mathematics

Article

Risk-Sensitive Maximum Principle for Controlled System with Delay

Peng Wang

School of Control Science and Engineering, Shandong University, Jinan 250061, China; wang-peng@mail.sdu.edu.cn

Abstract: Risk-sensitive maximum principle and verification theorem for controlled system with delay is obtained by virtue of classical convex variational technique. The prime feature in the research is that risk-sensitive parameter ϑ seriously affects adjoint equation and variational inequality. Moreover, a verification theorem of optimality is derived under some concavity conditions. An example is given to illustrate our theoretical result.

Keywords: maximum principle; risk-sensitive optimal control; stochastic differential equation with delay

MSC: 49N90; 49K45; 93C43; 93E20

Citation: Wang, P. Risk-Sensitive Maximum Principle for Controlled System with Delay. *Mathematics* **2023**, *11*, 1058. https://doi.org/10.3390/math11041058

Academic Editors: Zhibin Liang and Jingtao Shi

Received: 10 January 2023
Revised: 10 February 2023
Accepted: 16 February 2023
Published: 20 February 2023

1. Introduction

Risk-sensitive stochastic control problem (RSCP) is an important kind of control problem, which is closely related to differential game of exponential linear quadratic Gaussian (LQG) problem [1–5], and can be widely used in asset management to describe investors' risk attitude through a risk-sensitive parameter [6]. At first, dynamic programming principle (DPP) is mainly a tool to study RSCP. Ref. [7] established a maximum principle (MP) of RSCP based on large deviation theory in 1990. Since then, many studies on RSCP have focused on MP. In 2005, a new kind of risk-sensitive (RS) MP was derived in [8] by means of relationship between DPP and MP and a logarithmic transformation. On this basis, a general MP of partially observable and partial information RSCP was obtained in [9–11]. Ref. [12] studied an RSCP where the state system consisted of a jump diffusion. Refs. [13,14] established an MP for a mean-field (MF) partially observed RSCP and an MP for an MF type Markov regime-switching jump diffusion systems, respectively. Ref. [15] studied an RSCP in which the cost functional is given by a controlled backward stochastic differential equation (BSDE). The generalized risk-sensitive DPP of valued function was obtained.

Time delay is a familiar phenomenon and is used to describe historically relevant behaviors and phenomena in medicine, networking, and congestion fields, see [16–20]. Therefore, many models in the fields of economics, finance, and engineering are described by delayed stochastic systems, for example [21–23]. With wide application of stochastic systems with time delay, its optimization has always been a hot topic and has received more and more attention. In general, delayed systems are tricky to deal with due to its infinite dimensional state space structure and lack of Itô's formula. An MP for optimal control problem with delay was obtained in [24]. Ref. [25] introduced a novel type BSDE named anticipated BSDE(ABSDE). Ref. [26] derived an MP for a stochastic control delay system using the duality between SDDE and ABSDE. Refs. [27–29] studied optimal control problems of different forms of delay systems. Ref. [30] studied an optimal control problem for RS MF SDDE with partial information and obtained a stochastic MP. Refs. [31–33] investigated the different types of stochastic differential game. In many papers and references therein, including [8,13,30], cost functional is type of exponential-integral RS. This paper considers a class of utility functional named hyperbolic absolute risk aversion (HARA).

This cost functional has practical implications, where it can formulate a vital risk-sensitive optimal portfolio model.

In this paper, we investigate a kind of RSCP with delay and HARA expected utility functional with the exponent $\vartheta > 0$. The key contributions of this work are as follows:

(1) Different from the systems investigated in [24,30], the state system considered in this paper is stochastic and delayed. Moreover, the cost functional is an HARA expected utility functional with exponent $\vartheta > 0$, which can be used to describe some specific financial phenomena. Thus, the results of this paper can be applied to solve more financial problems.

(2) The existence of delay and the complexity of the cost functional cause some difficulties to handle the problem. Thus, a duality method and an expanding variable dimension method are adopted to obtain an MP. This is different from the methods in [3,4,8,9,24].

(3) The adjoint equation and maximum condition are greatly affected by parameter ϑ. Note that if $\vartheta = 1$, we can obtain results similar to [26]. Thus, our results are more general than [26].

The rest of this paper is organized as follows. A formulation of RSCP with delay is given in Section 2. An MP and verification theorem of optimal control are derived in Section 3. Further, we consider an RSCP with a general running cost functional, and obtain MP and verification theorem for an optimal control in Section 4. Applied derived results, an RS management problem of pension fund deferred surplus is solved in Section 5. The last, we conclude this research with a concluding statement.

2. Problem Formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete filtered probability space with a natural filtration $\{\mathcal{F}_r\}_{0\le r\le T}$ generated by a one-dimensional standard Brownian motion $\{B(r)\}_{0\le r\le T}$. $T > 0$ is the finite fixed time node. $\varsigma > 0$ is the time delay quantity. R represents a one-dimensional Euclidean space. There are brief notations below for simplification: $C[-\varsigma, 0] = \{m(\cdot) : [-\varsigma, 0] \to R | m(\cdot)$ is a continuous function$\}$, $L^2(-\varsigma, 0; U) = \{m(\cdot) : [-\varsigma, 0] \to U | m(\cdot)$ is a deterministic continuous function satisfying $\int_{-\delta}^{0} |m(r)|^2 dr < +\infty\}$, $L^2_{\mathcal{F}}(0, T; R) = \{m(\cdot) : \Omega \times [0, T] \to R | m(\cdot)$ is an $\mathcal{F}_t$-adapted process satisfying $\mathbb{E}\int_0^T |m(r)|^2 dr < +\infty\}$.

Consider an SDDE system

$$\begin{cases} dx^{\nu}(r) = f(r, x^{\nu}(r), x^{\nu}(r-\varsigma), \nu(r), \nu(r-\varsigma))dr \\ \qquad\qquad + g(r, x^{\nu}(r), x^{\nu}(r-\varsigma), \nu(r), \nu(r-\varsigma))dB(r), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r \in [0, T], \\ x^{\nu}(r) = \zeta(r), \quad \nu(r) = \eta(r), \quad r \in [-\varsigma, 0], \end{cases} \tag{1}$$

where $f, g : [0, T] \times R^4 \to R$, $\eta(\cdot) \in L^2(-\varsigma, 0; U)$; $\zeta(\cdot) \in C[-\varsigma, 0]$ is $\mathcal{F}_0$-measurable and satisfies $\mathbb{E}\sup_{-\varsigma\le r\le 0} |\zeta(r)|^2 < +\infty$.

Define an admissible control set $\mathcal{U}_{ad} = \{\nu(r) | \nu(r) \in L^2_{\mathcal{F}}(0, T; R),\ \nu(r) \in U,\ a.s.,\ r \in [0, T]$, and $\nu(r) = \eta(r), r \in [-\varsigma, 0]\}$, where $U \subseteq R$ is a non-empty, convex set.

A cost functional is

$$J(\nu(\cdot)) = \frac{1}{\vartheta}\mathbb{E}[\Theta(x^{\nu}(T))]^{\vartheta}, \tag{2}$$

where $\vartheta > 0$ is a risk-sensitivity index.

Problem 1 (RSC). *Find a $\mu(\cdot) \in \mathcal{U}_{ad}$ achieving*

$$J(\mu(\cdot)) = \max_{\nu(\cdot)\in\mathcal{U}_{ad}} J(\nu(\cdot))$$

associated with (1). Any $\mu(\cdot)$ exists, then $\mu(\cdot)$ is called an optimal control. The corresponding state is denoted by $x^{\mu}(\cdot)$.

3. MP and Verification Theorem

We need assumptions below.

Hypothesis 1 (H1). *f, g are continuously differentiable in $(x^\nu, x^\nu_\delta, \nu, \nu_\delta)$, and their derivatives are bounded.*

Hypothesis 2 (H2). *$\Theta : R \to [0, +\infty)$ is continuously differential with regard to x and $|\Theta| \leq K(1 + |x|)$ and derivative $|\Theta_x| \leq M$, where M is a positive constant.*

Hypothesis 3 (H3). *Let $\mathbb{E}[\Theta(x(T))]^{(2\vartheta-2)} < +\infty$, when $0 < \vartheta < 1$, and let $\mathbb{E}[x(T)]^{(2\vartheta-2)} < +\infty$, when $\vartheta > 1$.*

3.1. MP

Let $(\mu(\cdot), x^\mu(\cdot))$ be an optimal solution of Problem (RSC), and $\nu_1(\cdot) \in L^2_{\mathcal{F}}(0, T; R)$ be such that $\mu(\cdot) + \nu_1(\cdot) \in \mathcal{U}_{ad}$. Since U is convex, $\mu^\varepsilon(\cdot) = \mu(\cdot) + \varepsilon\nu_1(\cdot)$, $0 \leq \varepsilon \leq 1$ is also in $\mathcal{U}_{ad}$. Its corresponding trajectory is denoted by $x^\varepsilon(\cdot)$.

Introduce these signs for ease of notation.

$$\begin{aligned}
b^\varepsilon(r) &= b(r, x^\varepsilon(r), x^\varepsilon(r-\varsigma), \mu^\varepsilon(r), \mu^\varepsilon(r-\varsigma)),\\
b^\mu(r) &= b(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma)),\\
b^\mu_\phi(r) &= b_\phi(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma)),
\end{aligned}$$

where $b = f, g$ and $\phi = x^\nu, x^\nu_\varsigma, \nu, \nu_\varsigma$.

The equation of variation is

$$\begin{cases}
dx^\mu_1(r) = [f^\mu_x x^\mu_1(r) + f^\mu_{x_\varsigma}(r) x^\mu_1(r-\varsigma) + f^\mu_\nu(r)\nu_1(r) + f^\mu_{\nu_\varsigma}(r)\nu_1(r-\varsigma)]dr \\
\qquad + [g^\mu_x(r) x^\mu_1(r) + g^\mu_{x_\varsigma}(r) x^\mu_1(r-\varsigma) + g^\mu_\nu(r)\nu_1(r) + g^\mu_{\nu_\varsigma}(r)\nu_1(r-\varsigma)]dB(r), \\
\qquad\qquad r \in [0, T], \\
x^\mu_1(r) = 0,\ \nu_1(r) = 0, \quad r \in [-\varsigma, 0].
\end{cases} \tag{3}$$

By similar means in [26], we can obtain following result.

Lemma 1. *Suppose that (H1) is tenable, there is*

$$\lim_{\varepsilon \to 0} \sup_{0 \leq r \leq T} \mathbb{E}|\widetilde{x}(r)|^2 = 0, \tag{4}$$

where $\widetilde{x}(r) = \frac{x^\varepsilon(r) - x^\mu(r)}{\varepsilon} - x^\mu_1(r)$.

Similar to [34], we can obtain Lemma 2 by using Lemma 1 and Taylor's expansion.

Lemma 2. *Suppose that (H1)–(H3) is tenable, there is*

$$\mathbb{E}\{[\Theta(x^\mu(T))]^{\vartheta-1}\Theta^\mu_x(x(T))x^\mu_1(T)\} \leq 0, \tag{5}$$

where $x^\mu_1(T)$ satisfies (3).

A Hamiltonian function is defined as follows

$$H(r, x^\nu, x^\nu_\varsigma, \nu, \nu_\varsigma, m, n) = mf(r, x^\nu, x^\nu_\varsigma, \nu, \nu_\varsigma) + ng(r, x^\nu, x^\nu_\varsigma, \nu, \nu_\varsigma),$$

and use the notation $H^\mu_\phi = H_\phi(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), m(r), n(r))$, $\phi = \nu, \nu_\delta$.

Introduce an adjoint equation

$$\begin{cases} -dm(r) = [f_x^\mu(r)m(r) + g_x^\mu(r)n(r) + \mathbb{E}^{\mathcal{F}_r}[f_{x_\varsigma}^\mu(r+\varsigma)m(r+\varsigma) \\ \qquad\qquad + g_{x_\varsigma}^\mu(r+\varsigma)n(r+\varsigma)]dr - n(r)dB(r), \quad r \in [0,T], \\ \quad m(T) = [\Theta(x^\mu(T))]^{\vartheta-1}\Theta_x(x^\mu(T)),\ m(r) = 0,\ n(r) = 0, \quad r \in [T, T+\varsigma]. \end{cases} \tag{6}$$

Obviously, (6) admits a unique solution (see [25]).
The following is obtained by making used of Itô's formula

$$\begin{aligned} &\mathbb{E}\langle m(T), x_1^\mu(T)\rangle \\ =\ &\mathbb{E}\int_0^T \{-\mathbb{E}^{\mathcal{F}_t}[f_{x_\varsigma}^\mu(r+\varsigma)m(r+\varsigma)]x_1^\mu(r) - \mathbb{E}^{\mathcal{F}_r}[g_{x_\varsigma}^\mu(r+\varsigma)n(r+\varsigma)]x_1^\mu(r) \\ &+ m(r)f_{x_\varsigma}^\mu(r)x_1^\mu(r-\varsigma) + m(r)f_v^\mu(r)v_1(r) + m(r)f_{v_\varsigma}^\mu(r)v_1(r-\varsigma) + n(r)g_{x_\varsigma}^\mu(r)x_1^\mu(r-\varsigma) \\ &+ n(r)g_v^\mu(r)v_1(r) + n(r)g_{v_\varsigma}^\mu(r)v_1(r-\varsigma)\}dr. \end{aligned}$$

Combining special conditions, it yields

$$\begin{aligned} &\mathbb{E}\int_0^T \{m(r)f_{x_\varsigma}^\mu(r)x_1^\mu(r-\varsigma) - \mathbb{E}^{\mathcal{F}_r}[(f_{x_\varsigma}^\mu|_{r+\varsigma})m(r+\varsigma)]x_1^\mu(r)\}dr \\ =\ &\mathbb{E}\int_0^T m(r)f_{x_\varsigma}^\mu(r)x_1^\mu(r-\varsigma)dr - \mathbb{E}\int_\varsigma^{T+\varsigma} f_{x_\varsigma}^\mu(r)m(r)x_1^\mu(r-\varsigma)dr \\ =\ &\mathbb{E}\int_0^\varsigma m(r)f_{x_\varsigma}^\mu(r)x_1^\mu(r-\varsigma)dr - \mathbb{E}\int_T^{T+\varsigma} f_{x_\varsigma}^\mu(r)m(r)x_1^\mu(r-\varsigma)dr \\ =\ &0. \end{aligned}$$

Similarly, we derive

$$\mathbb{E}\int_0^T \{n(r)g_{x_\varsigma}^\mu(r)x_1^\mu(r-\varsigma) - \mathbb{E}^{\mathcal{F}_r}[(g_{x_\varsigma}^\mu|_{r+\varsigma})n(r+\varsigma)]x_1^\mu(r)\}dr = 0.$$

Using (5), it can be deduced

$$\mathbb{E}\int_0^T \{\langle H_v^\mu, v_1(r)\rangle + \langle H_{v_\varsigma}^\mu, v_1(r-\varsigma)\rangle\}dr \le 0,$$

and thus,

$$\langle H_v^\mu + \mathbb{E}^{\mathcal{F}_r}[H_{v_\varsigma}^\mu|_{r+\varsigma}], v - \mu(r)\rangle \le 0,\ \forall v \in U,\ a.e.,\ a.s.. \tag{7}$$

Then, we draw the desired conclusion.

Theorem 1 (MP: I). *Suppose that (H1)–(H3) is tenable. Let $(\mu(\cdot), x^\mu(\cdot))$ be an optimal solution of Problem (RSC), then, we assert (7).*

3.2. Verification Theorem

Next up, we will construct a verification theorem for optimality.
Introduce an additional hypothesis.

Hypothesis 4 (H4). $0 < \vartheta < 1$, *$H(r, \cdot, \cdot, \cdot, \cdot, m(r), n(r))$ is concave in $(x^v, x_\varsigma^v, v, v_\varsigma)$ and $\Theta(\cdot)$ is concave in x.*

Theorem 2 (Verification Theorem: I). *Suppose $\mu(\cdot) \in \mathcal{U}_{ad}$ and let $x^\mu(\cdot)$ be the corresponding trajectory. Suppose that $(m(\cdot), n(\cdot))$ is a solution to (6). If hypotheses (H1)–(H4) and maximum condition (7) hold for $\mu(\cdot)$, then $\mu(\cdot)$ is an optimal control for Problem (RSC).*

Proof. For any $\nu(\cdot) \in \mathcal{U}_{ad}$, denote by $x^{\nu}(\cdot)$ its matching state process. Calculate

$$\begin{aligned}
&J(\nu(\cdot)) - J(\mu(\cdot)) \\
&= \frac{1}{\vartheta}\mathbb{E}\{[\Theta(x^{\nu}(T))]^{\vartheta} - [\Theta(x^{\mu}(T))]^{\vartheta}\} \\
&\leq \frac{1}{\vartheta}\mathbb{E}\{\vartheta[\Theta(x^{\mu}(T))]^{\vartheta-1}\Theta_x(x^{\mu}(T))(x^{\nu}(T) - x^{\mu}(T))\} \\
&= \mathbb{E}\{(\Theta(x^{\mu}(T)))^{\vartheta-1}\Theta_x(x^{\mu}(T))(x^{\nu}(T) - x^{\mu}(T))\}.
\end{aligned}$$

It can be obtained with the help of Itô's formula as follows

$$\begin{aligned}
&\mathbb{E}\{(\Theta(x^{\mu}(T)))^{\vartheta-1}\Theta_x(x^{\mu}(T))(x^{\nu}(T) - x^{\mu}(T))\} \\
&= \mathbb{E}\int_0^T \{\langle m(r), f^{\nu}(r) - f^{\mu}(r)\rangle + \langle -f_x^{\mu}(r)m(r), x^{\nu}(r) - x^{\mu}(r)\rangle + \langle -g_x^{\mu}(r)n(r), x^{\nu}(r) - x^{\mu}(r)\rangle \\
&\quad - \langle \mathbb{E}^{\mathcal{F}_r}(f_x^{\mu}|_{r+\varsigma})m(r+\varsigma), x^{\nu}(r) - x^{\mu}(r)\rangle - \langle \mathbb{E}^{\mathcal{F}_r}(g_x^{\mu}|_{r+\varsigma})n(r+\varsigma), x^{\nu}(r) - x^{\mu}(r)\rangle \\
&\quad + \langle n(r), g^{\nu}(r) - g^{\mu}(r)\rangle\}dr \\
&= \mathbb{E}\int_0^T \{H(r, x^{\nu}(r), x^{\nu}(r-\varsigma), \nu(r), \nu(r-\varsigma), m(r), n(r)) - H(r, x^{\mu}(r), x^{\mu}(r-\varsigma), \mu(r), \\
&\quad \mu(r-\varsigma), m(r), n(r)) - \langle f_x^{\mu}(r)m(r), x^{\nu}(r) - x^{\mu}(r)\rangle - \langle g_x^{\mu}(r)n(r), x^{\nu}(r) - x^{\mu}(r)\rangle \\
&\quad - \langle \mathbb{E}^{\mathcal{F}_r}[(f_{x_\varsigma}^{\mu}|_{r+\varsigma})m(r+\varsigma)], x^{\nu}(r) - x^{\mu}(r)\rangle - \langle \mathbb{E}^{\mathcal{F}_r}[(g_{x_\varsigma}^{\mu}|_{r+\varsigma})n(r+\varsigma)], x^{\nu}(r) - x^{\mu}(r)\rangle\}dr \\
&\leq \mathbb{E}\int_0^T \{\langle H_x(r, x^{\mu}(r), x^{\mu}(r-\varsigma), \mu(r), \mu(r-\varsigma), m(r), n(r)), x^{\nu}(r) - x^{\mu}(r)\rangle \\
&\quad + \langle H_{x_\varsigma}(r, x^{\mu}(r), x^{\mu}(r-\varsigma), \mu(r), \mu(r-\varsigma), m(r), n(r)), x^{\nu}(r-\varsigma) - x^{\mu}(r-\varsigma)\rangle \\
&\quad + \langle H_{\nu}^{\mu}, \nu(r) - \mu(r)\rangle + \langle H_{\nu_\varsigma}^{\mu}, \nu(r-\varsigma) - \mu(r-\varsigma)\rangle - \langle f_x^{\mu}(r)m(r), x^{\nu}(r) - x^{\mu}(r)\rangle \\
&\quad - \langle g_x^{\mu}(r)n(r), x^{\nu}(r) - x^{\mu}(r)\rangle - \langle \mathbb{E}^{\mathcal{F}_r}[(f_{x_\varsigma}^{\mu}|_{r+\varsigma})m(r+\varsigma)], x^{\nu}(r) - x^{\mu}(r)\rangle \\
&\quad - \langle \mathbb{E}^{\mathcal{F}_r}[(g_{x_\varsigma}^{\mu}|_{r+\varsigma})n(r+\varsigma)], x^{\nu}(r) - x^{\mu}(r)\rangle\}dr \\
&\leq \mathbb{E}\int_0^T \{\langle H_x(r, x^{\mu}(r), x^{\mu}(r-\varsigma), \mu(r), \mu(r-\varsigma), m(r), n(r)), x^{\nu}(r) - x^{\mu}(r)\rangle \\
&\quad + \langle H_{x_\varsigma}(r, x^{\mu}(r), x^{\mu}(r-\varsigma), \mu(r), \mu(r-\varsigma), m(r), n(r)), x^{\nu}(r-\varsigma) - x^{\mu}(r-\varsigma)\rangle \\
&\quad - \langle f_x^{\mu}(r)m(r), x^{\nu}(r) - x^{\mu}(r)\rangle - \langle g_x^{\mu}(r)n(r), x^{\nu}(r) - x^{\mu}(r) \\
&\quad - \langle \mathbb{E}^{\mathcal{F}_r}[(f_{x_\varsigma}^{\mu}|_{r+\varsigma})m(r+\varsigma)], x^{\nu}(r) - x^{\mu}(r)\rangle - \langle \mathbb{E}^{\mathcal{F}_r}[(g_{x_\varsigma}^{\mu}|_{r+\varsigma})n(r+\varsigma)], x^{\nu}(r) - x^{\mu}(r)\rangle\}dr \\
&= 0.
\end{aligned}$$

Then we complete the proof. □

4. A General RSCP

A general RS cost functional is considered

$$J(\nu(\cdot)) = \frac{1}{\vartheta}\mathbb{E}[\int_0^T l(r, \nu(r), \nu(r-\varsigma))dr + \Theta(x^{\nu}(T))]^{\vartheta}, \quad \vartheta > 0. \tag{8}$$

Problem 2 (G-RSC). *The objective is to seek a $\mu(\cdot) \in \mathcal{U}_{ad}$ achieving*

$$J(\mu(\cdot)) = \max_{\nu(\cdot)\in\mathcal{U}_{ad}} J(\nu(\cdot)), \tag{9}$$

associated with (1).

We introduce the following assumptions.

Hypothesis 5 (H5). *$l : [0, T] \times U^2 \to R_+$ is continuously differentiable in ν, ν_ς, and the derivatives are bounded.*

Hypothesis 6 (H6). *If* $0 < \vartheta < 1$, *we assume*

$$\mathbb{E}[\int_0^T l(r, \nu(r), \nu(r-\varsigma))dr + \Theta(x^\nu(T))]^{2\vartheta-2} < +\infty;$$

if $\vartheta > 1$, *then*

$$\mathbb{E}[\int_0^T l(r, \nu(r), \nu(r-\varsigma))dr]^{2\vartheta-2} < +\infty.$$

4.1. MP

Let $(\mu(\cdot), x^\mu(\cdot))$ be an optimal solution to Problem (G-RSC). In order to obtain the desired results, we define an SDDE

$$\begin{cases} dy^\nu(r) = l(r, \nu(r), \nu(r-\varsigma))dr, \quad r \in [0, T], \\ y^\nu(r) = 0,\ \nu(r) = \eta(r), \quad r \in [-\varsigma, 0]. \end{cases} \tag{10}$$

Let $y^\varepsilon(\cdot)$ and $y^\mu(\cdot)$, respectively, correspond to $\mu^\varepsilon(\cdot)$ and $\mu(\cdot)$ through (10).
A variational equation is

$$\begin{cases} dy_1^\mu(r) = [l_\nu^\mu(r)\nu_1(r) + l_{\nu_\varsigma}^\mu(r)\nu_1(r-\varsigma)]dr, \quad r \in [0, T], \\ y_1^\mu(r) = 0,\ \nu_1(r) = 0, \quad r \in [-\varsigma, 0], \end{cases}$$

where $l_\phi^\mu(r) = l_\phi(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma)), \phi = \nu, \nu_\varsigma$. We can derive that

$$\lim_{\varepsilon\to 0} \sup_{0\le r\le T} \mathbb{E}|\widetilde{y}(r)|^2 = 0,$$

where $\widetilde{y}(r) = \frac{y^\varepsilon(r) - y^\mu(r)}{\varepsilon} - y_1^\mu(r)$.
Thus, Problem (G-RSC) is simplified as maximizing

$$J(\nu(\cdot)) = \frac{1}{\vartheta}\mathbb{E}[y^\nu(T) + \Theta(x^\nu(T))]^\vartheta, \quad \vartheta > 0 \tag{11}$$

associated with (1) and (10).
We can derive that from $J(\mu^\varepsilon(\cdot)) - J(\mu(\cdot)) \le 0$

$$\begin{aligned} &\frac{1}{\vartheta}\mathbb{E}[y^\varepsilon(T) + \Theta(x^\varepsilon(T))]^\vartheta - \frac{1}{\vartheta}\mathbb{E}[y^\mu(T) + \Theta(x^\mu(T))]^\vartheta \\ &= \frac{1}{\vartheta}\mathbb{E}[\vartheta(y^\mu(T) + \Theta(x^\mu(T)))^{\vartheta-1}\Theta_x(x^\mu(T))(x^\varepsilon(T) - x^\mu(T)) + \vartheta(y^\mu(T)\Theta(x^\mu(T)))^{\vartheta-1} \\ &\quad (y^\varepsilon(T) - y^\mu(T)) + o(\varepsilon)] \\ &= \varepsilon\mathbb{E}[(y^\mu(T) + \Theta(x^\mu(T)))^{\vartheta-1}\Theta_x(x^\mu(T))(x_1^\mu(T)] + \varepsilon\mathbb{E}[(y^\mu(T) + \Theta(x^\mu(T)))^{\vartheta-1}y_1^\mu(T)] + o(\varepsilon) \\ &\le 0. \end{aligned}$$

Divide both sides of the above inequality by ε, then take the limit $\varepsilon \to 0$, such that

$$\mathbb{E}\{[y^\mu(T) + \Theta(x^\mu(T))]^{\vartheta-1}\Theta_x(x^\mu(T))x_1^\mu(T)\} + \mathbb{E}\{[y^\mu(T) + \Theta(x^\mu(T))]^{\vartheta-1}y_1^\mu(T)\} \le 0. \tag{12}$$

Introduce the ABSDEs

$$\begin{cases} d\alpha(r) = \beta(r)dB(r), \quad r \in [0, T], \\ \alpha(T) = [y^\mu(T) + \Theta(x^\mu(T))]^{\vartheta-1},\ \alpha(r) = 0,\ \beta(r) = 0, \quad r \in [T, T+\varsigma], \end{cases} \tag{13}$$

and

$$\begin{cases} -d\psi(r) = [f_x^\mu(r)\psi(r) + g_x^\mu(r)\varphi(r) + \mathbb{E}^{\mathcal{F}_r}[(f_{x_\varsigma}^\mu|_{r+\varsigma})\psi(r+\varsigma) \\ \qquad\qquad + (g_{x_\varsigma}^\mu|_{r+\varsigma})\varphi(r+\varsigma)]dr - \varphi(r)dB(r), \quad r \in [0,T], \\ \quad \psi(T) = [y^\mu(T) + \Theta(x^\mu(T))]^{\vartheta-1}\Theta_x(x^\mu(T)), \quad \psi(r) = 0, \quad \varphi(r) = 0, \quad r \in [T, T+\varsigma]. \end{cases} \tag{14}$$

Obviously, we can obtain in virtue of Itô's formula

$$\begin{aligned} &\mathbb{E}\{[y^\mu(T) + \Theta(x^\mu(T))]^{\vartheta-1} y_1^\mu(T)\} \\ &= \mathbb{E}\int_0^T \{\alpha(r) l_{\nu_\varsigma}^\mu(r)\nu_1(r-\varsigma) + \alpha(r) l_\nu^\mu(r)\nu_1(r)\}dr, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\mathbb{E}\{[y^\mu(T) + \Theta(x^\mu(T))]^{\vartheta-1}\Theta_x(x^\mu(T)) x_1^\mu(T)\} \\ &= \mathbb{E}\int_0^T \{\psi(r) f_\nu^\mu(r)\nu_1(r) + \psi(r) f_{\nu_\varsigma}^\mu(r)\nu_1(r-\varsigma) + \varphi(r) g_\nu^\mu(r)\nu_1(r) + \varphi(r) g_{\nu_\varsigma}^\mu(r)\nu_1(r-\varsigma)\}dr. \end{aligned} \tag{16}$$

Combining (12), (15), and (16), we obtain

$$\begin{aligned} &\mathbb{E}\int_0^T \{\alpha(r) l_{\nu_\varsigma}^\mu(r)\nu_1(r-\varsigma) + \alpha(r) l_\nu^\mu(r)\nu_1(r) + \psi(r) f_\nu^\mu(r)\nu_1(r) + \psi(r) f_{\nu_\varsigma}^\mu(r)\nu_1(r-\varsigma) \\ &+ \varphi(r) g_\nu^\mu(r)\nu_1(r) + \varphi(r) g_{\nu_\varsigma}^\mu(r)\nu_1(r-\varsigma)\}dr \le 0. \end{aligned} \tag{17}$$

A Hamiltonian function is introduced as follows

$$\mathcal{H}(r, x^\nu, x_\varsigma^\nu, \nu, \nu_\varsigma, \psi, \varphi, \alpha) = H(r, x^\nu, x_\varsigma^\nu, \nu, \nu_\varsigma, \psi, \varphi) + \langle \alpha, l(r, \nu, \nu_\varsigma)\rangle. \tag{18}$$

For convenience, here are the following abbreviation

$$\begin{aligned} \mathcal{H}(r) &= \mathcal{H}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r)), \\ \mathcal{H}(r, \nu, \mu(r-\varsigma)) &= \mathcal{H}(r, x^\mu(r), x^\mu(r-\varsigma), \nu, \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r)), \\ \mathcal{H}(r, \mu(r), \nu) &= \mathcal{H}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \nu, \psi(r), \varphi(r), \alpha(r)). \end{aligned}$$

It follows from (17), (18), and Theorem 1 that

$$\langle \mathcal{H}_\nu^\mu + \mathbb{E}^{\mathcal{F}_r}[\mathcal{H}_{\nu_\varsigma}^\mu|_{r+\varsigma}], \nu - \mu(r)\rangle \le 0, \ \forall \nu \in U, \ a.e., \ a.s.. \tag{19}$$

Through above analysis, the following natural result is obtained.

Theorem 3 (MP: II). *Under hypotheses (H1), (H2), (H5), and (H6), if $(\mu(\cdot), x^\mu(\cdot))$ is an optimal solution to Problem (G-RSC), then, (19) holds.*

4.2. Verification Theorem

We need the assumption below.

Hypothesis 7 (H7). *l is differentiable in (ν, ν_ς), and $l(\cdot, \nu(\cdot), \nu(\cdot - \varsigma)) \in L^1_{\mathcal{F}}(0, T; R)$.*

Hypothesis 8 (H8). *$\mathcal{H}_v(t, x(t), x(t-\delta), v, u(t-\delta), \psi(t), \varphi(t), \alpha(t))$ and $\mathcal{H}_{v_\delta}(t, x(t), x(t-\delta), u(t), v, \psi(t), \varphi(t), \alpha(t))$ are continuous at $v = u(t)$ for any $t \in [0,T]$, and for all $(t, x, x_\delta, v, v_\delta) \in [0,T] \times R \times R \times U \times U$,*

$$(x, x_\delta, v, v_\delta) \to \mathcal{H}(t, x, x_\delta, v, v_\delta, \psi(t), \varphi(t), \alpha(t))$$

is concave, $x \to \Theta(x)$ is concave, and $0 < \vartheta < 1$.

Theorem 4 (Verification Theorem: II). *Let $u(\cdot) \in \mathcal{U}_{ad}$ be given such that $l_v^u(\cdot), l_{v_\delta}^u(\cdot) \in L^2_{\mathcal{F}}(0, T; R)$, $l_v(t, v, u(t-\delta)) \in L^1(\Omega, \mathcal{F}, P)$ and $l_v(t, u(t), v) \in L^1(\Omega, \mathcal{F}, P)$ hold. Let $(\alpha(\cdot),$*

$\beta(\cdot)$) and $(\psi(\cdot), \varphi(\cdot))$ be the solutions of adjoint equations (13) and (14). If (H1), (H2), (H6), (H7), and (H8) hold, and

$$\mathcal{H}(t) + \mathbb{E}^{\mathcal{F}_t}[\mathcal{H}(t)|_{t+\delta}] = \max_{v\in U}\{\mathcal{H}(t, v, u(t-\delta)) + \mathbb{E}^{\mathcal{F}_t}[\mathcal{H}(t, u(t), v)|_{t+\delta}]\}$$

hold for all $t \in [0, T]$. Then, $u(\cdot)$ is an optimal control for Problem (G-RSC).

Proof. For any $v(\cdot) \in \mathcal{U}_{ad}$, we consider

$$\begin{aligned}
&J(v(\cdot)) - J(u(\cdot))\\
&= \frac{1}{\vartheta}\mathbb{E}\{[y^v(T) + \Theta(x^v(T))]^\vartheta - [y(T) + \Theta(x(T))]^\vartheta\}\\
&\le \frac{1}{\vartheta}\mathbb{E}\{\vartheta[y(T) + \Theta(x(T))]^{\vartheta-1}\Theta_x(x(T))(x^v(T) - x(T))\\
&\quad + \vartheta[y(T) + \Theta(x(T))]^{\vartheta-1}(y^v(T) - y(T))\}\\
&= \mathbb{E}\{[y(T) + \Theta(x(T))]^{\vartheta-1}\Theta_x(x(T))(x^v(T) - x(T))\\
&\quad + [y(T) + \Theta(x(T))]^{\vartheta-1}(y^v(T) - y(T))\}\\
&= \mathbb{E}\{\psi(T)(x^v(T) - x(T)) + \alpha(T)(y^v(T) - y(T))\}.
\end{aligned} \tag{20}$$

Using Itô's formula, it yields

$$\begin{aligned}
&\mathbb{E}\{\psi(T)(x^\nu(T) - x^\mu(T))\}\\
&= -\mathbb{E}\int_0^T \langle f_x^\mu(r)\psi(r), x^\nu(r) - x^\mu(r)\rangle dr - \mathbb{E}\int_0^T \langle g_x^\mu(r)\varphi(r), x^\nu(r) - x^\mu(r)\rangle dr\\
&\quad - \mathbb{E}\int_0^T \langle \mathbb{E}^{\mathcal{F}_r}[(f_{x_\varsigma}^\mu|_{r+\varsigma})\psi(r+\varsigma), x^\nu(r) - x^\mu(r)\rangle dr\\
&\quad - \mathbb{E}\int_0^T \langle \mathbb{E}^{\mathcal{F}_r}[(g_{x_\varsigma}^\mu|_{r+\varsigma})\varphi(r+\varsigma)], x^\nu(r) - x^\mu(r)\rangle dr\\
&\quad + \mathbb{E}\int_0^T \langle \psi(r), f(r, x^\nu(r), x^\nu(r-\varsigma), \nu(r), \nu(r-\varsigma)) - f(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma))\rangle dr\\
&\quad + \mathbb{E}\int_0^T \langle \varphi(r), g(r, x^\nu(r), x^\nu(r-\varsigma), \nu(r), \nu(r-\varsigma)) - g(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma))\rangle dr
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\mathbb{E}\{\alpha(T)(y^\nu(T) - y^\mu(T))\}\\
&= \mathbb{E}\int_0^T \langle \alpha(r), l(r, \nu(r), \nu(r-\varsigma)) - l(r, \mu(r), \mu(r-\varsigma))\rangle dr.
\end{aligned} \tag{22}$$

Combining (20)–(22), we obtain

$$\begin{aligned}
&J(\nu(\cdot)) - J(\mu(\cdot))\\
&\le \mathbb{E}\int_0^T [\mathcal{H}(r, x^\nu(r), x^\nu(r-\varsigma), \nu(r), \nu(r-\varsigma), \psi(r), \varphi(r), \alpha(r))\\
&\quad - \mathcal{H}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r))]dr\\
&\quad - \mathbb{E}\int_0^T \langle f_x^\mu(r)\psi(r), x^\nu(r) - x^\mu(r)\rangle dr - \mathbb{E}\int_0^T \langle g_x^\mu(r)\varphi(r), x^\nu(r) - x^\mu(r)\rangle dr\\
&\quad - \mathbb{E}\int_0^T \langle \mathbb{E}^{\mathcal{F}_r}[(f_{x_\varsigma}^\mu|_{r+\varsigma})\psi(r+\varsigma), x^\nu(r) - x^\mu(r)\rangle dr\\
&\quad - \mathbb{E}\int_0^T \langle \mathbb{E}^{\mathcal{F}_r}[(g_{x_\varsigma}^\mu|_{r+\varsigma})\varphi(r+\varsigma)], x^\nu(r) - x^\mu(r)\rangle dr.
\end{aligned}$$

Noticing that $(x^\nu, x^\nu_\varsigma, \nu, \nu_\varsigma) \to \mathcal{H}(r, x^\nu, x^\nu_\varsigma, \nu, \nu_\varsigma, \psi(r), \varphi(r), \alpha(r))$ is concave, so there is

$$\begin{aligned}
&J(\nu(\cdot)) - J(\mu(\cdot)) \\
\leq\ & \mathbb{E}\int_0^T \langle \mathcal{H}_\nu(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r)), \nu(r) - \mu(r)\rangle dr \\
&+ \mathbb{E}\int_0^T \langle \mathcal{H}_{\nu_\varsigma}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r)), \nu(r-\varsigma) - \mu(r-\varsigma)\rangle dr \\
=\ & \mathbb{E}\int_0^T \langle \mathcal{H}_\nu(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r)), \nu(r) - \mu(r)\rangle dr \\
&+ \mathbb{E}\int_0^T \langle \mathbb{E}^{\mathcal{F}_r}[\mathcal{H}_{\nu_\varsigma}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r))|_{r+\varsigma}], \nu(r) - \mu(r)\rangle dr \\
=\ & \mathbb{E}\int_0^T \langle \mathcal{H}_\nu(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r)) \\
&+ \mathbb{E}^{\mathcal{F}_r}[\mathcal{H}_{\nu_\varsigma}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r))|_{r+\varsigma}], \nu(r) - \mu(r)\rangle dr.
\end{aligned}$$

Recalling that, for each $r \in [0,T], \nu \to \mathcal{H}(r, \nu, \mu(r-\varsigma)) + \mathbb{E}^{\mathcal{F}_r}[\mathcal{H}(r, \mu(r), \nu)|_{r+\varsigma}]$ is maximal at $\nu = \mu(r)$, and $\mathcal{H}_\nu(r, \nu, \mu(r-\varsigma))$ and $\mathbb{E}^{\mathcal{F}_r}[\mathcal{H}(r, \nu, \mu(r-\varsigma))|_{r+\varsigma}]$ are continuous in v for all $\omega \in \Omega$ uniformly, then we obtain

$$\begin{aligned}
&\{\mathcal{H}_\nu(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r)) \\
&\quad + \mathbb{E}^{\mathcal{F}_r}[\mathcal{H}_{\nu_\varsigma}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r))|_{r+\varsigma}]\} \times (\nu(r) - \mu(r)) \\
=&\{\frac{\partial}{\partial \nu}[\mathcal{H}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r)) \\
&\quad + \mathbb{E}^{\mathcal{F}_r}[\mathcal{H}(r, x^\mu(r), x^\mu(r-\varsigma), \mu(r), \mu(r-\varsigma), \psi(r), \varphi(r), \alpha(r))|_{r+\varsigma}]\} \times (\nu(r) - \mu(r)) \leq 0.
\end{aligned}$$

Then it hints that

$$J(\mu(\cdot)) = \max_{\nu(\cdot)\in\mathcal{U}_{ad}} J(\nu(\cdot)).$$

So the conclusion is confirmed. □

Obviously, the hypotheses in Theorems 3 and 4 are strict. When $\vartheta = 1$, Problem (G-RSC) degenerates to a usual risk-neutral optimal control problem, where we denote this problem by **Problem (G-RNC)**. In this case, the hypotheses on Θ and l can be simplified by

Hypothesis 9 (H9). *$l : [0,T] \times U^2 \to R$ and $\Theta : R \to R$ are, respectively, continuously differentiable in $(\nu, \nu_\varsigma, x^\nu)$, and the derivatives are bounded.*

Define a Hamiltonian function

$$\begin{aligned}
&\overline{\mathcal{H}}(r, x^\nu(r), x^\nu(r-\varsigma), \nu(r), \nu(r-\varsigma), \xi(r), \eta(r)) \\
&= H(r, x^\nu(r), x^\nu(r-\varsigma), \nu(r), \nu(r-\varsigma), \xi(r), \eta(r)) + l(r, \nu(r), \nu(r-\varsigma)),
\end{aligned}$$

where $(\xi(\cdot), \eta(\cdot))$ satisfies

$$\begin{cases}
-d\xi(r) = [f^\mu_x(r)\xi(r) + g^\mu_x(r)\eta(r) + \mathbb{E}^{\mathcal{F}_r}[(f^\mu_{x_\varsigma}|_{r+\varsigma})\xi(r+\varsigma) \\
\qquad\qquad + (g^\mu_{x_\varsigma}|_{r+\varsigma})\eta(r+\varsigma)]dr - \eta(r)dB(r), \quad r \in [0,T], \\
\xi(T) = \Theta_x(x^\mu(T)), \quad \xi(r) = 0, \quad \eta(r) = 0, \quad r \in [T, T+\varsigma].
\end{cases} \tag{23}$$

Assume (H1), (H8) hold and $\vartheta = 1$. We can obtain Theorem 5 by virtue of techniques in Theorem 1.

Theorem 5 (Risk-Neutral MP). *Presume that $\mu(\cdot)$ is an optimal control to Problem (G-RNC), and $x^\mu(\cdot)$ be the corresponding trajectory. Then, the maximum principle*

$$\langle \overline{\mathcal{H}}_{\nu}^{\mu} + \mathbb{E}^{\mathcal{F}_r}[\overline{\mathcal{H}}_{\nu_{\varsigma}}^{\mu}|_{r+\varsigma}], \nu - \mu(r)\rangle \leq 0,\ \forall \nu \in U,\ a.e.,\ a.s. \tag{24}$$

is supported.

Similarly, maximum condition (24) added to some concavity hypothesis is also a sufficient condition.

Introduce an additional hypothesis.

Hypothesis 10 (H10). *$\overline{\mathcal{H}}(r,\cdot,\cdot,\cdot,\cdot,\xi(r),\eta(r))$ is concave in $(x^{\nu}, x_{\varsigma}^{\nu}, \nu, \nu_{\varsigma})$ and $\Theta(\cdot)$ is concave in x^{ν}.*

Theorem 6 (Risk-Neutral Verification Theorem). *Suppose $\mu(\cdot) \in \mathcal{U}_{ad}$ and let $x^{\mu}(\cdot)$ be the corresponding trajectory, $(\xi(\cdot), \eta(\cdot))$ satisfy (23). If $\mu(\cdot)$ satisfies (H1), (H9), (H10), and maximum condition (24), hence $\mu(\cdot)$ is an optimal solution to Problem (G-RNC).*

5. Applications

Let us give an application example in this section.

Example 1. *There are two types of investment products to choose for a pension fund manager, named bond and stock. Additionally, their prices meet, respectively, $dM_0(s) = r(s)M_0(s)ds$ and $dM_1(s) = M_1(s)[\mu(s)ds + \sigma(s)dB(s)]$, where $r(\cdot)$ is return rate, $\mu(\cdot)$ is appreciation rate of return and $\sigma(\cdot)$ is volatility coefficient. Further, presume that $r(\cdot)$ and $\sigma(\cdot)$ are deterministic bounded functions, and $\mu(\cdot)$ is an $\mathcal{F}_r$-adapted bounded process. In addition, $\sigma(s)^{-1}$ properly exists and is bounded.*

We use $\theta(s)$ to represent the manager's investment amount in stocks, $x(s)$ to represent his wealth, whose initial value is $m > 0$. Further $k(x(s) - x(s-\varsigma))$ represents fund members' surplus premium, which depends on fund growth's performance over the past period for some $k > 0$. Hence, $x(\cdot)$ satisfies SDDE

$$\begin{cases} dx(s) = [(r(s)-k)x(s) + kx(s-\varsigma) + (\mu(s) - r(s))\theta(s)]dr + \theta(s)\sigma(s)dB(s), & s \in [0,T], \\ x(0) = m, \quad s \in [-\varsigma, 0]. \end{cases} \tag{25}$$

Represent $\mathcal{U}_{ad} = \{\theta(\cdot) \in L^2_{\mathcal{F}}(0,T;R) | \theta(s) \geq c_0,\ s \in [0,T]\}$ by admissible control set. Define the associated utility functional

$$J(\theta(\cdot)) = \mathbb{E}[\int_0^T Le^{-\beta s}\log\theta(s)ds + Kx(T)],$$

where $L, K > 0$, and β are discount factors.

Problem 3 (B). *The manager wants to achieve*

$$J(\theta^*(\cdot)) = \max_{\theta(\cdot)\in\mathcal{U}_{ad}} J(\theta(\cdot)).$$

Now, applying Theorem 5 and Theorem 6, we obtain the Hamiltonian function

$$\overline{\mathcal{H}}(s,x,x_{\varsigma},\xi,\eta) = \xi[(r(s)-k)x + kx_{\varsigma} + (\mu(s)-r(s))\theta(s)] + \eta\theta(s)\sigma(s) + Le^{-\beta s}\log\theta(s),$$

and adjoint equation

$$\begin{cases} -d\xi(s) = \{[(r(s)-k)\xi(s) + \mathbb{E}^{\mathcal{F}_s}[k\xi(s+\varsigma)]\}ds - \eta(s)dB(s), & s \in [0,T], \\ \xi(T) = K, \quad \xi(s) = 0, \quad s \in (T, T+\varsigma], \\ \eta(s) = 0, \quad s \in [T, T+\varsigma]. \end{cases} \tag{26}$$

We can solve (3) *by continuously Itô integrating on steps of length δ, i.e.,*

$$\begin{aligned}
&\xi(s) = Ke^{\int_0^T (r(s)-k)ds},\ \eta(s) = 0,\ s \in [T-\varsigma, T];\\
&\xi(s) = Ke^{\int_{T-\varsigma}^T (r(s)-k)ds} e^{\int_{T-2\delta}^s (r(t)-k)dt} + \int_{T-2\varsigma}^s Kke^{\int_{t+\varsigma}^T (r(u)-k)du} e^{\int_t^s (r(u)-k)du} dt,\\
&\eta(s) = 0,\ s \in [T-2\varsigma, T-\varsigma];\\
&\ \ \cdots
\end{aligned} \tag{27}$$

Then, an optimal investment amount $\theta^*(s)$ of Problem (B) is

$$\theta^*(s) = \begin{cases} \omega(s), & \text{if } \omega(s) \geq c_0, \\ c_0, & \text{if } \omega(s) < c_0, \end{cases} \tag{28}$$

where

$$\omega(s) = Le^{-\beta s} \xi(s)^{-1}. \tag{29}$$

We can give the following result by Theorem 6.

Proposition 1. *The optimal investment amount $\theta^*(s)$ of Problem (B) is given by (28), where $\omega(s)$ is defined by (29) and $\xi(s)$ is of the form* (27).

6. Conclusions

In this article, an MP of a kind of RSCP with delay and HARA utility is derived by using a dual method and a expanding variable dimension method. Moreover, a sufficient condition is also obtained under some concavity conditions. A pension fund management problem is used as an application illustration. The results develop those of [26,34].

We will consider possible extensions to the problem with partial information, mean-field, etc., in our future works. On the other hand, issues such as financial equations in quantum finance transformed into Hamiltonian form by variables (e.g., [35]) and its application for analyzing the phenomena of spontaneous symmetry breaking in Quantum Finance (e.g., [36]) are also interesting and worthy of study. Research on these problems is currently under way.

Funding: This research was funded by the National Key R&D Program of China under Grant No. 2022YFA1006103, the NSF of China under Grant No. 61925306, 61821004, and 11831010, and the NSF of Shandong Province under Grant No. ZR2020ZD24 and ZR2019ZD42.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank Professor Guangchen Wang for his careful reading and instructive suggestions for improving the quality of this paper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bensoussan, A.; Van Schuppen, J. Optimal control of partially observable stochastic systems with an exponential-of-integral performance index. *SIAM J. Control Optim.* **1985**, *23*, 599–613. [CrossRef]
2. Hata, H.; Sekine, J. Explicit solution to a certain non-ELQG risk-sensitive stochastic control problem. *Appl. Math. Optim* **2010**, *62*, 341–380. [CrossRef]
3. Lim, A.; Zhou, X. Risk-sensitive control with HARA utility. *IEEE Trans. Automat. Control* **2001**, *46*, 563–578. [CrossRef]
4. Fleming, W. Risk-sensitive stochastic control and differential games. *Commun. Inf. Syst.* **2006**, *6*, 161–178. [CrossRef]
5. Tembine, H.; Zhu, Q.; Basar, T. Risk-sensitive mean-field games. *IEEE Trans. Automat. Control* **2014**, *59*, 835–850. [CrossRef]
6. Davis, M.; Lleo, S. Jump-diffusion risk-sensitive asset management I: Diffusion factor model. *SIAM J. Financ. Math.* **2011**, *2*, 22–54. [CrossRef]
7. Whittle, P. A risk-sensitive maximum principle. *Systems Control Lett.* **1990**, *15*, 183–192. [CrossRef]
8. Lim, A.; Zhou, X. A new risk-sensitive maximum principle. *IEEE Trans. Automat. Control* **2005**, *50*, 958–966. [CrossRef]

9. Shi, J.; Wu, Z. Maximum principle for risk-sensitive stochastic optimal control problem and applications to finance. *Stoch. Anal. Appl.* **2012**, *30*, 997–1018. [CrossRef]
10. Wang, G.; Wu, Z. General maximum principles for partially observed risksensitive optimal control problems and applications to finance. *J. Optim. Theory Appl.* **2009**, *141*, 677–700. [CrossRef]
11. Huang, J.; Li, X.; Wang, G. Maximum principles for a class of partial information risk-sensitive optimal controls. *IEEE Trans. Automat. Control* **2010**, *55*, 1438–1443. [CrossRef]
12. Shi, J.; Wu, Z. A risk-sensitive stochastic maximum principle for optimal control of jump diffusions and its applications. *Acta Math. Sci. Ser. B* **2011**, *31*, 419–433.
13. Ma, H.; Liu, B. Maximum principle for partially observed risk-sensitive optimal control problems of mean-field type. *Eur. J. Control* **2016**, *32*, 16–23. [CrossRef]
14. Moon, J. Risk-sensitive maximum principle for stochastic optimal control of mean-field type Markov regime-switching jump-diffusion systems. *Int. J. Robust Nonlinear Control* **2021**, *31*, 2141–2167. [CrossRef]
15. Moon, J. Generalized risk-sensitive optimal control and Hamilton-Jacobi-Bellman equation. *IEEE Trans. Automat. Control* **2021**, *66*, 2319–2325. [CrossRef]
16. Zhu, B.; Han, B.; Lin, X. Existence results for a class of semilinear fractional partial differential equations with delay in Banach spaces. *J. Funct. Spaces* **2019**, *2019*, 6295019. [CrossRef]
17. Mohammed, S. *Stochastic Functional Differential Equations*; Pitman Advanced Publishing Program: El-Bireh, Palestine, 1984; Volume 99.
18. Mohammed, S. Stochastic differential equations with memory: Theory, examples and applications. In *Stochastic Analysis and Related Topics VI*; Progress in Probability ; Birkhuser: Boston, MA, USA, 1996; pp. 1–77.
19. Ramsey, R. A mathematical theory of savings. *Econ. J.* **1928**, *38*, 543–559. [CrossRef]
20. Gandolfo, G. *Economic Dynamics*; Springer: Berlin/Heidelberg, Germany, 1996.
21. Arriojas, M.; Hu, Y.; Mohammed, S.; Pap, G. A delayed black and scholes formula. *Stoch. Anal. Appl.* **2007**, *25*, 471–492. [CrossRef]
22. Ivanov, A.; Swishchuk, A. Optimal control of stochastic dierential delay equations with application in economics. *Russ. Math. Surv.* **2008**, *2*, 157–183.
23. Richard, J. Time-delay systems: An overview of some recent advances and open problems. *Automatica* **2003**, *39*, 1667–1694. [CrossRef]
24. Wang, P. Maximum principle for optimal control problem with delay. In Proceedings of the 2018 Chinese Automation Congress (CAC), Xi'an, China, 30 November–2 December 2018. [CrossRef]
25. Peng, S.; Yang, Z. Anticipated backward stochastic differential equation. *Ann. Probab.* **2009**, *37*, 877–902. [CrossRef]
26. Chen, L.; Wu, Z. Maximum principle for the stochastic optimal control problem with delay and application. *Automatica* **2010**, *46*, 1074–1080. [CrossRef]
27. Wu, S. Partially-observed maximum principle for backward stochastic differential delay equations. *IEEE-CAA J. Autom. Sin.* **2017**, *99*, 1–6. [CrossRef]
28. Li, N.; Wang, Y.; Wu, Z. An indefinite stochastic linear quadratic optimal control problem with delay and related forward-backward stochastic differential equations. *J. Optim. Theory Appl.* **2018**, *179*, 722–744. [CrossRef]
29. Hao, T.; Meng, Q. A second-order maximum principle for singular optimal controls with recursive utilities of stochastic delay systems. *Eur. J. Control* **2019**, *50*, 96–106. [CrossRef]
30. Ma, H.; Liu, B. Optimal control problem for risk-sensitive mean-field stochastic delay differential equation with partial information. *Asian J. Control* **2017**, *19*, 2097–2115. [CrossRef]
31. Moon, J.; Duncan, T.E.; Basar, T. Risk-sensitive zero-sum differential games. *IEEE Trans. Automat. Control* **2019**, *64*, 1503–1518. [CrossRef]
32. Moon, J. Necessary and sufficient conditions of risk-sensitive optimal control and differential games for stochastic differential delayed equations. *Int. J. Robust Nonlinear Control* **2019**, *29*, 4812–4827. [CrossRef]
33. Zhang, L. A BSDE approach to stochastic differential games involving impulse controls and HJBI equation. *J. Syst. Sci. Complex.* **2022**, *35*, 766–801. [CrossRef]
34. Wang, G.; Wu, Z. Stochastic maximum principle for a kind of risk-sensitive optimal control problem and application to portfolio choice. *Acta Automat. Sin.* **2007**, *33*, 1043–1047. [CrossRef]
35. Baaquie, B. *Quantum Finance: Path Integrals and Hamiltonians for Options and Interest Rates*; Cambridge University Press: Cambridge, UK, 2004.
36. Arraut, I.; Au, A.; Tse, A. Spontaneous symmetry breaking in quantum finance. *Europhys. Lett.* **2020**, *131*, 68003. [CrossRef]