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Graph Convergence, Algorithms, and Approximation of Common Solutions of a System of Generalized Variational Inclusions and Fixed-Point Problems

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Abstract: In this paper, under some new appropriate conditions imposed on the parameters and mappings involved in the proximal mapping associated with a general H-monotone operator, its Lipschitz continuity is proved and an estimate of its Lipschitz constant is computed. The main contribution of this work is the establishment of a new equivalence relationship between the graph convergence of a sequence of general strongly H-monotone mappings and their associated proximal mappings, respectively, to a given general strongly H-monotone mapping and its associated proximal mapping by using the notions of graph convergence and proximal mapping concerning a general strongly H-monotone mapping. By employing the concept of proximal mapping relating to general strongly H-monotone mapping, some iterative algorithms are proposed, and as an application of the obtained equivalence relationship mentioned above, a convergence theorem for approximating a common element of the set of solutions of a system of generalized variational inclusions involving general strongly H-monotone mappings and the set of fixed points of an $(\{a_n\}, \{b_n\}, \phi)$-total uniformly L-Lipschitzian mapping is proved. It is significant to emphasize that our results are new and improve and generalize many known corresponding results.

Keywords: system of generalized variational inclusions; $(\{a_n\}, \{b_n\}, \phi)$-total uniformly L-Lipschitzian mapping; general H-monotone operator; proximal mapping; graph convergence; fixed point; convergence analysis

MSC: 47H05; 47H09; 47J20; 47J22; 47J25; 49J40

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1. Introduction

It is a well-known truth that inequalities have always been of great importance for the development of many branches of mathematics. For this reason, the study of various kinds of inequalities in the context of different spaces has a long history and is the focus of the attention of researchers coming from mathematics, economics, and many other disciplines. The history of variational inequality started with a contact problem posed in 1979 by Signorini [1], and then Fichera [2] formulated this problem as a variational inequality by using this term for the first time. It has been recognized as a suitable mathematical model for dealing with many problems arising in different fields, such as optimization theory, partial differential equations, economic and transportation equilibria, engineering science, etc.

Because of its importance and wide applications in many important fields of science, over recent decades, the variational inequality problem has received a great deal of interest from the scientific community, and there has been major activity in proposing and analyzing various extensions of it. During the last few decades, the study of the variational inclusion problem as an important and significant extension of the classical variational inequality problem has gained noticeable importance, and various classes of variational inclusion problems have been intensively studied. For more details, we refer the reader to [3–14] and the references therein.

In recent decades, the introduction and study of efficient ways to find solutions of various kinds of variational inequalities/inclusions have received much attention from many authors due to the fact that one of the most interesting and important problems in the theory of variational inequalities/inclusions is the development of an efficient and implementable iterative algorithm for computing approximate solutions. Among the methods that have appeared in the literature, the method based on the resolvent operator technique as an extension of the projection method represents an important and useful tool for finding the approximation solutions of various types of variational inequalities/inclusions. There is a rich literature on solving different classes of variational inequality/inclusion problems by using the resolvent operator technique. For more information and relevant commentaries, the reader can refer to [5–18].

In the recent past, considerable attention was paid to the extension of the notion of monotone operators in the frameworks of different spaces. This is mainly because monotone operators have turned out to be an important tool in the study of various problems arising in the domains of optimization, nonlinear analysis, differential equations, and related fields; see, for example, [19,20].

Since the appearance of the theory of maximal monotonicity of operators defined on Banach spaces in the 1960s, due to its many diverse applications in the theory of partial differential equations, it has been intensively studied by many mathematicians. At the same time, the importance and indispensable role of maximal monotone operators in the theory of variational inequalities/inclusions have motivated many researchers to extend this notion and to introduce several interesting generalized monotone operators with relevant resolvent operators. Further information and details can be found in [5–8] and the references therein. By the same taken, in 2007, Xia and Huang [13] introduced a class of generalized monotone operators called general H-monotone operators; this class is essentially wider than the classes of maximal monotone operators in the framework of Banach spaces and H-monotone operators in the setting of Hilbert spaces according to Fang and Huang [5]. By defining the proximal mapping associated with a general H-monotone operator, they verified its Lipschitz continuity and computed an estimate of its Lipschitz constant under some appropriate conditions. They also considered a class of variational inclusions involving general H-monotone operators and constructed an iterative algorithm for finding the approximate solution of it in a Banach space setting. Finally, they studied a convergence analysis of the iterative sequence generated by their suggested iterative algorithm under some suitable conditions.

The concept of graph convergence was first introduced by Attouch [21] for functions and operators involving the classical resolvent operators of set-valued mappings in a Hilbert space setting. In recent times, this notion has attracted much attention, and there have been successful attempts to generalize it, though these were limited to the maximal monotone mappings in [21] for the generalized monotone operators that have appeared in the literature. For more details, see [9–12,21].

On the other hand, fixed-point theory—the study of which comes under the purview of the wider area of nonlinear functional analysis and which began almost a century ago in the field of algebraic topology—has gained importance because of its wide range of applicability for resolving diverse problems emanating from the theory of nonlinear differential equations, the theory of nonlinear integral equations, game theory, mathematical economics, control theory, and so forth. The recent rapid development of efficient techniques for computing fixed points has enormously increased the usefulness of the theory of fixed points for applications. Thus, fixed-point theory is becoming an increasingly invaluable tool in the arsenal of applied mathematics. As we know, many of the most important nonlinear problems of applied mathematics are reduced to solving a given equation, which, in turn, may be reduced to finding the fixed points of a certain operator. The existence of a close relation between variational inequality/inclusion problems and fixed-point problems has motivated many researchers to present a unified approach to these two different problems. For more details, we refer the readers to [15–18,22]. The study of nonexpansive mappings has a long history, and it is related to research on monotone and accretive operators. As a matter of fact, historically, the study of monotone and accretive operators, two classes of operators that naturally arise in the theory of differential equations, has led to the study of nonexpansive mappings. It is worth mentioning that the importance of the theory of nonexpansive mappings and its applications in fixed-point theory have been well documented in the literature. Over the last 50 years or so, the concept of nonexpansive mapping has attracted increasing attention, and many authors have made efforts to generalize this notion. Thereby, various extensions of it have been proposed and analyzed. For example, Goebel and Kirk [23] were the first to introduce the class of asymptotically nonexpansive mappings as an extension of the class of nonexpansive mappings and proved the existence of a fixed point for them under some appropriate conditions. Afterward, the improvement and generalization of this concept attracted and have continued to attract the interest of many authors, and a huge amount of literature has reported on applications, generalizations, and extensions of it; see, for example, [24–26]. Recently, Alber et al. [27] carried out a successful attempt to introduce a class of generalized nonexpansive mappings called total asymptotically nonexpansive mappings; this class is more general than the class of asymptotically nonexpansive mappings, and they studied methods of approximation of fixed points of mappings belonging to this class. In fact, their motivation for doing this was to unify various definitions related to the class of asymptotically nonexpansive mappings and their generalizations that have appeared in the literature, as well as to verify a general convergence theorem that is applicable to all of these classes of nonlinear mappings. Motivated and inspired by the works mentioned above, recently, Kiziltunc and Purtas [28] introduced the class of total uniformly L-Lipschitzian mappings, which can be viewed as a unifying framework for the classes of asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings, and several other classes of generalized nonexpansive mappings existing in the literature. In order to find more information and details about extensions of nonexpansive mappings along with several interesting illustrative examples, the reader can refer to [23–31].

Now, we briefly describe the contents of this paper. In Section 2, we recall the basic definitions and provide some preliminary results needed in the following. Section 3 is concerned with the introduction and formulation of a new system of generalized variational inclusions (SGVI) involving general strongly H-monotone mappings in the framework of Banach spaces. Under some appropriate conditions, the existence of a unique solution for the SGVI is proved. In Section 4, with the goal of finding a point belonging to the intersection of the set of solutions of the SGVI and the set of fixed points of a total uniformly L-Lipschitzian mapping, an iterative algorithm is proposed. The notions of graph convergence and proximal mapping relating to a general strongly H-monotone mapping are used, and a new equivalence relationship between the graph convergence of a sequence of general strongly H-monotone mappings and their associated proximal mappings, respectively, to a given general strongly H-monotone mapping and its associated proximal mapping is established. Finally, the strong convergence of the sequence generated by our suggested iterative algorithm to a common element of the set of solutions of the SGVI and the set of fixed points of a total uniformly L-Lipschitzian mapping is verified. Furthermore, as a consequence of our main results, we derive a conclusion that improves the main corresponding result presented in [13]. In fact, in our conclusion, the condition of the strict monotonicity of operator involvement in the corresponding variational inclusion problem of Xia and Huang [13] is not needed, and it is replaced by a more mild condition of monotonicity.

2. Preliminary Materials and Basic Results

Throughout the paper, unless otherwise specified, we use the following notations, terminology, and assumptions. Let B be a real Banach space and let B^* be its continuous dual space. The pairing between B and B^* is designated by $\langle ., . \rangle$, and the family of all of the nonempty subsets of B is denoted by 2^B. The value of a functional $x^* \in B^*$ at $x \in B$ is denoted by either $\langle x, x^* \rangle$ or $x^*(x)$, as is convenient. For the sake of simplicity, the norms of B and B^* are denoted by the symbol $\|.\|$.

For a given set-valued mapping $M : B \to 2^{B^*}$,

(i) the set Range(M) defined by

$$\text{Range}(M) = \{x^* \in B^* : \exists x \in B : (x, x^*) \in M\} = \bigcup_{x \in B} M(x)$$

is called the *range* of M;

(ii) the set Graph(M) defined by

$$\text{Graph}(M) = \{(x, x^*) \in B \times B^* : x^* \in M(x)\},$$

is called the *graph* of M.

Definition 1. *A normed space B is called strictly convex if the unit sphere in B is strictly convex, that is, the inequality $\|x + y\| < 2$ holds for all distinct unit vectors x and y in B. It is said to be smooth if for every unit vector x in B, there exists a unique $x^* \in B^*$ such that $\|x^*\| = \langle x, x^* \rangle = 1$.*

It is known that B is smooth if B^* is strictly convex and that B is strictly convex if B^* is smooth.

Definition 2. *A normed space B is said to be uniformly convex if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that if x and y are united vectors in B with $\|x - y\| \geq 2\varepsilon$, then the average $(x + y)/2$ has a norm of at most $1 - \delta$.*

Thus, a normed space is uniformly convex if for any two distinct points x and y on the unit sphere centered at the origin, the midpoint of the line segment joining x and y is never on the sphere, but is close to the sphere only if x and y are sufficiently close to each other.

The function $\delta_B : [0, 2] \to [0, 1]$ given by

$$\delta_B(\varepsilon) := \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in B, \|x\| = \|y\| = 1, \|x - y\| = \varepsilon\}$$

is called the *modulus of convexity* of B.

The function δ_B is continuous and increasing on the interval $[0, 2]$ and $\delta_B(0) = 0$. Clearly, in the light of the definition of the function δ_B, a normed space B is uniformly convex if and only if $\delta_B(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$.

In the particular case of an inner product space $\mathcal{H}$, we have $\delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$.

Definition 3. *A normed space B is said to be uniformly smooth if, for all $\varepsilon > 0$, there is a $\tau > 0$ such that if x and y are unit vectors in B with $\|x - y\| \leq 2\tau$, then the average $(x + y)/2$ has a norm of at least $1 - \varepsilon\tau$.*

The function $\rho_B : [0, +\infty) \to [0, +\infty)$ given by

$$\rho_B(\tau) = \sup\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in B, \|x\| = \|y\| = 1\}$$

is called the *modulus of smoothness* of B.

Note that the function ρ_B is convex, continuous, and increasing on the interval $[0, +\infty)$ and $\rho_B(0) = 0$. In addition, $\rho_B(\tau) \leq \tau$ for all $\tau \geq 0$. Invoking the definition of the function ρ_B, a normed space B is *uniformly smooth* if and only if $\lim\limits_{\tau \to 0} \frac{\rho_B(\tau)}{\tau} = 0$.

It should be remarked that in the definitions of $\delta_B(\varepsilon)$ and $\rho_B(\tau)$, we can also take the infimum and supremum over all vectors $x, y \in B$ with $\|x\|, \|y\| \leq 1$.

Any uniformly convex and any uniformly smooth Banach space is reflexive. A Banach space B is uniformly convex (or uniformly smooth) if and only if B^* is uniformly smooth (or uniformly convex).

The spaces l^p, L^p, and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, are uniformly convex, as well as uniformly smooth; see [32–34]. Meanwhile, the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$ can be found in [32–34].

Definition 4. *A single-valued mapping $A : B \to B^*$ is said to be*

(i) *monotone if*

$$\langle A(x) - A(y), x - y \rangle \geq 0, \quad \forall x, y \in B;$$

(ii) *strictly monotone if A is monotone, and equality holds if and only if $x = y$;*

(iii) *k-strongly monotone if there exists a constant $k > 0$ such that*

$$\langle A(x) - A(y), x - y \rangle \geq k\|x - y\|^2, \quad \forall x, y \in B;$$

(iv) *ϱ-Lipschiz continuous if there exists a constant $\varrho > 0$ such that*

$$\|A(x) - A(y)\| \leq \varrho\|x - y\|, \quad \forall x, y \in B.$$

Definition 5 ([13])**.** *Let B be a Banach space with the dual space B^*, and let $M : B \to 2^{B^*}$ be a set-valued mapping. M is said to be*

(i) *monotone if*

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (x, u), (y, v) \in \text{Graph}(M);$$

(ii) *maximal monotone if, for $x \in B$ and $u \in B^*$,*

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (y, v) \in \text{Graph}(M)$$

implies that $(x, u) \in \text{Graph}(M)$;

(iii) *γ-strongly monotone if there exists a constant $\gamma > 0$ such that*

$$\langle u - v, x - y \rangle \geq \gamma\|x - y\|^2, \quad \forall (x, u), (y, v) \in \text{Graph}(M).$$

Relying on Definition 5(ii), we note that M is a maximal monotone mapping if and only if M is monotone and there is no other monotone mapping whose graph contains strictly $\text{Graph}(M)$. The maximality is to be understood in terms of the inclusions of graphs. If $M : B \to 2^{B^*}$ is a maximal monotone mapping, then adding anything to its graph so as to obtain the graph of a new set-valued mapping destroys the monotonicity. In fact, the extended mapping is no longer monotone. In other words, for every pair $(x, u) \in B \times B^* \setminus \text{Graph}(M)$, there exists $(y, v) \in \text{Graph}(M)$ such that $\langle u - v, x - y \rangle < 0$. Consequently, a necessary and sufficient condition for a set-valued mapping $M : B \to 2^{B^*}$ to be maximal monotone is that for any $x \in B$ and $u \in B^*$, the property $\langle u - v, x - y \rangle \geq 0$, for all $(y, v) \in \text{Graph}(M)$, is equivalent to $u \in M(x)$.

Recall that the normalized duality mapping $J : B \to 2^{B^*}$ is defined by

$$J(x) = \{x^* \in B^* : \langle x, x^* \rangle = \|x^*\|\|x\|, \|x^*\| = \|x\|\}, \quad \forall x \in B.$$

We immediately observe that if $B = \mathcal{H}$, a Hilbert space, then J is the identity mapping on $\mathcal{H}$. At the same time, it is an immediate consequence of the Hahn–Banach theorem that $J(x)$ is nonempty for each $x \in B$.

Let B be a reflexive Banach space with the dual space B^*, and let $M : B \to 2^{B^*}$ be a set-valued mapping. Equivalently, we say that M is maximal monotone [35] if M is monotone and $(J + \lambda M)(B) = B^*$ for every $\lambda > 0$, where J is the normalized duality mapping.

The notion of an H-monotone operator was first introduced by Fang and Huang [5] in 2003 as follows.

Definition 6 ([5])**.** *Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued operator and let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. M is said to be H-monotone if M is monotone and $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$.*

It should be remarked that if $H = I$, the identity mapping on $\mathcal{H}$, then the definition of I-monotone operators is that of maximal monotone operators.

Afterward, replacing the Hilbert space $\mathcal{H}$ with a Banach space, Xia and Huang [13] introduced a larger class of monotone operators—the so-called general H-monotone mappings—as an extension of the class of H-monotone operators as follows.

Definition 7 ([13])**.** *Let B be a Banach space with the dual space B^*, let $H : B \to B^*$ be a single-valued mapping, and let $M : B \to 2^{B^*}$ be a set-valued mapping. M is said to be general H-monotone if M is monotone and $(H + \lambda M)(B) = B^*$ holds for every $\lambda > 0$.*

Note, in particular, that if $B = \mathcal{H}$ is a Hilbert space, then the class of general H-monotone operators coincides exactly with the class of H-monotone operators.

The following example illustrates that for a given mapping $H : B \to B^*$, a maximal monotone mapping need not be general H-monotone.

Example 1. *Let m and n be two arbitrary but fixed natural numbers such that n is even, and let $M_{m\times n}(\mathbb{C})$ be the vector space of all $m \times n$ matrices with complex entries over $\mathbb{C}$. Then,*

$$M_{m\times n}(\mathbb{C}) = \{A = (\ a_{lj}\ )|a_{lj} \in \mathbb{C}, l = 1,2,\ldots,m; j = 1,2,\ldots,n\}$$

is a Hilbert space with the inner product $\langle A, C\rangle := tr(A^*C)$ for all $A, C \in M_{m\times n}(\mathbb{C})$, where tr denotes the trace, that is, the sum of diagonal entries, and A^* denotes the Hermitian conjugate (or adjoint) of the matrix A, that is, $A^* = \overline{A^t}$, the complex conjugate of the transpose A. The inner product defined above induces a norm on $M_{m\times n}(\mathbb{C})$ as follows:

$$\|A\| = \Big(\sum_{l=1}^{m}\sum_{j=1}^{n}|a_{lj}|^2\Big)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{C}).$$

Taking into account that every finite-dimensional normed space is a Banach space, it follows that $(M_{m\times n}(\mathbb{C}), \|.\|)$ is a Banach space. For any $A = (\ a_{lj}\ ) \in M_{m\times n}(\mathbb{C})$, we have

$$A = (\ a_{lj}\ ) = \sum_{l=1}^{m}\sum_{j\in\Gamma}(A_{l(2j-1)(2j+1)} + A_{l(2j)(2j+2)}),$$

where $\Gamma = \{1, 3, \ldots, \frac{n-2}{2}\}$. Thereby, every $m \times n$ matrix $A \in M_{m\times n}(\mathbb{C})$ can be written as a linear combination of $\frac{mn}{2}$ matrices $A_{l(2j-1)(2j+1)}$ and $A_{l(2j)(2j+2)}$, where for each $l \in \{1, 2, \ldots, m\}$ and $j \in \Gamma$, $A_{l(2j-1)(2j+1)}$ is an $m \times n$ matrix with the entries $a_{l(2j-1)} = x_{l(2j-1)} + iy_{l(2j-1)}$ and $a_{l(2j+1)} = x_{l(2j+1)} + iy_{l(2j+1)}$ at the $(l, 2j-1)$ and $(l, 2j+1)$ places, respectively, and 0s everywhere else; $A_{l(2j)(2j+2)}$ is an $m \times n$ matrix with the entries $a_{l(2j)} = x_{l(2j)} + iy_{l(2j)}$ and

$a_{l(2j+2)} = x_{l(2j+2)} + iy_{l(2j+2)}$ *in the positions* $(l, 2j)$ *and* $(l, 2j+2)$*, respectively, and* 0*s elsewhere. For each* $l \in \{1, 2, \ldots, m\}$ *and* $j \in \Gamma$*, we obtain*

$$A_{l(2j-1)(2j+1)} + A_{l(2j)(2j+2)} = \begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & a_{l(2j-1)} & 0 & a_{l(2j+1)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & a_{l(2j)} & 0 & a_{l(2j+2)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} &= \frac{y_{l(2j-1)} + y_{l(2j+1)} - i(x_{l(2j-1)} + x_{l(2j+1)})}{2} W_{l(2j-1)(2j+1)} \\ &+ \frac{y_{l(2j-1)} - y_{l(2j+1)} - i(x_{l(2j-1)} - x_{l(2j+1)})}{2} W'_{l(2j-1)(2j+1)} \\ &+ \frac{y_{l(2j)} + y_{l(2j+2)} - i(x_{l(2j)} + x_{l(2j+2)})}{2} W_{l(2j)(2j+2)} \\ &+ \frac{y_{l(2j)} - y_{l(2j+2)} - i(x_{l(2j)} - x_{l(2j+2)})}{2} W'_{l(2j)(2j+2)}, \end{aligned}$$

where, for each $l \in \{1, 2, \ldots, m\}$ *and* $j \in \Gamma$*,* $W_{l(2j-1)(2j+1)}$ *is an* $m \times n$ *matrix in which the* $(l, 2j-1)$ *and* $(l, 2j+1)$ *entries are equal to i and all other entries are equal to zero,* $W'_{l(2j-1)(2j+1)}$ *is an* $m \times n$ *matrix with the entries i and* $-i$ *in the* $(l, 2j-1)$ *and* $(l, 2j+1)$ *positions, respectively, and there are* 0*s everywhere else;* $W_{l(2j)(2j+2)}$ *is an* $m \times n$ *matrix with the* $(l, 2j)$ *and* $(l, 2j+2)$ *entries i, and all other entries are equal to zero, and* $W'_{l(2j)(2j+2)}$ *is an* $m \times n$ *matrix with the entries i and* $-i$ *at the* $(l, 2j)$ *and* $(l, 2j+2)$ *places, respectively, and* 0*s elsewhere. Accordingly, for any* $A \in M_{m\times n}(\mathbb{C})$*, we have*

$$\begin{aligned} A &= \sum_{l=1}^{m} \sum_{j\in\Gamma} (A_{l(2j-1)(2j+1)} + A_{l(2j)(2j+2)}) \\ &= \sum_{l=1}^{m} \sum_{j\in\Gamma} \Big[\frac{y_{l(2j-1)} + y_{l(2j+1)} - i(x_{l(2j-1)} + x_{l(2j+1)})}{2} W_{l(2j-1)(2j+1)} \\ &\quad + \frac{y_{l(2j-1)} - y_{l(2j+1)} - i(x_{l(2j-1)} - x_{l(2j+1)})}{2} W'_{l(2j-1)(2j+1)} \\ &\quad + \frac{y_{l(2j)} + y_{l(2j+2)} - i(x_{l(2j)} + x_{l(2j+2)})}{2} W_{l(2j)(2j+2)} \\ &\quad + \frac{y_{l(2j)} - y_{l(2j+2)} - i(x_{l(2j)} - x_{l(2j+2)})}{2} W'_{l(2j)(2j+2)} \Big]. \end{aligned}$$

Therefore, the set

$$\Big\{ W_{l(2j-1)(2j+1)}, W'_{l(2j-1)(2j+1)}, W_{l(2j)(2j+2)}, W'_{l(2j)(2j+2)} : l = 1, 2, \ldots, m; j = 1, 3, \ldots, \frac{n-2}{2} \Big\}$$

spans the Hilbert space $M_{m\times n}(\mathbb{C})$. Taking $P_{l(2j-1)(2j+1)} := \frac{1}{\sqrt{2}}W_{l(2j-1)(2j+1)}$, $P'_{l(2j-1)(2j+1)} := \frac{1}{\sqrt{2}}W'_{l(2j-1)(2j+1)}$, $P_{l(2j)(2j+2)} := \frac{1}{\sqrt{2}}W_{l(2j)(2j+2)}$, and $P'_{l(2j)(2j+2)} := \frac{1}{\sqrt{2}}W'_{l(2j)(2j+2)}$, for each $l \in \{1,2,\dots,m\}$ and $j \in \Gamma$, it follows that the set

$$\mathfrak{B} = \left\{P_{l(2j-1)(2j+1)}, P'_{l(2j-1)(2j+1)}, P_{l(2j)(2j+2)}, P'_{l(2j)(2j+2)} : l = 1,2,\dots,m; j = 1,3,\dots,\frac{n-2}{2}\right\}$$

also spans the Hilbert space $M_{m\times n}(\mathbb{C})$. It can be easily shown that the set $\mathfrak{B}$ is linearly independent and orthonormal, so $\mathfrak{B}$ is an orthonormal basis for the Hilbert space $M_{m\times n}(\mathbb{C})$.

Let the mappings $H, M : M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ be defined, respectively, by $H(A) = -\alpha A + \theta P_{k(2s)(2s+2)} + \varsigma P'_{k(2s)(2s+2)}$ and $M(A) = \alpha A + \beta P_{k(2s-1)(2s+1)} + \gamma P'_{k(2s-1)(2s+1)}$ for all $A \in M_{m\times n}(\mathbb{C})$, where α is an arbitrary positive real constant, β, γ, θ, and ς are arbitrary nonzero real constants, and $k \in \{1,2,\dots,n\}$ and $s \in \Gamma$ are arbitrary but fixed natural numbers. Then, for all $A, C \in M_{m\times n}(\mathbb{C})$, this yields

$$\begin{aligned}\langle M(A) - M(C), A - C\rangle &= \langle \alpha A + \beta P_{k(2s-1)(2s+1)} + \gamma P'_{k(2s-1)(2s+1)} - \alpha C \\ &\quad - \beta P_{k(2s-1)(2s+1)} - \gamma P'_{k(2s-1)(2s+1)}, A - C\rangle \\ &= \alpha\langle A - C, A - C\rangle = \alpha\|A - C\|^2 \geq 0,\end{aligned}$$

that is, M is a monotone operator. Taking into account that for any $A \in M_{m\times n}(\mathbb{C})$ and every constant $\lambda > 0$,

$$\begin{aligned}(I + \lambda M)(A) &= A + \lambda\alpha A + \lambda\beta P_{k(2s-1)(2s+1)} + \lambda\gamma P'_{k(2s-1)(2s+1)} \\ &= (1 + \lambda\alpha)A + \lambda\beta P_{k(2s-1)(2s+1)} + \lambda\gamma P'_{k(2s-1)(2s+1)},\end{aligned}$$

where I is the identity operator on $B = M_{m\times n}(\mathbb{C})$, it follows that $(I + \lambda M)(M_{m\times n}(\mathbb{C})) = M_{m\times n}(\mathbb{C})$ for every constant $\lambda > 0$, which means that the mapping $I + \lambda M$ is surjective for every positive real constant λ. Consequently, M is a maximal monotone operator.

In virtue of the fact that for every $A \in M_{m\times n}(\mathbb{C})$,

$$\begin{aligned}(H + M)(A) &= \beta P_{k(2s-1)(2s+1)} + \gamma P'_{k(2s-1)(2s+1)} + \theta P_{k(2s)(2s+2)} + \varsigma P'_{k(2s)(2s+2)} \\ &= \begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & \frac{\beta+\gamma}{\sqrt{2}}i_{l(2j-1)} & \frac{\theta+\varsigma}{\sqrt{2}}i_{l(2j)} & \frac{\beta-\gamma}{\sqrt{2}}i_{l(2j+1)} & \frac{\theta-\varsigma}{\sqrt{2}}i_{l(2j+2)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix},\end{aligned}$$

we conclude that for any $A \in M_{m\times n}(\mathbb{C})$,

$$\begin{aligned}\|(H + M)(A)\| &= (\frac{\beta+\gamma}{\sqrt{2}})^2 + (\frac{\theta+\varsigma}{\sqrt{2}})^2 + (\frac{\beta-\gamma}{\sqrt{2}})^2 + (\frac{\theta-\varsigma}{\sqrt{2}})^2 \\ &= \beta^2 + \gamma^2 + \theta^2 + \varsigma^2 > 0.\end{aligned}$$

This fact implies that $\mathbf{0} \notin (H + M)(M_{m\times n}(\mathbb{C}))$, where $\mathbf{0}$ is the zero vector of the space $M_{m\times n}(\mathbb{C})$, that is, the zero $m \times n$ matrix. Thus, the mapping $H + M$ is not surjective, which ensures that the operator M is not general H-monotone.

Example 2. Let $H_2(\mathbb{C})$ be the set of all Hermitian matrices with complex entries. Let us recall that a square matrix A is said to be Hermitian (or self-adjoint) if it is equal to its own Hermitian conjugate, i.e., $A^* = \overline{A^t} = A$. In the light of the definition of a Hermitian 2×2

matrix, the condition $A^* = A$ implies that the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is Hermitian if and only if $a, d \in \mathbb{R}$ and $b = \bar{c}$. Thus,

$$H_2(\mathbb{C}) = \left\{ \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} | x, y, z, w \in \mathbb{R} \right\}.$$

Then, $H_2(\mathbb{C})$ is a subspace of $M_2(\mathbb{C})$, the space of all 2×2 matrices with complex entries, with respect to the operations of addition and scalar multiplication defined on $M_2(\mathbb{C})$, when $M_2(\mathbb{C})$ is considered a real vector space. By introducing the scalar product on $H_2(\mathbb{C})$ as $\langle A, C \rangle := \frac{1}{2} tr(AC)$ for all $A, C \in H_2(\mathbb{C})$, it can be easily seen that $\langle ., . \rangle$ is an inner product, i.e., $(H_2(\mathbb{C}), \langle ., . \rangle)$ is an inner product space. The inner product defined above induces a norm on $H_2(\mathbb{C})$ as follows:

$$\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{\frac{1}{2} tr(AA)} = \sqrt{x^2 + y^2 + \frac{1}{2}(z^2 + w^2)}, \quad \forall A \in H_2(\mathbb{C}).$$

Since $B = (H_2(\mathbb{C}), \|.\|)$ is a finite-dimensional normed space, it follows that it is a Banach space. Now, let the operators $H_1, H_2, M : H_2(\mathbb{C}) \to H_2(\mathbb{C})$ be defined, respectively, by

$$H_1(A) = \begin{pmatrix} z^k + \frac{\tau^z - 1}{\tau^z + 1} - \alpha z^l - \beta \sqrt[n]{z} & x^t - iy^t \\ x^t + iy^t & |w - \gamma| - |w - \theta| - \varrho w^m - \mu \sqrt[s]{w} \end{pmatrix},$$

$$H_2(A) = \begin{pmatrix} \delta |z| & x^p - iy^p \\ x^p + iy^p & \xi(w + \sin w) \end{pmatrix} \text{ and } M(A) = \begin{pmatrix} \alpha z^l + \beta \sqrt[n]{z} & \varsigma(\sqrt[q]{x} - i \sqrt[q]{y}) \\ \varsigma(\sqrt[q]{x} + i \sqrt[q]{y}) & \varrho w^m + \mu \sqrt[s]{w} \end{pmatrix},$$

for all $A = \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \in H_2(\mathbb{C})$, where $\alpha, \beta, \varrho, \mu, \varsigma, \delta, \xi$, and τ are arbitrary real constants, γ and θ are arbitrary nonzero real constants, k and t are arbitrary but fixed even natural numbers, and m, n, l, p, q and s are arbitrary but fixed odd natural numbers.

Then, for any $A = \begin{pmatrix} z_1 & x_1 - iy_1 \\ x_1 + iy_1 & w_1 \end{pmatrix}, C = \begin{pmatrix} z_2 & x_2 - iy_2 \\ x_2 + iy_2 & w_2 \end{pmatrix} \in H_2(\mathbb{C})$, it can be easily observed that

$$\begin{aligned} \langle M(A) - M(C), A - C \rangle = & \frac{\alpha}{2}(z_1^l - z_2^l)(z_1 - z_2) + \frac{\beta}{2}(\sqrt[n]{z_1} - \sqrt[n]{z_2})(z_1 - z_2) \\ & + \varsigma(\sqrt[q]{x_1} - \sqrt[q]{x_2})(x_1 - x_2) + \varsigma(\sqrt[q]{y_1} - \sqrt[q]{y_2})(y_1 - y_2) \\ & + \frac{\varrho}{2}(w_1^m - w_2^m)(w_1 - w_2) + \frac{\mu}{2}(\sqrt[s]{w_1} - \sqrt[s]{w_2})(w_1 - w_2) \geq 0, \end{aligned} \quad (1)$$

i.e., M is a monotone operator. We define the functions $f, g, h : \mathbb{R} \to \mathbb{R}$, respectively, as $f(v) := v^k + \frac{\tau^v - 1}{\tau^v + 1}, g(v) := |v - \gamma| - |v - \theta|$, and $h(v) := v^t + \varsigma \sqrt[q]{v}$ for all $v \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \in H_2(\mathbb{C})$, we get

$$\begin{aligned} (H_1 + M)(A) &= (H_1 + M)\left(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \right) \\ &= \begin{pmatrix} z^k + \frac{\tau^z - 1}{\tau^z + 1} & x^t + \varsigma \sqrt[q]{x} - i(y^t + \varsigma \sqrt[q]{y}) \\ x^t + \varsigma \sqrt[q]{x} + i(y^t + \varsigma \sqrt[q]{y}) & |w - \gamma| - |w - \theta| \end{pmatrix} \\ &= \begin{pmatrix} f(z) & h(x) - ih(y) \\ h(x) + ih(y) & g(w) \end{pmatrix}. \end{aligned}$$

Since k and t are even natural numbers, it is not difficult to see that $f(\mathbb{R}), h(\mathbb{R}) \neq \mathbb{R}$. Meanwhile, $g(\mathbb{R}) = [-|\gamma - \theta|, |\gamma - \theta|]$. This fact implies that $(H_1 + M)(H_2(\mathbb{C})) \neq H_2(\mathbb{C})$, i.e., the mapping $H_1 + M$ is not surjective, so M is not a general H_1-monotone operator.

Now, let $\lambda > 0$ be an arbitrary real constant and suppose that the functions $\widehat{f}, \widehat{g}, \widehat{h} : \mathbb{R} \to \mathbb{R}$ are defined, respectively, by $\widehat{f}(v) := \delta|v| + \alpha\lambda v^l + \beta\lambda\sqrt[n]{v}$, $\widehat{g}(v) := \xi(v + \sin v) + \varrho\lambda v^m + \mu\lambda\sqrt[s]{v}$, and $\widehat{h}(v) := v^p + \lambda\varsigma\sqrt[q]{v}$ for all $v \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \in H_2(\mathbb{C})$, we get

$$\begin{aligned}(H_2 + \lambda M)(A) &= (H_2 + \lambda M)\left(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}\right) \\ &= \begin{pmatrix} \delta|z| + \alpha\lambda z^l + \beta\lambda\sqrt[n]{z} & x^p + \lambda\varsigma\sqrt[q]{x} - i(y^p + \lambda\varsigma\sqrt[q]{y}) \\ x^p + \lambda\varsigma\sqrt[q]{x} + i(y^p + \lambda\varsigma\sqrt[q]{y}) & \xi(w + \sin w) + \varrho\lambda w^n + \mu\lambda\sqrt[s]{w} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{f}(z) & \widehat{h}(x) - i\widehat{h}(y) \\ \widehat{h}(x) + i\widehat{h}(y) & \widehat{g}(w) \end{pmatrix}.\end{aligned}$$

Since m, n, p, q, l, and s are odd natural numbers, it is easy to see that $\widetilde{f}(\mathbb{R}) = \widetilde{g}(\mathbb{R}) = \widetilde{h}(\mathbb{R}) = \mathbb{R}$, which implies that $(H_2 + \lambda M)(H_2(\mathbb{C})) = H_2(\mathbb{C})$, that is, the mapping $H_2 + \lambda M$ is surjective. Taking into account the arbitrariness in the choice of $\lambda > 0$, we conclude that M is a general H_2-monotone operator.

Remark 1. If $H = J$, the identity mapping on B, then the definition of general J-monotone mappings is that of maximal monotone operators. In fact, the class of general H-monotone mappings has a close relation with that of maximal monotone operators. This fact is illustrated by the following assertion.

Lemma 1 ([35] Proposition 2.1)**.** *Let B be a reflexive Banach space with the dual space B^*. Let $H : B \to B^*$ be a strictly monotone mapping, and let $M : B \to 2^{B^*}$ be a general H-monotone mapping. If $\langle u - v, x - y\rangle \geq 0$ holds for all $(y, v) \in \text{Graph}(M)$, then $u \in M(x)$, that is, M is maximal monotone.*

Invoking Example 1, for a given mapping $H : B \to B^*$, a maximal monotone mapping need not be general H-monotone. A natural question then arises—that of the conditions under which, for a given mapping $H : B \to B^*$, a maximal monotone mapping is general H-monotone.

To answer this question, we need the concepts presented in the next definition.

Definition 8. *Let B be a Banach space with the dual space B^*. A mapping $H : B \to B^*$ is said to be*

(i) *coercive if* $\lim\limits_{\|x\| \to +\infty} \frac{\langle H(x), x\rangle}{\|x\|} = +\infty$;
(ii) *hemi-continuous if, for any fixed $x, y, z \in B$, the function $t \mapsto \langle H(x + ty), z\rangle$ is continuous at 0^+;*
(iii) *bounded if $H(A)$ is a bounded subset of B^* for every bounded subset A of B.*

Now, an answer to the question raised above is given by the following result.

Lemma 2. *Let B be a reflexive Banach space with the dual space B^*, and let $H : B \to B^*$ be a bounded, coercive, hemi-continuous, and monotone mapping. If $M : B \to 2^{B^*}$ is a maximal monotone mapping, then M is general H-monotone.*

Proof. Taking into account that H is bounded, coercive, hemi-continuous, and monotone, by means of Theorem 4.5 on page 315 of Guo [36], it follows that $(H + \lambda M)(B) = B^*$ for every $\lambda > 0$. Accordingly, M is a general H-monotone mapping. □

Theorem 1. *Let B be a reflexive Banach space with the dual space B^*. Let $H : B \to B^*$ be a monotone mapping, and let $M : B \to 2^{B^*}$ be a β-strongly monotone mapping. Then, the mapping $(H + \lambda M)^{-1} : \mathrm{Range}(H + \lambda M) \to B$ is single-valued for every $\lambda > 0$.*

Proof. Let $\lambda > 0$ be an arbitrary real constant. For any given $z^* \in \mathrm{Range}(H + \lambda M)$, letting $x, y \in (H + \lambda M)^{-1}(z^*)$, we have $z^* = (H + \lambda M)(x) = (H + \lambda M)(y)$, which implies that $\lambda^{-1}(z^* - H(x)) \in M(x)$ and $\lambda^{-1}(z^* - H(y)) \in M(y)$. In light of the facts that H is monotone and M is β-strongly monotone, we conclude that

$$\lambda\beta\|x - y\|^2 \leq \lambda\langle\lambda^{-1}(z^* - H(x)) - \lambda^{-1}(z^* - H(y)), x - y\rangle + \langle H(x) - H(y), x - y\rangle = 0.$$

Since $\lambda, \beta > 0$, by utilizing the preceding inequality, it follows that $x = y$, which guarantees that the mapping $H + \lambda M$ from $\mathrm{Range}(H + \lambda M)$ into B is single-valued. This gives the desired result. □

In the rest of this paper, the notion of general strong H-monotonicity with constant β of the set-valued mapping $M : B \to 2^{B^*}$ will be used to mean that M is a β-strongly monotone mapping and $(H + \lambda M)(B) = B^*$ for every $\lambda > 0$, that is, M is a β-strongly and general H-monotone mapping.

The single-valuedness of the mapping $(H + \lambda M)^{-1} : B^* \to B$ immediately follows from the last result.

Corollary 1. *Let B be a reflexive Banach space with the dual space B^*. Furthermore, let $H : B \to B^*$ be a monotone mapping, and let $M : B \to 2^{B^*}$ be a general strongly H-monotone mapping with constant β. Then, the mapping $(H + \lambda M)^{-1} : B^* \to B$ is single-valued for every $\lambda > 0$.*

With the help of Corollary 1, we are able to define the proximal mapping $R^H_{M,\lambda}$ associated with H, M, and an arbitrary positive real constant λ as follows.

Definition 9. *Assume that B is a reflexive Banach space with the dual space B^*. Let $H : B \to B^*$ be a monotone mapping, and let $M : B \to 2^{B^*}$ be a general strongly H-monotone with constant β. For every real constant $\lambda > 0$, the proximal mapping $R^H_{M,\lambda} : B^* \to B$ is defined by*

$$R^H_{M,\lambda}(x^*) = (H + \lambda M)^{-1}(x^*), \quad \forall x^* \in B^*.$$

We now close this section with the following theorem, in which the necessary conditions for proving the Lipschitz continuity of the proximal mapping $R^H_{M,\lambda}$ and calculating its Lipschitz constant are stated.

Theorem 2. *Let B be a reflexive Banach space with the dual space B^*. Suppose, further, that $H : B \to B^*$ is a monotone mapping and that $M : B \to 2^{B^*}$ is a general strongly H-monotone mapping with constant β. Then, for any real constant $\lambda > 0$, the proximal mapping $R^H_{M,\lambda} : B^* \to B$ is $\frac{1}{\lambda\beta}$-Lipschitz continuous, i.e.,*

$$\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\| \leq \frac{1}{\lambda\beta}\|x^* - y^*\|, \quad \forall x^*, y^* \in B^*.$$

Proof. Since M is a general H-monotone mapping, for any given points $x^*, y^* \in B^*$ with $\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\| \neq 0$, we have

$$R^H_{M,\lambda}(x^*) = (H + \lambda M)^{-1}(x^*) \text{ and } R^H_{M,\lambda}(y^*) = (H + \lambda M)^{-1}(y^*),$$

which implies that

$$\lambda^{-1}\big(x^* - H(R^H_{M,\lambda}(x^*))\big) \in M(R^H_{M,\lambda}(x^*)) \text{ and } \lambda^{-1}\big(y^* - H(R^H_{M,\lambda}(y^*))\big) \in M(R^H_{M,\lambda}(y^*)).$$

Taking into consideration the fact that M is β-strongly monotone, we conclude that

$$\lambda^{-1}\langle x^* - H(R^H_{M,\lambda}(x^*)) - (y^* - H(R^H_{M,\lambda}(y^*))), R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\rangle$$
$$\geq \beta\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\|^2.$$

Owing to the fact that $\lambda^{-1} > 0$, from the preceding inequality, we get

$$\langle x^* - y^*, R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\rangle \geq \langle H(R^H_{M,\lambda}(x^*)) - H(R^H_{M,\lambda}(y^*)), R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\rangle$$
$$+ \lambda\beta\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\|^2.$$

In light of the fact that H is monotone, the last inequality implies that

$$\begin{aligned}
&\|x^* - y^*\|\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\| \\
&\geq \langle x^* - y^*, R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\rangle \\
&\geq \langle H(R^H_{M,\lambda}(x^*)) - H(R^H_{M,\lambda}(y^*)), R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\rangle \\
&\quad + \lambda\beta\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\|^2 \\
&\geq \lambda\beta\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\|^2.
\end{aligned}$$

By virtue of the fact that $\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\| \neq 0$, from the preceding inequality, it follows that

$$\|R^H_{M,\lambda}(x^*) - R^H_{M,\lambda}(y^*)\| \leq \frac{1}{\lambda\beta}\|x^* - y^*\|.$$

This completes the proof. □

Remark 2. It should be pointed out that by comparing Theorem 2 and the provided corresponding result in part (ii) of Theorem 3.2 in [13], it can be easily observed that Theorem 2 improves part (ii) of Theorem 3.2 in [13]. In fact, the Lipschitz continuity of the proximal mapping $R^H_{M,\lambda}$ and its Lipschitz constant in [13] are proved and calculated under the assumption of strict monotonicity for the mapping $H : B \to B^*$, whereas the same results are derived in Theorem 2 under the assumption of monotonicity of H, which is a weaker condition of strict monotonicity.

3. Formulation of the Problem: Existence and Uniqueness of a Solution

Let $p \in \mathbb{N}$ be an arbitrary constant; for each $i \in \{1, 2, \ldots, p\}$, let B_i be a real Banach space with the topological dual space B_i^*, and let $\langle ., .\rangle_i$ be the pairing between B_i and B_i^*. With slight abuse of notation, for each $i \in \{1, 2, \ldots, p\}$, we use the same symbol $\|.\|_i$ for the norms in B_i and B_i^*. Suppose, further, that $g_i : B_i \to B_i$, $A_i : \prod_{l=1}^{p} B_l \to B_i^*$ and $M_i : B_i \times B_i \to 2^{B_i^*}$ $(i = 1, 2, \ldots, p)$ are the mappings. We consider the problem of finding $(u_1, u_2, \ldots, u_p) \in \prod_{l=1}^{p} B_l$ such that

$$0 \in A_i(u_1, u_2, \ldots, u_p) + M_i(g_i(u_i), u_i), \tag{2}$$

which is called a *system of generalized variational inclusions* (SGVI) in real Banach spaces.

If $p = 1$, $B_1 = B$ is a real Banach space with the topological dual space B^* and the norm $\|.\|$, $g : B \to B$ and $A : B \to B^*$ are single-valued mappings, and $M : B \to 2^{B^*}$ is a univariate set-valued mapping; then, the SGVI (2) reduces to the following variational inclusion problem (VIP): Find $u \in B$ such that

$$0 \in A(u) + M(g(u)), \tag{3}$$

which was considered and studied by Xia and Huang [13].

We remark that for suitable choices of the mappings A_i, g_i, M_i and the underlying spaces B_i $(i = 1, 2, \dots, p)$, the SGVI (2) reduces to various classes of variational inclusions and variational inequalities; see, for example, [5–7,13,37–41] and the references therein.

The following assertion, which tells that the SGVI (2) is equivalent to a fixed-point problem, gives a characterization of the solution of the SGVI (2).

Lemma 3. *For each $i \in \{1, 2, \dots, p\}$, let B_i be a reflexive Banach space with the dual space B_i^*, and let $A_i : \prod_{l=1}^{p} B_l \to B_i^*$, $g_i : B_i \to B_i$, and $H_i : B_i \to B_i^*$ be the mappings such that for each $i \in \{1, 2, \dots, p\}$, H_i is a monotone mapping with $g_i(B_i) \cap \operatorname{dom} H_i \neq \emptyset$. Suppose, further, that for each $i \in \{1, 2, \dots, p\}$, $M_i : B_i \times B_i \to 2^{B_i^*}$ is a set-valued mapping such that for each $w_i \in B_i$, $M_i(., w_i) : B_i \to 2^{B_i^*}$ is a general strongly H_i-monotone mapping with constant β_i and $g_i(B_i) \cap \operatorname{dom} M_i(., w_i) \neq \emptyset$. Then, $(u_1, u_2, \dots, u_p) \in \prod_{l=1}^{p} B_l$ is a solution of the* SGVI (2) *if and only if $(u_1, u_2, \dots, u_p)$ satisfies*

$$g_i(u_i) = R^{H_i}_{M_i(.,u_i),\lambda_i}[H_i \circ g_i(u_i) - \lambda_i A_i(u_1, u_2, \dots, u_p)], \tag{4}$$

where $i = 1, 2, \dots, p$; $\lambda_i > 0$ are real constants; $H_i \circ g_i$ denotes the H_i composition g_i, and $R^{H_i}_{M_i(.,u_i),\lambda_i} = (H_i + \lambda_i M_i(., u_i))^{-1}$.

Proof. From Definition 9, it follows that $(u_1, u_2, \dots, u_p) \in \prod_{i=1}^{p} B_i$ is a solution of the SGVI (2) if and only if

$$0 \in A_i(u_1, u_2, \dots, u_p) + M_i(g_i(u_i), u_i),$$

$\Leftrightarrow$

$$H_i \circ g_i(u_i) - \lambda_i A_i(u_1, u_2, \dots, u_p) \in (H_i + \lambda_i M_i(., u_i))(g_i(u_i))$$

$\Leftrightarrow$

$$\begin{aligned} g_i(u_i) &= (H_i + \lambda_i M_i(., u_i))^{-1}[H_i \circ g_i(u_i) - \lambda_i A_i(u_1, u_2, \dots, u_p)] \\ &= R^{H_i}_{M_i(.,u_i),\lambda_i}[H_i \circ g_i(u_i) - \lambda_i A_i(u_1, u_2, \dots, u_p)], \end{aligned}$$

where $i = 1, 2, \dots, p$; $\lambda_i > 0$ are real constants; $H_i \circ g_i$ denotes the H_i composition g_i, and $R^{H_i}_{M_i(.,u_i),\lambda_i} = (H_i + \lambda_i M_i(., u_i))^{-1}$. □

As a direct consequence of the above assertion, we derive the following result.

Lemma 4. *Assume that B is a real Banach space with the dual space B^*, and $A, H : B \to B^*$ and $g : B \to B$ are the mappings such that $g(B) \cap \operatorname{dom} H \neq \emptyset$. Furthermore, let $M : B \to 2^{B^*}$ be a general strongly H-monotone mapping with constant β and $g(B) \cap \operatorname{dom} M \neq \emptyset$. Then, $u \in B$ is a solution of the* VIP (3) *if and only if u satisfies*

$$g(u) = R^H_{M,\lambda}[H \circ g(u) - \lambda A(u)],$$

where $\lambda > 0$ is an arbitrary real constant and $R^H_{M,\lambda} = (H + \lambda M(., u))^{-1}$.

Before proceeding to the main result of this paper, we need to give some specific notions and recall an efficient lemma.

Definition 10. *For each $i \in \{1,2,\ldots,p\}$, let B_i be a real Banach space with the topological dual space B_i^*. A mapping $A : \prod_{j=1}^{p} B_j \to B_i^*$ is said to be α_i-Lipschitz continuous in the ith argument if there exists a constant $\alpha_i > 0$ such that*

$$\begin{aligned}&\|A(u_1,u_2,\ldots,u_{i-1},u_i,u_{i+1},\ldots,u_p) - A(u_1,u_2,\ldots,u_{i-1},u_i',u_{i+1},\ldots,u_p)\|_i \\ &\le \alpha_i\|u_i - u_i'\|_i, \quad \forall u_i,u_i' \in B_i, u_j \in B_j (j=1,2,\ldots,p; j\neq i).\end{aligned}$$

Definition 11. *Let B be a real uniformly smooth Banach space with the dual space B^*, and let J be the normalized duality mapping from B into B^*. A mapping $g : B \to B$ is said to be k-strongly accretive if there exists a constant $k > 0$ such that*

$$\langle J(x-y), g(x)-g(y)\rangle \ge k\|x-y\|^2, \quad \forall x,y \in B.$$

Lemma 5 ([42])**.** *Let B be a uniformly smooth Banach space, and let J be the normalized duality mapping from B into B^*. Then, for all $x,y \in B$, we have*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle J(x+y),y\rangle$;

(ii) $\langle J(x)-J(y),x-y\rangle \le 2d^2(x,y)\rho_B(\frac{4\|x-y\|}{d(x,y)})$, *where* $d(x,y) = \sqrt{\frac{\|x\|^2+\|y\|^2}{2}}$.

Theorem 3. *For each $i \in \Gamma = \{1,2,\ldots,p\}$, let B_i be a real uniformly smooth Banach space with the dual space B_i^* and $\rho_{B_i}(t) \le C_i t^2$ for some $C_i > 0$. Suppose that for each $i \in \Gamma$, $g_i : B_i \to B_i$ is a k_i-strongly accretive and δ_i-Lipschitz continuous mapping, $H_i : B_i \to B_i^*$ is a monotone and s_i-Lipschitz continuous mapping with $g_i(B_i) \cap \operatorname{dom} H_i \neq \emptyset$, and the mapping $A_i : \prod_{l=1}^{p} B_l \to B_i^*$ is α_i-Lipschitz continuous in the ith argument and $\varsigma_{i,j}$-Lipschitz continuous in the jth argument $(j \in \Gamma, j \neq i)$. For each $i \in \Gamma$, let $M_i : B_i \times B_i \to 2^{B_i^*}$ be a set-valued mapping such that for each $w_i \in B_i$, $M_i(.,w_i) : B_i \to 2^{B_i^*}$ is a general strongly H_i-monotone mapping with constant β_i and $g_i(B_i) \cap \operatorname{dom} M_i(.,w_i) \neq \emptyset$. Assume, further, that for each $i \in \Gamma$, there exist constants $\varrho_i, \lambda_i > 0$ such that*

$$\|R^{H_i}_{M_i(.,x_i),\lambda_i}(z_i) - R^{H_i}_{M_i(.,y_i),\lambda_i}(z_i)\|_i \le \varrho_i\|x_i - y_i\|_i, \quad \forall x_i,y_i,z_i \in B_i \tag{5}$$

and

$$\begin{cases} \lambda_i > \dfrac{s_i\delta_i}{\beta_i-\alpha_i-\beta_i\left(\varrho_i+\sqrt{1-2k_i+64C_i\delta_i^2}+\sum\limits_{q\in\Gamma,q\neq i}\frac{\varsigma_{q,i}}{\beta_q}\right)}, \\ \beta_i-\alpha_i-\beta_i\left(\varrho_i+\sqrt{1-2k_i+64C_i\delta_i^2}+\sum\limits_{q\in\Gamma,q\neq i}\frac{\varsigma_{q,i}}{\beta_q}\right) > 0, \\ 1+64C_i\delta_i^2 > 2k_i. \end{cases} \tag{6}$$

Then, the SGVI (2) *admits a unique solution.*

Proof. We define, for each $i \in \Gamma = \{1,2,\ldots,p\}$, the mapping $F_i : \prod_{q=1}^{p} B_q \to B_i$ by

$$F_i(u_1,u_2,\ldots,u_p) = u_i - g_i(u_i) + R^{H_i}_{M_i(.,u_i),\lambda_i}[H_i \circ g_i(u_i) - \lambda_i A_i(u_1,u_2,\ldots,u_p)], \tag{7}$$

for all $(u_1,u_2,\ldots,u_p) \in \prod_{q=1}^{p} B_q$. Let $\|.\|_*$ be a function defined on $\prod_{q=1}^{p} B_q$ by

$$\|(u_1,u_2,\ldots,u_p)\|_* = \sum_{q=1}^{p}\|x_q\|_q, \quad \forall (u_1,u_2,\ldots,u_p) \in \prod_{q=1}^{p} B_q. \tag{8}$$

It can be easily observed that $(\prod_{q=1}^{p} B_q, \|.\|_*)$ is a Banach space. Assume, further, that the mapping $T : \prod_{q=1}^{p} B_q \to \prod_{q=1}^{p} B_q$ is defined by

$$T(u_1, u_2, \ldots, u_p) = (F_1(u_1, u_2, \ldots, u_p), \ldots, F_p(u_1, u_2, \ldots, u_p)), \tag{9}$$

for all $(x_1, x_2, \ldots, x_p) \in \prod_{q=1}^{p} B_q$. We now prove that T is a contraction mapping. For this end, we choose $(u_1, u_2, \ldots, u_p), (u'_1, u'_2, \ldots, u'_p) \in \prod_{q=1}^{p} B_q$ arbitrarily. Making use of (5), (7), and Theorem 2, it follows that for each $i \in \Gamma$,

$$\begin{aligned}
&\|T_i(u_1, u_2, \ldots, u_p) - T_i(u'_1, u'_2, \ldots, u'_p)\|_i \\
&\leq \|u_i - u'_i - (g_i(u_i) - g_i(u'_i))\|_i \\
&\quad + \|R^{H_i}_{M_i(.,u_i),\lambda_i}[H_i \circ g_i(u_i) - \lambda_i A_i(u_1, u_2, \ldots, u_p)] \\
&\quad - R^{H_i}_{M_i(.,u'_i),\lambda_i}[H_i \circ g_i(u_i) - \lambda_i A_i(u_1, u_2, \ldots, u_p)]\|_i \\
&\quad + \|R^{H_i}_{M_i(.,u'_i),\lambda_i}[H_i \circ g_i(u_i) - \lambda_i A_i(u_1, u_2, \ldots, u_p)] \\
&\quad - R^{H_i}_{M_i(.,u'_i),\lambda_i}[H_i \circ g_i(u'_i) - \lambda_i A_i(u'_1, u'_2, \ldots, u'_p)]\|_i \\
&\leq \|u_i - u'_i - (g_i(u_i) - g_i(u'_i))\|_i + \varrho_i\|u_i - u'_i\|_i \\
&\quad + \frac{1}{\lambda_i\beta_i}\|H_i \circ g_i(u_i) - H_i \circ g_i(u'_i) \\
&\quad - \lambda_i(A_i(u_1, u_2, \ldots, u_p) - A_i(u'_1, u'_2, \ldots, u'_p))\|_i \\
&\leq \|u_i - u'_i - (g_i(u_i) - g_i(u'_i))\|_i + \varrho_i\|u_i - u'_i\|_i \\
&\quad + \frac{1}{\lambda_i\beta_i}\|H_i \circ g_i(u_i) - H_i \circ g_i(u'_i)\|_i \\
&\quad + \frac{1}{\beta_i}\|A_i(u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_p) \\
&\quad - A_i(u_1, u_2, \ldots, u_{i-1}, u'_i, u_{i+1}, \ldots, u_p)\|_i \\
&\quad + \frac{1}{\beta_i}\sum_{j\in\Gamma, j\neq i}\|A_i(u_1, u_2, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_p) \\
&\quad - A_i(u_1, u_2, \ldots, u_{j-1}, u'_j, u_{j+1}, \ldots, u_p)\|_i.
\end{aligned} \tag{10}$$

Taking into account that for each $i \in \{1, 2, \ldots, k\}$, g_i is a k_i-strongly accretive and δ_i-Lipschitz continuous mapping and B_i is a uniformly smooth Banach space with $\rho_{B_i}(t) \leq C_i t^2$ for some $C_i > 0$, by Lemma 5, we get

$$\begin{aligned}
&\|u_i - u'_i - (g_i(u_i) - g_i(u'_i))\|_i^2 \\
&\leq \|u_i - u'_i\|_i^2 + 2\langle J_i(u_i - u'_i - (g_i(u_i) - g_i(u'_i))), -(g_i(u_i) - g_i(u'_i))\rangle_i \\
&= \|u_i - u'_i\|_i^2 - 2\langle J_i(u_i - u'_i), g_i(u_i) - g_i(u'_i)\rangle_i \\
&\quad + 2\langle J_i(u_i - u'_i - (g_i(u_i) - g_i(u'_i))) - J_i(u_i - u'_i), -(g_i(u_i) - g_i(u'_i))\rangle_i \\
&\leq \|u_i - u'_i\|_i^2 - 2k_i\|u_i - u'_i\|_i + 4d_i^2(u_i - u'_i - (g_i(u_i) - g_i(u'_i)), u_i - u'_i) \\
&\quad \times \rho_{B_i}\Big(\frac{4\|g_i(u_i) - g_i(u'_i)\|_i}{d_i(u_i - u'_i - (g_i(u_i) - g_i(u'_i)), u_i - u'_i)}\Big) \\
&\leq (1 - 2k_i + 64C_i\delta_i^2)\|u_i - u'_i\|_i^2,
\end{aligned} \tag{11}$$

where, for each $i \in \Gamma$, J_i is the normalized duality mapping from B_i to B_i^*.

From (11), it follows that for each $i \in \Gamma$,

$$\|u_i - u_i' - (g_i(u_i) - g_i(u_i'))\|_i \leq \sqrt{1 - 2k_i + 64C_i\delta_i^2}\|u_i - u_i'\|_i. \tag{12}$$

In light of the facts that for each $i \in \Gamma$, H_i is s_i-Lipschitz continuous, g_i is δ_i-Lipschitz continuous, and A_i is α_i-Lipschitz continuous in the ith argument and $\varsigma_{i,j}$-Lipschitz continuous in the jth argument $(j \in \Gamma, j \neq i)$, we conclude that for each $i \in \Gamma$,

$$\|H_i \circ g_i(u_i) - H_i \circ g_i(u_i')\|_i \leq s_i\delta_i\|u_i - u_i'\|_i, \tag{13}$$

$$\begin{aligned}&\|A_i(u_1, u_2, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_p)\\ &- A_i(u_1, u_2, \dots, u_{i-1}, u_i', u_{i+1}, \dots, u_p)\|_i \leq \alpha_i\|u_i - u_i'\|_i\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\|A_i(u_1, u_2, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_p)\\ &- A_i(u_1, u_2, \dots, u_{j-1}, u_j', u_{j+1}, \dots, u_p)\|_i \leq \varsigma_{i,j}\|u_j - u_j'\|_j.\end{aligned} \tag{15}$$

Substituting (12)–(15) into (10), for each $i \in \Gamma$, we get

$$\begin{aligned}&\|F_i(u_1, u_2, \dots, u_p) - F_i(u_1', u_2', \dots, u_p')\|_i\\ &\leq \sigma_i\|u_i - u_i'\|_i + \frac{1}{\beta_i}\sum_{j\in\Gamma, j\neq i}\varsigma_{i,j}\|u_j - u_j'\|_j,\end{aligned} \tag{16}$$

where, for each $i \in \Gamma$,

$$\sigma_i = \varrho_i + \sqrt{1 - 2k_i + 64C_i\delta_i^2} + \frac{s_i\delta_i + \lambda_i\alpha_i}{\lambda_i\beta_i}.$$

Making use of (9) and (16), we obtain

$$\begin{aligned}&\|T(u_1, u_2, \dots, u_p) - T(u_1', u_2', \dots, u_p')\|_*\\ &= \sum_{i=1}^{p}\|F_i(u_1, u_2, \dots, u_p) - F_i(u_1', u_2', \dots, u_p')\|_i\\ &\leq \sum_{i=1}^{p}\Big(\sigma_i\|u_i - u_i'\|_i + \frac{1}{\beta_i}\sum_{j\in\Gamma, j\neq i}\varsigma_{i,j}\|u_j - u_j'\|_j\Big)\\ &= \Big(\sigma_1 + \sum_{q=2}^{p}\frac{\varsigma_{q,1}}{\beta_q}\Big)\|u_1 - u_1'\|_1 + \Big(\sigma_2 + \sum_{q\in\Gamma, q\neq 2}\frac{\varsigma_{q,2}}{\beta_q}\Big)\|u_2 - u_2'\|_2\\ &\quad + \cdots + \Big(\sigma_p + \sum_{q=1}^{p-1}\frac{\varsigma_{q,p}}{\beta_q}\Big)\|u_p - u_p'\|_p\\ &\leq \varphi\sum_{i=1}^{p}\|u_i - u_i'\|_i = \varphi\|(u_1, u_2, \dots, u_p) - (u_1', u_2', \dots, u_p')\|_*,\end{aligned} \tag{17}$$

where $\varphi = \max\{\sigma_i + \sum_{q\in\Gamma, q\neq i}\frac{\varsigma_{q,i}}{\beta_q} : i = 1, 2, \dots, p\}$. Due to (6), it follows that $\varphi \in (0, 1)$, so (17) ensures that T is a contraction mapping. Consequently, invoking the Banach fixed-point theorem, there exists $(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_p) \in \prod_{q=1}^{p} B_q$ such that $T(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_p) = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_p)$. Thereby, (7) and (9) imply that $(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_p)$ satisfies Equation (4), that is, for each $i \in \Gamma$,

$$g_i(\hat{u}_i) = R^{H_i}_{M_i(.,\hat{u}_i),\lambda_i}[H_i \circ g_i(\hat{u}_i) - \lambda_i A_i(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_p)].$$

Now, Lemma 3 guarantees that $(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_p) \in \prod_{q=1}^{p} B_q$ is a unique solution of the SGVI (2). The proof is finished. □

As a direct consequence of the previous theorem, we obtain the following.

Corollary 2. *Suppose that B is a real uniformly smooth Banach space with the dual space B^* and $\rho_B(t) \leq Ct^2$ for some $C > 0$. Let $g : B \to B$ be a k-strongly accretive and δ-Lipschitz continuous mapping, let $H : B \to B^*$ be a monotone and s-Lipschitz continuous mapping with $g(B) \cap \operatorname{dom} H \neq \emptyset$, and let the mapping $A : B \to B^*$ be α-Lipschitz continuous. Assume, further, that $M : B \to 2^{B^*}$ is a general strongly H-monotone mapping with constant β and $g(B) \cap \operatorname{dom} M \neq \emptyset$. If there exists a constant $\lambda > 0$ such that*

$$\begin{cases} \lambda > \frac{s\delta}{\beta - \alpha - \beta\sqrt{1-2k+64C\delta^2}}, \\ \beta - \alpha - \beta\sqrt{1-2k+64C\delta^2} > 0, \\ 1 + 64C\delta^2 > 2k, \end{cases} \tag{18}$$

then the VIP (3) *has a unique solution.*

Given a real normed space B with a norm $\|.\|$, it is well known that a mapping $T : B \to B$ is said to be nonexpansive whenever $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in B$. Due to the existence of a deep and close relation between the classes of monotone and accretive operators, which naturally arise in the theory of differential equations, and the class of nonexpansive mappings, the notion of nonexpansive mapping had a rapid development and a prolific growth of its applications from the beginning. Because of the importance and active impact of nonexpansive mapping in fixed-point theory, the study of nonexpansive mappings in the frameworks of different spaces has been conducted extensively by many mathematicians in recent decades, and several generalizations and extensions of them have been introduced and analyzed. One of the first attempts in this direction was carried out by Goebel and Kirk [23], who defined the following notion of asymptotically nonexpansive mapping in 1972.

Definition 12 ([23])**.** *A mapping $T : B \to B$ is said to be asymptotically nonexpansive if there exists a sequence $\{a_n\} \subset (0, \infty)$ with $\lim_{n\to\infty} a_n = 0$ such that for all $x, y \in B$,*

$$\|T^n(x) - T^n(y)\| \leq (1 + a_n)\|x - y\|, \quad \forall n \in \mathbb{N}.$$

Efforts in this direction continued, and other generalized nonexpansive mappings were defined as follows.

Definition 13. *A nonlinear mapping $T : B \to B$ is said to be*

(i) *total asymptotically nonexpansive (also referred to as $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically nonexpansive in the literature)* [27] *if there exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in B$,*

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + a_n\phi(\|x - y\|) + b_n, \quad \forall n \in \mathbb{N}.$$

(ii) *total uniformly L-Lipschitzian (or $(\{a_n\}, \{b_n\}, \phi)$-total uniformly L-Lipschitzian)* [28] *if there exist a constant $L > 0$, nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$, and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for each $n \in \mathbb{N}$,*

$$\|T^n(x) - T^n(y)\| \leq L[\|x - y\| + a_n\phi(\|x - y\|) + b_n], \quad \forall x, y \in B.$$

It is significant to mention that every asymptotically nonexpansive mapping is total asymptotically nonexpansive with $b_n = 0$ (or equivalently $b_n = 0$ and $a_n = k_n - 1$) for all $n \in \mathbb{N}$ and $\phi(t) = t$ for all $t \geq 0$, but the converse is, in general, not true. This fact is shown in the next example.

Example 3. For $1 \leq p < \infty$, consider the classical space

$$l^p = \{x = \{x_n\}_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty, x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\},$$

consisting of all p-power summable sequences equipped with the p-norm $\|.\|_p$, which is defined on it by

$$\|x\|_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}}, \quad \forall x = \{x_n\}_{n\in\mathbb{N}} \in l^p.$$

Suppose, further, that B_{l^p} is the closed unit ball in the Banach space l^p, and let $X = \mathbb{R} \times l^p$ be endowed with the norm $\|.\|_X = |.|_{\mathbb{R}} + \|.\|_p$. Consider $B := [0, \alpha] \times B_{l^p}$ as a subset of X, where $\alpha \in (0, 1]$ is an arbitrary real constant. Suppose, further, that the self-mapping T of B is defined by

$$T(u, x) = \begin{cases} \beta(u, \widehat{x}), & \text{if } u \in [0, \alpha], \\ (0, \beta\widehat{x}), & \text{if } u = \alpha, \end{cases}$$

where $\widehat{x} = \{\widehat{x}_n\}_{n\in\mathbb{N}}$, with $\widehat{x}_i = 0$ for all $1 \leq i \leq q$, $\widehat{x}_{q+2i} = 0$ for all $i \in \mathbb{N}$,

$$\widehat{x}_{q+2i-1} = \begin{cases} \gamma|x_i|^{m_{\frac{i+2}{3}}}, & \text{if } i \in \{3r-2 | r = 1, 2, \ldots, \frac{t+2}{3}\}, \\ \frac{\gamma}{\sqrt[p]{2^{p+1}}}\big(|x_i|^{\lambda_{\frac{i+1}{3}}} - \sin|x_i|^{b_{\frac{i+1}{3}}}\big), & \text{if } i \in \{3r-1 | r = 1, 2, \ldots, \frac{t+2}{3}\}, \\ \frac{\gamma}{\sqrt[p]{2^{p+1}}}\big(\sin^{k_{\frac{i}{3}}} |x_i| - |x_i|^{s_{\frac{i}{3}}}\big), & \text{if } i \in \{3r | r = 1, 2, \ldots, \frac{t+2}{3}\}, \end{cases}$$

and $\widehat{x}_{q+2t+j} = \gamma x_{t+\frac{j+1}{2}}$ for all $j \in \{2l + 3 | l \in \mathbb{N}\}$; $\beta, \gamma \in (0, 1)$ are arbitrary real constants, $t \in \{3s - 2 | s \in \mathbb{N}\}$, and $q \geq t + 2$ and $m_i, k_i, \lambda_i, b_i, s_i \in \mathbb{N}$ $(i = 1, 2, \ldots, \frac{t+2}{3})$ are arbitrary but fixed natural numbers. Taking into account that the mapping T is discontinuous at the points (α, x) for all $x \in B_{l^p}$, it follows that T is not Lipschitzian, so it is not an asymptotically nonexpansive mapping. It can be easily observed that for all $(u, x), (v, y) \in [\alpha, 0) \times B_{l^p}$,

$$\begin{aligned} \|T(u,x) - T(v,y)\|_X \leq |u - v| + \gamma \max\{&\sum_{j=1}^{m_i} |x_{3i-2}|^{m_i - j}|y_{3i-2}|^{j-1}, \\ &\sum_{r=1}^{\lambda_i} |x_{3i-1}|^{\lambda_i - r}|y_{3i-1}|^{r-1}, \sum_{s'=1}^{b_i} |x_{3i-1}|^{b_i - s'}|y_{3i-1}|^{s'-1}, \\ &\sum_{r'=1}^{k_i} |x_{3i}|^{k_i - r'}|y_{3i}|^{r'-1}, \sum_{r''=1}^{s_i} |x_{3i}|^{s_i - r''}|y_{3i}|^{r''-1}, 1 : \\ &i = 1, 2, \ldots, \frac{t+2}{3}\}\|x - y\|_p + \beta \\ \leq\ & |u - v| + \gamma\varsigma\|x - y\|_p + \beta, \end{aligned} \tag{19}$$

where $\varsigma = \max\{m_i, \lambda_i, b_i, k_i, s_i : i = 1, 2, \ldots, \frac{t+2}{3}\}$. If $u \in [0, \alpha)$ and $v = \alpha$, then, in a similar fashion to that of the preceding analysis, owing to the fact that $0 < |u - v| \leq \alpha \leq 1$, we conclude that

$$\begin{aligned} \|T(u,x) - T(v,y)\|_X &\leq \beta(|u| + \sigma\iota\|x - y\|_p) \\ &\leq \beta(1 + \gamma\varsigma\|x - y\|_p) \\ &< |u - v| + \gamma\varsigma\|x - y\|_p + \beta. \end{aligned} \tag{20}$$

If $u = v = b$, then, by an argument similar to that of (19), for all $x \in B_{l^p}$, we get

$$\begin{aligned}\|T(u,x) - T(v,y)\|_X &= \beta\gamma\varsigma\|x-y\|_p \\ &< \beta(|u-v| + \gamma\varsigma\|x-y\|_p + 1) \\ &< |u-v| + \gamma\varsigma\|x-y\|_p + \beta.\end{aligned} \tag{21}$$

Making use of (19)–(21), we derive that for all $(u,x),(v,y) \in B$,

$$\begin{aligned}\|T(u,x) - T(v,y)\|_X &\le |u-v| + \gamma\varsigma\|x-y\|_p + \beta \\ &\le |u-v| + \|x-y\|_p + \gamma\varsigma(|u-v| + \|x-y\|_p) + \beta.\end{aligned} \tag{22}$$

For all $n \ge 2$ and $(u,x) \in [0,\alpha) \times B_{l^p}$, we have $T^n(u,x) = \beta^n(u,\widetilde{x})$, where

$$\begin{aligned}\widetilde{x} = (\, &\underbrace{0,0,\ldots,0}_{(2^n-1)q\ times}, \gamma^n|x_1|^{m_1}, \underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \Big(\frac{\gamma}{\sqrt[p]{2^{p+1}}}\Big)^n(|x_2|^{\lambda_1} - \sin|x_2|^{b_1}), \underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \\ &\Big(\frac{\gamma}{\sqrt[p]{2^{p+1}}}\Big)^n(\sin^{k_1}|x_3| - |x_3|^{s_1}), \underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \gamma^n|x_4|^{m_2}, \underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \Big(\frac{\gamma}{\sqrt[p]{2^{p+1}}}\Big)^n(|x_5|^{\lambda_2} \\ &- \sin|x_5|^{b_2}), \underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \Big(\frac{\gamma}{\sqrt[p]{2^{p+1}}}\Big)^n(\sin^{k_2}|x_6| - |x_6|^{s_2}), \ldots, \gamma^n|x_t|^{m_{\frac{t+2}{3}}}, \underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \\ &\Big(\frac{\gamma}{\sqrt[p]{2^{p+1}}}\Big)^n(|x_{t+1}|^{\lambda_{\frac{t+2}{3}}} - \sin|x_{t+1}|^{b_{\frac{t+2}{3}}}), \underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \Big(\frac{\gamma}{\sqrt[p]{2^{p+1}}}\Big)^n(\sin^{k_{\frac{t+2}{3}}}|x_{t+2}| - |x_{t+2}|^{s_{\frac{t+2}{3}}}), \\ &\underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \gamma^n x_{t+3}, \underbrace{0,0,\ldots,0}_{(2^n-1)\ times}, \gamma^n x_{t+4}, \ldots).\end{aligned}$$

Then, for all $(u,x),(v,y) \in [0,\alpha) \times B_{l^p}$ and $n \ge 2$, by an argument analogous to the previous one, one can prove that

$$\begin{aligned}\|T^n(u,x) - T^n(v,y)\|_X &\le \beta^n(|u-v| + \sigma^n\iota\|x-y\|_p) \\ &< |u-v| + \sigma^n\iota\|x-y\|_p + \beta^n.\end{aligned} \tag{23}$$

In the case where $u = [0,\alpha)$ and $v = \alpha$, for each $x \in B_{l^p}$ and $n \ge 2$, we have $T^n(u,x) = \beta^n(u,\widetilde{x})$ and $T^n(v,x) = (0,\beta^n\widetilde{x}) = \beta^n(0,\widetilde{x})$. Taking into account that $0 < |u-v| \le \alpha \le 1$, by the same argument as that used in (19), for all $x,y \in B_{l^p}$ and $n \ge 2$, it follows that

$$\begin{aligned}\|T^n(u,x) - T^n(v,y)\|_X &\le \beta^n(|u| + \gamma^n\varsigma\|x-y\|_p) \\ &\le \beta^n(\alpha + \gamma^n\varsigma\|x-y\|_p) \\ &\le \beta^n(1 + \gamma^n\varsigma\|x-y\|_p) \\ &< \gamma^n\varsigma\|x-y\|_p + \beta^n \\ &< |u-v| + \gamma^n\varsigma\|x-y\|_p + \beta^n.\end{aligned} \tag{24}$$

If $u = v = \alpha$, then for all $x \in B_{l^p}$ and $n \ge 2$, we have

$$T^n(u,x) = T^n(v,y) = (0,\beta^n\widetilde{x}) = \beta^n(0,\widetilde{x})$$

and

$$\begin{aligned}\|T^n(u,x) - T^n(v,y)\|_X &\le \beta^n\gamma^n\varsigma\|x-y\|_p \\ &< |u-v| + \gamma^n\varsigma\|x-y\|_p + \beta^n.\end{aligned} \tag{25}$$

Employing (23)–(25), we deduce that for all $(u,x),(v,y) \in B$ and $n \geq 2$,

$$\begin{aligned}\|T^n(u,x) - T^n(v,y)\|_X &< |u-v| + \gamma^n \varsigma \|x-y\|_p + \beta^n \\ &\leq |u-v| + \|x-y\|_p + \gamma^n \varsigma(|u-v| + \|x-y\|_p) + \beta^n.\end{aligned} \tag{26}$$

Now, making use of (22) and (26), it follows that for all $(u,x),(v,y) \in B$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|T^n(u,x) - T^n(v,y)\|_X &\leq |u-v| + \|x-y\|_p + \gamma^n \varsigma(|u-v| + \|x-y\|_p) + \beta^n \\ &= \|(u,x)-(v,y)\|_X + \gamma^n \varsigma \|(u,x)-(v,y)\|_X + \beta^n.\end{aligned} \tag{27}$$

Let us now take $\theta_n = \gamma^n$ and $\mu_n = \beta^n$ for each $n \in \mathbb{N}$. Since $0 < \gamma, \beta < 1$, we conclude that $\mu_n, \theta_n \to 0$, as $n \to \infty$. Let the mapping $\phi : [0,+\infty) \to [0,+\infty)$ be defined by $\phi(w) = \varsigma w$ for all $w \in [0,+\infty)$. Then, using (27), for all $(u,x),(v,y) \in B$ and $n \in \mathbb{N}$, we get

$$\|T^n(u,x) - T^n(v,yx)\|_X \leq \|(u,x)-(v,y)\|_X + \theta_n \phi(\|(u,x)-(v,y)\|_X) + \mu_n),$$

i.e., T is a $(\{\theta_n\},\{\mu_n\},\phi)$-total asymptotically nonexpansive mapping.

Here, it is to be noted that, for given nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, an $(\{a_n\},\{b_n\},\phi)$-total asymptotically nonexpansive mapping is $(\{a_n\},\{b_n\},\phi)$-total uniformly L-Lipschitzian with $L = 1$, but the converse is not necessarily true. In other words, the class of total asymptotically nonexpansive mappings is strictly contained within the class of total uniformly L-Lipschitzian mappings. To illustrate this fact, the following example is given.

Example 4. Let $B = \mathbb{R}$ be endowed with the Euclidean norm $\|.\| = |.|$, and let the self-mapping T of B be defined by

$$T(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0), \\ \frac{1}{p}, & \text{if } x \in (0, \frac{1}{p}) \cup (\frac{1}{p}, q), \\ p, & \text{if } x \in \{0, \frac{1}{p}\} \cup [q, +\infty), \end{cases}$$

where $q > 0$ and $p > \frac{q+\sqrt{q^2+4}}{2}$ are arbitrary real constants such that $pq > 1$. Owing to the fact that the mapping T is discontinuous at the points $x = 0, \frac{1}{p}, q$, it follows that T is not Lipschitzian, so it is not an asymptotically nonexpansive mapping. We pick $a_n = \frac{\sigma}{n}$ and $b_n = \frac{q}{\lambda^n}$ for each $n \in \mathbb{N}$, where $\sigma > 0$ and $\lambda \in (1, pq) \cup (pq, +\infty)$ are arbitrary real constants. We define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = \mu t^k$ for all $t \in \mathbb{R}^+$, where $k \in \mathbb{N}$ and $\mu \in (0, \frac{\lambda^k(p^2-pq-1)}{\sigma p(\lambda-1)^k q^k})$ are arbitrary constants. Taking $x = q$ and $y = \frac{q}{\lambda}$, we have $T(x) = p$ and $T(y) = \frac{1}{p}$. In view of the fact that $0 < \mu < \frac{\lambda^k(p^2-pq-1)}{\sigma p(\lambda-1)^k q^k}$, we deduce that

$$\begin{aligned}|T(x) - T(y)| = p - \frac{1}{p} &> q + \frac{\sigma\mu(\lambda-1)^k q^k}{\lambda^k} \\ &= \frac{(\lambda-1)q}{\lambda} + \frac{\sigma\mu(\lambda-1)^k q^k}{\lambda^k} + \frac{q}{\lambda} \\ &= |x-y| + \sigma\mu|x-y|^k + \frac{q}{\lambda} \\ &= |x-y| + a_1\phi(|x-y|) + b_1,\end{aligned}$$

which ensures that T is not an $(\{a_n\},\{b_n\},\phi)$-total asymptotically nonexpansive mapping.

However, for all $x, y \in B$, we get

$$|T(x)-T(y)| \le p \le \frac{\lambda p}{q}(|x-y|+\sigma\mu|x-y|^k+\frac{q}{\lambda}) \\ = \frac{\lambda p}{q}(|x-y|+a_1\phi(|x-y|)+b_1) \quad (28)$$

and for all $n \ge 2$, because $T^n(z) = p$ for all $z \in B$, we derive that for all $x, y \in B$,

$$|T^n(x)-T^n(y)| < \frac{\lambda p}{q}(|x-y|+\frac{\sigma\mu}{n}|x-y|^k+\frac{q}{\lambda^n}) \\ = \frac{\lambda p}{q}(|x-y|+a_n\phi(|x-y|)+b_n). \quad (29)$$

From (28) and (29), it follows that T is a $(\{\frac{\sigma}{n}\}, \{\frac{q}{\lambda^n}\}, \phi)$-total uniformly $\frac{\lambda p}{q}$-Lipschitzian mapping.

Lemma 6. *For each $i \in \{1,2\dots,p\}$, let B_i be a real Banach space with the topological dual space B_i^* and the norm $\|.\|_i$, and let $S_i : B_i \to B_i$ be an $(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i)$-total uniformly L_i-Lipschitzian mapping. Suppose, further, that Q and ϕ are self-mappings of $\prod_{i=1}^{p} B_i$ and $\mathbb{R}^+$, respectively, which are defined by*

$$Q(u_1,u_2,\dots,u_p) = (S_1u_1, S_2u_2,\dots,S_pu_p), \quad \forall (u_1,u_2,\dots,u_p) \in \prod_{i=1}^{p} B_i \quad (30)$$

and

$$\phi(t) = \max\{\phi_i(t) : i = 1,2,\dots,p\}, \quad \forall t \in \mathbb{R}^+. \quad (31)$$

Then, Q is a $(\{\sum_{i=1}^{p} a_{n,i}\}_{n=1}^{\infty}, \{\sum_{i=1}^{p} b_{n,i}\}_{n=1}^{\infty}, \phi)$-total uniformly $\max\{L_i : i = 1,2,\dots,p\}$-Lipschitzian mapping.

Proof. Taking into account that for each $i \in \{1,2,\dots,p\}$, S_i is an $(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i)$-total uniformly L-Lipschitzian mapping and $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function, for all $(u_1,u_2,\dots,u_p), (v_1,v_2,\dots,v_p) \in \prod_{i=1}^{p} B_i$ and $n \in \mathbb{N}$, we get

$$\begin{aligned}
&\|Q^n(u_1,u_2,\dots,u_p) - Q^n(v_1,v_2,\dots,v_p)\|_* \\
&= \|(S_1^n u_1, S_2^n u_2,\dots,S_p^n u_p) - (S_1^n v_1, S_2^n v_2,\dots,S_p^n v_p)\|_* \\
&= \|(S_1^n u_1 - S_1^n v_1, S_2^n u_2 - S_2^n v_2,\dots,S_p^n u_p - S_p^n v_p)\|_* \\
&= \sum_{i=1}^{p} \|S_i^n u_i - S_i^n v_i\|_i \\
&\le \sum_{i=1}^{p} L_i\big(\|u_i - v_i\|_i + a_{n,i}\phi_i(\|u_i - v_i\|_i) + b_{n,i}\big) \\
&\le \max\{L_i : i = 1,2,\dots,p\}\Big(\sum_{i=1}^{p}\|u_i - v_i\|_i + \sum_{i=1}^{p} a_{n,i}\phi(\|u_i - v_i\|_i) + \sum_{i=1}^{p} b_{n,i}\Big) \\
&\le \max\{L_i : i = 1,2,\dots,p\}\Big(\sum_{i=1}^{p}\|u_i - v_i\|_i + \sum_{i=1}^{p} a_{n,i}\phi\big(\sum_{j=1}^{p}\|u_j - v_j\|_j\big) + \sum_{i=1}^{p} b_{n,i}\Big) \\
&= \max\{L_i : i = 1,2,\dots,p\}\Big(\|(u_1,u_2,\dots,u_p) - (v_1,v_2,\dots,v_p)\|_* \\
&\quad + \sum_{i=1}^{p} a_{n,i}\phi(\|(u_1,u_2,\dots,u_p) - (v_1,v_2,\dots,v_p)\|_*) + \sum_{i=1}^{p} b_{n,i}\Big),
\end{aligned}$$

where $\|.\|_*$ is a norm on $\prod_{i=1}^{p} B_i$ defined by (8). Because of this fact, it follows that Q is a $(\{\sum_{i=1}^{p} a_{n,i}\}_{n=1}^{\infty}, \{\sum_{i=1}^{p} b_{n,i}\}_{n=1}^{\infty}, \phi)$-total uniformly $\max\{L_i : i = 1,2,\ldots,p\}$-Lipschitzian mapping. The proof is complete. $\square$

4. Iterative Algorithms, Graph Convergence, and an Application

Suppose that, for each $i \in \{1,2,\ldots,p\}$, B_i is a real Banach space with the topological dual space B_i^*, and let $S_i : B_i \to B_i$ be an $(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i)$-total uniformly L_i-Lipschitzian mapping. Assume, further, that Q and ϕ are self-mappings of $\prod_{i=1}^{p} B_i$ and $\mathbb{R}^+$ defined by (30) and (31), respectively. We denote the sets of all fixed points of S_i $(i = 1,2,\ldots,p)$ and Q, respectively, by $\mathrm{Fix}(S_i)$ and $\mathrm{Fix}(Q)$. At the same time, we denote by Ω_{SGVI} the set of all solutions of the SGVI (2), where, for each $i \in \{1,2,\ldots,p\}$, $M_i : B_i \times B_i \to 2^{B_i^*}$ is a set-valued mapping such that for each $w_i \in B_i$, $M_i(.,w_i) : B_i \to 2^{B_i^*}$ is a general strongly H_i-monotone mapping with constant β_i and $g_i(B_i) \cap \mathrm{dom}\, M_i(.,z_i) \neq \emptyset$. Using (30), we deduce that for any $(u_1,u_2,\ldots,u_p) \in \prod_{i=1}^{p} B_i$, $(u_1,u_2,\ldots,u_p) \in \mathrm{Fix}(Q)$ if and only if for each $i \in \{1,2,\ldots,p\}$, $u_i \in \mathrm{Fix}(S_i)$, that is, $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1,S_2,\ldots,S_p) = \prod_{i=1}^{p} \mathrm{Fix}(S_i)$. If $(u_1^*,u_2^*,\ldots,u_p^*) \in \mathrm{Fix}(Q) \cap \Omega_{\text{SGVI}}$, then, by invoking Lemma 3, it can be easily seen that for each $i \in \{1,2,\ldots,p\}$ and $n \in \mathbb{N}$,

$$\begin{aligned} u_i^* &= S_i^n u_i^* = u_i^* - g_i(u_i^*) + R^{H_i}_{M_i(.,u_i^*),\lambda_i}[H_i \circ g_i(u_i^*) - \lambda_i A_i(u_1^*,u_2^*,\ldots,u_p^*)] \\ &= S_i^n\big(u_i^* - g_i(u_i^*) + R^{H_i}_{M_i(.,u_i^*),\lambda_i}[H_i \circ g_i(u_i^*) - \lambda_i A_i(u_1^*,u_2^*,\ldots,u_p^*)]\big). \end{aligned} \tag{32}$$

The fixed-point formulation (32) enables us to suggest the following iterative algorithm.

Algorithm 1. *Let A_i, B_i, g_i $(i = 1,2,\ldots,p)$ be the same as in the* SGVI (2), *and let $H_{n,i} : B_i \to B_i^*$ and $M_{n,i} : B_i \times B_i \to 2^{B_i^*}$ $(n \geq 0)$ be the mappings such that for each $i \in \{1,2,\ldots,p\}$ and $n \geq 0$, $H_{n,i}$ is a monotone mapping with $g_i(B_i) \cap \mathrm{dom}\, H_{n,i} \neq \emptyset$, and for each $w_i \in B_i$, $M_{n,i}(.,w_i) : B_i \to 2^{B_i^*}$ is a general strongly $H_{n,i}$-monotone mapping with constant $\beta_{n,i}$ and $g_i(B_i) \cap \mathrm{dom}\, M_{n,i}(.,w_i) \neq \emptyset$. Suppose, further, that for each $i \in \{1,2,\ldots,p\}$, $S_i : B_i \to B_i$ is an $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$-total uniformly L_i-Lipschitzian mapping. For an arbitrarily chosen initial point $(u_{0,1},u_{0,2},\ldots,u_{0,p}) \in \prod_{i=1}^{p} B_i$, compute the iterative sequence $\{(u_{n,1},u_{n,2},\ldots,u_{n,p})\}_{n=0}^{\infty}$ in $\prod_{i=1}^{p} B_i$ with the iterative schemes:*

$$\begin{aligned} u_{n+1,i} = \alpha_n u_{n,i} + (1-\alpha_n) S_i^n \big(u_{n,i} - g_i(u_{n,i}) + R^{H_{n,i}}_{M_{n,i}(.,u_{n,i}),\lambda_{n,i}}[H_{n,i} \circ g_i(u_{n,i}) \\ - \lambda_{n,i} A_i(u_{n,1},u_{n,2},\ldots,u_{n,p})]\big), \end{aligned} \tag{33}$$

where $i = 1,2,\ldots,p$; $n \geq 0$; $\lambda_{n,i} > 0$ are real constants; $\{\alpha_n\}$ is a sequence in the interval $[0,1)$ such that $\limsup_n \alpha_n < 1$.

If $p = 1$ and $S_1 \equiv I_1$, the identity mapping on B_1, $A_1 = A$, $B_1 = B$, $g_1 = g$, $\lambda_{n,1} = \lambda$, $H_{n,1} = H$, $M_{n,1} = M : B \to 2^{B^*}$ is a general strongly H-monotone mapping with constant β and $g(B) \cap \mathrm{dom}\, M \neq \emptyset$, and $\alpha_n = 0$ for all $n \geq 0$, then Algorithm 1 collapses into the following iterative algorithm.

Algorithm 2. *Let A, B, g, H, and M be the same as in Lemma 4. For any given $u_0 \in B$, compute the iterative sequence $\{u_n\}_{n=0}^{\infty}$ in B with the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + R^H_{M,\lambda}[H \circ g(u_n) - \lambda A(u_n)],$$

where $n = 0, 1, 2, \ldots$, and $\lambda > 0$ is an arbitrary constant.

Definition 14 ([9,10])**.** *Let B be a real Banach space with the dual space B^*, and let $M_n, M : B \to 2^{B^*}$ $(n \geq 0)$ be set-valued mappings. We say that the sequence $\{M_n\}_{n=0}^{\infty}$ is graph-convergent to M, and we denote $M_n \xrightarrow{G} M$ if, for every point $(x, u^*) \in \mathrm{Graph}(M)$, there exists a sequence of points $(x_n, u_n^*) \in \mathrm{Graph}(M_n)$ such that $x_n \to x$ and $u_n^* \to u^*$ as $n \to \infty$.*

In the next theorem, the concepts of graph convergence and proximal mapping relating to a general strongly H-monotone mapping are used, and a new equivalence relationship between the graph convergence of a sequence of general strongly H-monotone mappings and their associated proximal mappings, respectively, to a given general strongly H-monotone mapping and its associated proximal mapping is established.

Theorem 4. *Let B be a real reflexive Banach space with the dual space B^*. Let $H_n, H : B \to B^*$ $(n \geq 0)$ be monotone mappings, and for each $n \geq 0$, let H_n be an s_n-Lipschitz continuous mapping. Suppose that $M_n, M : B \to 2^{B^*}$* $(n \geq 0)$ are general strongly H_n-monotone and general strongly H-monotone mappings with constants β_n and β, respectively. Assume that the sequences $\{\frac{1}{\beta_n}\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ are bounded, and assume that $\lim_{n\to\infty} H_n(x) = H(x)$ for any $x \in B$. Furthermore, let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of positive real constants converging to a positive real constant λ. Then, $M_n \xrightarrow{G} M$ if and only if $\lim_{n\to\infty} R^{H_n}_{M_n,\lambda_n}(z^*) = R^H_{M,\lambda}(z^*)$ for all $z \in B^*$, where $R^{H_n}_{M_n,\lambda_n} = (H_n + \lambda_n M_n)^{-1}$ $(n \geq 0)$ and $R^H_{M,\lambda} = (H + \lambda M)^{-1}$.

Proof. Suppose, first, that for all $z^* \in B^*$, we have $R^{H_n}_{M_n,\lambda_n}(z^*) \to R^H_{M,\lambda}(z^*)$ as $n \to \infty$. Then, for any $(x, u^*) \in \mathrm{Graph}(M)$, we have $x = R^H_{M,\lambda}[H(x) + \lambda u^*]$, so $R^{H_n}_{M_n,\lambda_n}[H(x) + \lambda u^*] \to x$ as $n \to \infty$. Taking $x_n = R^{H_n}_{M_n,\lambda_n}[H(x) + \lambda u^*]$ for each $n \geq 0$, it follows that $H(x) + \lambda u^* \in (H_n + \lambda_n M_n)(x_n)$. Thus, for each $n \geq 0$, one can choose $u_n^* \in M_n(x_n)$ such that $H(x) + \lambda u^* = H_n(x_n) + \lambda_n u_n^*$. Then, for each $n \geq 0$, we obtain

$$\begin{aligned} \|\lambda_n u_n^* - \lambda u^*\| = \|H_n(x_n) - H(x)\| &\leq \|H_n(x_n) - H_n(x)\| + \|H_n(x) - H(x)\| \\ &\leq s_n\|x_n - x\| + \|H_n(x) - H(x)\|. \end{aligned}$$

Taking into account that $\{s_n\}_{n=0}^{\infty}$ is a bounded sequence and that $x_n \to x$ and $H_n(x) \to H(x)$ as $n \to \infty$, it follows that $\lim_{n\to\infty} \lambda_n u_n^* = \lambda u^*$. At the same time, for all $n \geq 0$, we get

$$\lambda\|u_n^* - u^*\| = \|\lambda u_n^* - \lambda u^*\| \leq |\lambda_n - \lambda|\|u_n^*\| + \|\lambda_n u_n^* - \lambda u^*\|.$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lim_{n\to\infty} \lambda_n u_n^* = \lambda u^*$, we infer that the right-hand side of the preceding inequality tends to zero as $n \to \infty$, which ensures that $\lim_{n\to\infty} u_n^* = u^*$. Now, Definition 14 implies that $M_n \xrightarrow{G} M$.

Conversely, assume that $M_n \xrightarrow{G} M$, and let $z^* \in B^*$ be chosen arbitrarily but fixed. Taking into account that M is a general H-monotone mapping, we conclude that the range of $H + \lambda M$ is precisely B, which implies the existence of a point $(x, u^*) \in \mathrm{Graph}(M)$ such that $z^* = H(x) + \lambda u^*$. In light of Definition 14, there exists a sequence $\{(x_n, u_n^*)\}_{n=0}^{\infty} \subset \mathrm{Graph}(M_n)$ such that $x_n \to x$ and $u_n^* \to u^*$ as $n \to \infty$. In virtue of the facts that $(x, u^*) \in \mathrm{Graph}(M)$ and $(x_n, u_n^*) \in \mathrm{Graph}(M_n)$, we get

$$x = R^H_{M,\lambda}[H(x) + \lambda u^*] \text{ and } x_n = R^{H^n}_{M_n,\lambda_n}[H_n(x_n) + \lambda_n u_n^*]. \tag{34}$$

Let us now take $z_n^* = H_n(x_n) + \lambda_n u_n^*$ for each $n \geq 0$. By making use of Theorem 2 and (34) and with the help of the assumptions, we derive that, for all $n \geq 0$,

$$\begin{aligned}
\|R^{H_n}_{M_n,\lambda_n}(z^*) - R^{H}_{M,\lambda}(z^*)\| &\leq \|R^{H_n}_{M_n,\lambda_n}(z^*) - R^{H_n}_{M_n,\lambda_n}(z_n^*)\| + \|R^{H_n}_{M_n,\lambda_n}(z_n^*) - R^{H}_{M,\lambda}(z^*)\| \\
&\leq \frac{1}{\lambda_n\beta_n}\|z_n^* - z^*\| + \|x_n - x\| \\
&\leq \frac{1}{\lambda_n\beta_n}(\|H_n(x_n) - H_n(x)\| + \|H_n(x) - H(x)\| \\
&\quad + \|\lambda_n u_n^* - \lambda_n u^*\| + \|\lambda_n u^* - \lambda u^*\|) + \|x_n - x\| \\
&\leq (1 + \frac{s_n}{\lambda_n\beta_n})\|x_n - x\| + \frac{1}{\lambda_n\beta_n}\|H_n(x) - H(x)\| \\
&\quad + \frac{1}{\beta_n}\|u_n^* - u^*\| + \frac{|\lambda_n - \lambda|}{\lambda_n\beta_n}\|u^*\|.
\end{aligned} \tag{35}$$

Relying on the facts that $\lim\limits_{n\to\infty} \lambda_n = \lambda$ and the sequence $\{\frac{1}{\beta_n}\}_{n=0}^{\infty}$ is bounded, it follows that the sequence $\{\frac{1}{\lambda_n\beta_n}\}_{n=0}^{\infty}$ is also bounded. Due to the fact that $x_n \to x$, $u_n^* \to u^*$, and $\lambda_n \to \lambda$ as $n \to \infty$, we deduce that the right-hand side of (35) approaches zero as $n \to \infty$. Thereby, $R^{H_n}_{M_n,\lambda_n}(z^*) \to R^{H}_{M,\lambda}(z^*)$ as $n \to \infty$. This completes the proof. □

Now, as an application of the equivalence relationship obtained in the theorem above, we prove the strong convergence of the iterative sequence generated by Algorithm 1 to a common element of the two sets Ω_{SGVI} and $\mathrm{Fix}(Q)$, where $Q = (S_1, S_2, \ldots, S_p)$ is a self-mapping of $\prod\limits_{i=1}^{p} B_i$ defined by (30). For this aim, we need to give a significant lemma that plays a key role in its proof and can be obtained from Lemma 4 in [43] as a direct consequence.

Lemma 7. *Let $\{\varrho_n\}_{n=0}^{\infty}$, $\{\xi_n\}_{n=0}^{\infty}$, and $\{k_n\}_{n=0}^{\infty}$ be three real sequences of nonnegative numbers that satisfy the following conditions:*

(i) $\limsup\limits_{n} k_n < 1$;

(ii) $\varrho_{n+1} \leq k_n\varrho_n + \xi_n$, *for all* $n \geq 0$;

(iii) $\lim\limits_{n\to\infty} \xi_n = 0$.

Then, $\lim\limits_{n\to\infty} \varrho_n = 0$.

Theorem 5. *Let A_i, B_i, g_i, H_i and M_i ($i \in \Gamma = \{1, 2, \ldots, p\}$) be the same as in Theorem 3, and let all of the conditions of Theorem 3 hold. Suppose that $H_{n,i}, M_{n,i}, \lambda_{n,i}$, and S_i ($n \geq 0; i \in \Gamma$) are the same as in Algorithm 1 such that for each $i \in \Gamma$ and $n \geq 0$, $H_{n,i}$ is an $s_{n,i}$-Lipschitz continuous mapping. Assume that Q is a self-mapping of $\prod\limits_{i=1}^{p} B_i$ defined by (30) such that $\mathrm{Fix}(Q) \cap \Omega_{\mathrm{SGVI}} \neq \emptyset$. For each $i \in \Gamma$, let $\lim\limits_{n\to\infty} H_{n,i}(w_i) = H_i(w_i)$, $M_{n,i}(., w_i) \xrightarrow{G} M_i(., w_i)$ for any $w_i \in B_i$, $s_{n,i} \to s_i$, $\beta_{n,i} \to \beta_i$, as $n \to \infty$, and let $L_i(\varphi + 1) < 2$, where φ is the same as in (17). Furthermore, for each $i \in \Gamma$, let there exist a sequence $\{\varrho_{n,i}\}_{n=0}^{\infty}$ and a constant $\lambda_i > 0$ such that $\varrho_{n,i} \to \varrho_i$ and $\lambda_{n,i} \to \lambda_i$ as $n \to \infty$,*

$$\|R^{H_{n,i}}_{M_{n,i}(.,x_i),\lambda_{n,i}}(z_i) - R^{H_{n,i}}_{M_{n,i}(.,y_i),\lambda_{n,i}}(z_i)\|_i \leq \varrho_{n,i}\|x_i - y_i\|_i, \quad \forall x_i, y_i, z_i \in B_i, \tag{36}$$

and (6) is satisfied. Then, the iterative sequence $\{(u_{n,1}, u_{n,2}, \ldots, u_{n,p})\}_{n=0}^{\infty}$ generated by Algorithm 1 converges strongly to the only element $(u_1^*, u_2^*, \ldots, u_p^*) \in \mathrm{Fix}(Q) \cap \Omega_{\mathrm{SGVI}}$.

Proof. Since all of the conditions of Theorem 3 hold, it ensures the existence of a unique solution $(u_1^*, u_2^*, \ldots, u_p^*) \in \prod_{i=1}^{p} B_i$ for the SGVI (2). Then, in light of Lemma 3, we derive that for each $i \in \Gamma$,

$$g_i(u_i^*) = R^{H_i}_{M_i(.,u_i^*),\lambda_i}[H_i \circ g_i(u_i^*) - \lambda_i A_i(u_1^*, u_2^*, \ldots, u_p^*)]. \tag{37}$$

Taking into account that Ω_{SGVI} is a singleton set and $\text{Fix}(Q) \cap \Omega_{\text{SGVI}}$, it follows that for each $i \in \Gamma$, $u_i^* \in \text{Fix}(S_i)$. Then, by making use of (37), for each $n \geq 0$ and $i \in \Gamma$, we can write

$$\begin{aligned} u_i^* = \alpha_n u_i^* + (1-\alpha_n) S_i^n \big(u_i^* - g_i(u_i^*) + R^{H_i}_{M_i(.,u_i^*),\lambda_i}[H_i \circ g_i(u_i^*) \\ - \lambda_i A_i(u_1^*, u_2^*, \ldots, u_p^*)]\big), \end{aligned} \tag{38}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is the same as that in Algorithm 1. In light of the assumptions and by using Theorem 2, for each $i \in \Gamma$ and $n \geq 0$, one can prove that

$$\begin{aligned} &\|R^{H_{n,i}}_{M_{n,i}(.,u_i^*),\lambda_{n,i}}[H_{n,i} \circ g_i(u_{n,i}) - \lambda_{n,i} A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,p})] \\ &- R^{H_{n,i}}_{M_{n,i}(.,u_i^*),\lambda_{n,i}}[H_i \circ g_i(u_i^*) - \lambda_i A_i(u_1^*, u_2^*, \ldots, u_p^*)]\|_i \\ &\leq \frac{1}{\lambda_{n,i}\beta_{n,i}} \|H_{n,i} \circ g_i(u_{n,i}) - H_{n,i} \circ g_i(u_i^*)\|_i \\ &\quad + \frac{1}{\beta_{n,i}} \|A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,i-1}, u_{n,i}, u_{n,i+1}, \ldots, u_{n,p}) \\ &\quad - A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,i-1}, u_i^*, u_{n,i+1}, \ldots, u_{n,p})\|_i \\ &\quad + \frac{1}{\beta_{n,i}} \sum_{j \in \Gamma, j \neq i} \|A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,j-1}, u_{n,j}, u_{n,j+1}, \ldots, u_{n,p}) \\ &\quad - A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,j-1}, u_j^*, u_{n,j+1}, \ldots, u_{n,p})\|_i \\ &\quad + \frac{1}{\lambda_{n,i}\beta_{n,i}} \Big(\|H_{n,i} \circ g_i(u_i^*) - H_i \circ g_i(u_i^*)\|_i + |\lambda_{n,i} - \lambda_i| \|A_i(u_1^*, u_2^*, \ldots, u_p^*)\|_i \Big). \end{aligned} \tag{39}$$

In view of the fact that for each $i \in \Gamma$ and $n \geq 0$, g_i is δ_i-Lipschitz continuous, $H_{n,i}$ is $s_{n,i}$-Lipschitz continuous, and A_i is α_i-Lipschitz continuous in the ith argument and $\varsigma_{i,j}$-Lipschitz continuous in the jth argument $(j \in \Gamma; j \neq i)$, it follows that for each $n \geq 0$ and $i \in \Gamma$,

$$\|H_{n,i} \circ g_i(u_{n,i}) - H_{n,i} \circ g_i(u_i^*)\|_i \leq s_{n,i}\delta_i \|u_{n,i} - u_i^*\|_i, \tag{40}$$

$$\begin{aligned} &\|A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,i-1}, u_{n,i}, u_{n,i+1}, \ldots, u_{n,p}) \\ &- A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,i-1}, u_i^*, u_{n,i+1}, \ldots, u_{n,p})\|_i \leq \alpha_i \|u_{n,i} - u_i^*\|_i \end{aligned} \tag{41}$$

and

$$\begin{aligned} &\|A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,j-1}, u_{n,j}, u_{n,j+1}, \ldots, u_{n,p}) \\ &- A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,j-1}, u_j^*, u_{n,j+1}, \ldots, u_{n,p})\|_i \leq \varsigma_{i,j} \|u_{n,j} - u_j^*\|_i. \end{aligned} \tag{42}$$

Combining (39)–(42), we conclude that for each $i \in \Gamma$ and $n \geq 0$,

$$\begin{aligned} &\|R^{H_{n,i}}_{M_{n,i}(.,u_i^*),\lambda_{n,i}}[H_{n,i} \circ g_i(u_{n,i}) - \lambda_{n,i} A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,p})] \\ &- R^{H_{n,i}}_{M_{n,i}(.,u_i^*),\lambda_{n,i}}[H_i \circ g_i(u_i^*) - \lambda_i A_i(u_1^*, u_2^*, \ldots, u_p^*)]\|_i \\ &\leq \frac{s_{n,i}\delta_i}{\lambda_{n,i}\beta_{n,i}} \|u_{n,i} - u_i^*\|_i + \frac{\alpha_i}{\beta_{n,i}} \|u_{n,i} - u_i^*\|_i + \frac{1}{\beta_{n,i}} \sum_{j \in \Gamma, j \neq i} \varsigma_{i,j} \|u_{n,j} - u_j^*\|_j \\ &\quad + \frac{1}{\lambda_{n,i}\beta_{n,i}} (\|H_{n,i} \circ g_i(u_i^*) - H_i \circ g_i(u_i^*)\|_i + |\lambda_{n,i} - \lambda_i| \|A_i(u_1^*, u_2^*, \ldots, u_p^*)\|_i). \end{aligned} \tag{43}$$

Since for each $i \in \Gamma$, g_i is a k_i-strongly accretive and δ_i-Lipschitz continuous mapping and B_i is a uniformly smooth Banach space with $\rho_{B_i}(t) \leq C_i t^2$ for some $C_i > 0$, it is not hard to see that, for each $i \in \Gamma$ and $n \geq 0$,

$$\|u_{n,i} - u_i^* - (g_i(u_{n,i}) - g_i(u_i^*))\|_i \leq \sqrt{1 - 2k_i + 64C_i\delta_i^2}\|u_{n,i} - u_i^*\|_i. \tag{44}$$

Due to the assumptions and by making use of (33), (36), (38), (43), and (44), for each $n \geq 0$ and $i \in \Gamma$, one can deduce that

$$\begin{aligned}
\|u_{n+1,i} - u_i^*\|_i \leq{}& \alpha_n\|u_{n,i} - u_i^*\|_i + (1-\alpha_n)\|S_i^n(u_{n,i} - g_i(u_{n,i}) \\
&+ R^{H_{n,i}}_{M_{n,i}(.,u_{n,i}),\lambda_{n,i}}[H_{n,i} \circ g_i(u_{n,i}) - \lambda_{n,i}A_i(u_{n,1}, u_{n,2}, \ldots, u_{n,p})]) \\
&- S_i^n(u_i^* - g_i(u_i^*) + R^{H_i}_{M_i(.,u_i^*),\lambda_i}[H_i \circ g_i(u_i^*) - \lambda_i A_i(u_1^*, u_2^*, \ldots, u_p^*)])\|_i \\
\leq{}& \alpha_n\|u_{n,i} - u_i^*\|_i + (1-\alpha_n)L_i\Big(\sigma_{n,i}\|u_{n,i} - u_i^*\|_i \\
&+ \frac{1}{\beta_{n,i}}\sum_{j\in\Gamma, j\neq i}\varsigma_{i,j}\|u_{n,j} - u_j^*\|_j + \mu_{n,i} + a_{n,i}\phi_i(\sigma_{n,i}\|u_{n,i} - u_i^*\|_i \\
&+ \frac{1}{\beta_{n,i}}\sum_{j\in\Gamma, j\neq i}\varsigma_{i,j}\|u_{n,j} - u_j^*\|_j + \mu_{n,i}) + b_{n,i}\Big),
\end{aligned} \tag{45}$$

where, for each $n \geq 0$ and $i \in \Gamma$,

$$\begin{aligned}
&\sigma_{n,i} = \varrho_{n,i} + \sqrt{1 - 2k_i + 64C_i\delta_i^2} + \frac{s_{n,i}\delta_i + \lambda_{n,i}\alpha_i}{\lambda_{n,i}\beta_{n,i}}, \\
&\Delta_{n,i} = R^{H_{n,i}}_{M_{n,i}(.,u_i^*),\lambda_{n,i}}[H_i \circ g_i(u_i^*) - \lambda_i A_i(u_1^*, u_2^*, \ldots, u_p^*)] \\
&\quad - R^{H_i}_{M_i(.,u_i^*),\lambda_i}[H_i \circ g_i(u_i^*) - \lambda_i A_i(u_1^*, u_2^*, \ldots, u_p^*)], \\
&\mu_{n,i} = \frac{1}{\lambda_{n,i}\beta_{n,i}}(\|H_{n,i} \circ g_i(u_i^*) - H_i \circ g_i(u_i^*)\|_i \\
&\quad + |\lambda_{n,i} - \lambda_i|\|A_i(u_1^*, , u_2^*, \ldots, u_p^*)\|_i) + \|\Delta_{n,i}\|_i.
\end{aligned}$$

Let us assume that $L = \max\{L_i : i = 1, 2, \ldots, p\}$. Then, by making use of (45), it follows that for all $n \geq 0$,

$$\begin{aligned}
&\|(u_{n+1,1}, u_{n+1,2}, \ldots, u_{n+1,p}) - (u_1^*, u_2^*, \ldots, u_p^*)\|_* = \sum_{i=1}^p \|u_{n+1,i} - u_i^*\|_i \\
&\leq \alpha_n \sum_{i=1}^p \|u_{n,i} - u_i^*\|_i + (1-\alpha_n)L\Big((\sigma_{n,1} + \sum_{q=2}^p \frac{\varsigma_{q,1}}{\beta_{n,q}})\|u_{n,1} - u_1^*\|_1 \\
&+ (\sigma_{n,2} + \sum_{q\in\Gamma, q\neq 2}\frac{\varsigma_{q,2}}{\beta_{n,q}})\|u_{n,2} - u_2^*\|_2 + \cdots + (\sigma_{n,p} + \sum_{q=1}^{p-1}\frac{\varsigma_{q,p}}{\beta_{n,q}})\|u_{n,p} - u_p^*\|_p \\
&+ \sum_{i=1}^p \mu_{n,i} + \sum_{i=1}^p a_{n,i}\phi_i(\sigma_{n,i}\|u_{n,i} - u_i^*\|_i \\
&+ \frac{1}{\beta_{n,i}}\sum_{j\in\Gamma, j\neq i}\varsigma_{i,j}\|u_{n,j} - u_j^*\|_j + \mu_{n,i}) + \sum_{i=1}^p b_{n,i}\Big)
\end{aligned} \tag{46}$$

$$\leq \alpha_n \sum_{i=1}^{p} \|u_{n,i} - u_i^*\|_i + (1-\alpha_n) L\Big(\varphi(n) \sum_{i=1}^{p} \|u_{n,i} - u_i^*\|_i + \sum_{i=1}^{p} \mu_{n,i}$$
$$+ \sum_{i=1}^{p} a_{n,i} \phi(\sigma_{n,i} \|u_{n,i} - u_i^*\|_i + \frac{1}{\beta_{n,i}} \sum_{j\in\Gamma, j\neq i} \varsigma_{i,j} \|u_{n,j} - u_j^*\|_j + \mu_{n,i}) + \sum_{i=1}^{p} b_{n,i}\Big),$$

where ϕ is a self-mapping of $\mathbb{R}^+$ defined by (31), and for each $n \geq 0$,

$$\varphi(n) = \max\{\sigma_{n,i} + \sum_{q\in\Gamma, q\neq i} \frac{\varsigma_{q,i}}{\beta_{n,q}} : i = 1, 2, \ldots, p\}.$$

By virtue of the fact that for each $i \in \Gamma$, $\varrho_{n,i} \to \varrho_i$, $s_{n,i} \to s_i$, $\lambda_{n,i} \to \lambda_i$, and $\beta_{n,i} \to \beta_i$ as $n \to \infty$, we deduce that $\varphi(n) \to \varphi$ as $n \to \infty$. Then, for $\hat{\varphi} = \frac{\varphi+1}{2} \in (\varphi, 1)$, there exists $n_0 \geq 1$ such that $\varphi(n) < \hat{\varphi}$ for all $n \geq n_0$. Thus, from (46), it follows that for all $n \geq n_0$,

$$\begin{aligned}
&\|(u_{n+1,1}, u_{n+1,2}, \ldots, u_{n+1,p}) - (u_1^*, u_2^*, \ldots, u_p^*)\|_* \\
&\leq \alpha_n \sum_{i=1}^{p} \|u_{n,i} - u_i^*\|_i + (1-\alpha_n) L\Big(\hat{\varphi} \sum_{i=1}^{p} \|u_{n,i} - u_i^*\|_i \\
&\quad + \sum_{i=1}^{p} \mu_{n,i} + \sum_{i=1}^{p} a_{n,i} \phi(\sigma_{n,i} \|u_{n,i} - u_i^*\|_i \\
&\quad + \frac{1}{\beta_{n,i}} \sum_{j\in\Gamma, j\neq i} \varsigma_{i,j} \|u_{n,j} - u_j^*\|_j + \mu_{n,i}) + \sum_{i=1}^{p} b_{n,i}\Big) \\
&= \big(L\hat{\varphi} + (1 - L\hat{\varphi})\alpha_n\big) \|(u_{n,1}, u_{n,2}, \ldots, u_{n,p}) - (u_1^*, u_2^*, \ldots, u_p^*)\|_* \\
&\quad + (1-\alpha_n) L\Big(\sum_{i=1}^{p} (\mu_{n,i} + b_{n,i}) + \sum_{i=1}^{p} a_{n,i} \phi(\sigma_{n,i} \|u_{n,i} - u_i^*\|_i \\
&\quad + \frac{1}{\beta_{n,i}} \sum_{j\in\Gamma, j\neq i} \varsigma_{i,j} \|u_{n,j} - u_j^*\|_j + \mu_{n,i})\Big).
\end{aligned} \tag{47}$$

Letting $k_n = L\hat{\varphi} + (1 - L\hat{\varphi})\alpha_n$ for each $n \geq 0$ and taking into account that $L(\varphi + 1) < 2$ and $\limsup_n \alpha_n < 1$, we deduce that

$$\begin{aligned}
\limsup_n k_n &= \limsup_n \big(L\hat{\varphi} + (1 - L\hat{\varphi})\alpha_n\big) \\
&= L\hat{\varphi} + (1 - L\hat{\varphi}) \limsup_n \alpha_n \\
&< 1.
\end{aligned}$$

Relying on the fact that for each $i \in \Gamma$ and $w_i \in B_i$, $M_{n,i}(., w_i) \xrightarrow{G} M_i(., w_i)$, from Theorem 4, it follows that for each $i \in \Gamma$, $\|\Delta_{n,i}\|_i \to 0$ as $n \to \infty$. Meanwhile, the fact that for each $i \in \Gamma$, $\lambda_{n,i} \to \lambda_i$ and $H_{n,i}(w_i) \to H_i(w_i)$ for any $w_i \in B_i$ as $n \to \infty$ implies that for each $i \in \Gamma$, $\mu_{n,i} \to 0$ as $n \to \infty$. Considering the fact that for each $i \in \Gamma$, S_i is an $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$-total uniformly L_i-Lipschitzian mapping, in light of Definition 13(ix), for each $i \in \Gamma$, we have $a_{n,i}, b_{n,i} \to 0$ as $n \to \infty$. Due to these arguments and the fact that $\limsup_n k_n < 1$, assuming that $\varrho_n = \|(u_{n,1}, u_{n,2}, \ldots, u_{n,p}) - (u_1^*, u_2^*, \ldots, u_p^*)\|_*$ and $\xi_n = (1-\alpha_n) L\big(\sum_{i=1}^{p} (\mu_{n,i} + b_{n,i}) + \sum_{i=1}^{p} a_{n,i} \phi(\sigma_{n,i} \|u_{n,i} - u_i^*\|_i + \frac{1}{\beta_{n,i}} \sum_{j\in\Gamma, j\neq i} \varsigma_{i,j} \|u_{n,j} - u_j^*\|_j + \mu_{n,i})\big)$ for each $n \geq 0$, we infer that $\lim_{n\to\infty} \xi_n = 0$ and, thereby, all conditions of Lemma 7 are satisfied. By utilizing (47) and Lemma 7, we conclude that $\varrho_n \to 0$ as $n \to \infty$, i.e.,

$$(u_{n,1}, u_{n,2}, \ldots, u_{n,p}) \to (u_1^*, u_2^*, \ldots, u_p^*), \text{ as } n \to \infty.$$

Hence, the sequence $\{(u_{n,1}, u_{n,2}, \ldots, u_{n,p})\}_{n=0}^{\infty}$ generated by Algorithm 1 converges strongly to the unique solution of the SGVI (2), that is, the only element of $\in \text{Fix}(Q) \cap \Omega_{\text{SGVI}}$. The proof is finished. □

As a direct consequence of the theorem above, we have the following corollary.

Corollary 3. *Let A, B, g, H, and M be the same as in Corollary 2, and let all of the conditions of Corollary 2 hold. Suppose, further, that there exists a constant $\lambda > 0$ such that (18) holds. Then, the iterative sequence $\{u_n\}_{n=0}^{\infty}$ generated by Algorithm 2 converges strongly to the unique solution of the* VIP (3).

Remark 3. *By comparing Corollary 3 and ([13] and Theorem 3.4), it is significant to mention that Corollary 3 improves upon Theorem 3.4 in [13]. In fact, in ([13], Theorem 3.4), if we replace the strict monotonicity condition of $H : B \to B^*$ with a milder condition of monotonicity, then the strong convergence of the sequence $\{u_n\}_{n=0}^{\infty}$ generated by the iterative algorithm proposed in [13] can be obtained as it was derived and and as it appears in Corollary 3.*

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