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Adaptive Output Feedback Control for Constrained Switched Systems with Input Quantization

Shuyan Qi [1,2], Jun Zhao [1,3,*] and Li Tang [4]

[1] State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang 110819, China
[2] Key Laboratory of Data Analytics and Optimization for Smart Industry (Northeastern University), Ministry of Education, Shenyang 110819, China
[3] College of Information Science and Engineering, Northeastern University, Shenyang 110819, China
[4] College of Science, Liaoning University of Technology, Jinzhou 121000, China
* Correspondence: zhaojun@ise.neu.edu.cn

Abstract: This paper investigates adaptive output feedback control problem for switched uncertain nonlinear systems with input quantization, unmeasured system states and state constraints. Firstly, fuzzy logic systems are introduced to identify system uncertainties, then the fuzzy based observer is constructed to estimate unavailable states. Secondly, combing the backstepping technique and the barrier Lyapunov function method, an adaptive fuzzy output feedback control law is designed, which guarantees that all signals in the closed-loop system are bounded, the system output tracks the reference signal, and system states satisfy their corresponding constraint conditions. Finally, simulation results further show the good performances of the proposed control scheme.

Keywords: switched systems; fuzzy observer; input quantization; state constraints

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1. Introduction

With the development of social technology and industry, more and more control systems show the complex and multimode characteristics. Switched systems as a kind of hybrid systems have been widely concerned. In recent years, researches on switched systems have made some achievements. For switched linear systems, there are more related results [1–4]. In [5], an event-triggered based H_∞ control issue was investigated for a class of switched linear systems with bounded exogenous disturbances. In [6], the non-fragile finite-time extended dissipative control problem is studied for discrete-time switched linear systems under a switching law with average dwell-time. For switched nonlinear systems, the research results are relatively less [7–9]. For example, the multierror constraint control problem was first proposed under different coordinate transformations in [10], and was successfully solved by using dwell time method. In [11], an adaptive output feedback fault tolerant control problem was studied for switched nonstrict-feedback nonlinear systems based on the small-gain technique.

Similar to traditional control systems, it is often difficult to obtain the system states in the control process of switched systems. Then, it is necessary to design a corresponding state observer to eatimate the unmeasurable states [12–16]. However, the switching signal is the unique feature of switched systems compared to other systems. This also makes the design of controller and observer more difficult. As we know, the state observer of a switched system also contains the switching signal. Then, is the switching signal consistent with that of the original control system or not? Moreover, due to the existing of nonlinear structures, these directly affect the design of the controllers. In this paper, an observer design method in which the switching signal is consistent with that of the original control system is considered, which makes it easier to observe the unavailable states. Moreover, the unknown system dynamics are considered in this paper, then an adaptive fuzzy observer is designed. The fuzzy based approach is one of the artificial intelligence techniques widely used in various applications [17–19]. In [17], an innovative approach to reducing the mesh-induced error in CFD analysis of an impinging jet using fuzzy logic was proposed.

In addition, both the input quantification problem and the state constraint issue are also considered. These problems are common in the actual production process, so the research of this paper has a certain practical application value. At present, the research on these problems is almost focused on the nonswitched systems [20–24]. For example, in [21] the fault-tolerant attitude control of flexible spacecraft is investigated over digital communication channels, where a uniform quantizer is considered with respect to the sensor signals and controller indexes. An adaptive neural control method was proposed for a class of nonlinear time-varying delayed systems with time-varying full-state constraints in [22]. The related research results of switched systems are few and far between [25–27]. This is also one of the motivations of this paper.

Inspired by the above analysis, this paper focuses on switched nonlinear systems with the input quantification problem and the state constraint issue. The fuzzy based observer is constructed to estimate unmeasurable states. The barrier Lyapunov function method is used to ensure the system states satisfy the corresponding constraint conditions. An adaptive output feedback controller is designed that can offset the effect caused by the input quantification by using the backstepping technique. Finally, simulation results further show the good performances of the proposed control scheme.

Compared with the existing results of switched systems, the main contribution of this paper is that the adaptive output feedback control problem for switched systems is studied. As far as we know, this is the first time in this kind of system to ensure that the system states meet the constraint conditions, and to overcome the impact of quantized input at the same time. Combining the barrier Lyapunov function method with backstepping technology, the adaptive controller is designed, and these problems are successfully solved.

This paper is organized as follows. Section 2 shows the system description, the control objective, and the necessary assumption. In Section 3, an fuzzy based observer is designed to estimate the unavailable system states. Section 4 gives the design process of the adaptive controller by using barrier Lyapunov function method. Section 5 expresses the main result and analyzes the stability of the closed-loop system. A simulation example is employed to illustrate the effectiveness of the proposed control approach in Section 6. Finally, Section 7 provides the conclusions.

2. System Description

In this paper, the following switched uncertain nonlinear system is considered

$$\begin{cases} \dot{x}_1 = x_2 + f_{1,\sigma(t)}(x_1) + \rho_{1,\sigma(t)} \\ \vdots \\ \dot{x}_n = v_{\sigma(t)} + f_{n,\sigma(t)}(X) + \rho_{n,\sigma(t)} \\ y = x_1 \end{cases} \tag{1}$$

where $\overline{x}_i = [x_1, x_2, \cdots, x_i]^T$, $i = 1, \ldots, n-1$ and $X = [x_1, x_2, \cdots, x_n]^T$ are the system state vectors, $y \in R$ represents the system output. The function $\sigma(t) : [0, \infty) \to \Gamma = \{1, 2, \cdots, \pi\}$ is the switching signal, which is assumed to be a piecewise continuous (from the right) function of time, and π represents the number of subsystems. In addition, $\sigma(t) = r$ indicates that the rth subsystem is in working state. $v_r \in R$ stands for the system control input, which is also the output of the quantizer (the input quantization will be introduced later), $f_{i,r}(x)$, $i = 1, 2, \cdots, n$ is a smooth nonlinear function, which is unknown, $\rho_{i,r}(t)$ is the bounded disturbance, and satisfies $|\rho_{i,r}(t)| \leq \overline{\rho}_{i,r}$ with $\overline{\rho}_{i,r}$ being a constant. It is worth noting that x_i is unmeasurable, and only x_1 is available in this paper.

In addition, the input quantization is considered here, which is expressed as [28]

$$v_r(t) = \begin{cases} (1-\delta_r)q^r_{1,j}, & if\ q^r_{1,j} \le u_r(t) \le q^r_{1,j+1} \\ 0, & if\ 0 \le u_r(t) \le q^r_{1,1} \\ -U_r(-u_r(t)), & if\ u_r(t) < 0 \end{cases} \tag{2}$$

where $u_r(t)$ is the input for quantizer, $\Xi_r = \frac{1-\varsigma_r}{1+\varsigma_r}$ and $q^r_{1,j} = a\varsigma_r^{1-j} > 0$ with ς_r being the quantitative degree.

To facilitate the analysis, the input quantization (2) is rewritten as

$$v_r(t) = U_r(u_r(t)) = q^r_1(t)u_r(t) + q^r_2(t) \tag{3}$$

where

$$q^r_1(t) = \begin{cases} \frac{U_r(u_r(t))}{u_r(t)}, & |u_r(t)| \ge B_r \\ 1, & |u_r(t)| < B_r \end{cases} \tag{4}$$

$$q^r_2(t) = \begin{cases} 0, & |u_r(t)| \ge B_r \\ U_r(u_r(t)) - u_r(t), & |u_r(t)| < B_r \end{cases} \tag{5}$$

with $B_r > 0$ being a constant. Since the sign of the control input is invariant during quantization, it follows from (4) that $q^r_1(t) > 0$. To simplify writing, let $\varpi = 1/q^r_1(t)$, which is also positive.

Control Objectives: The control objective of this paper is to design an adaptive output feedback controller for the switched system (1) with input quantization (2) and state constraints under the arbitrarily switching law such that (1) all signals in the closed-loop system are bounded, (2) the tracking errors can converge to a small neighborhood of origin, (3) all system states do not violate their corresponding constraint conditions.

Assumption 1 [29]. *Assumed that the tracking signal y_d and its time derivatives up to the nth order are continuous and bounded.*

3. Fuzzy Observer Design

A. Description of Fuzzy Logic Systems

The FLSs consisting of If-Then rules is introduced to solve the unknown function problem in the controlled system:

R^q: If x_1 is F^q_1 ... and x_{n-1} is F^q_{n-1} and x_n is F^q_n, y is $B^q(q = 1, 2 \cdots \chi)$.

When $x = [x_1, \cdots x_n]^T$ is the input and y is the output, F^q_i and B^q are fuzzy sets in R, relevant to the fuzzy affiliation functions $\mu_{F^q_i}(x_i)$ and $\mu_{B^q}(y)$. χ is the fuzzy rule number. Then, the following formula be utilized to describe FLSs

$$y(x(t)) = \frac{\sum^{\chi}_{q=1} \bar{y}_q \prod^q_{i=1} \mu_{F^q_i}(x_i)}{\sum^{\chi}_{q=1} \left(\prod^n_{i=1} \mu_{F^q_i}(x_i)\right)}$$

where $\bar{y}_q = \max_{y \in R} \mu_{B^q}(y)$ refers to the maximum value of $\mu_{B^q}(y)$ on R.

Let

$$\varphi_q = \frac{\prod^n_{i=1} \mu_{F^q_i}(x_i)}{\sum^{\chi}_{q=1} \left(\prod^n_{i=1} \mu_{F^q_i}(x_i)\right)}$$

and denote $w = \left[\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_\chi\right]^T = [w_1, w_2, \cdots, w_\chi]^T$ and $\varphi^T(x) = [\varphi_1(x), \varphi_2(x), \cdots,$ $\varphi_\chi(x)]$, then FLSs can be expressed as $y(x) = w^T\varphi(x)$.

B. Fuzzy Observer Design

To solve the unavailable system state problem, this section introduces a fuzzy state observer. Firstly, rewrite the considered system (1) as follows

$$\dot{X} = AX + \xi y + \sum_{i=1}^{n} B_i \left[f_{i,\sigma(t)}(X_i) + \rho_{i,\sigma(t)}(t) \right] + Bv_{\sigma(t)} \tag{6}$$

where $A = \begin{bmatrix} -\xi_1 & & \\ \vdots & I_{n-1} & \\ -\xi_n & \cdots & 0 \end{bmatrix}$, $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$, $B_i = \left[\underbrace{0 \cdots 1}_{i} \cdots 0 \right]^T$

Select $\xi_i, i = 1, \ldots, n$ such that $A^T P + PA = -Q$ holds, where P and Q are all positive matrice.

The following adaptive fuzzy observer is designed

$$\begin{cases} \dot{\hat{X}} = A\hat{X} + \xi y + \sum\limits_{i=1}^{n} B_i \hat{f}_{i,\sigma(t)} \left(\hat{X}_i | \hat{w}_{i,\sigma(t)} \right) + Bv_{\sigma(t)} \\ \hat{y} = C\hat{X} \end{cases} \tag{7}$$

where $C = [1 \cdots 0 \cdots 0]$, $\hat{X} = [\hat{x}_1, \hat{x}_2, \hat{x}_3, \cdots, \hat{x}_n]^T$ is the estimation of X, and $\hat{X}_i = [\hat{x}_1, \hat{x}_2, \hat{x}_3, \cdots, \hat{x}_i]$. $\hat{w}_{i,r}$ is the estimation of $w^*_{i,r}$ with $w^*_{i,r}$ being the optimal vector. Let $\widetilde{w}_{i,r} = w^*_{i,r} - \hat{w}_{i,r}$ be the estimated error.

Based on the universal approximation of fuzzy logic systems, the uncertain nonlinear system functions are approximated by the following form:

$$\begin{cases} f_{i,r}(X_i | \hat{w}_{i,r}) = \hat{w}^T_{i,r} \varphi_i(X_i) \\ \hat{f}_{i,r}(\hat{X}_i | \hat{w}_{i,r}) = \hat{w}^T_{i,r} \varphi_i(\hat{X}_i) \end{cases} \tag{8}$$

Define the corresponding error as

$$\begin{cases} \varepsilon_{i,r} = f_{i,r}(X_i) - \hat{f}_{i,r} \left(\hat{X}_i | w^*_{i,r} \right) \\ \delta_{i,r} = f_{i,r}(X_i) - \hat{f}_{i,r} \left(\hat{X}_i | \hat{w}_{i,r} \right) \end{cases} \tag{9}$$

where $\delta_{i,r}$ and $\varepsilon_{i,r}, \quad i = 1, \ldots, n$ are corresponding error variables, which are bounded. That is to say, there exist constants $\overline{\delta}_{i,r}$ and $\overline{\varepsilon}_{i,r}$ such that $\delta_{i,r} \leq \overline{\delta}_{i,r}$ and $\varepsilon_{i,r} \leq \overline{\varepsilon}_{i,r}$.

Define the observation error as $e = X - \hat{X} = [e_1, e_2, \cdots, e_n]^T$.

Based on (1) and (8), the following results can be obtained

$$\begin{aligned} \dot{e} &= Ae + \sum_{i=1}^{n} B_i \left[f_{i,r}(X_i) - \hat{f}_{i,r} \left(\hat{X}_i | \hat{w}_{i,r} \right) + \rho_{i,r} \right] \\ &= Ae + \delta_r + D_r \end{aligned} \tag{10}$$

where $\delta_r = [\delta_{1,r}, \delta_{2,r}, \cdots, \delta_{n,r}]^T$, $D_r = [\rho_{1,r}, \rho_{2,r}, \cdots, \rho_{n,r}]^T$, and there are two constants $\overline{\delta}_r$ and $\overline{D}_r$ such that $\|D_r\| \leq \overline{D}_r$ and $\|\delta_r\| \leq \overline{\delta}_r$ hold.

4. Adaptive Fuzzy Controller

In this section, the backstepping technique and the barrier Lyapunov function method are used to construct the adaptive fuzzy controller. Firstly, the coordinate transformation is defined as

$$\begin{cases} z_1 = x_1 - y_d \\ z_2 = \hat{x}_2 - \alpha_1 - \dot{y}_d \\ \qquad \vdots \\ z_n = \hat{x}_n - \alpha_{n-1} - y_d^{(n-1)} \end{cases} \tag{11}$$

where y_d is the reference signal, $z_1, \ldots, z_n$ denote the transformation errors, $\alpha_1, \ldots, \alpha_{n-1}$ are virtual controllers, which are designed in later steps.

Step 1: According to (1), (8) and (12), the derivative of z_1 is

$$\dot{z}_1 = z_2 + \alpha_1 + e_2 + w_{1,r}^{*T}\varphi_1(\hat{x}_1) + \varepsilon_{1,r} + \rho_{1,r} \tag{12}$$

To solve the state constraint problem, the barrier Lyapunov function V_1 is selected as follows

$$V_1 = e^T Pe + \frac{1}{2}\log\frac{k_1^2(t)}{k_1^2(t) - z_1^2} + \frac{1}{2}\zeta_1^{-1}\widetilde{\Theta}_1^2 \tag{13}$$

where $k_m(t)$ is time-varying function, and $|z_m| <|k_m(t)|$, $m = 1, \cdots, n$. $\zeta_1 > 0$ is a design parameter. $\widetilde{\Theta}_m = \Theta_m - \hat{\Theta}_m, m = 1, \ldots, n$ represents the estimation error between Θ_m and $\hat{\Theta}_m$, $\hat{\Theta}_m$ stands for the estimate of Θ_m with $\Theta_m = \max\left\{\left\|w_{m,r}^*\right\|^2, r \in \Gamma\right\}$.

The derivative of V_1 is calculated as

$$\begin{aligned} \dot{V}_1 &= \dot{e}^T Pe + e^T P\dot{e} + \tfrac{z_1}{k_1^2(t)-z_1^2}\left(\dot{z}_1 - \tfrac{\dot{k}_1(t)}{k_1(t)}z_1\right) - \tfrac{1}{\zeta_1}\widetilde{\Theta}_1\dot{\hat{\Theta}}_1 \\ &= -e^T Qe + 2e^T P\,\delta_r + 2e^T PD_r \\ &\quad -\tfrac{1}{\zeta_1}\widetilde{\Theta}_1\dot{\hat{\Theta}}_1 - \tfrac{z_1}{k_1^2(t)-z_1^2}\Big(z_2 + \alpha_1 + e_2 \\ &\quad + w_{1,r}^{*T}\varphi_1(\hat{x}_1) + \varepsilon_{1,r} + \rho_{1,r} - \tfrac{\dot{k}_1(t)}{k_1(t)}z_1\Big) \end{aligned} \tag{14}$$

Consider the following facts

$$2e^T P\delta_r + 2e^T PD_r \le 2\|e\|^2 + \|P\|^2\overline{\delta}^2{}_r + \|P\|^2\overline{D}^2{}_r \tag{15}$$

$$\frac{z_1}{k_1^2(t) - z_1^2}(e_2 + \varepsilon_{1,r} + \rho_{1,r}) \le \frac{3}{2}\frac{z_1^2}{\left(k_1^2(t) - z_1^2\right)^2} + \frac{1}{2}\left(\overline{\varepsilon}_{1,r}^2 + \overline{\rho}_{1,r}^2 + \|e\|^2\right) \tag{16}$$

$$\frac{z_1}{k_1^2(t) - z_1^2}w_{1,r}^{*T}\varphi_1(\hat{x}_1) \le \frac{z_1^2\Theta_1}{2a_{1,r}^2\left(k_1^2(t) - z_1^2\right)^2} + \frac{a_{1,r}^2}{2} \tag{17}$$

where $a_{1,r} > 0$ is a design parameter. Then, according to (15)–(18) into (15), one has

$$\begin{aligned} \dot{V}_1 \le &-e^T Qe + \tfrac{5}{2}\|e\|^2 + \|P\|^2\overline{\delta}^2{}_r + \|P\|^2\overline{D}^2{}_r \\ &+ \tfrac{z_1}{k_1^2(t)-z_1^2}\Big(z_2 + \alpha_1 - \tfrac{\dot{k}_1(t)}{k_1(t)}z_1 + \tfrac{3}{2}\tfrac{z_1^2}{\left(k_1^2(t)-z_1^2\right)^2} \\ &+ \tfrac{z_1\Theta_1}{2a_{1,r}^2\left(k_1^2(t)-z_1^2\right)}\Big) - \tfrac{1}{\zeta_1}\widetilde{\Theta}_1\dot{\hat{\Theta}}_1 + \tfrac{1}{2}\left(\overline{\varepsilon}_{1,r}^2 + \overline{\rho}_{1,r}^2\right) + \tfrac{a_{1,r}^2}{2} \end{aligned} \tag{18}$$

Design the virtual control law as follows

$$\alpha_1 = -(K_1 + \overline{\kappa}_1)z_1 - \frac{3}{2}\frac{z_1}{k_1^2(t) - z_1^2} - \frac{z_1\hat{\Theta}_1}{2a_{1,\min}^2\left(k_1^2(t) - z_1^2\right)} \tag{19}$$

where $K_1 > 0$ is a design parameter, $a_{1,\min} = \min\{a_{1,r}, r \in \Gamma\}$, $\overline{\kappa}_1 = \sqrt{{m_1}^2 + p_1}$, $p_1 > 0$ and $m_1 = \frac{\dot{k}_1(t)}{k_1(t)}$.

Select the following adaptive law

$$\dot{\hat{\Theta}}_1 = -\Psi_1\hat{\Theta}_1 + \frac{\zeta_1 z_1^2}{2a_{1,\min}^2\left(k_1^2(t) - z_1^2\right)^2} \tag{20}$$

where $\Psi_1 > 0$ is a parameter.

Further, it holds that

$$\frac{\Psi_1}{\zeta_1}\widetilde{\Theta}_1\hat{\Theta}_1 \le \frac{\Psi_1}{2\zeta_1}\left(-\widetilde{\Theta}_1^2 + \Theta_1^2\right) \tag{21}$$

Substituting (20)–(22) into (19), one gets

$$\begin{aligned}\dot{V}_1 \le& -\left[\lambda_{\min}(Q) - \tfrac{5}{2}\right]\|e\|^2 + \tfrac{z_1 z_2}{k_1^2(t)-z_1^2} \\ &-K_1\tfrac{z_1^2}{\left(k_1^2(t)-z_1^2\right)^2} - \tfrac{\Psi_1}{2\zeta_1}\widetilde{\Theta}_1^2 + H_{1,r}\end{aligned} \tag{22}$$

where $H_{1,r} = \|P\|^2\overline{\delta}^2{}_r + \|P\|^2\overline{D}^2{}_r + \frac{1}{2}(\overline{\varepsilon}_{1,r}^2 + \overline{\rho}_{1,r}^2) + \frac{a_{1,r}^2}{2} + \frac{\Psi_1}{2\zeta_1}\Theta_1^2, r \in \Gamma$.

Step $i(2 \le i \le n-1)$: Based on (1) and (12), one obtains

$$\dot{z}_i = z_{i+1} + \alpha_i + \xi_i e_1 + w_{i,r}^{*T}\varphi_i\left(\overline{\hat{X}}_i\right) + \varepsilon_{i,r} - \delta_{i,r} - \dot{\alpha}_{i-1} \tag{23}$$

where $\overline{\hat{X}}_i = [\hat{x}_1, \quad \hat{x}_2, \quad \hat{x}_3, \quad \cdots, \quad \hat{x}_i]^T$,

$$\begin{aligned}\dot{\alpha}_{i-1} =& \sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial\hat{x}_m}(\hat{x}_{m+1}+\xi_m e_1 + w_{m,r}^{*T}\varphi_m\left(\overline{\hat{X}}_m\right) + \varepsilon_{m,r} \\ &-\delta_{m,r}) + \sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial\hat{\Theta}_m}\dot{\hat{\Theta}}_m + \sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial y_d^{(m-1)}}y_d^{(m)} \\ &+\tfrac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + e_2 + w_{1,r}^{*T}\varphi_1(\hat{x}_1) + \varepsilon_{1,r} + \rho_{1,r})\end{aligned}.$$

Select the barrier Lyapunov function V_i as

$$V_i = V_{i-1} + \frac{1}{2}\log\frac{k_i^2(t)}{k_i^2(t) - z_i^2} + \frac{1}{2}\zeta_i^{-1}\widetilde{\Theta}_i^2 \tag{24}$$

where $\zeta_i > 0$ represents the design parameter.

Taking the time derivative of V_i, one has

$$\begin{aligned}\dot{V}_1 =& \dot{V}_{i-1} + \tfrac{z_i}{k_i^2(t)-z_i^2}\left(\dot{z}_i - \tfrac{\dot{k}_i(t)}{k_i(t)}z_i\right) - \tfrac{1}{\zeta_i}\widetilde{\Theta}_i\dot{\hat{\Theta}}_i \\ =& \dot{V}_{i-1} + \tfrac{z_i}{k_i^2(t)-z_i^2}\left(z_{i+1} + \alpha_i + \xi_i e_1 + w_{i,r}^{*T}\varphi_i\left(\overline{\hat{X}}_i\right)\right. \\ &\left.+\varepsilon_{i,r} - \delta_{i,r} - \dot{\alpha}_{i-1} - \tfrac{\dot{k}_i(t)}{k_i(t)}z_i\right) - \tfrac{1}{\zeta_i}\widetilde{\Theta}_i\dot{\hat{\Theta}}_i \\ =& \dot{V}_{i-1} + \tfrac{z_i}{k_i^2(t)-z_i^2}\left(z_{i+1} + \alpha_i + \xi_i e_1 + w_{i,r}^{*T}\varphi_i\left(\overline{\hat{X}}_i\right)\right. \\ &+\varepsilon_{i,r} - \delta_{i,r} - \tfrac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + e_2 + w_{1,r}^{*T}\varphi_1(\hat{x}_1) + \varepsilon_{1,r} \\ &+\rho_{1,r}) - \sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial\hat{x}_m}(\hat{x}_{m+1} + \xi_m e_1 + w_{m,r}^{*T}\varphi_m\left(\overline{\hat{X}}_m\right) \\ &+\varepsilon_{m,r} - \delta_{m,r}) - \sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial\hat{\Theta}_m}\dot{\hat{\Theta}}_m - \sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial k_m(t)}\dot{k}_m(t) \\ &\left.-\sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial y_d^{(m-1)}}y_d^{(m)} - \tfrac{\dot{k}_i(t)}{k_i(t)}z_i\right) - \tfrac{1}{\zeta_i}\widetilde{\Theta}_i\dot{\hat{\Theta}}_i\end{aligned} \tag{25}$$

Applying Young's inequality, it derives that

$$\frac{z_i}{k_i^2(t) - z_i^2}w_{i,r}^{*T}\varphi_i\left(\overline{\hat{X}}_i\right) \le \frac{z_i^2\Theta_i}{2a_{i,r}^2\left(k_i^2(t) - z_i^2\right)^2} + \frac{a_{i,r}^2}{2} \tag{26}$$

where $a_{i,r} > 0$ is a design parameter.

$$-\frac{z_i}{k_i^2(t) - z_i^2}\sum_{m=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_m}w_{m,r}^{*T}\varphi_m\left(\overline{\hat{X}}_m\right) \le \frac{z_i^2}{2\left(k_i^2(t) - z_i^2\right)^2}\sum_{m=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial\hat{x}_m}\right)^2 + \frac{1}{2}\sum_{m=1}^{i-1}\Theta_m \tag{27}$$

$$-\frac{z_i}{k_i^2(t)-z_i^2}\sum_{m=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_m}(\varepsilon_{m,r}-\delta_{m,r}) \le \frac{z_i^2}{\left(k_i^2(t)-z_i^2\right)^2}\sum_{m=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial\hat{x}_m}\right)^2+\frac{1}{2}\sum_{m=1}^{i-1}\left(\bar{\varepsilon}^2{}_{m,r}+\bar{\delta}^2{}_{m,r}\right) \tag{28}$$

$$-\frac{z_i}{k_i^2(t)-z_i^2}\frac{\partial\alpha_{i-1}}{\partial y}w_{1,r}^{*T}\varphi_1(\hat{x}_1) \le \frac{z_i^2}{2\left(k_i^2(t)-z_i^2\right)^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2+\frac{1}{2}\Theta_1 \tag{29}$$

$$-\frac{z_i}{k_i^2(t)-z_i^2}\frac{\partial\alpha_{i-1}}{\partial y}(e_2+\varepsilon_{1,r}+\rho_{1,r}) \le \frac{3}{2}\frac{z_i^2}{\left(k_i^2(t)-z_i^2\right)^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2+\frac{1}{2}\left(\|e\|^2+\bar{\varepsilon}^2{}_{1,r}+\bar{\rho}^2{}_{1,r}\right) \tag{30}$$

$$\frac{z_i}{k_i^2(t)-z_i^2}(\varepsilon_{i,r}-\delta_{i,r}) \le \frac{z_i^2}{\left(k_i^2(t)-z_i^2\right)^2}+\frac{1}{2}\left(\bar{\varepsilon}^2{}_{i,r}+\bar{\delta}^2{}_{i,r}\right) \tag{31}$$

Combing (26)–(32), one gets

$$\begin{aligned}\dot{V}_i = \dot{V}_{i-1} &+ \tfrac{z_i}{k_i^2(t)-z_i^2}\Bigg(z_{i+1}+\alpha_i+\xi_i e_1-\tfrac{\dot{k}_i(t)}{k_i(t)}z_i\\ &-\sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial\hat{\Theta}_m}\dot{\hat{\Theta}}_m-\sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial\hat{x}_m}(\hat{x}_{m+1}+\xi_m e_1)-\sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial y_d^{(m-1)}}y_d^{(m)}\\ &-\sum_{m=1}^{i-1}\tfrac{\partial\alpha_{i-1}}{\partial k_m(t)}\dot{k}_{m-1}(t)-\tfrac{\partial\alpha_{i-1}}{\partial y}\hat{x}_2+\tfrac{z_i\Theta_i}{2a_{i,r}^2\left(k_i^2(t)-z_i^2\right)}\\ &+\tfrac{3z_i}{2\left(k_i^2(t)-z_i^2\right)}\sum_{m=1}^{i-1}\left(\tfrac{\partial\alpha_{i-1}}{\partial\hat{x}_m}\right)^2+\tfrac{z_i}{k_i^2(t)-z_i^2}\\ &+\tfrac{2z_i}{\left(k_i^2(t)-z_i^2\right)}\left(\tfrac{\partial\alpha_{i-1}}{\partial y}\right)^2\Bigg)+\tfrac{1}{2}\left(\|e\|^2+\bar{\varepsilon}^2{}_{i,r}+\bar{\rho}^2{}_{i,r}\right)\\ &+\tfrac{1}{2}\sum_{m=1}^{i-1}\left(\bar{\varepsilon}^2{}_{m,r}+\bar{\delta}^2{}_{m,r}\right)+\tfrac{a_{i,r}^2}{2}+\tfrac{1}{2}\Theta_1+\tfrac{1}{2}\left(\bar{\varepsilon}^2{}_{i,r}+\bar{\delta}^2{}_{i,r}\right)\\ &+\tfrac{1}{2}\sum_{m=1}^{i-1}\Theta_m-\tfrac{1}{\zeta_i}\tilde{\Theta}_i\dot{\hat{\Theta}}_i\end{aligned} \tag{32}$$

For control purposes, the virtual controller is designed as

$$\begin{aligned}\alpha_i = &-(K_i+\bar{\kappa}_i)z_i-\tfrac{z_i}{k_i^2(t)-z_i^2}-\tfrac{z_i\hat{\Theta}_i}{2a_{i,\min}^2\left(k_i^2(t)-z_i^2\right)}\\ &-\tfrac{k_i^2(t)-z_i^2}{k_{i-1}^2(t)-z_{i-1}^2}z_{i-1}-R_{i-1}\end{aligned} \tag{33}$$

where $K_i>0$ is a design parameter, $a_{i,\min}=\min\{a_{i,r},\ r\in\Gamma\}$, $\bar{\kappa}_i=\sqrt{m_i{}^2+p_i}$, $p_i>0$ and $m_i=\frac{\dot{k}_i(t)}{k_i(t)}$, $R_{i-1}=-\sum_{m=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_m}(\hat{x}_{m+1}+\xi_m e_1)-\sum_{m=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\Theta}_m}\dot{\hat{\Theta}}_m-\frac{\partial\alpha_{i-1}}{\partial y}\hat{x}_2-\sum_{m=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(m-1)}}y_d^{(m)}-\sum_{m=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial k_m(t)}\times\dot{k}_m(t)^2+2\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)\frac{z_i}{k_i^2(t)-z_i^2}+\frac{3z_i^2}{2\left(k_i^2(t)-z_i^2\right)^2}\sum_{m=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial\hat{x}_m}\right)^2+\xi_i e_1$.

And the adaptive law is chosen as

$$\dot{\hat{\Theta}}_i=-\Psi_i\hat{\Theta}_i+\frac{\zeta_i z_i^2}{2a_{i,\min}^2\left(k_i^2(t)-z_i^2\right)^2} \tag{34}$$

where $\Psi_i>0$ is a design parameter.

Consider that

$$\frac{\Psi_i}{\zeta_i}\tilde{\Theta}_i\hat{\Theta}_i \le \frac{\Psi_i}{2\zeta_i}\left(-\tilde{\Theta}_i^2+\Theta_i^2\right) \tag{35}$$

and substituting (34)–(36) into (33), one gets

$$\begin{aligned}\dot{V}_i \le &-\left[\lambda_{\min}(Q)-2-\tfrac{1}{2}i\right]\|e\|^2+\tfrac{z_i z_{i+1}}{k_i^2(t)-z_i^2}\\ &-\sum_{m=1}^{i}K_m\tfrac{z_m^2}{k_m^2(t)-z_m^2}-\sum_{m=1}^{i}\tfrac{\Psi_m}{2\zeta_m}\tilde{\Theta}_m^2+H_{i,r}\end{aligned} \tag{36}$$

where $H_{i,r} = H_{i-1,r} + \frac{\Theta_1}{2} + \frac{1}{2}\left(\bar{\varepsilon}_{i,r}^2 + \bar{\rho}_{i,r}^2 + \sum_{m=1}^{i}\left(\bar{\varepsilon}^2{}_{m,r} + \bar{\delta}^2{}_{m,r}\right)\right) + a_{i,r}^2 + \sum_{m=1}^{i-1}\Theta_m\Bigg) + \frac{\Psi_i}{2\zeta_i}\Theta_i^2$, $r \in \Gamma$.

Step n: Based on (1), (3) and (8), one gets

$$\dot{z}_n = q_1^r(t)u_r(t) + q_2^r(t) + \xi_n e_1 + w_{n,r}^{*T}\varphi_n\left(\hat{\overline{X}}_n\right) + \varepsilon_{n,r} - \delta_{n,r} - \dot{\alpha}_{n-1} \tag{37}$$

where $\overline{X}_n = [\hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \cdots,\ \hat{x}_n]^T$, $\dot{\alpha}_{n-1} = \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}\left(\hat{x}_{m+1} + \xi_m e_1 + w_{m,r}^{*T}\varphi_m\left(\hat{\overline{X}}_m\right) + \varepsilon_{m,r} - \delta_{m,r}\right) + \sum_{m=1}^{n-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(m-1)}}y_d^{(m)} + \sum_{m=1}^{n-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\Theta}_m}\dot{\hat{\Theta}}_m + \frac{\partial\alpha_{n-1}}{\partial y}(\hat{x}_2 + e_2 + w_{1,r}^{*T}\varphi_1(\hat{x}_1) + \varepsilon_{1,r} + \rho_{1,r}) + \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial k_m(t)}\dot{k}_m(t)$.

Choose the following barrier Lyapunov function

$$V_n = V_{n-1} + \frac{1}{2}\log\frac{k_n^2(t)}{k_n^2(t) - z_n^2} + \frac{1}{2}\zeta_n^{-1}\tilde{\Theta}_n^2 \tag{38}$$

where $\zeta_n > 0$ indicates a design parameter.

Differentiating V_n, one obtains

$$\begin{aligned}
\dot{V}_n &= \dot{V}_{n-1} + \tfrac{z_n}{k_n^2(t)-z_n^2}\left(\dot{z}_n - \tfrac{\dot{k}_n(t)}{k_n(t)}z_n\right) - \tfrac{1}{\zeta_n}\tilde{\Theta}_n\dot{\hat{\Theta}}_n \\
&= \dot{V}_{n-1} + \tfrac{z_n}{k_n^2(t)-z_n^2}\Big(q_1^r(t)u_r(t) + q_2^r(t) + \xi_n e_1 \\
&\quad + w_{n,r}^{*T}\varphi_n\left(\hat{\overline{X}}_n\right) + \varepsilon_{n,r} - \delta_{n,r} - \dot{\alpha}_{n-1} - \tfrac{\dot{k}_n(t)}{k_n(t)}z_n\Big) - \tfrac{1}{\zeta_n}\tilde{\Theta}_n\dot{\hat{\Theta}}_n \\
&= \dot{V}_{n-1} + \tfrac{z_n}{k_n^2(t)-z_n^2}\Big(q_1^r(t)u_r(t) + q_2^r(t) + \xi_n e_1 + w_{n,r}^{*T}\varphi_n\left(\hat{\overline{X}}_n\right) \\
&\quad + \varepsilon_{n,r} - \delta_{n,r} - \sum_{m=1}^{n-1}\tfrac{\partial\alpha_{n-1}}{\partial\hat{x}_m}\left(\hat{x}_{m+1} + \xi_m e_1 + w_{m,r}^{*T}\varphi_m\left(\hat{\overline{X}}_m\right) + \varepsilon_{m,r}\right. \\
&\quad \left. - \delta_{m,r}\right) - \sum_{m=1}^{n-1}\tfrac{\partial\alpha_{i-1}}{\partial\hat{\Theta}_m}\dot{\hat{\Theta}}_m - \tfrac{\partial\alpha_{n-1}}{\partial y}\left(\hat{x}_2 + e_2 + w_{1,r}^{*T}\varphi_1(\hat{x}_1) + \varepsilon_{1,r}\right. \\
&\quad \left. + \rho_{1,r}\right) - \sum_{m=1}^{n-1}\tfrac{\partial\alpha_{n-1}}{\partial y_d^{(m-1)}}y_d^{(m)} - \sum_{m=1}^{n-1}\tfrac{\partial\alpha_{i-1}}{\partial k_m(t)}\dot{k}_m(t) - \tfrac{\dot{k}_n(t)}{k_n(t)}z_n\Big) - \tfrac{1}{\zeta_n}\tilde{\Theta}_n\dot{\hat{\Theta}}_n
\end{aligned} \tag{39}$$

Consider the following inequalities

$$\frac{z_n}{k_n^2(t) - z_n^2}w_{n,r}^{*T}\varphi_n\left(\hat{\overline{X}}_n\right) \le \frac{z_n^2\Theta_n}{2a_{n,r}^2\left(k_n^2(t) - z_n^2\right)^2} + \frac{a_{n,r}^2}{2} \tag{40}$$

where $a_{n,r} > 0$ is a design parameter.

$$-\frac{z_n}{k_n^2(t) - z_n^2}\sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}w_{m,r}^{*T}\varphi_m\left(\hat{\overline{X}}_m\right) \le \frac{1}{2}\sum_{m=1}^{n-1}\Theta_m + \frac{z_n^2}{2\left(k_n^2(t) - z_n^2\right)^2}\sum_{m=1}^{n-1}\left(\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}\right)^2 \tag{41}$$

$$-\frac{z_n}{k_n^2(t) - z_n^2}\sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}(\varepsilon_{m,r} - \delta_{m,r}) \le \frac{z_n^2}{\left(k_n^2(t) - z_n^2\right)^2}\sum_{m=1}^{n-1}\left(\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}\right)^2 + \frac{1}{2}\sum_{m=1}^{n-1}\left(\bar{\varepsilon}^2{}_{m,r} + \bar{\delta}^2{}_{m,r}\right) \tag{42}$$

$$-\frac{z_n}{k_n^2(t) - z_n^2}\frac{\partial\alpha_{n-1}}{\partial y}w_{1,r}^{*T}\varphi_1(\hat{x}_1) \le \frac{z_n^2}{2\left(k_n^2(t) - z_n^2\right)^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^2 + \frac{1}{2}\Theta_n \tag{43}$$

$$-\frac{z_n}{k_n^2(t) - z_n^2}\frac{\partial\alpha_{n-1}}{\partial y}(e_2 + \varepsilon_{1,r} + \rho_{1,r}) \le \frac{3}{2}\frac{z_n^2}{\left(k_n^2(t) - z_n^2\right)^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^2 + \frac{1}{2}\left(\|e\|^2 + \bar{\varepsilon}^2{}_{1,r} + \bar{\rho}^2{}_{1,r}\right) \tag{44}$$

$$\tfrac{z_n}{k_n^2(t)-z_n^2}(\varepsilon_{n,r} - \delta_{n,r}) \le \tfrac{z_n^2}{\left(k_n^2(t)-z_n^2\right)^2} + \tfrac{1}{2}\left(\bar{\varepsilon}^2{}_{n,r} + \bar{\delta}^2{}_{n,r}\right) \tag{45}$$

According (40), (41)–(46), one has

$$\begin{aligned}\dot{V}_n = & \dot{V}_{n-1} + \frac{z_n}{k_n^2(t)-z_n^2}\Bigg(-\sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}(\hat{x}_{m+1}+\xi_m e_1) + q_1^r(t)u_r(t) + q_2^r(t) \\ & +\xi_n e_1 - \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\Theta}_m}\dot{\hat{\Theta}}_m - \sum_{m=1}^{i-1}\frac{\partial\alpha_{n-1}}{\partial y_d^{(m-1)}}y_d^{(m)} - \frac{\partial\alpha_{n-1}}{\partial y}\hat{x}_2 + \frac{3z_n}{2\left(k_n^2(t)-z_n^2\right)} \\ & \times\sum_{m=1}^{n-1}\left(\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}\right)^2 + \frac{2z_n}{k_n^2(t)-z_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^2 - \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial k_m(t)}\dot{k}_m(t) - \frac{\dot{k}_n(t)}{k_n(t)}z_n \\ & + \frac{z_n\Theta_n}{2a_{n,r}^2\left(k_n^2(t)-z_n^2\right)} + \frac{z_n}{k_n^2(t)-z_n^2}\Bigg) + \frac{1}{2}\left(\|e\|^2 + \bar{\varepsilon}^2{}_{1,r} + \bar{\rho}^2{}_{1,r}\right) + \frac{a_{n,r}^2}{2} \\ & + \frac{1}{2}\Theta_1 + \frac{1}{2}\left(\bar{\varepsilon}^2{}_{n,r} + \bar{\delta}^2{}_{n,r}\right) \quad + \frac{1}{2}\sum_{m=1}^{n-1}\Theta_m - \frac{1}{\zeta_n}\tilde{\Theta}_n\dot{\tilde{\Theta}}_n + \frac{1}{2}\sum_{m=1}^{n-1}\left(\bar{\varepsilon}^2{}_{m,r} + \bar{\delta}^2{}_{m,r}\right)\end{aligned} \tag{46}$$

Design the adaptive output feedback control law

$$\begin{aligned}u_r = & [-(K_n+\bar{\kappa}_n)z_n - R_{n-1} - \frac{z_n\hat{\Theta}_n}{2a_{n,\min}^2\left(k_n^2(t)-z_n^2\right)} \\ & - \frac{z_n}{k_n^2(t)-z_n^2} - q_2^r(t) - \frac{k_n^2(t)-z_n^2}{k_{n-1}^2(t)-z_{n-1}^2}z_{n-1}]\varpi\end{aligned} \tag{47}$$

where $K_n > 0$ is design parameter, $a_{n,\min} = \min\{a_{n,r},\ r\in\Gamma\}$, $\bar{\kappa}_n = \sqrt{m_n{}^2 + p_n}$, $p_n > 0$ and $m_n = \frac{\dot{k}_n(t)}{k_n(t)}$, $R_{n-1} = -\sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}(\hat{x}_{m+1}+\xi_m e_1) - \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\Theta}_m}\dot{\hat{\Theta}}_m - \frac{\partial\alpha_{n-1}}{\partial y}\hat{x}_2 - \sum_{m=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(m-1)}}y_d^{(m)} - \sum_{m=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial k_m(t)}\times\dot{k}_m(t) + \xi_n e_1 + \frac{3z_n}{2\left(k_n^2(t)-z_n^2\right)}\sum_{m=1}^{n-1}\left(\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}\right)^2 + \frac{z_n}{k_n^2(t)-z_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial\hat{x}_m}\right)^2 + \frac{2z_n}{k_n^2(t)-z_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^2$.

Design the adaptive law

$$\dot{\hat{\Theta}}_n = -\Psi_n\hat{\Theta}_n + \frac{\zeta_n z_n^2}{2a_{n,\min}^2\left(k_n^2(t)-z_n^2\right)^2} \tag{48}$$

where $\Psi_n > 0$ is a design parameter.

The following inequalities always hold

$$\frac{\Psi_n}{\zeta_n}\tilde{\Theta}_n\hat{\Theta}_n \le \frac{\Psi_n}{2\zeta_n}\left(-\tilde{\Theta}_n^2 + \Theta_n^2\right) \tag{49}$$

According to (47)–(50), one obtains

$$\dot{V}_n \le -\left[\lambda_{\min}(Q) - 2 - \frac{1}{2}n\right]\|e\|^2 - \sum_{m=1}^{n}\frac{\Psi_m}{2\zeta_m}\tilde{\Theta}_m^2 - \sum_{m=1}^{n}K_m\frac{z_m^2}{k_m^2(t)-z_m^2} + H_{n,r} \tag{50}$$

where $H_{n,r} = H_{n-1,r} + \frac{\Theta_1}{2} + \frac{1}{2}\left(\bar{\varepsilon}_{1,r}^2 + \bar{\rho}_{1,r}^2 + \sum_{m=1}^{n}\left(\bar{\varepsilon}^2{}_{m,r} + \bar{\delta}^2{}_{m,r}\right)\ \right) + a_{n,r}^2 + \sum_{m=1}^{i-1}\Theta_m\Bigg) + \frac{\Psi_n}{2\zeta_n}\Theta_n^2, \quad r\in\Gamma$.

5. Stability Analysis

To facilitate the description, the following notation is defined

$$\Lambda = \min\{\mu,\ K_m,\ \Psi_m : m = 1,\ \ldots,\ n\} \tag{51}$$

$$H = \max_{r\in\Gamma}\{H_{n,r}\} \tag{52}$$

where $\mu = \lambda_{\min}\left(Q - 2 - \frac{1}{2}n\right)/\lambda_{\min}(P)$.

Theorem 1. *Consider the switched uncertain nonlinear system (1) with input quantization (2), unmeasurable states and state contraints, under Assumption 1, the adaptive output feedback* *controller in (48) is constructed by using the barrier Lyapunov function method and the backstepping technique, which can guarantee that all signals in the closed-loop system are bounded, the tracking error converges to a small neighborhood of origin, and none of the states in the system conflict with their corresponding constraints.*

Proof. Select the barrier Lyapunov function as follows

$$V = V_n = e^T Pe + \frac{1}{2}\sum_{i=1}^{n} \log \frac{k_i^2(t)}{k_i^2(t) - z_i^2} + \frac{1}{2}\sum_{i=1}^{n} \zeta_i^{-1}\widetilde{\Theta}_i^2 \tag{53}$$

According to the above analysis and the definitions in (52) and (53), one has

$$\dot{V} \leq -\Lambda V + H \tag{54}$$

Integrating (55) over the interval $[0, t]$, one obtains

$$V(t) \leq \left(V(0) - \frac{H}{\Lambda}\right)e^{-\Lambda t} + \frac{H}{\Lambda} \tag{55}$$

Based on (54) and (56), one deduces that

$$|z_i| \leq \sqrt{k_i^2(t) - \frac{k_i^2(t)}{e^{2(V(0)-\frac{H}{\Lambda})e^{-\Lambda t}+\frac{H}{\Lambda}}}}, \quad i = 1, \ldots, n \tag{56}$$

Similarly, it can be derived that

$$\|e\| \leq \sqrt{\frac{\left(V(0) - \frac{H}{\Lambda}\right)e^{-\Lambda t} + \frac{H}{\Lambda}}{\lambda_{\min}(P)}}, i = 1, \ldots, n \tag{57}$$

From (54) and (56), one easily knows that transformation errors z_i, observer errors e_i and estimation errors $\widetilde{\Theta}_i (i = 1, \cdots n)$ are all bounded for the bounded initial values. The tracking error z_1 is bounded, this implies that the system output tracks the reference signal. Due to the boundedness of Θ_i and $\widetilde{\Theta}_i (i = 1, \cdots n)$, then $\hat{\Theta}_i (i = 1, \cdots n)$ is also bounded. According to the definitions of $z_i (i = 1, \cdots, n)$, $e_i (i = 1, \cdots, n)$, $\alpha_i (i = 1, \cdots, n-1)$, u_r, and Assumption 1, it can be seen that $x_i (i = 1, \cdots, n)$, $\hat{x}_i (i = 1, \cdots, n)$, $\alpha_i (i = 1, \cdots, n-1)$, and u_r are all bounded. Thus, all signals in the closed-loop system are all bounded. Moreover, in the proposed control method, the barrier Lyapunov function is employed such that the system states satisfy the corresponding states. □

Remark 1. *It is worth noting that when* $|u_r(t)| < B_r$ *and* $q_1^r(t) = 1$ *are bounded,* $U_r(u_r(t))$ *is also bounded. This can be seen from (4). Then, by observing (5), one can easily know that* $q_2^r(t)$ *is bounded, namely* $\left|q_2^r(t)\right| \leq Y_r$*, where* Y_r *is a positive constant. Therefore, in the future work, the boundedness case of* $q_2^r(t)$ *can be considered.*

Remark 2. *In this paper, there are many design parameters in the control, which may affect the system performance. In order to ensure that the system has a good performance, the selection of these design parameters are important. P and Q are positive matrices which should be satisfy the Riccati-like equation. From (58), it is easy to know that the value of the tracking error is associated with* K_m*,* Ψ_m *and* $a_{m,min} (m = 1, 2, \cdots n)$*. For lager* K_m*,* Ψ_m *and smaller* $a_{m,min}$*, the tracking error are smaller. At the same time the other signals of the system are smaller. For the other parameters, it is enough to choose them satisfy their corresponding conditions, such as positive.*

Remark 3. *Compared with the existing results in* [30–32]*, in which the mode-dependent Lyapunov function is chosen, a common Lyapunov function which is mode-independent is selected in this paper. This because of that the arbitrary switching law is employed here. Actually, in engineering problems,* *switching signals are often unknown or unable to be determined in advance, so the stability of switched systems under arbitrary switching signals is of special significance. Thus, in this paper, the arbitrary switching law is selected. If a switched system can be stabilized under the arbitrary signal, then the subsequent research (such as robust control, etc.) does not need to pay attention to the impact of switching signals on the system stability.*

6. Simulation Example

In this section, a simulation experiment is used to show the effectiveness of the proposed adaptive fuzzy output feedback control approach. Consider the following uncertain nonlinear switched system

$$\begin{cases} \dot{x}_1 = x_2 + f_{1,\sigma(t)} + \rho_{1,\sigma(t)} \\ \dot{x}_2 = v_{\sigma(t)} + f_{2,\sigma(t)} + \rho_{2,\sigma(t)} \\ y = x_1 \end{cases} \tag{58}$$

where $\sigma(t) \in \{1,2\}$ stands for the switching signal. When $\sigma(t) = 1$, the external disturbance are chosen as $\rho_{11} = 0.02\cos(t)$, and $\rho_{21} = 0.015\sin(t)$, the nonlinear functionss are $f_{11} = 0.012x_1 e^{-0.11x_1}$ and $f_{21} = 0.11x_2 + x_1^3$. When $\sigma(t) = 2$, the external disturbances are $\rho_{12} = 0.01\sin(t)$ and $\rho_{22} = 0.02\sin(0.2t)$, and the nonlinear functions are chosen as $f_{12} = 0.021x_1 e^{-x_1^2}$ and $f_{22} = 0.016\sin(x_2)e^{-x_1}$.

Moeover, the input quantization is selected as

$$v_r(t) = \begin{cases} (1-\delta_r)q_{1,j}^r, & if\ q_{1,j}^r \le u_r(t) \le q_{1,j+1}^r \\ 0, & if\ \ 0 \le u_r(t) \le q_{1,1}^r \\ -U_r(-u_r(t)), & if\ \ u_r(t) < 0 \end{cases}$$

where $r = 1,2$. In addition, to solve the problem of unmeasurable states, the following fuzzy-based observer is designed

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \hat{w}_{1,\sigma(t)}^T \varphi_1(X_1) + \xi_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 = v_{\sigma(t)} + \hat{w}_{2,\sigma(t)}^T \varphi_2(X_2) + \xi_2(x_1 - \hat{x}_1) \\ \hat{y} = \hat{x}_1 \end{cases}$$

and the adaptive virtual controller and the adaptive quantization controller are chosen as

$$\alpha_1 = -(K_1 + \overline{\kappa}_1)z_1 - \frac{3}{2}\frac{z_1}{k_1^2(t) - z_1^2} - \frac{z_1\hat{\Theta}_1}{2a_{1,\min}^2\left(k_1^2(t) - z_1^2\right)}$$

$$u_r = [-(K_2 + \overline{\kappa}_2)z_2 - R_1 - \tfrac{z_2\hat{\Theta}_2}{2a_{2,\min}^2\left(k_2^2(t)-z_2^2\right)} \\ - \tfrac{z_2}{k_2^2(t)-z_2^2} - q_2^r(t) - \tfrac{k_2^2(t)-z_2^2}{k_1^2(t)-z_1^2}z_1]\varpi$$

$$\dot{\hat{\Theta}}_1 = -\Psi_1\hat{\Theta}_1 + \frac{\zeta_1 z_1^2}{2a_{1,\min}^2\left(k_1^2(t) - z_1^2\right)^2}$$

$$\dot{\hat{\Theta}}_2 = -\Psi_2\hat{\Theta}_2 + \frac{\zeta_2 z_2^2}{2a_{2,\min}^2\left(k_2^2(t) - z_2^2\right)^2}$$

In this simulation, the reference signal is selected as $y_d = 0.5\sin(2t)$, the associated initial value is set as $x_1(0) = 0.01$, $x_2(0) = 0.1$, $\hat{x}_1(0) = 0.02$, $\hat{x}_2(0) = 0.2$, $\hat{\Theta}_1(0) = 0.01$, $\hat{\Theta}_2(0) = 0.02$, $k_1(t) = 0.8 + 0.1\sin(0.5t)$, $k_2(t) = 2 + 0.1\sin(2t)$. The relevant design parameters are chosen as $K_1 = 20, K_2 = 50, \xi_1 = 15, \xi_2 = 25, \Psi_1 = 1, \Psi_2 = 0.5, \zeta_1 = 0.01$, $\zeta_2 = 0.02, a_{1,\min} = 0.2, a_{2,\min} = 0.5, p_1 = 10, p_2 = 20$.

Figures 1–7 show the simulation results. Figure 1 gives the switching signal. Figure 2 depicts the trajectories of the system output, its constraint bounds and the tracking signal. From which, one sees that the reference signal can be followed well, and the system output successfully satisfies its constraint condition. Figure 3 describes the trajectories of state x_2 and its constraint condition. We can find that x_2 is limited within prespecified range. Figure 4 expresses the curves of the state x_1 and $\hat{x}_1$, and their estimated error e_1, respectively. Figure 5 shows the curves of the state x_2 and $\hat{x}_2$, and their estimated error e_2, respectively. From Figures 4 and 5, we can conclude that the system state x_1 and x_2 are well observed. Figure 6 gives the curves of the adaptive laws $\hat{\Theta}_1$, $\hat{\Theta}_2$, and the input quantization u is presented in Figure 7. Figures 6 and 7 immplies that $\hat{\Theta}_1$, $\hat{\Theta}_2$ and u are all bounded. From the simulation results above, it can be concluded that by using the proposed control approach, the control objectives are achieved, thus it is effective.

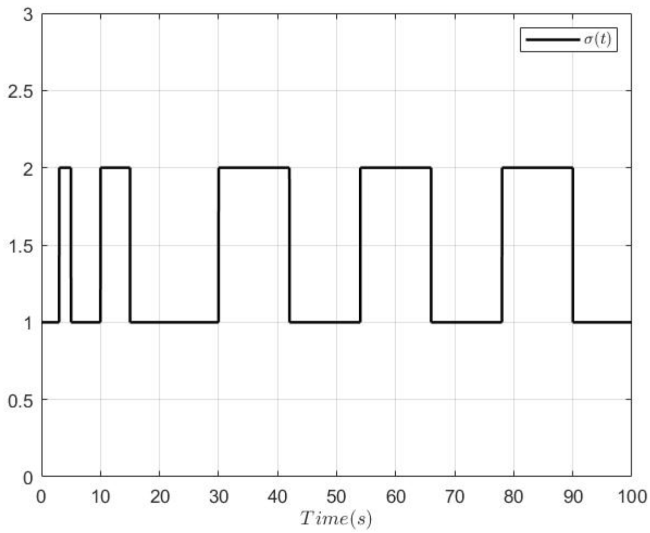

Figure 1. The switching signal.

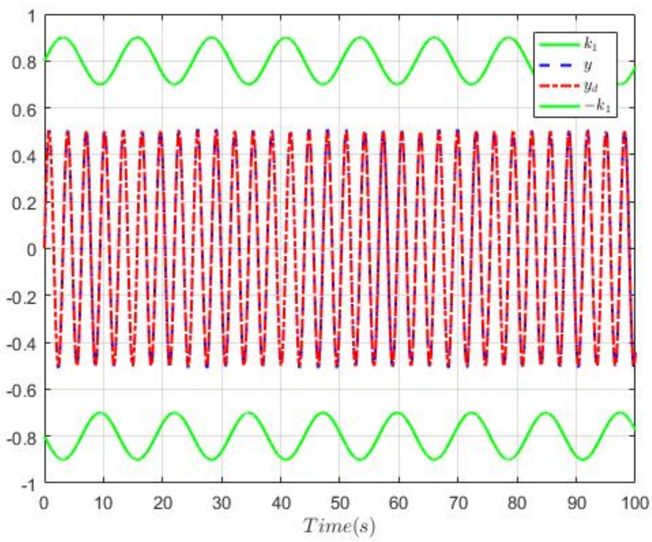

Figure 2. Curves of tracking results.

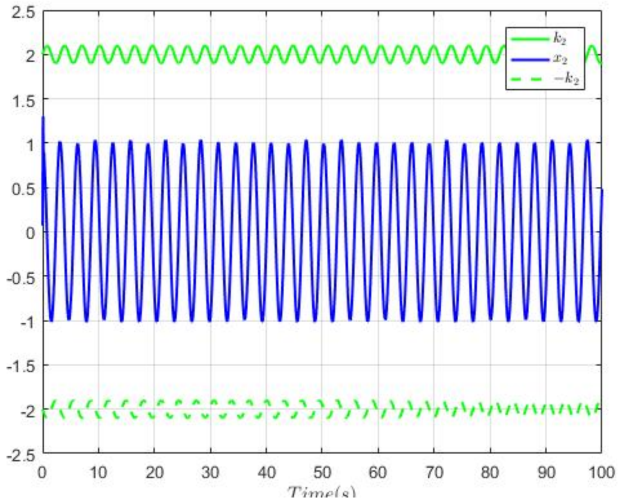

Figure 3. Trajectories of system states x_2 and its constraint conditions.

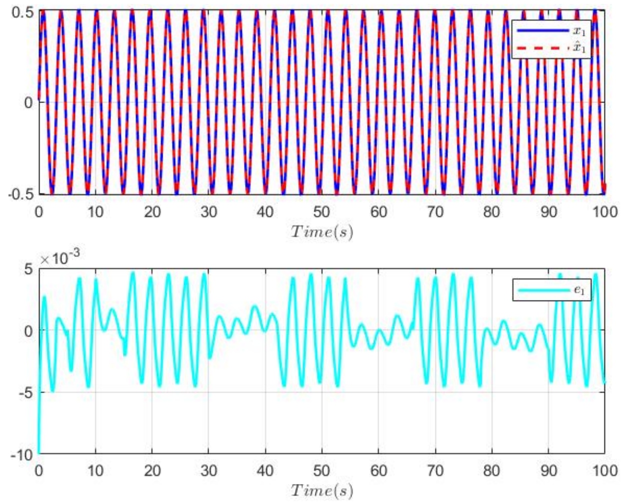

Figure 4. Trajectories of the system state x_1, the observer state $\hat{x}_1$, and their estimation error.

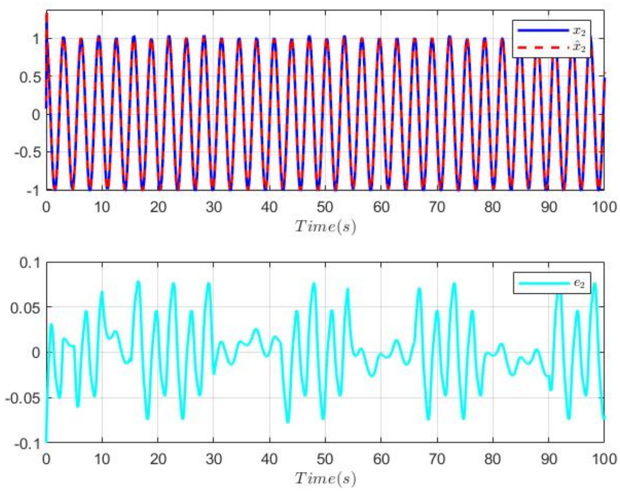

Figure 5. Trajectories of the system state x_2, the observer state $\hat{x}_2$, and their estimation error.

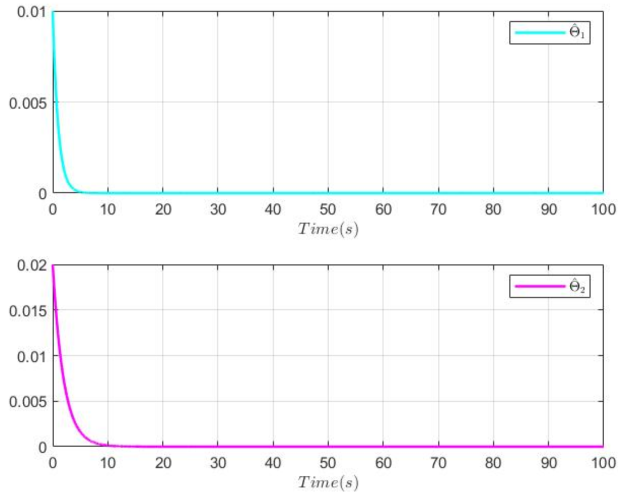

Figure 6. Curves of $\hat{\Theta}_1$ and $\hat{\Theta}_2$.

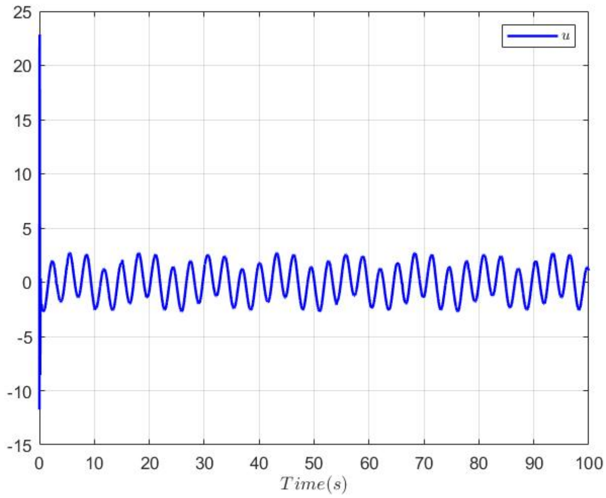

Figure 7. The trajectory of input quantization u.

7. Conclusions

This paper focuses on the input quantization, state constraints and unmeasured system states issues for switched nonlinear systems. For the unmeasured states, a fuzzy-based state observer is developed to estimate them. Then, the barrier Lyapunov function approach is employed to guarantee that the system state does not exceed the corresponding constraint bound. Using the backstepping technique, an adaptive control law is designed that can offset the impact of input quantification. However, due to the application of backstepping method, the problem of computation explosion may be caused. Therefore, how to reduce the amount of computation will be studied in future research.

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References

1. Ren, H.L.; Zong, G.D.; Li, T.S. Event-triggered finite-time control for networked switched linear systems with asynchronous switching. *IEEE Trans. Syst. Man Cybern. Syst.* **2018**, *48*, 1874–1884. [CrossRef]
2. Yang, D.; Li, X.D.; Qiu, J.L. Output tracking control of delayed switched systems via state-dependent switching and dynamic output feedback. *Nonlinear Anal. Hybrid Syst.* **2019**, *32*, 294–305. [CrossRef]
3. Zong, G.; Yang, D.; Lam, J.; Song, X. Fault-tolerant control of switched LPV systems: A bumpless transfer approach. *IEEE/ASME Trans. Mechatron.* **2021**, *27*, 1436–1446.
4. Liu, C.; Liu, X.Y. Stability of switched systems with time-varying delays under state-dependent switching. *Mathematics* **2022**, *10*, 2722. [CrossRef]
5. Qi, Y.W.; Zeng, P.Y.; Bao, W. Event-triggered and self-triggered H_∞ control of uncertain switched linear systems. *IEEE Trans. Syst. Man Cybern. Syst.* **2018**, *50*, 1442–1454. [CrossRef]
6. Xia, J.; Gao, H.; Liu, M.; Zhuang, G.; Zhang, B. Non-fragile finite-time extended dissipative control for a class of uncertain discrete time switched linear systems. *J. Frankl. Inst.* **2018**, *355*, 3031–3049. [CrossRef]
7. Liu, L.; Liu, Y.J.; Chen, A.; Tong, S.; Chen, C.L.P. Integral barrier Lyapunov function-based adaptive control for switched nonlinear systems. *Sci. China Inf. Sci.* **2020**, *63*, 1–14. [CrossRef]
8. Zhao, X.; Wang, X.; Ma, L.; Zong, G. Fuzzy approximation based asymptotic tracking control for a class of uncertain switched nonlinear systems. *IEEE Trans. Fuzzy Syst.* **2019**, *28*, 632–644. [CrossRef]
9. Sun, K.; Mou, S.; Qiu, J.; Wang, T.; Gao, H. Adaptive fuzzy control for nontriangular structural stochastic switched nonlinear systems with full state constraints. *IEEE Trans. Fuzzy Syst.* **2018**, *27*, 1587–1601. [CrossRef]
10. Liu, L.; Liu, Y.J.; Tong, S.C. Fuzzy-based multierror constraint control for switched nonlinear systems and its applications. *IEEE Trans. Fuzzy Syst.* **2018**, *27*, 1519–1531. [CrossRef]
11. Ma, L.; Xu, N.; Zhao, X.; Zong, G.; Huo, X. Small-gain technique-based adaptive neural output-feedback fault-tolerant control of switched nonlinear systems with unmodeled dynamics. *IEEE Trans. Syst. Man Cybern. Syst.* **2020**, *51*, 7051–7062. [CrossRef]
12. Deng, W.X.; Yao, J.Y. Extended-state-observer-based adaptive control of electrohydraulic servomechanisms without velocity measurement. *IEEE/ASME Trans. Mechatron.* **2019**, *25*, 1151–1161. [CrossRef]
13. Xiao, W.; Cao, L.; Li, H.; Lu, R. Observer-based adaptive consensus control for nonlinear multi-agent systems with time-delay. *Sci. China Inf. Sci.* **2020**, *63*, 1–17. [CrossRef]
14. Zhu, J.W.; Gu, C.Y.; Ding, S.X.; Zhang, W.-A.; Wang, X.; Yu, L. A new observer-based cooperative fault-tolerant tracking control method with application to networked multiaxis motion control system. *IEEE Trans. Ind. Electron.* **2020**, *68*, 7422–7432. [CrossRef]
15. Yao, H.J.; Gao, F.Z. Design of observer and dynamic output feedback control for fuzzy networked systems. *Mathematics* **2023**, *11*, 148. [CrossRef]
16. Zhang, H.G.; Liu, Y.; Wang, Y.C. Observer-based finite-time adaptive fuzzy control for nontriangular nonlinear systems with full-state constraints. *IEEE Trans. Cybern.* **2020**, *51*, 1110–1120. [CrossRef]
17. Sosnowski, M.; Krzywanski, J.; Scurek, R. A fuzzy logic approach for the reduction of mesh-induced error in CFD analysis: A case study of an impinging jet. *Entropy* **2019**, *21*, 1047. [CrossRef]
18. Krzywanski, J. Heat transfer performance in a superheater of an industrial CFBC using fuzzy logic-based methods. *Entropy* **2019**, *21*, 919. [CrossRef]
19. Xiao, F. A hybrid fuzzy soft sets decision making method in medical diagnosis. *IEEE Access* **2018**, *6*, 25300–25312. [CrossRef]
20. Tang, R.Q.; Su, H.S.; Zou, Y.; Yang, X.S. Finite-time synchronization of Markovian coupled neural networks with delays via intermittent quantized control: Linear programming approach. *IEEE Trans. Neural Netw. Learn. Systems* **2021**, *33*, 5268–5278. [CrossRef]
21. Liu, Q.; Liu, M.; Duan, G. Adaptive fuzzy backstepping control for attitude stabilization of flexible spacecraft with signal quantization and actuator faults. *Sci. China Inf. Sci.* **2021**, *64*, 116. [CrossRef]
22. Li, D.; Chen, C.L.P.; Liu, Y.J.; Tong, S. Neural network controller design for a class of nonlinear delayed systems with time-varying full-state constraints. *IEEE Trans. Neural Netw. Learn. Syst.* **2019**, *30*, 2625–2636. [CrossRef] [PubMed]
23. Xia, J.; Zhang, J.; Sun, W.; Zhang, B.; Wang, Z. Finite-time adaptive fuzzy control for nonlinear systems with full state constraints. *IEEE Trans. Syst. Man Cybern. Syst.* **2018**, *49*, 1541–1548. [CrossRef]
24. Sun, R.H.; Tang, L.; Liu, Y.J. Boundary controller design for a class of horizontal belt transmission system with boundary vibration constraint. *Mathematics* **2022**, *10*, 1391. [CrossRef]
25. Yang, X.S.; Cao, J.D.; Xu, C.; Feng, J.W. Finite-time stabilization of switched dynamical networks with quantized couplings via quantized controller. *Sci. China Technol. Sci.* **2018**, *61*, 299–308. [CrossRef]
26. Zou, Y.; Su, H.S.; Tang, R.Q.; Yang, X.S. Finite-time bipartite synchronization of switched competitive neural networks with time delay via quantized control. *ISA Trans.* **2022**, *125*, 156–165. [CrossRef] [PubMed]
27. Yang, X.S.; Wan, X.X.; Cheng, Z.S.; Cao, J.D.; Liu, Y.; Rutkowski, L. Synchronization of switched discrete-time neural networks via quantized output control with actuator fault. *IEEE Trans. Neural Netw. Learn. Syst.* **2021**, *32*, 4191–4201. [CrossRef] [PubMed]
28. Wang, C.; Wen, C.; Lin, Y.; Wang, W. Decentralized adaptive tracking control for a class of interconnected nonlinear systems with input quantization. *Automatica* **2017**, *81*, 359–368. [CrossRef]
29. Liu, Y.J.; Lu, S.M.; Tong, S.C. Neural network controller design for an uncertain robot with time-varying output constraint. *IEEE Trans. Syst. Man Cybern. Syst.* **2017**, *47*, 2060–2068. [CrossRef]

30. He, C.; Tang, R.; Lam, H.K.; Cao, J. Mode-dependent event-triggered output control for switched TS fuzzy systems with stochastic switching. *IEEE Trans. Fuzzy Syst.* **2022**, *2022*, 1–11. [CrossRef]
31. Wang, H.; Yang, X.; Xiang, Z.; Tang, R. Synchronization of switched neural networks via attacked mode-dependent event-triggered control and its application in image encryption. *IEEE Trans. Cybern.* **2022**, *2022*, 1–10. [CrossRef]
32. Zhang, M.; Yang, X.; Xiang, Z.; Sun, Y. Monotone decreasing LKF method for secure consensus of second-order mass with delay and switching topology. *Syst. Control. Lett.* **2023**, *172*, 105436. [CrossRef]