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Article

Algebraic Methods for Achieving Super-Resolution by Digital Antenna Arrays

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Abstract: The actual modern problem of developing and improving measurement and observation systems (including robotic ones) is to increase the volume and quality of the information received. Increasing the angle resolution to values significantly exceeding the Rayleigh criterion, i.e. achieving super-resolution is one of important ways to solve the problem. Angular super-resolution which makes it possible to detail images of research objects and their individual fragments, improves the quality of solutions to detection, recognition and identification problems, increases the range of such systems. In many papers methods developed by authors to achieve a super-resolution based on approximate solutions of inverse problems in the form of Fredholm integral equation of the first kind of convolution type called algebraic are presented. The methods used, as well as their varieties, make it possible to reduce solutions of inverse problems posed to solving sets of linear algebraic equations (SLAE). This paper presents results of further improvement of algebraic methods based on intelligent analysis of received signals. It is shown that their use in systems based on digital antenna arrays makes it possible to increase the achieved degree of exceeding the Rayleigh criterion. In the course of numerical experiments with a mathematical model, the stability of the solutions obtained and their adequacy were confirmed. The numerical results obtained open the following possibilities: (1) obtaining images of studied objects with a resolution exceeding the Rayleigh criterion by 4 to 10 times, (2) determining the angular coordinates of individual small-sized objects as part of multi-element complex objects (group targets), (3) clarifying boundaries of extended objects and their individual elements, (4) localizing individual bright objects on a smoothly inhomogeneous reflective background. Applying presented new methods does not require a significant computing power, what allows you to work in a real time mode using relatively simple and inexpensive computing devices. The ways of further improvement of presented algebraic methods for solving applied inverse problems are described.

Keywords: Rayleigh criterion; angular super-resolution; integral equation of convolution type; parametrization of inverse problems; stability of inverse problems; conditionality numbers

MSC: 45B05

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1. Introduction

Many articles and reports at conferences in the USA, China, France, Italy, Great Britain, and Russia are devoted to achieving angular super-resolution. The methods of MUSIC (multiple signal classification) [1–3], ESPRIT [4], the deconvolution method [5,6], the maximum entropy method (MME) of Berg [7], Borgiotti–Lagunas [8], Capon [9], the maximum likelihood method [10], and others [11] have been studied in detail. New methods are being developed for solving problems of forming images of objects in the radio range with inaccurately specified information due to the presence of random components of measurement data [12]. These methods successfully solve one-dimensional problems. The performances of most algorithms based on them are not enough to work in real-time modes. Most methods (including the well-known MUSIC and ESPRIT) are applicable only for narrow-band signals and are ineffective when using broadband signals. This article proposes a new method for achieving super-resolution. This method allows for increasing the achievable level of the angular resolution.

The developed algebraic methods and their varieties [13–20] are free from the listed disadvantages, and allow obtaining solutions with super-resolution when using wide- and ultra-wide-band signals as well as solving two-dimensional problems [21]. Numerical experiments have shown that algebraic methods have significantly higher noise immunity [17–19].

The most important property of these methods involves the use of a priori information of the source signals, which allows increasing the stability of solutions to which the super-resolution problem reduces. Algebraic methods are quite simple, which makes it possible to implement high-speed algorithms even when relatively cheap computing devices are used. This allows, unlike many other methods [22–28], to apply them in real-time modes.

After the introduction, the article considers the formulation of the inverse problem of achieving a super-solution. The following is a brief justification of the algebraic method for solving the problem with additional conditions. The next section sets and solves the problem of increasing the degree of super-resolution based on a new mathematical method of digital signal processing for the antenna array.

2. Problem Statement

The achievement of angular super-resolution is based on solving inverse problems described by one- and two-dimensional Fredholm integral equations of the first kind of convolution type

$$U(\alpha,\varphi) = \int\limits_{\Omega} f(\alpha-\alpha',\varphi-\varphi')\,I(\alpha',\varphi')\,d\alpha'\,d\varphi', \tag{1}$$

where $I(\alpha,\varphi)$ is the angular distribution of the reflected signal amplitude, $f(\alpha,\varphi)$ is the two-dimensional radiation pattern (RP) of the antenna system, $U(\alpha,\varphi)$ is the envelope of the received signal dependence when scanning the viewing sector Ω. During an active location, $f(\alpha,\varphi)$ should be considered the product of the transmitting and receiving radiation patterns, while at passive and semi-active locations, $f(\alpha,\varphi)$ is the receiving RP. The task is to find an approximate solution of the integral Equation (1) with respect to $I(\alpha,\varphi)$, i.e., it reduces to the solution of the inverse problem.

In direct measuring, signal $U(\alpha,\varphi)$ describes a real distribution $I(\alpha,\varphi)$ with an angular resolution on the angles α and φ not exceeding the Rayleigh criterion, i.e., $\theta_r = \lambda/D$, where λ is the wavelength, D is the size of the antenna system. The task is to find an algorithm for digital signal processing $U(\alpha,\varphi)$ that provides $I(\alpha,\varphi)$ with a resolution better than θ_r, i.e., with super-resolution.

Considering $I(\alpha,\varphi)$ as an unknown function and $U(\alpha,\varphi)$ as the given one, we arrive at (1) in the form of the Fredholm equation of the first kind of convolution type. The task is inverse and ill-posed according to Hadamard.

3. An Algebraic Method for Finding an Approximate Solution

We give a brief description of an algebraic method for achieving super-resolution in one- and two-dimensional problems of radar, remote sensing, and hydroacoustics [12–17] based on an approximate solution (1). The RP of the antenna system, the received signal $U(\alpha,\varphi)$, and the angular distribution of the reflected signal $I(\alpha,\varphi)$ are described by physically realizable functions from the Hilbert space $L_2(\Omega)$ of quadratically integrable functions in a bounded two-dimensional domain Ω. The boundaries of the domain Ω are selected when constructing a solution for each specific problem, based on the analysis of the values of $U(\alpha,\varphi)$, which decreases to almost zero when angles are removed from the values corresponding to the maximum. The values of $I(\alpha,\varphi)$ at the boundaries of the selected areas are naturally considered zero.

The solution $I(\alpha, \varphi)$ is represented as a decomposition according to the chosen complete orthonormal system of $g_m(\alpha, \varphi)$ functions in the area of the source location Ω.

$$I(\alpha, \varphi) = \sum_{m=1}^{\infty} b_m g_m(\alpha, \varphi) \approx \sum_{m=1}^{M} b_m g_m(\alpha, \varphi). \tag{2}$$

An important question is the choice of the system of functions. The $g_m(\alpha, \varphi)$ system is determined based on a priori information about the solution, as well as reasonable assumptions about the properties of the solution. For example, in the remote sensing problem, if it is known that reflection occurs from a smoothly inhomogeneous surface with the possible presence of highly reflective small-sized inclusions, it is advisable to use wavelets. If the radar object under study is a collection of individual objects with small angular dimensions that are not resolved by direct observation, step functions should be used, etc. Thus, the problem is mathematically reduced to solving the inverse problem (1) with an additional condition: an approximate solution is sought on sets of functions defined by sets of introduced basic functions $g_m(\alpha, \varphi)$. To increase the resolution by N times, N basic functions should be used. The number of selected functions is limited by the stability of the solutions obtained and may vary with different approaches.

If the area Ω corresponds to the width of the beam, then the Rayleigh criterion corresponds to one function in (2), then for $M > 1$, we obtain a solution with super-resolution. The degree of the desired super-resolution increases with increasing the number of functions used M in (2). The described approach allows parameterization of inverse problems and reduces them to the search for expansion coefficients b_m, $m = 1, 2, \ldots, M$ from (2). Indeed, using (2), we obtain

$$U(\alpha, \varphi) = \sum_{m=1}^{\infty} b_m h_m(\alpha, \varphi) \approx \sum_{m=1}^{M} b_m h_m(\alpha, \varphi), \tag{3}$$

where

$$h_m(\alpha, \varphi) = \int_{\Omega} f(\alpha - \alpha', \varphi - \varphi')\, g_m(\alpha', \varphi')\, d\alpha'\, d\varphi'. \tag{4}$$

The b_m coefficients can be found by minimizing the standard deviation δ^2 of the final sum on the right side (3) of the signal under study $U(\alpha, \varphi)$ in the scanning sector, i.e., from the set of equations

$$\int_{\Omega} U(\alpha, \varphi) h_j(\alpha, \varphi) h_m(\alpha, \varphi)\, d\alpha\, d\varphi \approx \sum_{m=1}^{M} b_m \int_{\Omega} h_j(\alpha, \varphi) h_m(\alpha, \varphi)\, d\alpha\, d\varphi, \quad j = 1, 2, \ldots, M, \tag{5}$$

or in matrix notation, by entering the **B**-vector of coefficients b_j, and

$$V_j = \int_{\Omega} U(\alpha, \varphi) h_j(\alpha, \varphi)\, d\alpha\, d\varphi, \quad G_{jm} = \int_{\Omega} h_j(\alpha, \varphi) h_m(\alpha, \varphi)\, d\alpha\, d\varphi, \quad j, m = 1, 2, \ldots, M, \tag{6}$$

$$\mathbf{V} = \mathbf{G}\,\mathbf{B}. \tag{7}$$

As a result, the inverse problem is reduced to solving a system of linear algebraic equations (SLAE) (5)–(7).

The choice of the system of $g_m(\alpha, \varphi)$ functions used to represent the solution is carried out on the basis of a priori information about the solution or a mathematical model of the object under study is used.

The main feature of Equation (7) is its poor conditionality because the unstable inverse problem is solved. The conditioning number of matrices **G**—as experiments on mathematical models show—increase exponentially with the increasing N. Considering this factor, the solutions of SLAE (7) are constructed as iterative processes with a sequential increase in the number M of functions used in (2). This makes it possible to approach the maximum achievable level of angular super-resolution for a specific task. The iterative process is convergent since it is easy to show that adding a new term in (3) improves the approximation to U.

4. Over-Resolution by Digital Antenna Arrays

Numerical experiments have shown that the SLAE of form (7), due to the poor conditionality, makes it possible to obtain 2–3, under favorable conditions; 5–7 $g_m(\alpha, \varphi)$ functions in the representation (2) without significant distortion of the real distribution $I(\alpha, \varphi)$, which exceeds the Rayleigh criterion by 2–8 times.

We justify the possibility of further increasing the level of the achieved super-resolution by modern systems based on digital antenna arrays (DAA) using a new modification of the algebraic method.

Consider a linear equidistant DAA of $2N + 1$ elements with a distance d between neighboring emitters. We consider the amplitude RP of individual DAA emitters to be weakly directional and equal to a constant within the scanning area. Then the radiation pattern of the antenna array in the direction α_0 is

$$f(\alpha_0 - \alpha) = \sum_{n=-N}^{N} J_n \exp\big(-ikdn\sin\alpha\big) = \sum_{n=-N}^{N} \exp\big(ikdn(\sin\alpha_0 - \sin\alpha)\big), \tag{8}$$

where $k = 2\pi/\lambda$ and J_n are the complex amplitudes of the current on the n-th emitter $(n = -N, \ldots, N)$ with accuracy up to a constant equal in the case under consideration to $\exp\big(ikdn\sin\alpha_0\big)$. The angles α_0 and α lay in the same plane with the linear DAA and are counted from the direction perpendicular to it.

The signal (1) received when scanning along the angle α for a DAA with a directional diagram in form (8) is the sum of

$$U(\alpha) = \sum_{n=-N}^{N} C_n \exp\big(-ikdn\sin\alpha\big), \tag{9}$$

with

$$C_n = \int_{\Omega} \exp\big(ikdn\sin\alpha'\big)\, I(\alpha')\, d\alpha', \qquad n = 0, \pm 1, \ldots, \pm N. \tag{10}$$

Here, C_n is the signal received by the individual emitter of the DAA.

It is known that, in this case, solution $I(\alpha)$ up to the normalizing factor can be represented in the form:

$$I(\alpha) = \sum_{n=-N}^{N} \exp\big(-ikdn\sin\alpha\big) C_n + \psi(\alpha), \tag{11}$$

where $\psi(\alpha)$ is an arbitrary function orthogonal to all exponents (9) on the segment $[-\lambda/d; \lambda/d]$.

In contrast to the above-known method (3)–(7), we will use the fact that in DAA, signals received by each emitter can be represented digitally. Using (10), the solution representation in form (2), and taking into account that signal C is a complex conjugate, $C_{-n} = C_n^*$, we obtain

$$D_n = Re(C_n) \approx \sum_{m=1}^{M} b_m \int_{\Omega} \cos\big(ikdn\sin\alpha'\big) g_n(\alpha')\, d\alpha', \quad n = 0, \ldots, N. \tag{12}$$

Next, by entering a rectangular matrix $\mathbf{H}$ with elements

$$H_{nm} = \int_{\Omega} \cos\big(ikdn\sin\alpha'\big) g_m(\alpha')\, d\alpha', \quad n, m = 0, \ldots, N, \tag{13}$$

we come to a new SLAE with respect to the vector of unknown expansion coefficients **B**

$$\mathbf{D} = \mathbf{H}\,\mathbf{B}. \tag{14}$$

With the same dimension, the conditionality matrix numbers **G** from (7) are significantly larger than those of matrices **H** from (14). Greater stability of the system (14) solutions allows increasing the number of functions in the solution (2) representation. This makes it possible to increase the degree of super-resolution, as well as successfully process signals with higher levels of random components.

The solution of the entire problem is constructed in the form of an iterative process with sequentially increasing M—the number of $g_m(\alpha)$ functions in (2) and, hence, in (12)–(14).

At the beginning of the process, at $M < N$, the system (14) is reconditioned. In this case, as is usually recommended, a pseudo-solution is sought—a vector that minimizes the norm of the discrepancy of the system of equations. Thus, the problem with solving SLAE (14) is replaced by the problem of finding the global minimum of the function $f(\mathbf{B}) = \|\mathbf{HB} - \mathbf{D}\|$, where $\|\mathbf{A}\|$ is the norm of vector **A**. The norm is the sum of squares of vector components, and the described pseudo-solution search is an application of the least squares method.

For $M = N$, we obtain a normal system. The SLAE solution is carried out by the numerical LU decomposition method (lower–upper decomposition by Tadeusz Banachiewicz, 1938) [28], which is based on the Gaussian sequential exclusion algorithm. In this case, the results are accurate up to numerical rounding errors.

An important advantage of the method lies in its speed because for the most time-consuming operations—calculating the elements of the matrices **H**—the use of the received signal (vector **D**) is not required. When solving inverse problems of image reconstruction with the same matrix **H**, but with a different **D**, it is enough to calculate the LU decomposition of **H**. The speed increases even more if there is a database of a specific measuring system in the form of LU decompositions of **H** for the most commonly used systems of $g_m(\alpha, \varphi)$ functions. The application of this approach makes it possible to ensure the operation of autonomous robotic systems and real-time remote control systems.

The achieved level of super-resolution is limited by the instability of SLAE solutions (14). With increasing matrix **H** dimensions, the problems become incorrect, since negligibly small rounding errors in the calculation process lead to unacceptable errors in solutions.

An additional increase in the level of the achieved super-resolution is possible when carrying out the regularization of problems based, for example, on more complete a priori information about the solution, which is often available in real problems [17,19].

For $M > N$, the system (14) is underdetermined. To obtain the only adequate solution to the problem, it is necessary to redefine it by adding some a priori data about the solution, i.e., about the unknown vector **B**.

Usually, the solution of an underdetermined SLAE (14) is considered to be a normal pseudo-solution, i.e., one solution that has a minimum norm $\|\mathbf{B}\|_{\min}$. This solution is found when minimizing the norm **B** on a previously obtained family of SLAE solutions.

Generalization of the developed approach (8)–(14) to two-dimensional problems does not cause fundamental difficulties.

5. Numerical Results

In a mathematical model, during numerical experiments, the results of image reconstruction with conventional flat antenna arrays by the algebraic method (3)–(7) and the DAA of the same dimension were compared using the presented modification of the algebraic method. The classical problem of the angular resolutions of two point objects located within the beam width of DAA 0.5, i.e., not resolved by direct observation, was investigated.

Figure 1 shows the results of the image reconstructions for two identical signal sources shown as a thin solid line (Chart 1). The angular distance between the sources was $0.9\,\theta_{0.5}$.

Step functions were chosen to represent solutions as $g_m(\alpha)$ functions. The legends in the result figures are given in the text. The dashed polylines (Chart 2) here and in the following figures show the solutions found by the usual algebraic methods (3)–(7). Two objects were resolved, and the localization of each object was $1/6$ of the beam width $\theta_{0.5}$.

A solid bold polyline (Chart 3) shows the solution obtained on the basis of a modification of the algebraic method (8)–(14). The achieved localization of two objects was $(1/30)\theta_{0.5}$ with almost accurate determination of their angular coordinates.

When trying to improve localization, i.e., when using more than M functions in (2), distortions appear in the solutions in the forms of false sources, the amplitude values and their numbers increase rapidly with the increase in the number of $g_m(\alpha)$ functions used.

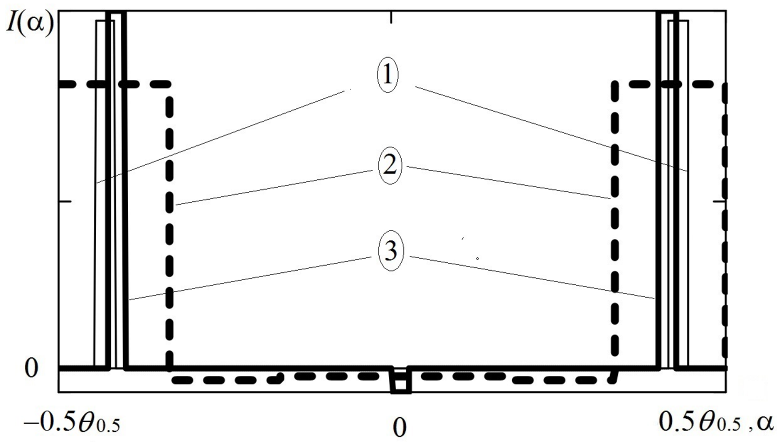

Figure 1. Image reconstruction of two point sources: 1-signal sources; 2-solution found by the usual algebraic method; 3-solution obtained on the basis of a modification of the algebraic method.

When the objects under study approach, the quality of the approximate solutions obtained begins to deteriorate. Figure 2 shows the found images $I(\alpha)$ at a distance of $0.2\,\theta_{0.5}$ in the same notation as in Figure 1. For illustration, an upper curve (Chart 4) is added showing the change in the amplitude of a received signal $U(\alpha)$ when scanning the sector. Both methods still allow one to resolve objects. The localization of objects has not changed, however, when using the algebraic method (3)–(7), a false source with an amplitude up to $1/3$ of the amplitude of the true source appeared.

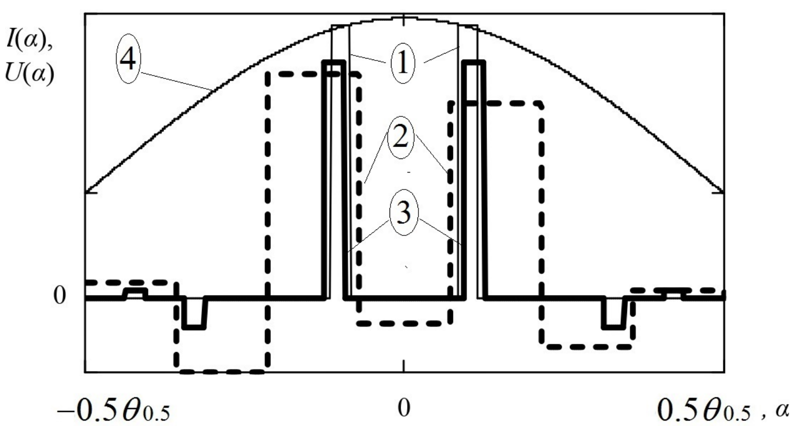

Figure 2. Image reconstruction of point sources at a distance of $0.2\,\theta_{0.5}$: 1-signal sources; 2-solution found by the usual algebraic method; 3-solution obtained on the basis of a modification of the algebraic method; 4-change in the amplitude of a received signal $U(\alpha)$ when scanning the sector.

With the further convergence of objects, more functions are required for an adequate representation of the solution, and, consequently, the stability of solutions decreases.

Figure 3 shows the solutions obtained with the highest possible resolution by the two methods compared with a very small angular distance between objects of $0.1\,\theta_{0.5}$ (a solid thin polyline, Chart 1). The algebraic method (3)–(7) did not allow resolving objects (dotted polyline, Chart 2).

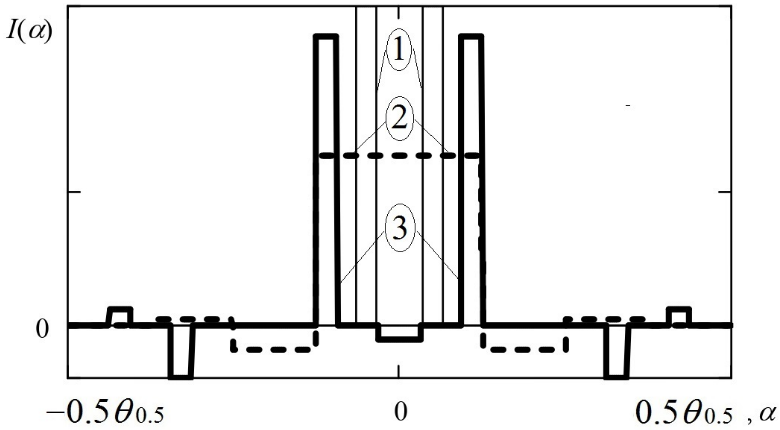

Figure 3. Image reconstruction of point sources at a distance of $0.1\,\theta_{0.5}$: 1-signal sources; 2-solution found by the usual algebraic method; 3-solution obtained on the basis of a modification of the algebraic method.

The modification of the method (8)–(14), which allows using an underdetermined system, made it possible to resolve objects based on a normal pseudo-solution, providing a ten-fold excess of the Rayleigh criterion (bold polyline, Chart 3). The resulting error in determining the angular position of the objects was $0.06\,\theta_{0.5}$.

Thus, DAA-based systems, due to a new signal processing method, can significantly increase the level of angular super-resolution compared to known methods with satisfactory image quality. The proposed modification makes it possible to successfully restore images of complex objects with super-resolution, including smoothly inhomogeneous regions and areas with large gradients of the amplitude distribution of the reflected signal.

Thus, Figure 4 shows the solutions with the maximally achieved resolution obtained by the algebraic method and its newly presented modification. The true distribution $I(\alpha)$ is presented in the form of a solid thin curve (Chart 1); in the form of a dotted curve (Chart 2)—the solution found by the algebraic method is found, in the form of a bold curve (Chart 3)—the solution for the DAA based on (10)–(14) is found. If the first solution as a whole correctly reflects the general averaged form of the distribution $I(\alpha)$, then the second allows one to highlight the details of the image, albeit with slight distortions.

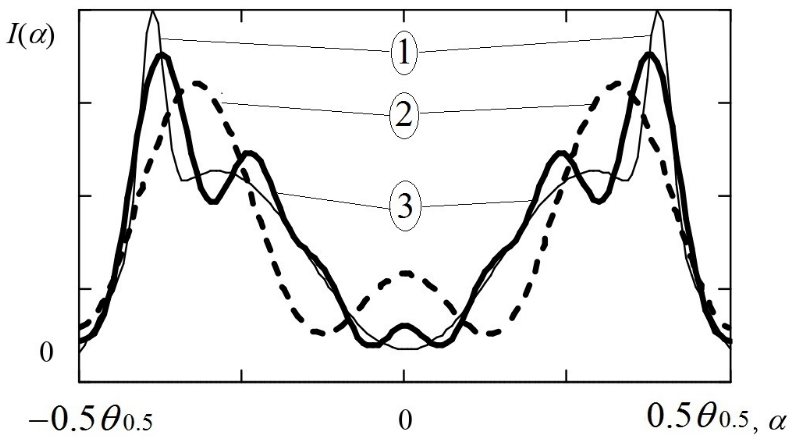

Figure 4. The original and reconstructed image of a complex object: 1-true distribution $I(\alpha)$; 2-solution found by the algebraic method; 3-solution for the DAA.

6. Conclusions

This article proposes a new method for the digital processing of measurement results, allowing technical systems based on digital antenna arrays (DAA) to increase the achieved level of angular super-resolution. The method is based on the intelligent analysis of signals received by each element of DAA and solving inverse problems described by the Fredholm integral equation of the first kind of convolution type with additional conditions. The results of numerical studies have shown that, compared with other varieties of algebraic methods, the proposed signal processing has greater noise immunity and allows obtaining solutions with high levels of angular super-resolutions, with good image recovery quality of complex objects. The performances of algorithms based on the presented method allow using them in real-time autonomous and remotely controlled robotic systems.

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Abbreviations

The following abbreviations are used in this manuscript:

DAA	digital antenna array(s)
RP	radiation pattern
SLAE	sets of linear algebraic equations
MUSIC	multiple signal classification
ESPRIT	estimation of signal parameters via rotational invariant techniques
MME	method of maximum entropy
LU	lower–upper

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