\alpha_5), & \alpha_5^* & = & xy^2\alpha_5, \\ \alpha_2^* & = & x^2y\alpha_2 & \alpha_4^* & = & xy^2\alpha_4, & \alpha_6^* & = & x^2y\alpha_6, \end{array}$

for ϕ_2: $\begin{array}{lllllllll} \alpha_1^* & = & y(y\alpha_3 + u\alpha_4 + z\alpha_5) & \alpha_3^* & = & x(x\alpha_1 + v\alpha_2 + t\alpha_6), & \alpha_5^* & = & x^2y\alpha_6, \\ \alpha_2^* & = & xy^2\alpha_4 & \alpha_4^* & = & x^2y\alpha_2, & \alpha_6^* & = & xy^2\alpha_5, \end{array}$

We are interested only in the cases with

$$(\alpha_2, \alpha_5) \neq (0,0), (\alpha_4, \alpha_6) \neq (0,0).$$

1. $(\alpha_2, \alpha_4) = (0,0)$, then $\alpha_5 \neq 0$, $\alpha_6 \neq 0$ and choosing $x = 1$, $y = \frac{\alpha_6}{\alpha_5}$, $t = -\frac{\alpha_1}{\alpha_6}$, $u = -\frac{\alpha_3\alpha_6}{\alpha_5^2}$, we have the representative $\langle \nabla_5 + \nabla_6 \rangle$;
2. $(\alpha_2, \alpha_4) \neq (0,0)$, then without loss of generality, we can suppose $\alpha_4 \neq 0$ and choosing $v = -\frac{y\alpha_3 + u\alpha_5}{\alpha_4}$, we have $\alpha_3^* = 0$.
   (a) $\alpha_6 = \alpha_2 = \alpha_1 = 0$, then we have the family of representatives $\langle \nabla_4 + \alpha\nabla_5 \rangle_{\alpha\neq 0}$;
   (b) $\alpha_6 = \alpha_2 = 0$, $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_1}$, $y = 1$, we have the family of representatives $\langle \nabla_1 + \nabla_4 + \alpha\nabla_5 \rangle_{\alpha\neq 0}$;
   (c) $\alpha_6 = 0$, $\alpha_2 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_2}{\alpha_4}$, $z = -\frac{\alpha_1}{\alpha_2}$, we have the family of representatives $\langle \nabla_2 + \nabla_4 + \alpha\nabla_5 \rangle$;
   (d) $\alpha_6 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_6}{\alpha_4}$, $z = 0$, $t = -\frac{\alpha_1}{\alpha_6}$, we have the family of representatives $\langle \beta\nabla_2 + \nabla_4 + \alpha\nabla_5 + \nabla_6 \rangle_{(\alpha,\beta)\neq(0,0)}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_5 + \nabla_6 \rangle, \langle \nabla_4 + \alpha\nabla_5 \rangle_{\alpha\neq 0}, \langle \nabla_1 + \nabla_4 + \alpha\nabla_5 \rangle_{\alpha\neq 0}, \langle \nabla_2 + \nabla_4 + \alpha\nabla_5 \rangle,$$
$$\langle \beta\nabla_2 + \nabla_4 + \alpha\nabla_5 + \nabla_6 \rangle_{(\alpha,\beta)\neq(0,0)}^{O(\alpha,\beta)\simeq O(\beta^{-1},\alpha^{-1})},$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{77}, \mathrm{B}_{78}^{\alpha\neq 0}, \mathrm{B}_{79}^{\alpha\neq 0}, \mathrm{B}_{80}, \mathrm{B}_{81}^{(\alpha,\beta)\neq(0,0)}.$$

2.3.9. Central Extensions of $\mathfrak{N}_{13}$

Let us use the following notations:

$$\begin{array}{lll} \nabla_1 = [\Delta_{21}], & \nabla_2 = [\Delta_{22}], & \nabla_3 = [\Delta_{14} + \Delta_{23}], \\ \nabla_4 = [\Delta_{13} - 2\Delta_{14} - \Delta_{24}], & \nabla_5 = [\Delta_{32} - \Delta_{41}], & \nabla_6 = [\Delta_{31} - 2\Delta_{32} + \Delta_{42}]. \end{array}$$

Take $\theta = \sum\limits_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{13})$. The automorphism group of $\mathfrak{N}_{13}$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ z & u & x^2 & 0 \\ t & v & 0 & x^2 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 & x & 0 & 0 \\ x & 0 & 0 & 0 \\ z & u & -x^2 & 2x^2 \\ t & v & 0 & x^2 \end{pmatrix}.$$

Since

$$\phi_1^T \begin{pmatrix} 0 & 0 & \alpha_4 & \alpha_3 - 2\alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & -\alpha_4 \\ \alpha_6 & \alpha_5 - \alpha_6 & 0 & 0 \\ -\alpha_5 & \alpha_6 & 0 & 0 \end{pmatrix} \phi_1 = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_4^* & \alpha_3^* - 2\alpha_4^* \\ \alpha_1^* - \alpha^{**} & \alpha_2^* + \alpha^* + 2\alpha^{**} & \alpha_3^* & -\alpha_4^* \\ \alpha_6^* & \alpha_5^* - \alpha_6^* & 0 & 0 \\ -\alpha_5^* & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{13})$ on the subspace $\langle \sum\limits_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\text{for } \phi_1: \begin{array}{lll} \alpha_1^* & = & x(x\alpha_1 + (v+z)\alpha_3 - (t-u+2v)\alpha_4 - (v-z)\alpha_5 + (t+u-z)\alpha_6), \\ \alpha_2^* & = & x(x\alpha_2 + (-t+u-2v)\alpha_3 + (2t-2u+3v-z)\alpha_4 + \\ & & \qquad (t+u-2z)\alpha_5 - (2t+u-v-z)\alpha_6), \\ \alpha_3^* & = & x^3\alpha_3, \\ \alpha_4^* & = & x^3\alpha_4, \\ \alpha_5^* & = & x^3\alpha_5, \\ \alpha_6^* & = & x^3\alpha_6. \end{array}$$

$$\text{for } \phi_2: \begin{array}{lll} \alpha_1^* & = & x(x\alpha_1 + (t+u)\alpha_3 - (2t+v-z)\alpha_4 - (t-u)\alpha_5 - (u-v-z)\alpha_6), \\ \alpha_2^* & = & -x(2x\alpha_1 + x\alpha_2 + (2u-v+z)\alpha_3 - (t+u)\alpha_4 - (2t-v-z)\alpha_5 + (t-u+z)\alpha_6), \\ \alpha_3^* & = & -x^3\alpha_4, \\ \alpha_4^* & = & -x^3\alpha_3, \\ \alpha_5^* & = & -x^3(2\alpha_5 - \alpha_6), \\ \alpha_6^* & = & -x^3(\alpha_5 - 2\alpha_6). \end{array}$$

We are interested only in the cases with

$$(\alpha_3, \alpha_4, \alpha_5, \alpha_6) \neq (0,0,0,0).$$

1. $(\alpha_5, \alpha_6) = (0,0)$, then without loss of generality, we can suppose $\alpha_3 \neq 0$. Let us consider the following subcases:
   (a) $\alpha_3 = \alpha_4,\ \alpha_1 = -\alpha_2$, then choosing $x = 1,\ u = v = z = 0, t = \frac{\alpha_1}{\alpha_3}$, we have the representative $\langle \nabla_3 + \nabla_4 \rangle$;
   (b) $\alpha_3 = \alpha_4,\ \alpha_1 \neq -\alpha_2$, then choosing $x = \frac{\alpha_1+\alpha_2}{\alpha_3}$, $u = v = z = 0$, $t = \frac{(\alpha_1+\alpha_2)\alpha_1}{\alpha_3^2}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_4 \rangle$;
   (c) $\alpha_3 \neq \alpha_4$, then choosing $x = 1$, $u = v = 0$, $z = \frac{\alpha_1(2\alpha_4-\alpha_3)+\alpha_2\alpha_4}{(\alpha_3-\alpha_4)^2}$, $t = \frac{\alpha_2\alpha_3+\alpha_1\alpha_4}{(\alpha_3-\alpha_4)^2}$, we have the family of representatives $\langle \nabla_3 + \alpha\nabla_4 \rangle_{\alpha\neq 1}$;

2. $(\alpha_5, \alpha_6) \neq (0,0)$, then without loss of generality, we can suppose $\alpha_6 \neq 0$. Let us consider the following subcases:
   (a) $\alpha_4 = \alpha_6,\ \alpha_3 = \alpha_5$, then choosing
   $$x = 1, z = 0, u = \frac{\alpha_1(\alpha_3 - 2\alpha_6) - \alpha_2\alpha_6}{\alpha_6^2}, v = \frac{\alpha_1(2\alpha_3 - 3\alpha_6) - 2\alpha_2\alpha_6}{2\alpha_6^2},$$
   we have the family of representatives $\langle \alpha\nabla_3 + \nabla_4 + \alpha\nabla_5 + \nabla_6 \rangle$;
   (b) $\alpha_4 = \alpha_6,\ \alpha_3 \neq \alpha_5$, then choosing $x = 1,\ z = v = 0,\ t = -\frac{x(\alpha_1(\alpha_3+\alpha_5-3\alpha_6)-2\alpha_2\alpha_6)}{2(\alpha_3-\alpha_5)\alpha_6}$ $u = -\frac{\alpha_1}{2\alpha_6}$, we have the family of representatives $\langle \alpha\nabla_3 + \nabla_4 + \beta\nabla_5 + \nabla_6 \rangle_{\beta\neq\alpha}$;
   (c) $\alpha_4 - \alpha_6 \neq 0$, then choosing $t = \frac{x\alpha_1+(v+z)\alpha_3+u\alpha_4-2v\alpha_4-v\alpha_5+z\alpha_5+u\alpha_6-z\alpha_6}{\alpha_4-\alpha_6}$, we can suppose $\alpha_1^* = 0$, and consider following subcases:
      i. $\alpha_6(2\alpha_3 + \alpha_6) = \alpha_4(2\alpha_5 + \alpha_6)$.
         A. $\alpha_6 = 2\alpha_5$, then choosing $x = 1,\ z = 0,\ v = -\frac{\alpha_2}{2\alpha_5}$, we have the family of representatives $\left\langle (\alpha - \frac{1}{2})\nabla_3 + \alpha\nabla_4 + \frac{1}{2}\nabla_5 + \nabla_6 \right\rangle_{\alpha\neq1}$;
         B. $\alpha_6 \neq 2\alpha_5,\ \alpha_4 = 0,\ 4\alpha_5^2 - 4\alpha_5\alpha_6 + 5\alpha_6^2 = 0, \alpha_2 = 0$, then we have the representative $\left\langle -\frac{1}{2}\nabla_3 + (\frac{1}{2} \pm i)\nabla_5 + \nabla_6 \right\rangle$;
         C. $\alpha_6 \neq 2\alpha_5,\ \alpha_4 = 0,\ 4\alpha_5^2 - 4\alpha_5\alpha_6 + 5\alpha_6^2 = 0,\ \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_6}$, we have the representative $\left\langle \nabla_2 - \frac{1}{2}\nabla_3 + (\frac{1}{2} \pm i)\nabla_5 + \nabla_6 \right\rangle$;
         D. $\alpha_6 \neq 2\alpha_5,\ \alpha_4(\alpha_6 - 2\alpha_5)^2 = \alpha_6(4\alpha_5\alpha_6 - 4\alpha_5^2 - 5\alpha_6^2),\ 4\alpha_5^2 \neq 4\alpha_5\alpha_6 + 5\alpha_6^2$, then choosing $x = 1,\ v = -\frac{2\alpha_2\alpha_6}{4\alpha_5^2 - 4\alpha_5\alpha_6 + 5\alpha_6^2}$, we have the family of representatives
         $$\left\langle -\frac{(3+\alpha+4\alpha^3)}{(1-2\alpha)^2}\nabla_3 - \frac{(5-4\alpha+4\alpha^2)}{(1-2\alpha)^2}\nabla_4 + \alpha\nabla_5 + \nabla_6 \right\rangle_{\alpha\neq\frac{1}{2},\frac{1}{2}\pm i};$$
         E. $\alpha_6 \neq 2\alpha_5,\ \alpha_4(\alpha_6 - 2\alpha_5)^2 + \alpha_6(4\alpha_5^2 - 4\alpha_5\alpha_6 + 5\alpha_6^2) \neq 0$, then choosing $x = 1,\ z = \frac{4\alpha_2\alpha_6^2}{\alpha_4(2\alpha_5-\alpha_6)^2+\alpha_6(4\alpha_5^2-4\alpha_5\alpha_6+5\alpha_6^2)},\ v = 0$, we have the family of representatives
         $$\left\langle \frac{1}{2}(\beta(1 + 2\alpha) + \alpha)\nabla_3 + \beta\nabla_4 + \alpha\nabla_5 + \nabla_6 \right\rangle_{\alpha\neq\frac{1}{2},\ \beta\neq1,-\frac{(5-4\alpha+4\alpha^2)}{(1-2\alpha)^2}}.$$
      ii. $\alpha_6(2\alpha_3 + \alpha_6) \neq \alpha_4(2\alpha_5 + \alpha_6)$, then choosing $u = \frac{\alpha_2(\alpha_4-\alpha_6)}{\alpha_6(2\alpha_3+\alpha_6)-\alpha_4(2\alpha_5+\alpha_6)}$, $x = 1, z = v = 0$, we have the family of representatives
      $$\langle \gamma\nabla_3 + \beta\nabla_4 + \alpha\nabla_5 + \nabla_6 \rangle_{\gamma\neq\frac{\beta(2\alpha+1)-1}{2},\ \beta\neq1}.$$

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_4 \rangle, \langle \nabla_3 + \alpha\nabla_4 \rangle^{O(\alpha)\simeq O(\alpha^{-1})}, \langle \nabla_1 + \nabla_3 + \nabla_4 \rangle, \langle \nabla_2 - \tfrac{1}{2}\nabla_3 + (\tfrac{1}{2} \pm i)\nabla_5 + \nabla_6 \rangle,$$
$$\langle \gamma\nabla_3 + \alpha\nabla_4 + \beta\nabla_5 + \nabla_6 \rangle^{O(\alpha,\beta,\gamma)\simeq O(\frac{\gamma}{\beta-2},\frac{1-2\beta}{2-\beta},\frac{\alpha}{\beta-2})},$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{82}, \mathrm{B}_{83}^{\alpha}, \mathrm{B}_{84}, \mathrm{B}_{85}, \mathrm{B}_{86}, \mathrm{B}_{87}^{\alpha,\beta,\gamma}.$$

2.3.10. Central Extensions of $\mathfrak{N}_{14}^0$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{23}], \quad \nabla_4 = [\Delta_{13} + \Delta_{24}],$$
$$\nabla_5 = [\Delta_{32}], \quad \nabla_6 = [\Delta_{42}], \quad \nabla_7 = [\Delta_{14}].$$

Take $\theta = \sum_{i=1}^{7} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{14}^0)$. The automorphism group of $\mathfrak{N}_{14}^0$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ w & u & y^2 & 0 \\ t & v & yz & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_4 & \alpha_7 \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 \\ 0 & \alpha_5 & 0 & 0 \\ 0 & \alpha_6 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_4^* & \alpha_7^* \\ \alpha_2^* & \alpha^{**} & \alpha_3^* & \alpha_4^* \\ 0 & \alpha_5^* & 0 & 0 \\ 0 & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{14}^0)$ on the subspace $\langle \sum_{i=1}^{7} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{7} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rcl}
\alpha_1^* & = & x(x\alpha_1 + w\alpha_4 + t\alpha_7), \\
\alpha_2^* & = & xz\alpha_1 + xy\alpha_2 + wy\alpha_3 + (ty + wz)\alpha_4 + tz\alpha_7, \\
\alpha_3^* & = & y(y^2\alpha_3 + 2yz\alpha_4 + z^2\alpha_7)), \\
\alpha_4^* & = & xy(y\alpha_4 + z\alpha_7), \\
\alpha_5^* & = & y^2(y\alpha_5 + z\alpha_6), \\
\alpha_6^* & = & xy^2\alpha_6, \\
\alpha_7^* & = & x^2y\alpha_7.
\end{array}$$

We are interested only in the cases with

$$(\alpha_3, \alpha_4, \alpha_5) \neq (0,0,0), (\alpha_4, \alpha_6, \alpha_7) \neq (0,0,0).$$

1. $\alpha_7 \neq 0$, then choosing $z = -\frac{y\alpha_4}{\alpha_7}$, $t = -\frac{x\alpha_1 + w\alpha_4}{\alpha_7}$, we have $\alpha_1^* = \alpha_4^* = 0$. Thus, we can suppose $\alpha_1 = \alpha_4 = 0$ and consider following subcases:
   (a) $\alpha_3 \neq 0$, then choosing $x = 1$, $y = \sqrt{\frac{\alpha_7}{\alpha_3}}$, $w = -\frac{\alpha_2}{\alpha_3}$, we have the family of representatives $\langle \nabla_3 + \beta\nabla_4 + \gamma\nabla_6 + \nabla_7 \rangle$;
   (b) $\alpha_3 = 0$, then $\alpha_5 \neq 0$.
      i. $\alpha_2 = 0$, then choosing $x = 1$, $y = \sqrt{\frac{\alpha_7}{\alpha_5}}$, we have the family of representatives $\langle \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle$;
      ii. $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_7}$, $y = \frac{\alpha_2}{\sqrt{\alpha_5\alpha_7}}$, we have the family of representatives $\langle \nabla_2 + \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle$.
2. $\alpha_7 = 0$, $\alpha_4 \neq 0$, then choosing $z = -\frac{y\alpha_3}{2\alpha_4}$, $t = \frac{x(\alpha_1\alpha_3 - \alpha_2\alpha_4)}{\alpha_4^2}$, $w = -\frac{x\alpha_1}{\alpha_4}$, we have $\alpha_1^* = \alpha_2^* = \alpha_3^* = 0$ and consider following subcases:
   (a) $\alpha_5 = 0$, then we have the family of representatives $\langle \nabla_4 + \beta\nabla_6 \rangle$;
   (b) $\alpha_5 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_4}{\alpha_5}$, have the family of representatives $\langle \nabla_4 + \nabla_5 + \beta\nabla_6 \rangle$.
3. $\alpha_7 = \alpha_4 = 0$, then $\alpha_6 \neq 0$ and choosing $z = -\frac{y\alpha_5}{\alpha_6}$, we have $\alpha_5^* = 0$. Thus we obtain that $\alpha_5 = 0$, which implies $\alpha_3 \neq 0$. Then choosing $w = -\frac{x\alpha_2}{\alpha_3}$, we have $\alpha_2 = 0$ and obtain the representatives $\langle \nabla_3 + \nabla_6 \rangle$ and $\langle \nabla_1 + \nabla_3 + \nabla_6 \rangle$ depending on whether $\alpha_1 = 0$ or not.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_3 + \beta\nabla_4 + \gamma\nabla_6 + \nabla_7 \rangle^{O(\beta,\gamma) \simeq O(\beta,-\gamma)}, \langle \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle^{O(\beta) \simeq O(-\beta)},$$
$$\langle \nabla_2 + \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle^{O(\beta) \simeq O(-\beta)}, \langle \nabla_4 + \beta\nabla_6 \rangle, \langle \nabla_4 + \nabla_5 + \beta\nabla_6 \rangle, \langle \nabla_3 + \nabla_6 \rangle,$$
$$\langle \nabla_1 + \nabla_3 + \nabla_6 \rangle.$$

2.3.11. Central Extensions of $\mathfrak{N}_{14}^{\alpha\neq 0}$

Let us use the following notations:

$$\begin{array}{lll} \nabla_1 = [\Delta_{11}], & \nabla_2 = [\Delta_{21}], & \nabla_3 = [\Delta_{23}], \\ \nabla_4 = [\Delta_{13} + \Delta_{24}], & \nabla_5 = [\Delta_{32}], & \nabla_6 = [\alpha\Delta_{31} + \Delta_{42}]. \end{array}$$

Take $\theta = \sum\limits_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{14}^{\alpha\neq 0})$. The automorphism group of $\mathfrak{N}_{14}^{\alpha\neq 0}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ w & u & y^2 & 0 \\ t & v & (1+\alpha)yz & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_4 & 0 \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 \\ \alpha\alpha_6 & \alpha_5 & 0 & 0 \\ 0 & \alpha_6 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_4^* & 0 \\ \alpha_2^* + \alpha\alpha^* & \alpha^{**} & \alpha_3^* & \alpha_4^* \\ \alpha\alpha_6^* & \alpha_5^* & 0 & 0 \\ 0 & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{14}^{\alpha\neq 0})$ on the subspace $\langle \sum\limits_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rcl} \alpha_1^* &=& x(x\alpha_1 + w(\alpha_4 + \alpha\alpha_6)), \\ \alpha_2^* &=& xz(1-\alpha)\alpha_1 + xy\alpha_2 + wy\alpha_3 + \\ && (ty + wz - ux\alpha)\alpha_4 - wy\alpha\alpha_5 - \alpha(ty - ux + wz\alpha)\alpha_6, \\ \alpha_3^* &=& y^2(y\alpha_3 + z(2+\alpha)\alpha_4), \\ \alpha_4^* &=& xy^2\alpha_4, \\ \alpha_5^* &=& y^2(y\alpha_5 + z(1+2\alpha)\alpha_6), \\ \alpha_6^* &=& xy^2\alpha_6. \end{array}$$

We are interested only in the cases with

$$(\alpha_4, \alpha_6) \neq (0,0).$$

1. $\alpha_4 = 0$, then $\alpha_6 \neq 0$ and choosing $w = -\frac{x\alpha_1}{\alpha\alpha_6}$, $t = \frac{x(\alpha\alpha_6(y\alpha_2 + u\alpha\alpha_6) - \alpha_1(y\alpha_3 - y\alpha\alpha_5 - z\alpha\alpha_6))}{y\alpha^2\alpha_6^2}$, we have $\alpha_1^* = \alpha_2^* = 0$. Thus, we can suppose $\alpha_1 = \alpha_2 = 0$ and consider following subcases:
   (a) $\alpha = -\frac{1}{2}$, $\alpha_5 = 0$, $\alpha_3 = 0$, then we have the representative $\langle \nabla_6 \rangle_{\alpha=-\frac{1}{2}}$;
   (b) $\alpha = -\frac{1}{2}$, $\alpha_5 = 0$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_6}$, $y = 1$, we have the representative $\langle \nabla_3 + \nabla_6 \rangle_{\alpha=-\frac{1}{2}}$;
   (c) $\alpha = -\frac{1}{2}$, $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_6}$, $y = 1$, we have the family of representatives $\langle \beta\nabla_3 + \nabla_5 + \nabla_6 \rangle_{\alpha=-\frac{1}{2}}$;
   (d) $\alpha \neq -\frac{1}{2}$, $\alpha_3 = 0$, then choosing $y = 1$, $z = -\frac{\alpha_5}{\alpha_6(1+2\alpha)}$, we have the representative $\langle \nabla_6 \rangle_{\alpha\neq-\frac{1}{2}}$;
   (e) $\alpha \neq -\frac{1}{2}$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_6}$, $y = 1$, $z = -\frac{\alpha_5}{\alpha_6(1+2\alpha)}$, we have the representative $\langle \nabla_3 + \nabla_6 \rangle_{\alpha\neq-\frac{1}{2}}$;
2. $\alpha_4 \neq 0$, then consider following subcases:
   (a) $\alpha = -2$, $\alpha_4 = 2\alpha_6$, then choosing $z = \frac{2y\alpha_1}{3\alpha_4}$, $t = -\frac{x(\alpha_1\alpha_5 + \alpha_2\alpha_6)}{4\alpha_6^2}$, $u = w = 0$, we can suppose $\alpha_2^* = \alpha_5^* = 0$ and consider following subcases:
      i. $\alpha_1 = \alpha_3 = 0$, we have the representative $\left\langle \nabla_4 + \frac{1}{2}\nabla_6 \right\rangle_{\alpha=-2}$;

ii. $\alpha_1 = 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_4},\ y = 1$, we have the representative $\left\langle \nabla_3 + \nabla_4 + \frac{1}{2}\nabla_6 \right\rangle_{\alpha=-2}$;

iii. $\alpha_1 \neq 0,\ \alpha_3 = 0$, then choosing $x = -\frac{\alpha_6}{\alpha_1},\ y = 1$, we have the representative $\langle \nabla_1 - 2\nabla_4 - \nabla_6 \rangle_{\alpha=-2}$;

iv. $\alpha_1 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_3^2}{\alpha_6^3},\ y = \frac{\alpha_1\alpha_3}{\alpha_6^2}$, we have the representative $\langle \nabla_1 + \nabla_3 + 2\nabla_4 + \nabla_6 \rangle_{\alpha=-2}$.

(b) $\alpha = -2, \alpha_4 \neq 2\alpha_6$, then choosing $w = -\frac{x\alpha_1}{\alpha_4 - 2\alpha_6}$, we can suppose $\alpha_1^* = 0$ and consider following subcases:

i. $\alpha_4 = \alpha_6, \alpha_3 = 0$, then choosing $x = 1,\ z = \frac{y\alpha_5}{3\alpha_4},\ t = -\frac{\alpha_2}{3\alpha_4}$, we have the representative $\langle \nabla_4 + \nabla_6 \rangle_{\alpha=-2}$;

ii. $\alpha_4 = \alpha_6, \alpha_3 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_3}{\alpha_4}\ z = \frac{y\alpha_5}{3\alpha_4},\ t = -\frac{\alpha_2}{3\alpha_4}$, we have the representative $\langle \nabla_3 + \nabla_4 + \nabla_6 \rangle_{\alpha=-2}$;

iii. $\alpha_4 \neq \alpha_6$, then choosing $t = 0,\ u = -\frac{y\alpha_2}{2(\alpha_4 - \alpha_6)}$, we have $\alpha_2^* = 0$.

A. $\alpha_6 \neq 0,\ \alpha_3 = 0$, then choosing $y = 1,\ z = \frac{\alpha_5}{3\alpha_6}$, we have the family of representatives $\langle \nabla_4 + \beta\nabla_6 \rangle_{\alpha=-2,\beta\neq 0,1}$;

B. $\alpha_6 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_4},\ y = 1,\ z = \frac{\alpha_5}{3\alpha_6}$, we have the family of representatives $\langle \nabla_3 + \nabla_4 + \beta\nabla_6 \rangle_{\alpha=-2,\beta\neq 0,1}$;

C. $\alpha_6 = 0,\ \alpha_5 = \alpha_3 = 0$, then we have the representative $\langle \nabla_4 \rangle_{\alpha=-2}$;

D. $\alpha_6 = 0,\ \alpha_5 = 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_4},\ y = 1$, we have the representative $\langle \nabla_3 + \nabla_4 \rangle_{\alpha=-2}$;

E. $\alpha_6 = 0,\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_4},\ y = 1$, we have the family of representatives $\langle \beta\nabla_3 + \nabla_4 + \nabla_5 \rangle_{\alpha=-2}$.

(c) $\alpha \neq -2$, then choosing $z = -\frac{y\alpha_3}{(2+\alpha)\alpha_4}$, we can suppose $\alpha_3^* = 0$ and consider following subcases:

i. $\alpha_4 + \alpha\alpha_6 = 0$, then choosing $t = -\frac{x\alpha_2}{2\alpha_4},\ u = 0,\ w = 0$, we can suppose $\alpha_2^* = 0$ and consider following subcases:

A. $\alpha_1 = \alpha_5 = 0$, then we have the family of representatives $\left\langle \nabla_4 - \frac{1}{\alpha}\nabla_6 \right\rangle$;

B. $\alpha_1 = 0,\ \alpha_5 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_4}{\alpha_5}$, we have the family of representatives $\left\langle \nabla_4 + \nabla_5 - \frac{1}{\alpha}\nabla_6 \right\rangle$;

C. $\alpha_1 \neq 0,\ \alpha_5 = 0$, then choosing $x = -\frac{\alpha_6}{\alpha_1},\ y = 1$, we have the family of representatives $\langle \nabla_1 + \alpha\nabla_4 - \nabla_6 \rangle$;

D. $\alpha_1 \neq 0,\ \alpha_5 \neq 0$, then choosing $x = -\frac{\alpha_1\alpha_5^2}{\alpha_6^3},\ y = \frac{\alpha_1\alpha_5}{\alpha_6^2}$, we have the family of representatives $\langle \nabla_1 + \alpha\nabla_4 + \nabla_5 - \nabla_6 \rangle$.

ii. $\alpha_4 + \alpha\alpha_6 \neq 0$, then choosing $w = -\frac{x\alpha_1}{\alpha_4 + \alpha\alpha_6}$, we can suppose $\alpha_1^* = 0$ and consider following subcases:

A. $\alpha_6 = \alpha_4, \alpha = 1, \alpha_2 = \alpha_5 = 0$, then we have the representative $\langle \nabla_4 + \nabla_6 \rangle_{\alpha=1}$;

B. $\alpha_6 = \alpha_4, \alpha = 1, \alpha_2 = 0, \alpha_5 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_4 + \nabla_5 + \nabla_6 \rangle_{\alpha=1}$;

C. $\alpha_6 = \alpha_4, \alpha = 1, \alpha_2 \neq 0, \alpha_5 = 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_6 \rangle_{\alpha=1}$;

D. $\alpha_6 = \alpha_4, \alpha = 1, \alpha_2 \neq 0, \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_2\alpha_5}{\alpha_4^2},\ y = \frac{\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_5 + \nabla_6 \rangle_{\alpha=1}$;

E. $\alpha_6 = \alpha_4, \alpha \neq 1, \alpha_5 = 0$, then choosing $x = 1,\ t = \frac{\alpha_2}{(\alpha-1)\alpha_4}$, we have the representative $\langle \nabla_4 + \nabla_6 \rangle_{\alpha\neq -1,1}$;

F. $\alpha_6 = \alpha_4, \alpha \neq 1, \alpha_5 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_4}{\alpha_5},\ t = \frac{\alpha_2}{(\alpha-1)\alpha_4}$, we have the representative $\langle \nabla_4 + \nabla_5 + \nabla_6 \rangle_{\alpha \neq -1,1}$;

G. $\alpha_6 \neq \alpha_4, \alpha_5 = 0$, then choosing $y = 1,\ t = 0,\ u = \frac{\alpha_2}{\alpha(\alpha_4 - \alpha_6)}$, we have the family of representatives $\langle \nabla_4 + \beta \nabla_6 \rangle_{\beta \neq 1}$;

H. $\alpha_6 \neq \alpha_4,\ \alpha_5 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_4}{\alpha_5},\ t = 0,\ u = \frac{\alpha_2 \alpha_4}{\alpha \alpha_5(\alpha_4 - \alpha_6)}$, we have the family of representatives $\langle \nabla_4 + \nabla_5 + \beta \nabla_6 \rangle_{\beta \neq 1}$;

Summarizing all cases of the central extension of the algebra $\mathfrak{N}_{14}^{\alpha}$, we have the following distinct orbits:

in case of $\alpha = -\frac{1}{2}$:

$$\langle \beta \nabla_3 + \nabla_5 + \nabla_6 \rangle,$$

in case of $\alpha = 1$:

$$\langle \nabla_2 + \nabla_4 + \nabla_6 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_5 + \nabla_6 \rangle,$$

in case of $\alpha = 0$:

$$\langle \nabla_3 + \beta \nabla_4 + \gamma \nabla_6 + \nabla_7 \rangle^{O(\beta,\gamma) \simeq O(\beta,-\gamma)}, \langle \nabla_2 + \nabla_5 + \beta \nabla_6 + \nabla_7 \rangle^{O(\beta) \simeq O(-\beta)},$$
$$\langle \nabla_5 + \beta \nabla_6 + \nabla_7 \rangle^{O(\beta) \simeq O(-\beta)}, \langle \nabla_1 + \nabla_3 + \nabla_6 \rangle,$$

in case of $\alpha = -2$:

$$\langle \nabla_1 + \nabla_3 + 2\nabla_4 + \nabla_6 \rangle, \langle \nabla_3 + \nabla_4 + \beta \nabla_6 \rangle, \langle \beta \nabla_3 + \nabla_4 + \nabla_5 \rangle,$$

for any $\alpha \in \mathbb{C}$:

$$\langle \nabla_3 + \nabla_6 \rangle, \langle \nabla_4 + \beta \nabla_6 \rangle, \langle \nabla_4 + \nabla_5 + \beta \nabla_6 \rangle_{\alpha \neq -2}, \langle \nabla_6 \rangle_{\alpha \neq 0}, \langle \nabla_4 - \tfrac{1}{\alpha} \nabla_6 \rangle_{\alpha \neq 0},$$
$$\langle \nabla_4 + \nabla_5 - \tfrac{1}{\alpha} \nabla_6 \rangle_{\alpha \neq 0}, \langle \nabla_1 + \alpha \nabla_4 - \nabla_6 \rangle_{\alpha \neq 0}, \langle \nabla_1 + \alpha \nabla_4 + \nabla_5 - \nabla_6 \rangle_{\alpha \neq 0,-2},$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{88}^{\beta}, \mathrm{B}_{89}, \mathrm{B}_{90}, \mathrm{B}_{91}^{\beta,\gamma}, \mathrm{B}_{92}^{\beta}, \mathrm{B}_{93}^{\beta}, \mathrm{B}_{94}, \mathrm{B}_{95}, \mathrm{B}_{96}^{\beta}, \mathrm{B}_{97}^{\beta}, \mathrm{B}_{98}^{\alpha}, \mathrm{B}_{99}^{\alpha,\beta},$$
$$\mathrm{B}_{100}^{\alpha \neq -2,\beta}, \mathrm{B}_{101}^{\alpha \neq 0}, \mathrm{B}_{102}^{\alpha \neq 0}, \mathrm{B}_{103}^{\alpha \neq 0}, \mathrm{B}_{104}^{\alpha \neq 0}, \mathrm{B}_{105}^{\alpha \neq 0,-2}.$$

2.4. One-Dimensional Central Extensions of Four-Dimensional Three-Step Nilpotent Bicommutative Algebras

2.4.1. The Description of Second Cohomology Space

In the following Table 2, we give the description of the second cohomology space of four-dimensional three-step nilpotent bicommutative algebras.

Table 2. The list of non-two-step nilpotent four-dimensional bicommutative algebras.

$\mathcal{B}_{01}^4$	:	$e_1e_1 = e_2$	$e_2e_1 = e_3$		
$\mathrm{H}^2(\mathcal{B}_{01}^4) = \Big\langle [\Delta_{12}], [\Delta_{14}], [\Delta_{31}], [\Delta_{41}], [\Delta_{44}] \Big\rangle$					
$\mathcal{B}_{02}^4(\alpha)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = \alpha e_3$	
$\mathrm{H}^2(\mathcal{B}_{02}^4(\alpha)) = \Big\langle [\Delta_{14}], [\Delta_{21}], [\Delta_{13} + \alpha\Delta_{22} + \alpha\Delta_{31}], [\Delta_{41}], [\Delta_{44}] \Big\rangle$					
$\mathcal{B}_{04}^4(\alpha)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = \alpha e_4$	$e_3e_3 = e_4$
$\mathrm{H}^2(\mathcal{B}_{04}^4(\alpha)) = \Big\langle [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \Big\rangle$					

Table 2. *Cont.*

$\mathcal{B}^4_{05}$: $e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4 \quad e_3e_3 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{05}) = \Big\langle [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \Big\rangle$	
$\mathcal{B}^4_{06}(\alpha \neq 0)$: $e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_1e_3 = e_4 \quad e_2e_1 = \alpha e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{06}(\alpha)) = \Big\langle [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{14} + \alpha\Delta_{22} + \alpha\Delta_{23} + \alpha\Delta_{41}], [\Delta_{33}] \Big\rangle$	
$\mathcal{B}^4_{07}$: $e_1e_1 = e_2 \quad e_2e_1 = e_4 \quad e_3e_3 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{07}) = \Big\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{31}], [\Delta_{33}] \Big\rangle$	
$\mathcal{B}^4_{08}$: $e_1e_1 = e_2 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{08}) = \Big\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}], [\Delta_{23} + \Delta_{41}] \Big\rangle$	
$\mathcal{B}^4_{09}$: $e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_3e_1 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{09}) = \Big\langle [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}], [\Delta_{14} + \Delta_{32}] \Big\rangle$	
$\mathcal{B}^4_{10}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4 \quad e_3e_2 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{10}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{22}], [\Delta_{32}] \Big\rangle$	
$\mathcal{B}^4_{11}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_3e_2 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{11}) = \Big\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}], [\Delta_{32}], [\Delta_{14} + \Delta_{33} + \Delta_{42}] \Big\rangle$	
$\mathcal{B}^4_{12}$: $e_1e_1 = e_4 \quad e_1e_2 = e_3 \quad e_2e_1 = e_4 \quad e_3e_2 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{12}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{22}], [\Delta_{32}] \Big\rangle$	
$\mathcal{B}^4_{13}$: $e_1e_2 = e_3 \quad e_2e_1 = e_4 \quad e_3e_2 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{13}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{22}], [\Delta_{32}] \Big\rangle$	
$\mathcal{B}^4_{14}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4 \quad e_2e_2 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{14}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{22}], [\Delta_{32}] \Big\rangle$	
$\mathcal{B}^4_{15}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{15}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{22}], [\Delta_{32}] \Big\rangle$	
$\mathcal{B}^4_{16}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_2e_2 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{16}) = \Big\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}], [\Delta_{32}], [\Delta_{14} + \Delta_{23}] \Big\rangle$	
$\mathcal{B}^4_{17}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{17}) = \Big\langle [\Delta_{11}], [\Delta_{14}], [\Delta_{21}], [\Delta_{22}], [\Delta_{32}] \Big\rangle$	
$\mathcal{B}^4_{18}$: $e_1e_1 = e_4 \quad e_1e_2 = e_3 \quad e_3e_2 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{18}) = \Big\langle [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{32}], [\Delta_{31} + \Delta_{42}] \Big\rangle$	
$\mathcal{B}^4_{19}$: $e_1e_2 = e_3 \quad e_3e_2 = e_4$	
$\mathrm{H}^2(\mathcal{B}^4_{19}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{42}] \Big\rangle$	

2.4.2. Central Extensions of $\mathcal{B}^4_{01}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{14}], \quad \nabla_3 = [\Delta_{31}], \quad \nabla_4 = [\Delta_{41}], \quad \nabla_5 = [\Delta_{44}].$$

Take $\theta = \sum\limits_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}_{01}^4)$. The automorphism group of $\mathcal{B}_{01}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & xy & x^3 & t \\ u & 0 & 0 & r \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & 0 & \alpha_2 \\ 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 \\ \alpha_4 & 0 & 0 & \alpha_5 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & 0 & \alpha_2^* \\ \alpha^{**} & 0 & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 \\ \alpha_4^* & 0 & 0 & \alpha_5^* \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}_{01}^4)$ on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rcl} \alpha_1^* &=& x^3\alpha_1, \\ \alpha_2^* &=& r(x\alpha_2 + u\alpha_5), \\ \alpha_3^* &=& x^4\alpha_3, \\ \alpha_4^* &=& tx\alpha_3 + rx\alpha_4 + ru\alpha_5, \\ \alpha_5^* &=& r^2\alpha_5. \end{array}$$

We are interested only in the cases with

$$(\alpha_2, \alpha_4, \alpha_5) \neq (0,0,0), \alpha_3 \neq 0.$$

Since $\alpha_3 \neq 0$, then choosing $t = -\frac{r(x\alpha_4 + u\alpha_5)}{x\alpha_3}$, we have $\alpha_4^* = 0$.

1. If $\alpha_5 \neq 0$, then choosing $u = -\frac{x\alpha_2}{\alpha_5}$, we have $\alpha_2^* = 0$.
   (a) $\alpha_1 = 0$, then choosing $x = 1,\ r = \sqrt{\frac{\alpha_3}{\alpha_5}}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$;
   (b) $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_3},\ r = \frac{\alpha_1^2}{\alpha_3\sqrt{\alpha_3\alpha_5}}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_5 \rangle$.
2. If $\alpha_5 = 0$, then $\alpha_2 \neq 0$.
   (a) $\alpha_1 = 0$, then choosing $x = 1,\ r = \frac{\alpha_3}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_3 \rangle$;
   (b) $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_3},\ r = \frac{\alpha_1^3}{\alpha_2\alpha_3^2}$, we have the representative $\langle \nabla_1 + \nabla_2 + \nabla_3 \rangle$.

Therefore, we have the following distinct orbits

$$\langle \nabla_3 + \nabla_5 \rangle, \quad \langle \nabla_1 + \nabla_3 + \nabla_5 \rangle, \quad \langle \nabla_2 + \nabla_3 \rangle, \quad \langle \nabla_1 + \nabla_2 + \nabla_3 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{106}, \mathrm{B}_{107}, \mathrm{B}_{108}, \mathrm{B}_{109}.$$

2.4.3. Central Extensions of $\mathcal{B}_{02}^{\alpha}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{14}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{13} + \alpha\Delta_{22} + \alpha\Delta_{31}], \quad \nabla_4 = [\Delta_{41}], \quad \nabla_5 = [\Delta_{44}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}_{02}^{\alpha})$. The automorphism group of $\mathcal{B}_{02}^{\alpha}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & (1+\alpha)xy & x^3 & t \\ u & 0 & 0 & r \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_3 & \alpha_1 \\ \alpha_2 & \alpha\alpha_3 & 0 & 0 \\ \alpha\alpha_3 & 0 & 0 & 0 \\ \alpha_4 & 0 & 0 & \alpha_5 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_3^* & \alpha_1^* \\ \alpha_2^* + \alpha\alpha^{**} & \alpha\alpha_3^* & 0 & 0 \\ \alpha\alpha_3^* & 0 & 0 & 0 \\ \alpha_4^* & 0 & 0 & \alpha_5^* \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}_{02}^{\alpha})$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned} \alpha_1^* &= rx\alpha_1 + tx\alpha_3 + ru\alpha_5, \\ \alpha_2^* &= x^2(x\alpha_2 - y\alpha(\alpha-1)\alpha_3), \\ \alpha_3^* &= x^4\alpha_3, \\ \alpha_4^* &= tx\alpha\alpha_3 + rx\alpha_4 + ru\alpha_5, \\ \alpha_5^* &= r^2\alpha_5. \end{aligned}$$

We are interested only in the cases with

$$\alpha_3 \neq 0, \ (\alpha_1, \alpha_4, \alpha_5) \neq (0,0,0).$$

$\alpha_3 \neq 0$, then choosing $t = -\frac{r(x\alpha_1+u\alpha_5)}{x\alpha_3}$, we have $\alpha_1^* = 0$. Now we consider following cases:

1. $\alpha_5 = 0$, then $\alpha_4 \neq 0$.
   (a) $\alpha \in \{0,1\}$, $\alpha_2 = 0$, then choosing $x = 1$, $r = \frac{\alpha_3}{\alpha_4}$, we have the representative $\langle \nabla_3 + \nabla_4 \rangle$;
   (b) $\alpha \in \{0,1\}$, $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3}$, $r = \frac{\alpha_2^3}{\alpha_3^2\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle$;
   (c) $\alpha \notin \{0,1\}$, then choosing $x = 1$, $y = \frac{\alpha_2}{(\alpha-1)\alpha\alpha_3}$, $r = \frac{\alpha_3}{\alpha_4}$, we have the representative $\langle \nabla_3 + \nabla_4 \rangle$.
2. $\alpha_5 \neq 0$.
   (a) $\alpha = 1$, $\alpha_4 = \alpha_2 = 0$, then choosing $x = 1$, $r = \sqrt{\frac{\alpha_3}{\alpha_5}}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$;
   (b) $\alpha = 1$, $\alpha_4 = 0$, $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3}$, $r = \frac{\alpha_2^2}{\alpha_3\sqrt{\alpha_3\alpha_5}}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5 \rangle$;
   (c) $\alpha = 1$, $\alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\sqrt{\alpha_3\alpha_5}}$, $r = \frac{\alpha_4^2}{\alpha_5\sqrt{\alpha_3\alpha_5}}$, we have the family of representatives $\langle \beta\nabla_2 + \nabla_3 + \nabla_4 + \nabla_5 \rangle^{O(\beta)\simeq O(-\beta)}$;
   (d) $\alpha = 0$, $\alpha_2 = 0$, then choosing $x = 1$, $r = \sqrt{\frac{\alpha_3}{\alpha_5}}$, $u = -\frac{\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$;
   (e) $\alpha = 0$, $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3}$, $r = \frac{\alpha_2^2}{\alpha_3\sqrt{\alpha_3\alpha_5}}$ $u = -\frac{\alpha_2\alpha_4}{\alpha_3\alpha_5}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5 \rangle$;
   (f) $\alpha \notin \{0,1\}$, then choosing $x = 1$, $y = \frac{\alpha_2}{(\alpha-1)\alpha\alpha_3}$, $r = \sqrt{\frac{\alpha_3}{\alpha_5}}$, $u = \frac{\alpha_4}{(\alpha-1)\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$.

Summarizing all cases, we have the following distinct orbits
in case of $\alpha = 0$: $\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle$, $\langle \nabla_2 + \nabla_3 + \nabla_5 \rangle$, in case of $\alpha = 1$

$$\langle \beta\nabla_2 + \nabla_3 + \nabla_4 + \nabla_5 \rangle^{O(\beta)\simeq O(-\beta)}, \quad \langle \nabla_2 + \nabla_3 + \nabla_4 \rangle, \quad \langle \nabla_2 + \nabla_3 + \nabla_5 \rangle,$$

in case of $\alpha \in \mathbb{C}$ $\langle \nabla_3 + \nabla_4 \rangle$, $\langle \nabla_3 + \nabla_5 \rangle$, which gives the following new algebras (see Section 3, as we are interested in non-commutative algebras, we do not consider B^1_{115}):

$$B_{110}, B_{111}, B^{\beta}_{112}, B_{113}, B_{114}, B^{\alpha\neq 1}_{115}, B^{\alpha}_{116}.$$

2.4.4. Central Extensions of $\mathcal{B}^4_{06}(\alpha \neq 0)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{13}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{31}], \quad \nabla_4 = [\Delta_{14} + \alpha\Delta_{22} + \alpha\Delta_{23} + \alpha\Delta_{41}], \quad \nabla_5 = [\Delta_{33}].$$

Take $\theta = \sum\limits_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}^4_{06}(\alpha \neq 0))$. The automorphism group of $\mathcal{B}^4_{06}(\alpha \neq 0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & 0 & x^2 & 0 \\ u & x((1+\alpha)y+z) & v & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_4 \\ \alpha_2 & \alpha\alpha_4 & \alpha\alpha_4 & 0 \\ \alpha_3 & 0 & \alpha_5 & 0 \\ \alpha\alpha_4 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_1^* + \alpha^{**} & \alpha_4^* \\ \alpha_2^* + \alpha\alpha^{**} & \alpha\alpha_4^* & \alpha\alpha_4^* & 0 \\ \alpha_3^* & 0 & \alpha_5^* & 0 \\ \alpha\alpha_4^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}^4_{06}(\alpha \neq 0))$ on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned} \alpha_1^* &= x(x^2\alpha_1 + (v - x((1+\alpha)y+z))\alpha_4 + xz\alpha_5), \\ \alpha_2^* &= x^2(x\alpha_2 + \alpha((1-\alpha)y+z)\alpha_4), \\ \alpha_3^* &= x\big(x^2\alpha_3 + v\alpha\alpha_4 + xz\alpha_5\big), \\ \alpha_4^* &= x^4\alpha_4, \\ \alpha_5^* &= x^4\alpha_5. \end{aligned}$$

We are interested only in the cases with $\alpha_4 \neq 0$.

Choosing $v = \frac{x(y\alpha\alpha_4(2\alpha\alpha_4+\alpha_5-\alpha\alpha_5)-x\alpha\alpha_1\alpha_4-x\alpha_2(\alpha_4-\alpha_5))}{\alpha\alpha_4^2}$, $z = y(\alpha-1) - \frac{x\alpha_2}{\alpha\alpha_4}$, we have $\alpha_1^* = \alpha_2^* = 0$.

1. $2\alpha^2\alpha_4 = (\alpha-1)^2\alpha_5$, $\alpha_3 = 0$, then $\alpha \neq 1$ and we have the family of representatives $\left\langle \nabla_4 + \frac{2\alpha^2}{(\alpha-1)^2}\nabla_5 \right\rangle$;
2. $2\alpha^2\alpha_4 = (\alpha-1)^2\alpha_5$, $\alpha_3 \neq 0$, then $\alpha \neq 1$ and choosing $x = \frac{\alpha_3}{\alpha_4}$, we have the family of representatives $\left\langle \nabla_3 + \nabla_4 + \frac{2\alpha^2}{(\alpha-1)^2}\nabla_5 \right\rangle$;
3. $2\alpha^2\alpha_4 \neq (\alpha-1)^2\alpha_5$, then choosing $x = 1$, $y = -\frac{x\alpha_3}{2\alpha^2\alpha_4-(\alpha-1)^2\alpha_5}$, we have the family of representatives $\langle \nabla_4 + \beta\nabla_5 \rangle_{\beta \neq \frac{2\alpha^2}{(\alpha-1)^2}}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_4 + \beta\nabla_5 \rangle, \langle \nabla_3 + \nabla_4 + \tfrac{2\alpha^2}{(\alpha-1)^2}\nabla_5 \rangle_{\alpha\neq 1},$$

which gives the following new algebras (see Section 3):

$$B^{\alpha\neq 0,\beta}_{117}, B^{\alpha\neq 0,1}_{118}.$$

2.4.5. Central Extensions of $\mathcal{B}_{08}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{31}], \quad \nabla_4 = [\Delta_{33}], \quad \nabla_5 = [\Delta_{23} + \Delta_{41}].$$

Take $\theta = \sum\limits_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}_{08}^4)$. The automorphism group of $\mathcal{B}_{08}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & 0 & x^2 & 0 \\ u & x(y+z) & v & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & \alpha_5 & 0 \\ \alpha_3 & 0 & \alpha_4 & 0 \\ \alpha_5 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha^{**} & 0 \\ \alpha_2^* + \alpha^{**} & 0 & \alpha_5^* & 0 \\ \alpha_3^* & 0 & \alpha_4^* & 0 \\ \alpha_5^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}_{08})$ on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllllll} \alpha_1^* & = & x^3\alpha_1, & \alpha_2^* & = & x^2(x\alpha_2 - z\alpha_4 + 2z\alpha_5), & \alpha_3^* & = & x(x^2\alpha_3 + xz\alpha_4 + v\alpha_5), \\ \alpha_4^* & = & x^4\alpha_4, & \alpha_5^* & = & x^4\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_5 \neq 0$. Choosing $v = -\frac{x(x\alpha_3 + z\alpha_4)}{\alpha_5}$, we have $\alpha_3^* = 0$.

1. $\alpha_4 = 2\alpha_5$, $\alpha_2 = \alpha_1 = 0$, then we have the representative $\langle 2\nabla_4 + \nabla_5 \rangle$;
2. $\alpha_4 = 2\alpha_5$, $\alpha_2 = 0$, $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_5}$, we have the representative $\langle \nabla_1 + 2\nabla_4 + \nabla_5 \rangle$;
3. $\alpha_4 = 2\alpha_5$, $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_5}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + 2\nabla_4 + \nabla_5 \rangle$;
4. $\alpha_4 \neq 2\alpha_5$, $\alpha_1 = 0$, then choosing $x = 1$, $z = \frac{\alpha_2}{\alpha_4 - 2\alpha_5}$, we have the family of representatives $\langle \alpha\nabla_4 + \nabla_5 \rangle_{\alpha \neq 2}$;
5. $\alpha_4 \neq 2\alpha_5$, $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_5}$, $z = \frac{\alpha_1\alpha_2}{(\alpha_4 - 2\alpha_5)\alpha_5}$, we have the family of representatives $\langle \nabla_1 + \alpha\nabla_4 + \nabla_5 \rangle_{\alpha \neq 2}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_1 + \alpha\nabla_4 + \nabla_5 \rangle, \langle \alpha\nabla_4 + \nabla_5 \rangle, \langle \alpha\nabla_1 + \nabla_2 + 2\nabla_4 + \nabla_5 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{119}^{\alpha}, \mathrm{B}_{120}^{\alpha}, \mathrm{B}_{121}^{\alpha}.$$

2.4.6. Central Extensions of $\mathcal{B}_{09}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{13}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{31}], \quad \nabla_4 = [\Delta_{33}], \quad \nabla_5 = [\Delta_{14} + \Delta_{32}].$$

Take $\theta = \sum\limits_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{N}_{09}^4)$. The automorphism group of $\mathcal{N}_{09}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & 0 & x^2 & 0 \\ u & x(y+z) & v & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_5 \\ \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & \alpha_5 & \alpha_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_1^* & \alpha_5^* \\ \alpha_2^* & 0 & 0 & 0 \\ \alpha_3^* + \alpha^{**} & \alpha_5^* & \alpha_4^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{09}^4)$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rclrclrcl} \alpha_1^* & = & x(x^2\alpha_1 + xz\alpha_4 + v\alpha_5), & \alpha_2^* & = & x^3\alpha_2, & & & \\ \alpha_3^* & = & x^2(x\alpha_3 + z(\alpha_4 - 2\alpha_5)), & \alpha_4^* & = & x^4\alpha_4, & \alpha_5^* & = & x^4\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_5 \neq 0$. Choosing $v = -\frac{x(x\alpha_1 + z\alpha_4)}{\alpha_5}$, we have $\alpha_1^* = 0$.

1. $\alpha_4 = 2\alpha_5,\ \alpha_3 = \alpha_2 = 0$, then we have the representative $\langle 2\nabla_4 + \nabla_5 \rangle$;
2. $\alpha_4 = 2\alpha_5,\ \alpha_3 = 0,\ \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_5}$, we have the representative $\langle \nabla_2 + 2\nabla_4 + \nabla_5 \rangle$;
3. $\alpha_4 = 2\alpha_5,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_5}$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_3 + 2\nabla_4 + \nabla_5 \rangle$;
4. $\alpha_4 \neq 2\alpha_5,\ \alpha_2 = 0$, then choosing $x = 1,\ z = -\frac{\alpha_3}{\alpha_4 - 2\alpha_5}$, we have the family of representatives $\langle \alpha\nabla_4 + \nabla_5 \rangle_{\alpha \neq 2}$;
5. $\alpha_4 \neq 2\alpha_5,\ \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_5},\ z = -\frac{\alpha_2\alpha_3}{(\alpha_4 - 2\alpha_5)\alpha_5}$, we have the family of representatives $\langle \nabla_2 + \alpha\nabla_4 + \nabla_5 \rangle_{\alpha \neq 2}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \alpha\nabla_2 + \nabla_3 + 2\nabla_4 + \nabla_5 \rangle, \quad \langle \alpha\nabla_4 + \nabla_5 \rangle, \quad \langle \nabla_2 + \alpha\nabla_4 + \nabla_5 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{122}^{\alpha}, \mathrm{B}_{123}^{\alpha}, \mathrm{B}_{124}^{\alpha}.$$

2.4.7. Central Extensions of $\mathcal{B}_{11}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{22}], \quad \nabla_4 = [\Delta_{32}], \quad \nabla_5 = [\Delta_{14} + \Delta_{33} + \Delta_{42}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}_{11}^4)$. The automorphism group of $\mathcal{B}_{11}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & 0 & x^2 & 0 \\ u & x(y+z) & v & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_5 \\ \alpha_2 & \alpha_3 & 0 & 0 \\ 0 & \alpha_4 & \alpha_5 & 0 \\ 0 & \alpha_5 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha^{**} & \alpha_5^* \\ \alpha_2^* & \alpha_3^* & 0 & 0 \\ 0 & \alpha_4^* + \alpha^{**} & \alpha_5^* & 0 \\ 0 & \alpha_5^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}_{11})$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rclrclrcl} \alpha_1^* & = & x(x\alpha_1 + y\alpha_5), & \alpha_2^* & = & x^2\alpha_2, & & & \\ \alpha_3^* & = & x(x\alpha_3 + z\alpha_5), & \alpha_4^* & = & x^3\alpha_4, & \alpha_5^* & = & x^4\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_5 \neq 0$. Choosing $y = -\frac{x\alpha_1}{\alpha_5}$, $z = -\frac{x\alpha_3}{\alpha_5}$, we have $\alpha_1^* = \alpha_3^* = 0$.

1. $\alpha_4 = \alpha_2 = 0$, then we have the representative $\langle \nabla_5 \rangle$;
2. $\alpha_4 = 0,\ \alpha_2 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_2}{\alpha_5}}$, we have the representative $\langle \nabla_2 + \nabla_5 \rangle$;
3. $\alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5}$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_4 + \nabla_5 \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_5 \rangle, \langle \nabla_2 + \nabla_5 \rangle, \langle \alpha\nabla_2 + \nabla_4 + \nabla_5 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{125}, \mathrm{B}_{126}, \mathrm{B}_{127}.$$

2.4.8. Central Extensions of $\mathcal{B}^4_{16}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{22}], \quad \nabla_4 = [\Delta_{14} + \Delta_{23}], \quad \nabla_5 = [\Delta_{32}].$$

Take $\theta = \sum\limits_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}^4_{16})$. The automorphism group of $\mathcal{B}^4_{16}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ 0 & y & x^3 & 0 \\ u & v & xy & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_4 & 0 \\ 0 & \alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha^{**} & \alpha_4^* \\ \alpha_2^* & \alpha_3^* + \alpha^{**} & \alpha_4^* & 0 \\ 0 & \alpha_5^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}^4_{16})$ on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllll} \alpha_1^* & = & x(x\alpha_1 + u\alpha_4), & \alpha_2^* & = & x^3\alpha_2, & \\ \alpha_3^* & = & x^2(x^2\alpha_3 + y\alpha_5), & \alpha_4^* & = & x^5\alpha_4, \quad \alpha_5^* \ = \ x^5\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_4 \neq 0$. Choosing $u = -\frac{x\alpha_1}{\alpha_4}$, we have $\alpha_1^* = 0$.

1. $\alpha_5 = \alpha_3 = \alpha_2 = 0$, then we have the representative $\langle \nabla_4 \rangle$;
2. $\alpha_5 = \alpha_3 = 0,\ \alpha_2 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_2}{\alpha_4}}$, we have the representative $\langle \nabla_2 + \nabla_4 \rangle$;
3. $\alpha_5 = 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_4}$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_3 + \nabla_4 \rangle$;
4. $\alpha_5 \neq 0,\ \alpha_2 = 0$, then choosing $x = 1,\ y = -\frac{\alpha_3}{\alpha_5}$, we have the family of representatives $\langle \nabla_4 + \alpha\nabla_5 \rangle_{\alpha \neq 0}$;
5. $\alpha_5 \neq 0,\ \alpha_2 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_2}{\alpha_4}},\ y = -\frac{\alpha_2\alpha_3}{\alpha_4\alpha_5}$, we have the family of representatives $\langle \nabla_2 + \nabla_4 + \alpha\nabla_5 \rangle_{\alpha \neq 0}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_4 + \alpha\nabla_5 \rangle, \langle \nabla_2 + \nabla_4 + \alpha\nabla_5 \rangle, \langle \alpha\nabla_2 + \nabla_3 + \nabla_4 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}^{\alpha}_{128}, \mathrm{B}^{\alpha}_{129}, \mathrm{B}^{\alpha}_{130}.$$

2.4.9. Central Extensions of $\mathcal{B}^4_{17}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{14}], \quad \nabla_3 = [\Delta_{21}], \quad \nabla_4 = [\Delta_{22}], \quad \nabla_5 = [\Delta_{32}].$$

Take $\theta = \sum\limits_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}^4_{17})$. The automorphism group of $\mathcal{B}^4_{17}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & z & xy & 0 \\ u & v & xz & x^2y \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_2 \\ \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha^{**} & \alpha_2^* \\ \alpha_3^* & \alpha_4^* & 0 & 0 \\ 0 & \alpha_5^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}^4_{17})$ on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rclrclrcl} \alpha_1^* &=& x(x\alpha_1 + u\alpha_2), & \alpha_2^* &=& x^3y\alpha_2, & \alpha_3^* &=& xy\alpha_3, \\ \alpha_4^* &=& y(y\alpha_4 + z\alpha_5), & \alpha_5^* &=& xy^2\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_2 \neq 0$. Choosing $v = -\frac{x\alpha_1}{\alpha_2}$, we have $\alpha_1^* = 0$.

1. $\alpha_5 = \alpha_4 = \alpha_3 = 0$, then we have the representative $\langle \nabla_2 \rangle$;
2. $\alpha_5 = \alpha_4 = 0$, $\alpha_3 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_3}{\alpha_2}}$, $y = 1$, we have the representative $\langle \nabla_2 + \nabla_3 \rangle$;
3. $\alpha_5 = 0$, $\alpha_4 \neq 0$, $\alpha_3 = 0$, then choosing $x = 1$, $y = \frac{\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4 \rangle$;
4. $\alpha_5 = 0$, $\alpha_4 \neq 0$, $\alpha_3 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_3}{\alpha_2}}$, $y = \frac{\alpha_3}{\alpha_4}\sqrt{\frac{\alpha_3}{\alpha_2}}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle$;
5. $\alpha_5 \neq 0$, $\alpha_3 = 0$, then choosing $x = 1$, $y = \frac{\alpha_2}{\alpha_5}$, $z = -\frac{\alpha_2\alpha_4}{\alpha_5^2}$, we have the representative $\langle \nabla_2 + \nabla_5 \rangle$;
6. $\alpha_5 \neq 0$, $\alpha_3 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_3}{\alpha_2}}$, $y = \frac{\alpha_3}{\alpha_5}$, $z = -\frac{\alpha_3\alpha_4}{\alpha_5^2}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5 \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_2 \rangle, \langle \nabla_2 + \nabla_4 \rangle, \langle \nabla_2 + \nabla_3 \rangle, \langle \nabla_2 + \nabla_3 + \nabla_4 \rangle, \langle \nabla_2 + \nabla_5 \rangle, \langle \nabla_2 + \nabla_3 + \nabla_5 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{131}, \mathrm{B}_{132}, \mathrm{B}_{133}, \mathrm{B}_{134}, \mathrm{B}_{135}, \mathrm{B}_{136}.$$

2.4.10. Central Extensions of $\mathcal{B}^4_{18}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{13}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{22}], \quad \nabla_4 = [\Delta_{32}], \quad \nabla_5 = [\Delta_{31} + \Delta_{42}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}^4_{18})$. The automorphism group of $\mathcal{B}^4_{18}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x^2 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ y & 0 & x^2 & 0 \\ u & z & xy & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & 0 \\ \alpha_2 & \alpha_3 & 0 & 0 \\ \alpha_5 & \alpha_4 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{**} & \alpha^* & \alpha_1^* & 0 \\ \alpha_2^* & \alpha_3^* & 0 & 0 \\ \alpha_5^* & \alpha_4^* + \alpha^{**} & 0 & 0 \\ 0 & \alpha_5^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}^4_{18})$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rclrclrcl} \alpha_1^* & = & x^5\alpha_1, & \alpha_2^* & = & x^3\alpha_2, & \alpha_3^* & = & x(x\alpha_3 + z\alpha_5), \\ \alpha_4^* & = & -x^2y\alpha_1 + x^4\alpha_4, & \alpha_5^* & = & x^5\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_5 \neq 0$. Choosing $z = -\frac{x\alpha_3}{\alpha_5}$, we have $\alpha_3^* = 0$.

1. $\alpha_1 = \alpha_4 = \alpha_2 = 0$, then we have the representative $\langle \nabla_5 \rangle$;
2. $\alpha_1 = \alpha_4 = 0,\ \alpha_2 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_2}{\alpha_5}}$, we have the representative $\langle \nabla_2 + \nabla_5 \rangle$;
3. $\alpha_1 = 0,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5}$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_4 + \nabla_5 \rangle$;
4. $\alpha_1 \neq 0,\ \alpha_2 = 0$, then choosing $x = 1,\ y = \frac{\alpha_4}{\alpha_1}$, we have the representative $\langle \alpha\nabla_1 + \nabla_5 \rangle_{\alpha \neq 0}$;
5. $\alpha_1 \neq 0,\ \alpha_2 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_2}{\alpha_5}},\ y = \frac{\alpha_2\alpha_4}{\alpha_1\alpha_5}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_5 \rangle_{\alpha \neq 0}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \alpha\nabla_2 + \nabla_4 + \nabla_5 \rangle, \langle \alpha\nabla_1 + \nabla_5 \rangle, \langle \alpha\nabla_1 + \nabla_2 + \nabla_5 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}^{\alpha}_{137}, \mathrm{B}^{\alpha}_{138}, \mathrm{B}^{\alpha}_{139}.$$

2.4.11. Central Extensions of $\mathcal{B}^4_{19}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{13}], \quad \nabla_3 = [\Delta_{21}], \quad \nabla_4 = [\Delta_{22}], \quad \nabla_5 = [\Delta_{42}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{B}^4_{19})$. The automorphism group of $\mathcal{B}^4_{19}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ z & 0 & xy & 0 \\ u & v & yz & xy^2 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_2^* & 0 \\ \alpha_3^* & \alpha_4^* & 0 & 0 \\ 0 & \alpha^{**} & 0 & 0 \\ 0 & \alpha_5^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{B}^4_{19})$ on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rclrclrcl} \alpha_1^* &=& x(x\alpha_1 + z\alpha_2), & \alpha_2^* &=& x^2 y\alpha_2, & \alpha_3^* &=& xy\alpha_3, \\ \alpha_4^* &=& y(y\alpha_4 + v\alpha_5), & \alpha_5^* &=& xy^3\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_5 \neq 0$. Choosing $v = -\frac{y\alpha_4}{\alpha_5}$, we have $\alpha_4^* = 0$.

1. $\alpha_2 = \alpha_1 = \alpha_3 = 0$, then we have the representative $\langle \nabla_5 \rangle$;
2. $\alpha_2 = \alpha_1 = 0,\ \alpha_3 \neq 0$, then choosing $y = \sqrt{\frac{\alpha_3}{\alpha_5}}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$;
3. $\alpha_2 = 0,\ \alpha_1 \neq 0,\ \alpha_3 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_1},\ y = 1$, we have the representative $\langle \nabla_1 + \nabla_5 \rangle$;
4. $\alpha_2 = 0,\ \alpha_1 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3\sqrt{\alpha_3}}{\alpha_1\sqrt{\alpha_5}},\ y = \sqrt{\frac{\alpha_3}{\alpha_5}}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_5 \rangle$;
5. $\alpha_2 \neq 0,\ \alpha_3 = 0$, then choosing $x = 1,\ y = \sqrt{\frac{\alpha_2}{\alpha_5}},\ z = -\frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_5 \rangle$;
6. $\alpha_2 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \sqrt{\frac{\alpha_3}{\alpha_5}},\ z = -\frac{\alpha_1\alpha_3}{\alpha_2^2}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5 \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_5 \rangle, \langle \nabla_3 + \nabla_5 \rangle, \langle \nabla_1 + \nabla_5 \rangle, \langle \nabla_1 + \nabla_3 + \nabla_5 \rangle, \langle \nabla_2 + \nabla_5 \rangle, \langle \nabla_2 + \nabla_3 + \nabla_5 \rangle,$$

which gives the following new algebras (see Section 3):

$$\mathrm{B}_{140}, \mathrm{B}_{141}, \mathrm{B}_{142}, \mathrm{B}_{143}, \mathrm{B}_{144}, \mathrm{B}_{145}.$$

2.5. Two-Dimensional Central Extensions of Three-Dimensional Nilpotent Bicommutative Algebras

2.5.1. The Description of Second Cohomology Spaces of Three-Dimensional Nilpotent Bicommutative Algebras

In the following Table 3, we give the description of the second cohomology space of two-generated three-dimensional nilpotent bicommutative algebras.

Table 3. The list of two-step nilpotent three-dimensional bicommutative algebras.

$\mathcal{B}^{3*}_{01}$	:	$e_1e_1 = e_2$
$\mathrm{H}^2_{com}(\mathcal{B}^{3*}_{01}) = \Big\langle [\Delta_{12} + \Delta_{21}], [\Delta_{13} + \Delta_{31}], [\Delta_{33}] \Big\rangle$		
$\mathrm{H}^2_{bicom}(\mathcal{B}^{3*}_{01}) = \mathrm{H}^2_{com}(\mathcal{B}^{*}_{01}) \oplus \Big\langle [\Delta_{21}], [\Delta_{31}] \Big\rangle$		
$\mathcal{B}^{3*}_{02}$	:	$e_1e_1 = e_3 \quad e_2e_2 = e_3$
$\mathrm{H}^2_{bicom}(\mathcal{B}^{3*}_{02}) = \Big\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$		
$\mathcal{B}^{3*}_{03}$	:	$e_1e_2 = e_3 \quad e_2e_1 = -e_3$
$\mathrm{H}^2(\mathcal{B}^{3*}_{03}) = \Big\langle [\Delta_{11}], [\Delta_{12}], [\Delta_{22}] \Big\rangle$		
$\mathcal{B}^{3*}_{04}(\alpha \neq 0)$	:	$e_1e_1 = \alpha e_3 \quad e_2e_1 = e_3 \quad e_2e_2 = e_3$
$\mathrm{H}^2(\mathcal{B}^{3*}_{04}(\alpha \neq 0)) = \Big\langle [\Delta_{11}], [\Delta_{12}], [\Delta_{21}] \Big\rangle$		
$\mathcal{B}^{3*}_{04}(0)$	:	$e_1e_2 = e_3$
$\mathrm{H}^2(\mathcal{B}^{*}_{04}(0)) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{32}] \Big\rangle$		

2.5.2. Central Extensions of $\mathcal{B}^{3*}_{01}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12} + \Delta_{21}], \quad \nabla_2 = [\Delta_{13} + \Delta_{31}], \quad \nabla_3 = [\Delta_{21}], \quad \nabla_4 = [\Delta_{31}], \quad \nabla_5 = [\Delta_{33}].$$

The automorphism group of $\mathcal{B}_{01}^{3*}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ u & x^2 & w \\ z & 0 & y \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1+\alpha_3 & 0 & 0 \\ \alpha_2+\alpha_4 & 0 & \alpha_5 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* \\ \alpha_1^*+\alpha_3^* & 0 & 0 \\ \alpha_2^*+\alpha_4^* & 0 & \alpha_5^* \end{pmatrix},$$

the action of $\text{Aut}(\mathcal{B}_{01}^{3*})$ on subspace $\Big\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \Big\rangle$ is given by $\Big\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \Big\rangle$, where

$$\begin{array}{lllllllll} \alpha_1^* &=& x^3\alpha_1, & \alpha_2^* &=& wx\alpha_1 + xy\alpha_2 + yz\alpha_5, & \alpha_3^* &=& x^3\alpha_3, \\ \alpha_4^* &=& x(w\alpha_3 + y\alpha_4), & \alpha_5^* &=& y^2\alpha_5. \end{array}$$

We are interested only in two-dimensional central extensions and consider the vector space generated by the following two cocycles:

$$\theta_1 = \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3 + \alpha_4\nabla_4 + \alpha_5\nabla_5 \text{ and } \theta_2 = \beta_1\nabla_1 + \beta_2\nabla_2 + \beta_4\nabla_4 + \beta_5\nabla_5.$$

Our aim is to find only central extensions with $(\alpha_3, \alpha_4, \beta_3, \beta_4) \neq 0$. Hence, we have the following cases.

1. $\alpha_3 \neq 0$, then we have

$$\begin{array}{llllll} \alpha_1^* &=& x^3\alpha_1, & \beta_1^* &=& x^3\beta_1, \\ \alpha_2^* &=& wx\alpha_1 + xy\alpha_2 + yz\alpha_5, & \beta_2^* &=& wx\beta_1 + xy\beta_2 + yz\beta_5, \\ \alpha_3^* &=& x^3\alpha_3, & \beta_3^* &=& 0, \\ \alpha_4^* &=& x(w\alpha_3 + y\alpha_4), & \beta_4^* &=& xy\beta_4, \\ \alpha_5^* &=& y^2\alpha_5, & \beta_5^* &=& y^2\beta_5. \end{array}$$

   (a) $\beta_5 \neq 0$, then we can suppose $\alpha_5 = 0$ and choosing $w = -\frac{y\alpha_4}{\alpha_3}, z = -\frac{x(\alpha_4\beta_1 - \alpha_3\beta_2)}{\alpha_3\beta_5}$, we have $\alpha_4^* = \beta_2^* = 0$. Thus, we can assume $\alpha_4 = \beta_2 = 0$ and consider following subcases:
      i. $\alpha_2 = \beta_4 = \beta_1 = 0$, then we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_5 \rangle$;
      ii. $\alpha_2 = \beta_4 = 0,\ \beta_1 \neq 0$, then choosing $x = \sqrt[3]{\beta_5\beta_1^{-1}},\ y = 1$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \nabla_5 \rangle$;
      iii. $\alpha_2 = 0,\ \beta_4 \neq 0,\ \beta_1 = 0$ then choosing $x = \beta_5\beta_4^{-1},\ y = 1$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_4 + \nabla_5 \rangle$;
      iv. $\alpha_2 = 0,\ \beta_4 \neq 0,\ \beta_1 \neq 0$, then choosing $x = \beta_4^2\beta_1^{-1}\beta_5^{-1},\ y = \beta_4^3\beta_1^{-1}\beta_5^{-2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \nabla_4 + \nabla_5 \rangle$;
      v. $\alpha_2 \neq 0,\ \beta_4 = \beta_1 = 0$, then choosing $x = 1,\ y = \alpha_3\alpha_2^{-1}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \nabla_5 \rangle$;
      vi. $\alpha_2 \neq 0,\ \beta_4 = 0,\ \beta_1 \neq 0$, then choosing $x = \alpha_2^2\beta_1\alpha_3^{-2}\beta_5^{-1}$, $y = \alpha_2^3\beta_1^2\alpha_3^{-3}\beta_5^{-2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \nabla_1 + \nabla_5 \rangle$;
      vii. $\alpha_2 \neq 0,\ \beta_4 \neq 0$, then choosing $x = \alpha_2\beta_4\alpha_3^{-1}\beta_5^{-1}, y = \alpha_2\beta_4^2\alpha_3^{-1}\beta_5^{-2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \beta\nabla_1 + \nabla_4 + \nabla_5 \rangle$;

   (b) $\beta_5 = 0, \beta_4 \neq 0$.
      i. $\alpha_5 = \beta_1 = 0,\ \alpha_1\beta_4 \neq \alpha_3\beta_2$, then choosing $y = 1,\ w = \frac{\alpha_4\beta_2 - \alpha_2\beta_4}{\alpha_1\beta_4 - \alpha_3\beta_2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \beta\nabla_2 + \nabla_4 \rangle_{\alpha \neq \beta}$;

ii. $\alpha_5 = \beta_1 = 0$, $\alpha_1\beta_4 = \alpha_3\beta_2$, $\alpha_2\alpha_3 = \alpha_1\alpha_4$, then choosing $y = 1$, $w = -\alpha_4\alpha_3^{-1}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \alpha\nabla_2 + \nabla_4 \rangle$;

iii. $\alpha_5 = \beta_1 = 0, \alpha_1\beta_4 = \alpha_3\beta_2, \alpha_2\alpha_3 \neq \alpha_1\alpha_4$ then choosing $x = \alpha_4\beta_2 - \alpha_2\beta_4$, $y = -\alpha_3\beta_4(\alpha_4\beta_2 - \alpha_2\beta_4)$, and $w = \alpha_4\beta_4(\alpha_4\beta_2 - \alpha_2\beta_4)$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \alpha\nabla_2 + \nabla_4 \rangle$;

iv. $\alpha_5 = 0$, $\beta_1 \neq 0$, then choosing

$$x = 1,\ y = \frac{\alpha_3}{\beta_4},$$
$$w = \frac{(\alpha_3\beta_2+\alpha_4\beta_1-\alpha_1\beta_4)-\sqrt{(\alpha_3\beta_2+\alpha_4\beta_1-\alpha_1\beta_4)^2-4\alpha_3\beta_1(\alpha_4\beta_2-\alpha_2\beta_4)}}{\alpha_3\beta_4},$$

we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \beta\nabla_2 + \nabla_4 \rangle$;

v. $\alpha_5 \neq 0$, $\beta_1 = 0$, then choosing

$$x = \alpha_5, y = -\sqrt{\alpha_3}\alpha_5, z = 0 \text{ and } w = -\frac{\sqrt{\alpha_3}\alpha_5(\alpha_4\beta_2-\alpha_2\beta_4)}{\alpha_3\beta_2-\alpha_1\beta_4},$$

we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3 + \nabla_5, \beta\nabla_2 + \nabla_4 \rangle$;

vi. $\alpha_5 \neq 0$, $\beta_1 \neq 0$, then choosing

$$x = \frac{\alpha_3\beta_4^2}{\alpha_5\beta_1^2}, y = \frac{\alpha_3^2\beta_4^3}{\alpha_5^2\beta_1^3}, w = -\frac{\alpha_3^2\beta_2\beta_4^3}{\alpha_5^2\beta_1^4} \text{ and } z = \frac{\alpha_3(\alpha_1\beta_2-\alpha_2\beta_1)\beta_4^2}{\alpha_5^2\beta_1^3},$$

we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3 + \nabla_5, \nabla_1 + \nabla_4 \rangle$;

(c) $\beta_5 = 0, \beta_4 = 0,\ \beta_1 \neq 0$, then we can suppose $\alpha_1 = 0$ and consider following subcases:

i. $\alpha_5 = 0$, then choosing $w = -\frac{y\beta_2}{\beta_1}$, we have $\beta_2^* = 0$.

A. if $\alpha_2 = \alpha_4 = 0$, then we have a split algebra;

B. if $\alpha_2 = 0,\ \alpha_4 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_3}{\alpha_4}$, we have the representative $\langle \nabla_3 + \nabla_4, \nabla_1 \rangle$;

C. if $\alpha_2 \neq 0$, then choosing $x = 1, y = \frac{\alpha_3}{\alpha_2}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \alpha\nabla_4, \nabla_1 \rangle$;

ii. $\alpha_5 \neq 0$, $\beta_2 = 0$, then choosing $x = \alpha_5$, $y = \sqrt{\alpha_3}\alpha_5$, $z = -\alpha_2$ and $w = 0$, we have the representative $\langle \nabla_3 + \nabla_5, \nabla_1 \rangle$;

iii. $\alpha_5 \neq 0$, $\beta_2 \neq 0$, then choosing

$$x = \frac{\alpha_3\beta_2^2}{\alpha_5\beta_1^2}, y = \frac{\alpha_3^2\beta_2^3}{\alpha_5^2\beta_1^3}, z = \frac{\alpha_3\beta_2^2(\alpha_2\beta_1-\alpha_1\beta_2)}{\alpha_5^2\beta_1^3} \text{ and } w = 0,$$

we have the representative $\langle \nabla_3 + \nabla_5, \nabla_1 + \nabla_2 \rangle$.

(d) $\beta_5 = \beta_4 = \beta_1 = 0, \beta_2 \neq 0$, then we can suppose $\alpha_2 = 0$ and choosing $w = -\frac{y\alpha_4}{\alpha_3}$, we have $\alpha_4^* = 0$. Thus, we have following subcases:

i. if $\alpha_5 = 0$, then we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_2 \rangle$;

ii. if $\alpha_5 \neq 0$, then choosing $x = 1,\ y = \sqrt{\alpha_3\alpha_5^{-1}}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3 + \nabla_5, \nabla_2 \rangle$.

2. $\alpha_3 = 0, \alpha_4 \neq 0$, then we can suppose $\beta_4 = 0$.

$$\begin{array}{rclrcl}
\alpha_1^* & = & x^3\alpha_1, & \beta_1^* & = & x^3\beta_1, \\
\alpha_2^* & = & wx\alpha_1 + xy\alpha_2 + yz\alpha_5, & \beta_2^* & = & wx\beta_1 + xy\beta_2 + yz\beta_5, \\
\alpha_3^* & = & 0, & \beta_3^* & = & 0, \\
\alpha_4^* & = & xy\alpha_4, & \beta_4^* & = & 0, \\
\alpha_5^* & = & y^2\alpha_5, & \beta_5^* & = & y^2\beta_5.
\end{array}$$

(a) $\beta_5 \neq 0$, then we can suppose $\alpha_5 = 0$ and choosing $z = -\frac{x(w\beta_1+y\beta_2)}{y\beta_5}$, we have $\beta_2^* = 0$. Thus, we have following subcases:

i. if $\beta_1 = 0$, then $\alpha_1 \neq 0$ and choosing $x = 1,\ y = \frac{\alpha_1}{\alpha_4},\ w = -\frac{\alpha_2}{\alpha_4}$ we have the representative $\langle \nabla_1 + \nabla_4, \nabla_5 \rangle$;

ii. if $\beta_1 \neq 0, \alpha_1 = 0$ then choosing $x = 1,\ y = \sqrt{\frac{\beta_1}{\beta_5}}$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_4, \nabla_1 + \nabla_5 \rangle$;

iii. if $\beta_1 \neq 0, \alpha_1 \neq 0$ then choosing $x = \frac{\alpha_4^2\beta_1}{\alpha_1^2\beta_5},\ y = \frac{\alpha_2^3\beta_1^2}{\alpha_1^3\beta_5^2},\ w = -\frac{\alpha_2^4\beta_1^2}{\alpha_1^4\beta_5^2}$, we have the representative $\langle \nabla_1 + \nabla_4, \nabla_1 + \nabla_5 \rangle$.

(b) $\beta_5 = 0, \beta_1 \neq 0$, then we can suppose $\alpha_1 = 0$ and choosing $w = -\frac{y\beta_2}{\beta_1}$, we have $\beta_2^* = 0$. Thus, we have following subcases:

i. if $\alpha_5 = 0$, then we have the family of representatives $\langle \alpha\nabla_2 + \nabla_4, \nabla_1 \rangle$;

ii. if $\alpha_5 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_4}{\alpha_5},\ z = -\frac{\alpha_2}{\alpha_5}$, we have the representative $\langle \nabla_4 + \nabla_5, \nabla_1 \rangle$.

(c) $\beta_5 = \beta_1 = 0, \beta_2 \neq 0$, then we can suppose $\alpha_2 = 0$. Since in case of $\alpha_1 = 0$, we have a split extension, we can assume $\alpha_1 \neq 0$, Thus, we have following subcases:

i. if $\alpha_5 = 0$, then choosing $x = 1,\ y = \frac{\alpha_1}{\alpha_4}$, we have the representative $\langle \nabla_1 + \nabla_4, \nabla_2 \rangle$;

ii. if $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_4^2}{\alpha_1\alpha_5},\ y = \frac{\alpha_4^3}{\alpha_1\alpha_5}$, we have the representative $\langle \nabla_1 + \nabla_4 + \nabla_5, \nabla_2 \rangle$.

Now we have the following distinct orbits:

$$\begin{gathered}
\langle \alpha\nabla_1 + \nabla_3, \nabla_5 \rangle,\ \langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \nabla_5 \rangle,\ \langle \alpha\nabla_1 + \nabla_3, \nabla_4 + \nabla_5 \rangle, \\
\langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \nabla_4 + \nabla_5 \rangle,\ \langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \nabla_5 \rangle,\ \langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \nabla_1 + \nabla_5 \rangle, \\
\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \beta\nabla_1 + \nabla_4 + \nabla_5 \rangle,\ \langle \alpha\nabla_1 + \nabla_3, \beta\nabla_2 + \nabla_4 \rangle, \\
\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \alpha\nabla_2 + \nabla_4 \rangle,\ \langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \beta\nabla_2 + \nabla_4 \rangle, \\
\langle \alpha\nabla_1 + \nabla_3 + \nabla_5, \beta\nabla_2 + \nabla_4 \rangle,\ \langle \alpha\nabla_1 + \nabla_3 + \nabla_5, \nabla_1 + \nabla_4 \rangle,\ \langle \nabla_1, \nabla_3 + \nabla_4 \rangle, \\
\langle \nabla_1, \nabla_2 + \nabla_3 + \alpha\nabla_4 \rangle,\ \langle \nabla_1, \nabla_3 + \nabla_5, \rangle,\ \langle \nabla_1 + \nabla_2, \nabla_3 + \nabla_5, \rangle,\ \langle \alpha\nabla_1 + \nabla_3, \nabla_2 \rangle, \\
\langle \alpha\nabla_1 + \nabla_3 + \nabla_5, \nabla_2 \rangle,\ \langle \nabla_1 + \nabla_4, \nabla_5 \rangle,\ \langle \nabla_1 + \nabla_5, \alpha\nabla_2 + \nabla_4 \rangle,\ \langle \nabla_1 + \nabla_4, \nabla_1 + \nabla_5 \rangle, \\
\langle \nabla_1, \alpha\nabla_2 + \nabla_4 \rangle,\ \langle \nabla_1, \nabla_4 + \nabla_5 \rangle,\ \langle \nabla_1 + \nabla_4, \nabla_2 \rangle,\ \langle \nabla_1 + \nabla_4 + \nabla_5, \nabla_2 \rangle.
\end{gathered}$$

Hence, we have the following new five-dimensional nilpotent bicommutative algebras (see Section 3):

$$\begin{gathered}
\mathrm{B}_{146}^{\alpha}, \mathrm{B}_{147}^{\alpha}, \mathrm{B}_{148}^{\alpha}, \mathrm{B}_{149}^{\alpha}, \mathrm{B}_{150}^{\alpha}, \mathrm{B}_{151}^{\alpha}, \mathrm{B}_{152}^{\alpha,\beta}, \mathrm{B}_{153}^{\alpha,\beta}, \mathrm{B}_{154}^{\alpha}, \mathrm{B}_{155}^{\alpha,\beta}, \mathrm{B}_{156}^{\alpha,\beta}, \mathrm{B}_{157}^{\alpha}, \mathrm{B}_{158}, \mathrm{B}_{159}^{\alpha}, \mathrm{B}_{160}, \mathrm{B}_{161}, \\
\mathrm{B}_{162}^{\alpha}, \mathrm{B}_{163}^{\alpha}, \mathrm{B}_{164}, \mathrm{B}_{165}^{\alpha}, \mathrm{B}_{166}, \mathrm{B}_{167}^{\alpha}, \mathrm{B}_{168}, \mathrm{B}_{169}, \mathrm{B}_{170}.
\end{gathered}$$

2.5.3. Central Extensions of $\mathcal{B}_{04}^{3*}(0)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{13}], \quad \nabla_3 = [\Delta_{21}], \quad \nabla_4 = [\Delta_{22}], \quad \nabla_5 = [\Delta_{32}].$$

The automorphism group of $\mathcal{B}_{04}^{3*}(0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & t & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ \alpha_3 & \alpha_4 & 0 \\ 0 & \alpha_5 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_2^* \\ \alpha_3^* & \alpha_4^* & 0 \\ 0 & \alpha_5^* & 0 \end{pmatrix},$$

the action of $\mathrm{Aut}(\mathcal{B}_{04}^{3*}(0))$ on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i\nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{5} \alpha_i^*\nabla_i \rangle$, where

$$\begin{aligned}
\alpha_1^* &= x(x\alpha_1 + z\alpha_2), & \alpha_2^* &= x^2y\alpha_2, & \alpha_3^* &= xy\alpha_3, \\
\alpha_4^* &= y(y\alpha_4 + t\alpha_5), & \alpha_5^* &= xy^2\alpha_5.
\end{aligned}$$

We are interested only in $(\alpha_2,\alpha_5)\neq(0,0)$ and consider the vector space generated by the following two cocycles:

$$\theta_1=\alpha_1\nabla_1+\alpha_2\nabla_2+\alpha_3\nabla_3+\alpha_4\nabla_4+\alpha_5\nabla_5 \text{ and } \theta_2=\beta_1\nabla_1+\beta_3\nabla_3+\beta_4\nabla_4+\beta_5\nabla_5.$$

1. $\alpha_2\neq 0$, then we have

$$\begin{array}{lllllll}
\alpha_1^* &=& x(x\alpha_1+z\alpha_2), & \beta_1^* &=& x^2\beta_1,\\
\alpha_2^* &=& x^2y\alpha_2, & \beta_2^* &=& 0,\\
\alpha_3^* &=& xy\alpha_3, & \beta_3^* &=& xy\beta_3,\\
\alpha_4^* &=& y(y\alpha_4+t\alpha_5), & \beta_4^* &=& y(y\beta_4+t\beta_5),\\
\alpha_5^* &=& xy^2\alpha_5, & \beta_5^* &=& xy^2\beta_5.
\end{array}$$

(a) $\beta_5\neq 0$, then we can suppose $\alpha_5^*=0$ and choosing $z=-\frac{x\alpha_1}{\alpha_2}$, $t=-\frac{y\beta_4}{\beta_5}$, we have $\alpha_1^*=\beta_4^*=0$. Thus, we have following subcases:

i. $\alpha_3=\alpha_4=\beta_3=\beta_1=0$, then we have the representative $\langle\nabla_2,\nabla_5\rangle$;

ii. $\alpha_3=\alpha_4=\beta_3=0$, $\beta_1\neq 0$, then choosing $x=\frac{\beta_5}{\beta_1}$, $y=1$, we have the representative $\langle\nabla_2,\nabla_1+\nabla_5\rangle$;

iii. $\alpha_3=\alpha_4=0$ $\beta_3\neq 0$, $\beta_1=0$, then choosing $y=\frac{\beta_3}{\beta_5}$, we have the representative $\langle\nabla_2,\nabla_3+\nabla_5\rangle$;

iv. $\alpha_3=\alpha_4=0$, $\beta_3\neq 0$, $\beta_1\neq 0$, then choosing $x=\frac{\beta_3^2}{\beta_1\beta_5}$, $y=\frac{\beta_3}{\beta_5}$, we have the representative $\langle\nabla_2,\nabla_1+\nabla_3+\nabla_5\rangle$;

v. $\alpha_3=0$, $\alpha_4\neq 0$, $\beta_3=\beta_1=0$, then choosing $x=1$, $y=\frac{\alpha_2}{\alpha_4}$, we have the representative $\langle\nabla_2+\nabla_4,\nabla_5\rangle$;

vi. $\alpha_3=0$, $\alpha_4\neq 0$, $\beta_3=0$, $\beta_1\neq 0$, then choosing $x=\sqrt[3]{\frac{\alpha_4^2\beta_1}{\alpha_2^2\beta_5}}$, $y=\sqrt[3]{\frac{\alpha_4\beta_1^2}{\alpha_2\beta_5^2}}$, we have the representative $\langle\nabla_2+\nabla_4,\nabla_1+\nabla_5\rangle$;

vii. $\alpha_3=0$, $\alpha_4\neq 0$, $\beta_3\neq 0$, then choosing $x=\sqrt{\frac{\alpha_4\beta_3}{\alpha_2\beta_5}}$, $y=\frac{\beta_3}{\beta_5}$, we have the family of representatives $\langle\nabla_2+\nabla_4,\alpha\nabla_1+\nabla_3+\nabla_5\rangle$;

viii. $\alpha_3\neq 0$, $\alpha_4=\beta_1=\beta_3=0$, then choosing $x=\frac{\alpha_3}{\alpha_2}$, we have the representative $\langle\nabla_2+\nabla_3,\nabla_5\rangle$;

ix. $\alpha_3\neq 0$, $\alpha_4=\beta_1=0$, $\beta_3\neq 0$, then choosing $x=\frac{\alpha_3}{\alpha_2}$, $y=\frac{\beta_3}{\beta_5}$, we have the representative $\langle\nabla_2+\nabla_3,\nabla_3+\nabla_5\rangle$;

x. $\alpha_3\neq 0$, $\alpha_4=0$, $\beta_1\neq 0$, then choosing $x=\frac{\alpha_3}{\alpha_2}$, $y=\sqrt{\frac{\alpha_3\beta_1}{\alpha_2\beta_5}}$, we have the family of representatives $\langle\nabla_2+\nabla_3,\nabla_1+\alpha\nabla_3+\nabla_5\rangle^{O(\alpha)\simeq O(-\alpha)}$;

xi. $\alpha_3\neq 0$, $\alpha_4\neq 0$, then choosing $x=\frac{\alpha_3}{\alpha_2}$, $y=\frac{\alpha_3^2}{\alpha_2\alpha_4}$, we have the family of representatives $\langle\nabla_2+\nabla_3+\nabla_4,\alpha\nabla_1+\beta\nabla_4+\nabla_5\rangle$.

(b) $\beta_5=0,\beta_4\neq 0$, then choosing $t=0$, $z=\frac{x(\alpha_4\beta_1-\alpha_1\beta_4)}{\alpha_2\beta_4}$, we can suppose $\alpha_1^*=\alpha_4^*=0$ and have following subcases:

i. $\alpha_3=\alpha_5=\beta_3=\beta_1=0$, then we have the representative $\langle\nabla_2,\nabla_4\rangle$;

ii. $\alpha_3=\alpha_5=\beta_3=0,\beta_1\neq 0$, then choosing $x=1$, $y=\sqrt{\frac{\beta_1}{\beta_4}}$, we have the representative $\langle\nabla_2,\nabla_1+\nabla_4\rangle$;

iii. $\alpha_3=\alpha_5=0$, $\beta_3\neq 0$, then choosing $x=1$, $y=\frac{\beta_3}{\beta_4}$, we have the family of representatives $\langle\nabla_2,\alpha\nabla_1+\nabla_3+\nabla_4\rangle$;

iv. $\alpha_3=0$, $\alpha_5\neq 0$, then choosing $x=\frac{\alpha_5}{\alpha_2}$, $y=1$, we have the family of representatives $\langle\nabla_2+\nabla_5,\alpha\nabla_1+\beta\nabla_3+\nabla_4\rangle$;

v. $\alpha_3\neq 0$, $\alpha_5=\beta_3=\beta_1=0$, then choosing $x=\frac{\alpha_3}{\alpha_2}$, we have the representative $\langle\nabla_2+\nabla_3,\nabla_4\rangle$;

vi. $\alpha_3 \neq 0,\ \alpha_5 = \beta_3 = 0,\ \beta_1 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3\sqrt{\beta_1}}{\alpha_2\sqrt{\beta_4}}$, we have the representative $\langle \nabla_2 + \nabla_3, \nabla_1 + \nabla_4 \rangle$;

vii. $\alpha_3 \neq 0,\ \alpha_5 = 0,\ \beta_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3\beta_3}{\alpha_2\beta_4}$, we have the family of representatives $\langle \nabla_2 + \nabla_3, \alpha\nabla_1 + \nabla_3 + \nabla_4 \rangle$;

viii. $\alpha_3 \neq 0,\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3}{\alpha_5}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \nabla_5, \alpha\nabla_1 + \beta\nabla_3 + \nabla_4 \rangle$.

(c) $\beta_5 = \beta_4 = 0,\ \beta_3 \neq 0$, then choosing $z = \frac{x(\alpha_3\beta_1 - \alpha_1\beta_3)}{\alpha_2\beta_3}$, we can suppose $\alpha_1^* = \alpha_3^* = 0$ and have following subcases:

i. $\alpha_5 = \beta_1 = \alpha_4 = 0$, then we have the representative $\langle \nabla_2, \nabla_3 \rangle$;

ii. $\alpha_5 = \beta_1 = 0,\ \alpha_4 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4, \nabla_3 \rangle$;

iii. $\alpha_5 = 0,\ \beta_1 \neq 0,\ \alpha_4 = 0$ then choosing $x = 1,\ y = \frac{\beta_1}{\beta_3}$, we have the representative $\langle \nabla_2, \nabla_1 + \nabla_3 \rangle$;

iv. $\alpha_5 = 0,\ \beta_1 \neq 0,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4\beta_1}{\alpha_2\beta_3},\ y = \frac{\alpha_4\beta_1^2}{\alpha_2\beta_3^2}$, we have the representative $\langle \nabla_2 + \nabla_4, \nabla_1 + \nabla_3 \rangle$;

v. $\alpha_5 \neq 0$, then choosing $y = 1,\ x = \frac{\alpha_5}{\alpha_2},\ t = -\frac{\alpha_4}{\alpha_5}$, we have the family of representatives $\langle \nabla_2 + \nabla_5, \alpha\nabla_1 + \nabla_3 \rangle$.

(d) $\beta_5 = \beta_4 = \beta_3 = 0,\ \beta_1 \neq 0$, then we can suppose $\alpha_1^* = 0$ and consider following subcases:

i. $\alpha_5 = \alpha_4 = \alpha_3 = 0$, then we have the representative $\langle \nabla_2, \nabla_1 \rangle$;

ii. $\alpha_5 = \alpha_4 = 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_3, \nabla_1 \rangle$;

iii. $\alpha_5 = 0,\ \alpha_4 \neq 0,\ \alpha_3 = 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4, \nabla_1 \rangle$;

iv. $\alpha_5 = 0,\ \alpha_4 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3^2}{\alpha_2\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4, \nabla_1 \rangle$;

v. $\alpha_5 \neq 0,\ \alpha_3 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_2},\ y = 1,\ t = -\frac{\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_2 + \nabla_5, \nabla_1 \rangle$;

vi. $\alpha_5 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3}{\alpha_5},\ t = -\frac{\alpha_3\alpha_4}{\alpha_5^2}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5, \nabla_1 \rangle$.

2. $\alpha_2 = 0$, then $\alpha_5 \neq 0$ and we have

$$
\begin{array}{rclrcl}
\alpha_1^* & = & x^2\alpha_1, & \beta_1^* & = & x^2\beta_1, \\
\alpha_2^* & = & 0, & \beta_2^* & = & 0, \\
\alpha_3^* & = & xy\alpha_3, & \beta_3^* & = & xy\beta_3, \\
\alpha_4^* & = & y(y\alpha_4 + t\alpha_5), & \beta_4^* & = & y^2\beta_4, \\
\alpha_5^* & = & xy^2\alpha_5, & \beta_5^* & = & 0.
\end{array}
$$

(a) $\beta_1 \neq 0$, then choosing $t = \frac{y(\alpha_1\beta_4 - \alpha_4\beta_1)}{\alpha_5\beta_1}$, we can suppose $\alpha_1^* = \alpha_4^* = 0$ and have following subcases:

i. $\alpha_3 = \beta_4 = \beta_3 = 0$, then we have the representative $\langle \nabla_5, \nabla_1 \rangle$;

ii. $\alpha_3 = \beta_4 = 0,\ \beta_3 \neq 0$, then choosing $x = 1, y = \frac{\beta_1}{\beta_3}$, we have the representative $\langle \nabla_5, \nabla_1 + \nabla_3 \rangle$;

iii. $\alpha_3 = 0,\ \beta_4 \neq 0$, then choosing $x = 1, y = \sqrt{\frac{\beta_1}{\beta_4}}$, we have the family of representatives $\langle \nabla_5, \nabla_1 + \alpha\nabla_3 + \nabla_4 \rangle^{O(\alpha)\simeq O(-\alpha)}$;

iv. $\alpha_3 \neq 0,\ \beta_4 = \beta_3 = 0$, then choosing $x = 1, y = \frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5, \nabla_1 \rangle$;

v. $\alpha_3 \neq 0,\ \beta_4 = 0,\ \beta_3 \neq 0$, then choosing $x = \frac{\alpha_3\beta_3}{\alpha_5\beta_1}, y = \frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5, \nabla_1 + \nabla_3 \rangle$;

vi. $\alpha_3 \neq 0,\ \beta_4 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_5}\sqrt{\frac{\beta_3}{\beta_5}}, y = \frac{\alpha_3}{\alpha_5}$, we have the family of representatives $\langle \nabla_3 + \nabla_5, \nabla_1 + \alpha\nabla_3 + \nabla_4 \rangle^{O(\alpha)\simeq O(-\alpha)}$.

(b) $\beta_1 = 0,\ \beta_3 \neq 0$, then choosing $t = \frac{y(\alpha_3\beta_4 - \alpha_4\beta_3)}{\alpha_5\beta_3}$, we can suppose $\alpha_3^* = \alpha_4^* = 0$ and have following subcases:

i. $\alpha_1 = \beta_4 = 0$, then we have the representative $\langle \nabla_5, \nabla_3 \rangle$;

ii. $\alpha_1 = 0,\ \beta_4 \neq 0$, then choosing $x = 1, y = \frac{\beta_3}{\beta_4}$, we have the representative $\langle \nabla_5, \nabla_3 + \nabla_4 \rangle$;

iii. $\alpha_1 \neq 0,\ \beta_4 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_1}, y = 1$, we have the representative $\langle \nabla_1 + \nabla_5, \nabla_3 \rangle$;

iv. $\alpha_1 \neq 0,\ \beta_4 \neq 0$, then choosing $x = \frac{\alpha_1\beta_4^2}{\alpha_5\beta_3^2}, y = \frac{\alpha_1\beta_4}{\alpha_5\beta_3}$, we have the representative $\langle \nabla_1 + \nabla_5, \nabla_3 + \nabla_4 \rangle$.

(c) $\beta_1 = \beta_3 = 0$, then $\beta_4 \neq 0$, and we can suppose $\alpha_4^* = 0$. Consider following subcases:

i. $\alpha_1 = \alpha_3 = 0$, then we have the representative $\langle \nabla_5, \nabla_4 \rangle$;

ii. $\alpha_1 = 0,\ \alpha_3 \neq 0$, then choosing $x = 1, y = \frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5, \nabla_4 \rangle$;

iii. $\alpha_1 \neq 0,\ \alpha_3 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_1}, y = 1$, we have the representative $\langle \nabla_1 + \nabla_5, \nabla_4 \rangle$;

iv. $\alpha_1 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3^2}{\alpha_1\alpha_5}, y = \frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_5, \nabla_4 \rangle$.

Now we have the following distinct orbits:

$$\begin{gathered}
\langle \nabla_2, \nabla_5 \rangle, \langle \nabla_2, \nabla_1 + \nabla_5 \rangle, \langle \nabla_2, \nabla_3 + \nabla_5 \rangle, \langle \nabla_2, \nabla_1 + \nabla_3 + \nabla_5 \rangle, \langle \nabla_2 + \nabla_4, \nabla_5 \rangle, \\
\langle \nabla_2 + \nabla_4, \nabla_1 + \nabla_5 \rangle, \langle \nabla_2 + \nabla_4, \alpha\nabla_1 + \nabla_3 + \nabla_5 \rangle, \langle \nabla_2 + \nabla_3, \nabla_5 \rangle, \langle \nabla_2 + \nabla_3, \nabla_3 + \nabla_5 \rangle, \\
\langle \nabla_2 + \nabla_3, \nabla_1 + \alpha\nabla_3 + \nabla_5 \rangle^{O(\alpha)\simeq O(-\alpha)}, \langle \nabla_2 + \nabla_3 + \nabla_4, \alpha\nabla_1 + \beta\nabla_4 + \nabla_5 \rangle, \langle \nabla_2, \nabla_4 \rangle, \\
\langle \nabla_2, \nabla_1 + \nabla_4 \rangle, \langle \nabla_2, \alpha\nabla_1 + \nabla_3 + \nabla_4 \rangle, \langle \nabla_2 + \nabla_5, \alpha\nabla_1 + \beta\nabla_3 + \nabla_4 \rangle, \langle \nabla_2 + \nabla_3, \nabla_4 \rangle, \\
\langle \nabla_2 + \nabla_3, \nabla_1 + \nabla_4 \rangle, \langle \nabla_2 + \nabla_3, \alpha\nabla_1 + \nabla_3 + \nabla_4 \rangle, \langle \nabla_2 + \nabla_3 + \nabla_5, \alpha\nabla_1 + \beta\nabla_3 + \nabla_4 \rangle, \\
\langle \nabla_2, \nabla_3 \rangle, \langle \nabla_2 + \nabla_4, \nabla_3 \rangle, \langle \nabla_2, \nabla_1 + \nabla_3 \rangle, \langle \nabla_2 + \nabla_4, \nabla_1 + \nabla_3 \rangle, \langle \nabla_2 + \nabla_5, \alpha\nabla_1 + \nabla_3 \rangle, \\
\langle \nabla_2, \nabla_1 \rangle, \langle \nabla_2 + \nabla_3, \nabla_1 \rangle, \langle \nabla_2 + \nabla_4, \nabla_1 \rangle, \langle \nabla_2 + \nabla_3 + \nabla_4, \nabla_1 \rangle, \langle \nabla_2 + \nabla_5, \nabla_1 \rangle, \\
\langle \nabla_2 + \nabla_3 + \nabla_5, \nabla_1 \rangle, \langle \nabla_5, \nabla_1 \rangle, \langle \nabla_5, \nabla_1 + \nabla_3 \rangle, \langle \nabla_5, \nabla_1 + \alpha\nabla_3 + \nabla_4 \rangle^{O(\alpha)\simeq O(-\alpha)}, \\
\langle \nabla_3 + \nabla_5, \nabla_1 \rangle, \langle \nabla_3 + \nabla_5, \nabla_1 + \nabla_3 \rangle, \langle \nabla_3 + \nabla_5, \nabla_1 + \alpha\nabla_3 + \nabla_4 \rangle^{O(\alpha)\simeq O(-\alpha)}, \langle \nabla_5, \nabla_3 \rangle, \\
\langle \nabla_5, \nabla_3 + \nabla_4 \rangle, \langle \nabla_1 + \nabla_5, \nabla_3 \rangle, \langle \nabla_1 + \nabla_5, \nabla_3 + \nabla_4 \rangle, \langle \nabla_5, \nabla_4 \rangle, \langle \nabla_3 + \nabla_5, \nabla_4 \rangle, \\
\langle \nabla_1 + \nabla_5, \nabla_4 \rangle, \langle \nabla_1 + \nabla_3 + \nabla_5, \nabla_4 \rangle.
\end{gathered}$$

Hence, we have the following new five-dimensional nilpotent bicommutative algebras (see Section 3):

$$\begin{gathered}
B_{171}, B_{172}, B_{173}, B_{174}, B_{175}, B_{176}, B_{177}^{\alpha}, B_{178}, B_{179}, B_{180}^{\alpha}, B_{181}^{\alpha,\beta}, B_{182}, \\
B_{183}, B_{184}^{\alpha}, B_{185}^{\alpha,\beta}, B_{186}, B_{187}, B_{188}^{\alpha}, B_{189}^{\alpha,\beta}, B_{78}^{0}, B_{79}^{0}, B_{190}, B_{191}, B_{192}^{\alpha}, B_{193}, B_{194}, B_{195}, B_{196}, B_{197}, \\
B_{198}, B_{199}, B_{200}, B_{201}^{\alpha}, B_{202}, B_{203}, B_{204}^{\alpha}, B_{205}, B_{57}^{0,0,0}, B_{206}, B_{56}^{0}, B_{101}^{0}, B_{207}, B_{104}^{0}, B_{105}^{0}.
\end{gathered}$$

3. Classification Theorem for Five-Dimensional Bicommutative Algebras

The algebraic classification of complex five-dimensional bicommutative algebras consists of two parts:

1. Five-dimensional algebras with identity $xyz = 0$ (also known as two-step nilpotent algebras) are the intersection of all varieties of algebras defined by a family of polynomial identities of degree three or more; for example, it is in the intersection of associative, Zinbiel, Leibniz, Novikov, bicommutative, etc, algebras. All these algebras can be obtained as central extensions of zero-product algebras. The geometric

classification of two-step nilpotent algebras is given in [9]. It is the reason why we are not interested in it.

2. Five-dimensional nilpotent (non-two-step nilpotent) bicommutative algebras, which are central extensions of nilpotent bicommutative algebras with nonzero products of a smaller dimension. These algebras are classified by several steps:
   (a) Complex split five-dimensional bicommutative algebras are classified in [13];
   (b) Complex non-split five-dimensional nilpotent commutative associative algebras are listed in [27];
   (c) Complex one-generated five-dimensional nilpotent bicommutative algebras are classified in [14];
   (d) Complex non-split non-one-generated five-dimensional nilpotent non-commutative bicommutative algebras are classified in Theorem (see below).

Theorem 2. *Let $\mathbb{B}$ be a complex non-split non-one-generated five-dimensional nilpotent (non-*2*-step nilpotent) non-commutative bicommutative algebra. Then $\mathbb{B}$ is isomorphic to one algebra from the following list:*

B_{01}	:	$e_1e_1 = e_2$	$e_1e_4 = e_5$	$e_2e_1 = e_5$	
		$e_3e_3 = e_5$	$e_4e_1 = e_5$		
B_{02}	:	$e_1e_1 = e_2$	$e_2e_1 = e_5$	$e_3e_4 = e_5$	$e_4e_3 = -e_5$
B_{03}	:	$e_1e_1 = e_2$	$e_1e_3 = e_5$	$e_2e_1 = e_5$	$e_3e_1 = e_5$
		$e_4e_3 = e_5$	$e_4e_4 = e_5$		
B_{04}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = \alpha e_5$	
		$e_3e_3 = e_5$	$e_4e_1 = e_5$		
B_{05}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = e_5$	
		$e_3e_1 = e_5$	$e_3e_3 = e_5$	$e_4e_4 = e_5$	
B_{06}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = e_5$	
		$e_3e_1 = e_5$	$e_3e_4 = e_5$	$e_4e_3 = e_5$	
B_{07}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_5$	$e_2e_1 = (\alpha+1)e_5$	
		$e_3e_3 = e_5$	$e_4e_4 = e_5$		
B_{08}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = \alpha e_5$	
		$e_3e_4 = e_5$	$e_4e_3 = -e_5$		
B_{09}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = -e_5$	
		$e_3e_1 = e_5$	$e_3e_4 = e_5$	$e_4e_3 = -e_5$	
$B_{10}^{\alpha,\lambda}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_5$	$e_2e_1 = (\alpha+1)e_5$	
		$e_3e_3 = \lambda e_5$	$e_4e_3 = e_5$	$e_4e_4 = e_5$	
B_{11}^{λ}	:	$e_1e_1 = e_2$	$e_1e_2 = \frac{-1+\sqrt{1-4\lambda}}{2}e_5$	$e_3e_3 = \lambda e_5$	$e_4e_3 = e_5$
		$e_4e_4 = e_5$	$e_2e_1 = \frac{1+\sqrt{1-4\lambda}}{2}e_5$	$e_3e_1 = e_5$	
$B_{12}^{\lambda\neq\frac{1}{4}}$	:	$e_1e_1 = e_2$	$e_1e_2 = \frac{-1-\sqrt{1-4\lambda}}{2}e_5$	$e_3e_3 = \lambda e_5$	$e_4e_3 = e_5$
		$e_4e_4 = e_5$	$e_2e_1 = \frac{1-\sqrt{1-4\lambda}}{2}e_5$	$e_3e_1 = e_5$	
B_{13}	:	$e_1e_1 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_5$	$e_2e_2 = e_4$
		$e_2e_4 = e_5$	$e_2e_4 = e_5$	$e_3e_1 = e_5$	
B_{14}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_5$	$e_1e_3 = \alpha e_5$	$e_2e_1 = e_5$
		$e_2e_2 = e_4$	$e_2e_4 = -(1+\alpha)e_5$	$e_3e_1 = (1+\alpha)e_5$	$e_4e_2 = -\alpha e_5$
B_{15}^{α}	:	$e_1e_1 = e_3$	$e_1e_3 = \alpha e_5$	$e_2e_2 = e_4$	
		$e_2e_4 = e_5$	$e_3e_1 = (1+\alpha)e_5$	$e_4e_2 = e_5$	
$B_{16}^{\alpha,\beta}$	:	$e_1e_1 = e_3$	$e_1e_3 = \alpha e_5$	$e_2e_2 = e_4$	
		$e_2e_4 = \beta e_5$	$e_3e_1 = (1+\alpha)e_5$	$e_4e_2 = (1+\beta)e_5$	
B_{17}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_4e_4 = e_5$
B_{18}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_4 = e_5$	$e_4e_4 = e_5$		
B_{19}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_1 = e_5$	$e_4e_4 = e_5$		
B_{20}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_1 = e_5$	$e_2e_4 = e_5$	$e_4e_4 = e_5$	
B_{21}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_5$
		$e_2e_2 = e_5$	$e_2e_4 = \alpha e_5$	$e_4e_4 = e_5$	
B_{22}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_2 = e_5$	$e_2e_4 = \alpha e_5$	$e_4e_4 = e_5$	
B_{23}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_2 = e_5$	$e_4e_1 = e_5$	$e_4e_2 = e_5$	

B_{24}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_2 = e_5$
		$e_2e_4 = -e_5$	$e_4e_1 = e_5$	$e_4e_2 = e_5$	
B_{25}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_4 = \alpha e_5$	$e_4e_1 = e_5$	$e_4e_2 = e_5$	
B_{26}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_1 = e_5$	$e_4e_2 = e_5$		
B_{27}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_2 = e_5$	$e_2e_4 = -e_5$	$e_4e_2 = e_5$	
B_{28}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_4 = \alpha e_5$	$e_4e_2 = e_5$		
B_{29}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_4e_1 = e_5$
B_{30}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_2 = e_5$	$e_4e_1 = e_5$		
B_{31}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_4 = e_5$	$e_4e_1 = e_5$		
B_{32}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_4 = e_5$
B_{33}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_3e_1 = e_5$	
		$e_3e_2 = e_5$	$e_4e_4 = e_5$		
B_{34}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_4 = e_5$	
B_{35}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3 + e_5$	$e_1e_4 = \alpha e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_4 = e_5$	
B_{36}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3 + e_5$	$e_2e_2 = e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_4 = e_5$	
B_{37}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3 + e_5$	$e_1e_4 = e_5$	$e_2e_2 = e_5$
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_4 = e_5$	
B_{38}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = \alpha e_5$	$e_2e_2 = e_5$
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_4 = e_5$	
B_{39}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_4 = e_5$	
B_{40}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = e_5$
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_4 = e_5$	
B_{41}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = \alpha e_5$
		$e_2e_2 = e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_4 = e_5$
B_{42}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	
		$e_2e_4 = e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	
B_{43}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3 + e_5$	$e_1e_4 = e_5$	
		$e_2e_4 = e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	
B_{44}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	$e_2e_4 = e_5$
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
B_{45}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3 + e_5$	$e_2e_4 = e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$		
B_{46}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3 + e_5$	$e_2e_4 = e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
B_{47}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = e_5$
		$e_2e_4 = e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = \alpha e_5$
$B_{48}^{\alpha,\beta}$	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = \beta e_5$	$e_2e_4 = e_5$
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = \alpha e_5$	
B_{49}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3 + e_5$	$e_1e_4 = -e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
B_{50}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = \alpha e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
B_{51}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = \alpha e_5$
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
B_{52}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$		
B_{53}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	
		$e_2e_2 = e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	

B_{54}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	
		$e_1e_4=e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	
B_{55}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_1e_4=e_5$
		$e_2e_2=e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	
B_{56}^{α}	:	$e_1e_1=e_5$	$e_1e_2=e_3$	$e_2e_1=e_4$	
		$e_2e_2=-e_3$	$e_2e_4=\alpha e_5$	$e_3e_2=e_5$	
$B_{57}^{\alpha,\beta,\gamma}$	:	$e_1e_2=e_3$	$e_1e_3=\gamma e_5$	$e_2e_1=e_4$	$e_2e_2=-e_3$
		$e_2e_3=-\gamma e_5$	$e_2e_4=\alpha e_5$	$e_3e_2=e_5$	$e_4e_1=\beta e_5$
$B_{58}^{(\alpha,\beta)\neq(0,0)}$	:	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_2e_1=e_4$	$e_2e_2=-e_3$
		$e_2e_3=-e_5$	$e_2e_4=\alpha e_5$	$e_4e_1=\beta e_5$	
$B_{59}^{\alpha\neq0}$	:	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_2e_1=e_4$	
		$e_2e_2=-e_3+e_5$	$e_2e_3=-e_5$	$e_4e_1=\alpha e_5$	
$B_{60}^{\alpha\neq1,\beta\neq0}$	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=e_5$	$e_1e_4=\beta e_5$
		$e_2e_1=-\alpha e_3$	$e_2e_2=-e_4$	$e_2e_3=-\alpha e_5$	$e_2e_4=-\beta e_5$
$B_{61}^{\alpha\neq1,\beta\neq0}$	:	$e_1e_1=e_3$	$e_1e_2=e_4+e_5$	$e_1e_4=\beta e_5$	$e_2e_1=-\alpha e_3$
		$e_2e_2=-e_4$	$e_2e_4=-\beta e_5$	$e_3e_1=e_5$	
$B_{62}^{\alpha\neq1,(\beta,\gamma)\neq(0,0),\delta}$	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\delta e_5$	$e_1e_4=\gamma e_5$
		$e_2e_1=-\alpha e_3$	$e_2e_2=-e_4$	$e_2e_3=-\alpha\delta e_5$	$e_2e_4=-\gamma e_5$
		$e_3e_1=e_5$	$e_4e_2=\beta e_5$		
$B_{63}^{\alpha\neq0}$	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\alpha e_5$	
		$e_2e_1=e_5$	$e_2e_2=-e_4$	$e_4e_2=e_5$	
$B_{64}^{\alpha\neq0,\beta}$	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\alpha e_5$	$e_1e_4=\beta e_5$
		$e_2e_2=-e_4$	$e_2e_4=-\beta e_5$	$e_4e_2=e_5$	
B_{65}	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=e_5$	$e_1e_4=-e_5$
		$e_2e_1=-e_3$	$e_2e_2=-e_4$	$e_2e_3=-e_5$	$e_2e_4=e_5$
B_{66}	:	$e_1e_1=e_3$	$e_1e_2=e_4+e_5$	$e_1e_3=e_5$	$e_1e_4=-e_5$
		$e_2e_1=-e_3$	$e_2e_2=-e_4$	$e_2e_3=-e_5$	$e_2e_4=e_5$
B_{67}	:	$e_1e_1=e_3$	$e_1e_2=e_4+e_5$	$e_1e_3=e_5$	$e_1e_4=-e_5$
		$e_2e_1=-e_3-e_5$	$e_2e_2=-e_4$	$e_2e_3=-e_5$	$e_2e_4=e_5$
B_{68}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\alpha e_5$	$e_1e_4=-\alpha e_5$
		$e_2e_1=-e_3$	$e_2e_2=-e_4$	$e_2e_3=-\alpha e_5$	$e_2e_4=\alpha e_5$
		$e_3e_1=e_5$	$e_3e_2=-e_5$	$e_4e_1=-e_5$	$e_4e_2=e_5$
B_{69}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\alpha e_5$	$e_1e_4=-\alpha e_5$
		$e_2e_1=-e_3+e_5$	$e_2e_2=-e_4$	$e_2e_3=-\alpha e_5$	$e_2e_4=\alpha e_5$
		$e_3e_1=e_5$	$e_3e_2=-e_5$	$e_4e_1=-e_5$	$e_4e_2=e_5$
B_{70}	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=e_5$	$e_2e_1=-e_3$
		$e_2e_2=-e_4$	$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=-e_5$
		$e_4e_1=-e_5$	$e_4e_2=e_5$		
B_{71}	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=e_5$	$e_2e_1=-e_3+e_5$
		$e_2e_2=-e_4$	$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=-e_5$
		$e_4e_1=-e_5$	$e_4e_2=e_5$		
B_{72}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\alpha e_5$	
		$e_1e_4=e_5$	$e_2e_1=-e_3$	$e_2e_2=-e_4$	
		$e_2e_3=-\alpha e_5$	$e_2e_4=-e_5$	$e_3e_1=e_5$	
B_{73}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_4+e_5$	$e_1e_3=\alpha e_5$	
		$e_1e_4=e_5$	$e_2e_1=-e_3$	$e_2e_2=-e_4$	
		$e_2e_3=-\alpha e_5$	$e_2e_4=-e_5$	$e_3e_1=e_5$	
$B_{74}^{\alpha,\beta}$	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\alpha e_5$	$e_1e_4=\beta e_5$
		$e_2e_1=-e_3$	$e_2e_2=-e_4$	$e_2e_3=-\alpha e_5$	$e_2e_4=-\beta e_5$
		$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_1=e_5$	
B_{75}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\alpha e_5$	
		$e_2e_1=-e_3$	$e_2e_2=-e_4$	$e_2e_3=-\beta e_5$	
		$e_3e_1=-2e_5$	$e_3e_2=e_5$	$e_4e_1=e_5$	
B_{76}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_4$	$e_1e_3=\alpha e_5$	$e_1e_4=e_5$
		$e_2e_1=-e_3$	$e_2e_2=-e_4$	$e_2e_3=-\alpha e_5$	$e_2e_4=-e_5$
		$e_3e_1=-2e_5$	$e_3e_2=e_5$	$e_4e_1=e_5$	

B_{77}	:	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_5$	$e_4e_1 = e_5$
B_{78}^{α}	:	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_2e_4 = e_5$	$e_3e_2 = \alpha e_5$
B_{79}^{α}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	
		$e_2e_4 = e_5$	$e_3e_2 = \alpha e_5$		
B_{80}	:	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_4$	
		$e_2e_4 = e_5$	$e_3e_2 = \alpha e_5$		
$B_{81}^{(\alpha,\beta)\neq(0,0)}$	:	$e_1e_2 = e_3$	$e_1e_3 = \beta e_5$	$e_2e_1 = e_4$	
		$e_2e_4 = e_5$	$e_3e_2 = \alpha e_5$	$e_4e_1 = e_5$	
B_{82}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = -2e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = 2e_3 + e_4$	$e_2e_4 = -e_5$	
B_{83}^{α}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = \alpha e_5$	$e_1e_4 = (1-2\alpha)e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = 2e_3 + e_4$	$e_2e_3 = e_5$	$e_2e_4 = -\alpha e_5$
B_{84}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = -e_5$
		$e_2e_1 = -e_3 + e_5$	$e_2e_2 = 2e_3 + e_4$	$e_2e_3 = e_5$	$e_2e_4 = -e_5$
B_{85}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_4 = -\frac{1}{2}e_5$	$e_2e_1 = -e_3$
		$e_2e_2 = 2e_3 + e_4 + e_5$	$e_2e_3 = -\frac{1}{2}e_5$	$e_3e_1 = e_5$	$e_3e_2 = (i - \frac{3}{2})e_5$
		$e_4e_1 = (-\frac{1}{2} - i)e_5$	$e_4e_2 = e_5$		
B_{86}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_4 = -\frac{1}{2}e_5$	$e_2e_1 = -e_3$
		$e_2e_2 = 2e_3 + e_4 + e_5$	$e_2e_3 = -\frac{1}{2}e_5$	$e_3e_1 = e_5$	$e_3e_2 = (-\frac{3}{2} - i)e_5$
		$e_4e_1 = (i - \frac{1}{2})e_5$	$e_4e_2 = e_5$		
$B_{87}^{\alpha,\beta,\gamma}$	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = \alpha e_5$	$e_1e_4 = (\gamma - 2\alpha)e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = 2e_3 + e_4$	$e_2e_3 = \gamma e_5$	$e_2e_4 = -\alpha e_5$
		$e_3e_1 = e_5$	$e_3e_2 = (\beta - 2)e_5$	$e_4e_1 = -\beta e_5$	$e_4e_2 = e_5$
B_{88}^{α}	:	$e_1e_2 = e_4$	$e_2e_1 = -\frac{1}{2}e_4$	$e_2e_2 = e_3$	$e_2e_3 = \alpha e_5$
		$e_3e_2 = e_5$	$e_3e_1 = -\frac{1}{2}e_5$	$e_4e_2 = e_5$	
B_{89}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = e_4 + e_5$	$e_2e_2 = e_3$
		$e_2e_4 = e_5$	$e_3e_1 = e_5$	$e_4e_2 = e_5$	
B_{90}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = e_4 + e_5$	$e_2e_2 = e_3$
		$e_2e_4 = e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_2 = e_5$
$B_{91}^{\alpha,\beta}$	:	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	$e_1e_4 = e_5$	$e_2e_2 = e_3$
		$e_2e_3 = e_5$	$e_2e_4 = \alpha e_5$	$e_4e_2 = \beta e_5$	
B_{92}^{α}	:	$e_1e_2 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = e_5$	
		$e_2e_2 = e_3$	$e_3e_2 = e_5$	$e_4e_2 = \alpha e_5$	
B_{93}^{α}	:	$e_1e_2 = e_4$	$e_1e_4 = e_5$	$e_2e_2 = e_3$	
		$e_3e_2 = e_5$	$e_4e_2 = \alpha e_5$		
B_{94}	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_2e_2 = e_3$	
		$e_2e_3 = e_5$	$e_4e_2 = e_5$		
B_{95}	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_1e_3 = 2e_5$	
		$e_2e_1 = -2e_4$	$e_2e_2 = e_3$	$e_2e_3 = e_5$	
		$e_2e_4 = 2e_5$	$e_3e_1 = -2e_5$	$e_4e_2 = e_5$	
B_{96}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = -2e_4$	$e_2e_2 = e_3$
		$e_2e_3 = e_5$	$e_2e_4 = e_5$	$e_3e_1 = -2\alpha e_5$	$e_4e_2 = \alpha e_5$
B_{97}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = -2e_4$	$e_2e_2 = e_3$
		$e_2e_3 = \alpha e_5$	$e_2e_4 = e_5$	$e_3e_2 = e_5$	
B_{98}^{α}	:	$e_1e_2 = e_4$	$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$	
		$e_3e_1 = \alpha e_5$	$e_4e_2 = e_5$		
$B_{99}^{\alpha,\beta}$	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$
		$e_2e_4 = e_5$	$e_3e_1 = \alpha\beta e_5$	$e_4e_2 = \beta e_5$	
$B_{100}^{\alpha\neq-2,\beta}$	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$
		$e_2e_4 = e_5$	$e_3e_1 = \alpha\beta e_5$	$e_3e_2 = e_5$	$e_4e_2 = \beta e_5$
B_{101}^{α}	:	$e_1e_2 = e_4$	$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$	
		$e_3e_1 = \alpha e_5$	$e_4e_2 = e_5$		
$B_{102}^{\alpha\neq0}$	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$
		$e_2e_4 = e_5$	$e_3e_1 = -e_5$	$e_4e_2 = -\frac{1}{\alpha}e_5$	
$B_{103}^{\alpha\neq0}$	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$
		$e_2e_4 = e_5$	$e_3e_1 = -e_5$	$e_3e_2 = e_5$	$e_4e_2 = -\frac{1}{\alpha}e_5$

B_{104}^{α}	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	$e_2e_1 = \alpha e_4$
		$e_2e_2 = e_3$	$e_2e_4 = \alpha e_5$	$e_3e_1 = -\alpha e_5$	$e_4e_2 = -e_5$
$B_{105}^{\alpha\neq -2}$	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	
		$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$	$e_2e_4 = \alpha e_5$	
		$e_3e_1 = -\alpha e_5$	$e_3e_2 = e_5$	$e_4e_2 = -e_5$	
B_{106}	:	$e_1e_1 = e_2$	$e_2e_1 = e_3$	$e_3e_1 = e_5$	$e_4e_4 = e_5$
B_{107}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = e_3$	
		$e_3e_1 = e_5$	$e_4e_4 = e_5$		
B_{108}	:	$e_1e_1 = e_2$	$e_1e_4 = e_5$	$e_2e_1 = e_3$	$e_3e_1 = e_5$
B_{109}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_1e_4 = e_5$	
		$e_2e_1 = e_3$	$e_3e_1 = e_5$		
B_{110}	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_1 = e_5$	$e_4e_1 = e_5$		
B_{111}	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_1 = e_5$	$e_4e_4 = e_5$		
B_{112}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_3 + \alpha e_5$
		$e_2e_2 = e_5$	$e_3e_1 = e_5$	$e_4e_1 = e_5$	$e_4e_4 = e_5$
B_{113}	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_3 + e_5$
		$e_2e_2 = e_5$	$e_3e_1 = e_5$	$e_4e_1 = e_5$	
B_{114}	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_3 + e_5$
		$e_2e_2 = e_5$	$e_3e_1 = e_5$	$e_4e_4 = e_5$	
$B_{115}^{\alpha\neq 1}$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_3$
		$e_2e_2 = \alpha e_5$	$e_3e_1 = \alpha e_5$	$e_4e_1 = e_5$	
$B_{116}^{\alpha\neq 1}$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_3$
		$e_2e_2 = \alpha e_5$	$e_3e_1 = \alpha e_5$	$e_4e_4 = e_5$	
$B_{117}^{\alpha\neq 0,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	
		$e_1e_4 = e_5$	$e_2e_1 = \alpha e_4$	$e_2e_2 = \alpha e_5$	
		$e_2e_3 = \alpha e_5$	$e_3e_3 = \beta e_5$	$e_4e_1 = \alpha e_5$	
$B_{118}^{\alpha\neq 0,1}$	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
		$e_2e_1 = \alpha e_4$	$e_2e_2 = \alpha e_5$	$e_2e_3 = \alpha e_5$	$e_3e_1 = e_5$
		$e_3e_3 = \frac{2\alpha^2}{(\alpha-1)^2}e_5$	$e_4e_1 = \alpha e_5$		
B_{119}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_1e_3 = e_4$	$e_2e_1 = e_4$
		$e_2e_3 = e_5$	$e_3e_3 = \alpha e_5$	$e_4e_1 = e_5$	
B_{120}^{α}	:	$e_1e_1 = e_2$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	
		$e_2e_3 = e_5$	$e_3e_3 = \alpha e_5$	$e_4e_1 = e_5$	
B_{121}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_5$	$e_1e_3 = e_4$	$e_2e_1 = e_4 + e_5$
		$e_2e_3 = e_5$	$e_3e_3 = 2e_5$	$e_4e_1 = e_5$	
B_{122}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = \alpha e_5$
		$e_3e_1 = e_4 + e_5$	$e_3e_2 = e_5$	$e_3e_3 = 2e_5$	
B_{123}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = e_5$	
		$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_3e_3 = \alpha e_5$	
B_{124}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = e_5$
		$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_3e_3 = \alpha e_5$	
B_{125}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	
		$e_3e_2 = e_4$	$e_3e_3 = e_5$	$e_4e_2 = e_5$	
B_{126}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = e_5$
		$e_3e_2 = e_4$	$e_3e_3 = e_5$	$e_4e_2 = e_5$	
B_{127}^{α}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = \alpha e_5$
		$e_3e_2 = e_4 + e_5$	$e_3e_3 = e_5$	$e_4e_2 = e_5$	
B_{128}^{α}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	
		$e_2e_2 = e_4$	$e_2e_3 = e_5$	$e_3e_2 = \alpha e_5$	
B_{129}^{α}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = e_5$
		$e_2e_2 = e_4$	$e_2e_3 = e_5$	$e_3e_2 = \alpha e_5$	
B_{130}^{α}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = \alpha e_5$
		$e_2e_2 = e_4 + e_5$	$e_2e_3 = e_5$		
B_{131}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	
B_{132}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_2e_2 = e_5$
B_{133}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = e_5$

B_{134}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	
		$e_2e_1 = e_5$	$e_2e_2 = e_5$		
B_{135}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_3e_2 = e_5$
B_{136}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	
		$e_2e_1 = e_5$	$e_3e_2 = e_5$		
B_{137}^{α}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_2e_1 = \alpha e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_4 + e_5$	$e_4e_2 = e_5$	
B_{138}^{α}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = \alpha e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_4$	$e_4e_2 = e_5$	
B_{139}^{α}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = \alpha e_5$	$e_2e_1 = e_5$
		$e_3e_1 = e_5$	$e_3e_2 = e_4$	$e_4e_2 = e_5$	
B_{140}	:	$e_1e_2 = e_3$	$e_3e_2 = e_4$	$e_4e_2 = e_5$	
B_{141}	:	$e_1e_2 = e_3$	$e_2e_1 = e_5$	$e_3e_2 = e_4$	$e_4e_2 = e_5$
B_{142}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_3e_2 = e_4$	$e_4e_2 = e_5$
B_{143}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	
		$e_3e_2 = e_4$	$e_4e_2 = e_5$		
B_{144}	:	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_3e_2 = e_4$	$e_4e_2 = e_5$
B_{145}	:	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_5$	
		$e_3e_2 = e_4$	$e_4e_2 = e_5$		
B_{146}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_2e_1 = (\alpha+1)e_4$	$e_3e_3 = e_5$
B_{147}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_2e_1 = (\alpha+1)e_4 + e_5$	$e_3e_3 = e_5$
B_{148}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_2e_1 = (\alpha+1)e_4$	
		$e_3e_1 = e_5$	$e_3e_3 = e_5$		
B_{149}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_2e_1 = (\alpha+1)e_4 + e_5$	
		$e_3e_1 = e_5$	$e_3e_3 = e_5$		
B_{150}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = e_4$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = e_4$	$e_3e_3 = e_5$	
B_{151}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_1e_3 = e_4$	
		$e_2e_1 = (\alpha+1)e_4 + e_5$	$e_3e_1 = e_4$	$e_3e_3 = e_5$	
$B_{152}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + \beta e_5$	$e_1e_3 = e_4$	
		$e_2e_1 = (\alpha+1)e_4 + \beta e_5$	$e_3e_1 = e_4 + e_5$	$e_3e_3 = e_5$	
$B_{153}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = \beta e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = \beta e_5$	$e_3e_3 = e_5$	
B_{154}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = e_4 + \alpha e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = e_4 + (\alpha+1)e_5$		
$B_{155}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_1e_3 = \beta e_5$	
		$e_2e_1 = (\alpha+1)e_4 + e_5$	$e_3e_1 = (\beta+1)e_5$		
$B_{156}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = \beta e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = (\beta+1)e_5$	$e_3e_3 = e_4$	
B_{157}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_2e_1 = (\alpha+1)e_4 + e_5$	
		$e_3e_1 = e_5$	$e_3e_3 = e_4$		
B_{158}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_4 + e_5$	$e_3e_1 = e_5$
B_{159}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = e_4 + e_5$	$e_3e_1 = \alpha e_5$		
B_{160}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_4 + e_5$	$e_3e_3 = e_5$
B_{161}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4 + e_5$	$e_3e_1 = e_4$	$e_3e_3 = e_5$	
B_{162}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = e_5$		
B_{163}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = e_5$	$e_3e_3 = e_4$	
B_{164}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_4$	
		$e_3e_1 = e_4$	$e_3e_3 = e_5$		
B_{165}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	
		$e_2e_1 = e_4$	$e_3e_1 = (\alpha+1)e_5$	$e_3e_3 = e_4$	
B_{166}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4 + e_5$	$e_2e_1 = e_4 + e_5$	
		$e_3e_1 = e_4$	$e_3e_3 = e_5$		

B^{α}_{167}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	
		$e_2e_1 = e_4$	$e_3e_1 = (\alpha+1)e_5$		
B_{168}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_4$	
		$e_3e_1 = e_5$	$e_3e_3 = e_5$		
B_{169}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = e_4$	$e_3e_1 = e_4 + e_5$		
B_{170}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = e_4$	$e_3e_1 = e_4 + e_5$	$e_3e_3 = e_4$	
B_{171}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_3e_2 = e_5$	
B_{172}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_3e_2 = e_5$
B_{173}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_5$	$e_3e_2 = e_5$
B_{174}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_5$	$e_3e_2 = e_5$		
B_{175}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_4$	$e_3e_2 = e_5$
B_{176}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_2 = e_4$	$e_3e_2 = e_5$		
B^{α}_{177}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_5$	$e_2e_2 = e_4$	$e_3e_2 = e_5$	
B_{178}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_3e_2 = e_5$
B_{179}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4 + e_5$	$e_3e_2 = e_5$
B^{α}_{180}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4 + \alpha e_5$	$e_3e_2 = e_5$		
$B^{\alpha,\beta}_{181}$	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$
		$e_2e_2 = e_4 + \beta e_5$	$e_3e_2 = e_5$		
B_{182}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_5$	
B_{183}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_5$
B^{α}_{184}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_5$	$e_2e_2 = e_5$		
$B^{\alpha,\beta}_{185}$	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = \beta e_5$	$e_2e_2 = e_5$	$e_3e_2 = e_4$	
B_{186}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_2e_2 = e_5$
B_{187}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4$	$e_2e_2 = e_5$		
B^{α}_{188}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4 + e_5$	$e_2e_2 = e_5$		
$B^{\alpha,\beta}_{189}$	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4 + \beta e_5$	$e_2e_2 = e_5$	$e_3e_2 = e_4$	
B_{190}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_5$
B_{191}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_5$	$e_2e_2 = e_4$		
B^{α}_{192}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_5$	$e_3e_2 = e_4$		
B_{193}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
B_{194}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$
B_{195}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_4$
B_{196}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4$	$e_2e_2 = e_4$		
B_{197}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_3e_2 = e_4$
B_{198}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4$	$e_3e_2 = e_4$		
B_{199}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_3e_2 = e_4$	
B_{200}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	$e_3e_2 = e_4$
B^{α}_{201}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = \alpha e_5$	
		$e_2e_2 = e_5$	$e_3e_2 = e_4$		
B_{202}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_4$
B_{203}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_4 + e_5$	$e_3e_2 = e_4$

B_{204}^{α}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_4 + \alpha e_5$	
		$e_2e_2 = e_5$	$e_3e_2 = e_4$		
B_{205}	:	$e_1e_2 = e_3$	$e_2e_1 = e_5$	$e_3e_2 = e_4$	
B_{206}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	$e_3e_2 = e_4$
B_{207}	:	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_2e_2 = e_5$	$e_3e_2 = e_4$

Note that B_{116}^{1} *is a commutative algebra and*

$$\begin{gathered}
B_{12}^{\frac{1}{4}} \simeq B_{11}^{\frac{1}{4}}, B_{16}^{\alpha,\beta} \simeq B_{16}^{\beta,\alpha}, B_{35}^{\alpha} \simeq B_{35}^{-\alpha}, B_{38}^{\alpha} \simeq B_{38}^{-\alpha}\ B_{41}^{\alpha} \simeq B_{41}^{-\alpha}, \\
B_{57}^{0,0,\gamma\neq0} \simeq B_{185}^{0,\frac{1}{\gamma}}, B_{58}^{(0,0)} \simeq B_{184}^{0}, B_{59}^{0} \simeq B_{188}^{0}, B_{60}^{0,0} \simeq B_{156}^{-1,-1}, B_{60}^{1,0} \simeq B_{93}, B_{60}^{\alpha\neq0,1;0} \simeq B_{184}^{\frac{1}{1-\alpha}}, \\
B_{61}^{0,0} \simeq B_{152}^{0,0}, B_{61}^{\alpha\neq0,0} \simeq B_{204}^{1-\alpha}, B_{63}^{0} \simeq B_{201}^{1}, B_{64}^{0,\gamma\neq0} \simeq B_{185}^{(\alpha^2,\alpha),\alpha\neq0}, B_{64}^{0,0} \simeq B_{201}^{1}, \\
B_{81}^{(0,0)} \simeq B_{192}^{0}, B_{81}^{\alpha,\beta} \simeq B_{81}^{\frac{1}{\beta},\frac{1}{\alpha}}, B_{83}^{\alpha} \simeq B_{83}^{\frac{1}{\alpha}}, B_{87}^{\alpha,\beta,\gamma} \simeq B_{87}^{\frac{\gamma}{\beta-2},\frac{1-2\beta}{2-\beta},\frac{\alpha}{\beta-2}}, \\
B_{91}^{\alpha,\beta} \simeq B_{91}^{\alpha,-\beta}\ B_{92}^{\alpha} \simeq B_{92}^{-\alpha}, B_{93}^{\alpha} \simeq B_{93}^{-\alpha}, B_{100}^{-2,0} \simeq B_{97}^{0}, B_{100}^{-2,\beta\neq0} \simeq B_{99}^{-2,\beta\neq0}, \\
B_{105}^{-2} \simeq B_{104}^{-2}. B_{112}^{\beta} \simeq B_{112}^{-\beta}, B_{180}^{\alpha} \simeq B_{180}^{-\alpha}, B_{201}^{\alpha} \simeq B_{201}^{-\alpha}, B_{204}^{\alpha} \simeq B_{204}^{-\alpha}.
\end{gathered}$$

Author Contributions: Conceptualization, I.K.; Methodology, I.K. and A.K.; Software, K.A., I.K. and A.K.; Validation, K.A., I.K. and A.K.; Formal analysis, K.A., I.K. and A.K.; Investigation, K.A., I.K. and A.K.; Resources, K.A., I.K. and A.K.; Data curation, K.A., I.K. and A.K.; Writing—original draft, K.A., I.K. and A.K.; Writing—review & editing, K.A., I.K. and A.K.; Visualization, K.A., I.K. and A.K.; Supervision, I.K. and A.K.; Project administration, I.K.; Funding acquisition, I.K. All authors have read and agreed to the published version of the manuscript.

Funding: The first part of this work is supported by FCT UIDB/MAT/00212/2020 and UIDP/MAT/00212/2020. The second part of this work is supported by the Russian Science Foundation under grant 22-11-00081.

Acknowledgments: The authors thank Thiago Castilho de Mello and Ivan Gonzales Gargate for their active stimulation to write this paper. The first part of this work is supported by FCT UIDB/MAT/00212/2020 and UIDP/MAT/00212/2020. The second part of this work is supported by the Russian Science Foundation under grant 22-11-00081.

Conflicts of Interest: The authors declare no conflict of interest.

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