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A Novel Method for Generating the M-Tri-Basis of an Ordered Γ-Semigroup

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Abstract: In this paper, we discuss the hypothesis that an ordered Γ-semigroup can be constructed on the M-left(right)-tri-basis. In order to generalize the left(right)-tri-basis using Γ-semigroups and ordered semigroups, we examined M-tri-ideals from a purely algebraic standpoint. We also present the form of the M-tri-ideal generator. We investigated the M-left(right)-tri-ideal using the ordered Γ-semigroup. In order to obtain their properties, we used M-left(right)-tri-basis. It was possible to generate a M-left(right)-tri-basis from elements and their subsets. Throughout this paper, we will present an interesting example of order $\preceq_{mlt}$ $(\preceq_{mrt})$, which is not a partial order of $\mathcal{S}$. Additionally, we introduce the notion of quasi-order. As an example, we demonstrate the relationship between M-left(right)-tri-basis and partial order.

Keywords: left tri-ideal; right tri-ideal; M-left-tri-basis; M-right-tri-basis; quasi-order; partial order

MSC: 06B10; 20M25; 16Y60

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1. Introduction

Several applications of algebraic structures can be found in mathematics. Generalizing the ideals of algebraic structures and ordered algebraic structures plays an important role, making them available for further study and application. Mathematicians studied bi-ideals, quasi-ideals, and interior ideals during 1950–1980. However, during 1950–2019, it was only mathematicians who studied their applications. In fact, the notion of one-sided ideals of rings and semigroups can be regarded as a generalization of the notion of ideals of rings and semigroups, as is the notion of quasi-ideals of semigroups and rings. In general, semigroups are generalizations of rings and groups. In semigroup theory, certain band decompositions are useful for studying semigroup structure. A new field in mathematics could be opened up by this research, one that aims to use semigroups of bi-ideals of semirings with semilattices that are additively reduced. The many different ideals associated with Γ-semigroups [1] and Γ-semirings [2] have been described by several researchers. Partially ordered relation " $\preceq$ " satisfies the conditions of reflexivity, antisymmetry, and transitivity. There are different classes of semigroups and Γ-semigroups based on bi-ideals that have been described by researchers in [3–6]. Munir [7] introduced new ideals in the form of M-bi-ideals over semigroups in 2018. An ordered semigroup is a generalization of a semigroup with a partially ordered relation constructed on a semigroup, so that the relation fits with the operation. An algebraic structure such as the ordered Γ-semigroup was introduced by Sen et al. in 1993 [8] and has been studied by several authors [9–12].

For an ordered semigroup $\mathcal{S}$ and subsemigroup $\mathcal{A}$ of $\mathcal{S}$, $\mathcal{A}^m = \mathcal{A}.\mathcal{A}\ldots\mathcal{A}(m - times)$, where m is a positive integer. Clearly, for any subsemigroup $\mathcal{A}$ of ordered semigroup $\mathcal{S}$, $\mathcal{A}^n \subseteq \mathcal{A}$ for all positive integers n, which are similarly right case. Hence, $\mathcal{A}^r \subseteq \mathcal{A}^t$ for all positive integers r and t, such that $r \geq t$, but the converse is not true. As a generalization of the bi-ideal of semirings and semigroups, a tri-ideal of semirings and semigroups can be characterized as a generalization of the bi-ideal. In the context of Γ-semigroups, an ordered Γ-semigroup is an extension of the Γ-semigroup. In contrast to the notion of the tri-ideal of semigroups, the notion of the tri-ideal of an ordered semigroup is a general form of the notion of the tri-ideal of semigroups. In semigroup theory, the M-tri-ideal is a generalization of the tri-ideal. Similarly, an ordered M-tri-ideal is a generalization of an ordered tri-ideal. In this paper, we describe the basic properties of the M-tri-basis from an algebraic standpoint. The fact that semigroups can be generalized to Γ-semigroups and Γ-semigroups to ordered Γ-semigroups is a result of these facts. It was work by Jantanan et al. that introduced the concept of bi-basis of ordered Γ-semigroups in 2022. We further describe the relationship between partial order and bi-basis [13]. As recently discussed in Palanikumar et al. [14–16], algebraic structures such as semigroups, semirings, ternary semigroups, and ternary semirings are all ideals and the generators of these structures are ideals. Rao introduced the tri-ideals of semigroups and semirings in [17,18]. Our paper extends a bi-basis of an ordered Γ-semigroup into a M-bi-basis of an ordered Γ-semigroup. We also generalize the tri-ideal of an ordered Γ-semigroup to an M-tri-basis of an ordered Γ-semigroup. The notion of almost bi-ideals and almost quasi-ideals of ordered semigroups is discussed in Sudaporn et al. [19]. The novel concept of M-bi-basis generators of an ordered Γ-semigroup is introduced by Palanikumar et al. [20]. Susmita et al. have discussed some important properties of bi-ideals of an ordered semigroups [21].

This paper discusses several important classical results for M-tri-basis and Γ-semigroups characterized by M-tri-ideals and M-tri-basis. Furthermore, we demonstrate how the elements and subsets of an ordered Γ-semigroup yield the M-tri-ideal and basis. This paper extends the notion of Γ-semigroup information into ordered Γ-semigroup information. The paper is divided into five sections. Section 1 is the introduction. There is a brief description of an ordered Γ-semigroup in Section 2, as well as relevant definitions and results. A numerical example of an M-left-tri-basis generator can be found in Section 3. As part of Section 4, a numerical example is given for the M-right-tri-basis generator concept. Our conclusions are provided in Section 5. In this paper, our purpose is to describe:

1. The generator of the M-tri-ideal for an ordered Γ-semigroup;
2. To interact, the order relation " $\preceq$ " based on the M-tri-basis should not be a partial order.
3. For example, the subset of an M-tri-basis is not an M-tri-basis itself.

2. Basic Concepts

It is assumed throughout this article that $\mathcal{S}$ denotes a Γ-semigroup, unless stated otherwise.

Definition 1 ([1])**.** *Let $\mathcal{S}$ and Γ be any two non-empty sets. Then, $\mathcal{S}$ is called a Γ-semigroup from $\mathcal{S} \cdot \Gamma \cdot \mathcal{S} \to \mathcal{S}$, which maps $(f_1, \pi, f_2) \to f_1 \cdot \pi \cdot f_2$, satisfying the condition $(f_1 \cdot \pi \cdot f_2) \cdot \theta \cdot f_3 = f_1 \cdot \pi \cdot (f_2 \cdot \theta \cdot f_3)$ for all $f_1, f_2, f_3 \in \mathcal{S}$ and $\pi, \theta \in \Gamma$.*

Definition 2 ([8])**.** *The algebraic system $(\mathcal{S}, \Gamma, \preceq)$ is said to be an ordered Γ-semigroup if it satisfies the following conditions:*

1. *$\mathcal{S}$ is a Γ-semigroup,*
2. *" $\preceq$ " is a relation from a partially ordered set (poset) $\mathcal{S}$,*
3. *If $s^{''} \preceq s^{'''}$, then $s^{''}\pi s^{'} \preceq s^{'''}\pi s^{'}$ and $s^{'}\pi s^{''} \preceq s^{'}\pi s^{'''}$ for any $s^{'}, s^{''}, s^{'''} \in \mathcal{S}$ and $\pi \in \Gamma$.*

Remark 1 ([8])**.** *Let $\mathcal{G}^{'}$ and $\mathcal{G}^{''}$ be any two subsets of $\mathcal{S}$. Then, the following properties hold:*

1. *$G^{'}\Gamma G^{''} = \{x^{'}\pi x^{''} | x^{'} \in G^{'}, x^{''} \in G^{''}, \pi \in \Gamma\}$,*
2. *$(G^{'}] = \{s \in \mathcal{S} | s \preceq x' \text{ for some } x' \in G^{'}\}$,*
3. *$G^{'} \sqsubseteq (G^{'}]$,*
4. *If $G^{'} \sqsubseteq G^{''}$, then $(G^{'}] \sqsubseteq (G^{''}]$ and $(G^{'}]\Gamma(G^{''}] \sqsubseteq (G^{'}\Gamma G^{''}]$.*

Definition 3 ([17]). *Let $\mathcal{S}$ be a Γ-semigroup and $\mathcal{G}$ be a subset of $\mathcal{S}$ called left-tri-ideal(right-tri-ideal) (or LTI and RTI, respectively) if it satisfies the following conditions:*

1. $\mathcal{G}$ *is a* Γ*-subsemigroup,*
2. $\mathcal{G}\Gamma\mathcal{S}\Gamma\mathcal{G}\Gamma\mathcal{G} \sqsubseteq \mathcal{G}$ $(\mathcal{G}\Gamma\mathcal{G}\Gamma\mathcal{S}\Gamma\mathcal{G} \sqsubseteq \mathcal{G})$,

Lemma 1 ([18]). *Let $\mathcal{S}$ be a Γ-semiring, $\mathcal{G}$ a subset of $\mathcal{S}$, and $a \in S$. Then, the following statements hold:*

1. $\langle G\rangle_{lt} = \mathcal{G} \cup \mathcal{G}\Gamma\mathcal{G} \cup \mathcal{G}\Gamma S\Gamma\mathcal{G}\Gamma\mathcal{G}$ *is the smallest* Γ*-LTI of* S *containing* $\mathcal{G}$*,*
2. $\langle G\rangle_{rt} = \mathcal{G} \cup \mathcal{G}\Gamma\mathcal{G} \cup \mathcal{G}\Gamma\mathcal{G}\Gamma S\Gamma\mathcal{G}$ *is the smallest* Γ*-RTI of* $\mathcal{S}$ *containing* $\mathcal{G}$*,*
3. $\langle a\rangle_{lt} = a \cup a\Gamma a \cup a\Gamma\mathcal{S}\Gamma a\Gamma a$ *is the smallest* Γ*-LTI of* $\mathcal{S}$ *containing "a",*
4. $\langle a\rangle_{rt} = a \cup a\Gamma a \cup a\Gamma a\Gamma\mathcal{S}\Gamma a$ *is the smallest* Γ*-RTI of* $\mathcal{S}$ *containing "a".*

Definition 4 ([18]). *(i) Let $\mathcal{S}$ be an ordered semigroup. A subsemigroup $\mathcal{G}$ of $\mathcal{S}$ is called an M-left-ideal of $\mathcal{S}$ if $\mathcal{S}^M\mathcal{G} \sqsubseteq \mathcal{G}$ and $(\mathcal{G}] = \mathcal{G}$, where M is a positive integer that is not necessarily one.*
(ii) A subsemigroup $\mathcal{G}$ of $\mathcal{S}$ is called a M-right-ideal of $\mathcal{S}$ if $\mathcal{G}\mathcal{S}^M \sqsubseteq \mathcal{G}$ and $(\mathcal{G}] = \mathcal{G}$, where M is a positive integer that is not necessarily one.

Definition 5 ([18]). *Let $\mathcal{G}$ be a subsemigroup of an ordered semigroup $\mathcal{G}$. Then,*
(i) The M-left-ideal generated by $\mathcal{G}$ is $(\mathcal{G})_{ml} = (\mathcal{G} \cup S^M\mathcal{G}]$.
(ii) The M-right-ideal generated by $\mathcal{G}$ is $(\mathcal{G})_{mr} = (\mathcal{G} \cup \mathcal{G}S^M]$.

Definition 6 ([7]). *Let $\mathcal{G}$ be a subset of $\mathcal{S}$, which is called an M-bi-ideal of semigroup S if it satisfies the following conditions:*

1. $\mathcal{G}$ *is a* Γ*-subsemigroup,*
2. $\mathcal{G} \cdot \mathcal{S}^M \cdot \mathcal{G} \sqsubseteq \mathcal{G}$*, where M is a positive integer.*

Definition 7 ([13]). *Let $\mathcal{G}$ be a subset of $\mathcal{S}$ that is called a bi-basis of $\mathcal{S}$ if it satisfies the following conditions:*

1. $\mathcal{S} = \langle\mathcal{G}\rangle_b$.
2. *If* $\mathcal{F} \sqsubseteq \mathcal{G}$ *such that* $\mathcal{S} = \langle\mathcal{F}\rangle_b$*, then* $\mathcal{F} = \mathcal{G}$.

Definition 8. *Let $\mathcal{G}$ be the subset of $\mathcal{S}$ that is called an M-bi-basis of $\mathcal{S}$ if satisfies the following conditions:*

1. $\mathcal{S} = \langle\mathcal{G}\rangle_{mb}$.
2. *If* $\mathcal{F} \sqsubseteq \mathcal{G}$ *such that* $\mathcal{S} = \langle\mathcal{F}\rangle_{mb}$*, then* $\mathcal{F} = \mathcal{G}$.

3. *M*-LTB Generator

In this paper, we present some results on the *M*-left-tri-ideal (*M*-LTI) generator, based on an ordered Γ-semigroup.

Definition 9. *Let $\mathcal{G}$ be the subset of $\mathcal{S}$ called an M-LTI of $\mathcal{S}$ if it satisfies the following conditions:*

1. $\mathcal{G}$ *is a* Γ*-subsemigroup,*
2. $\mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}\ \ (M - times)) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \sqsubseteq \mathcal{G}$*, where M is a positive integer,*
3. *If* $g \in \mathcal{G}$ *and* $s \in \mathcal{S}$ *such that* $s \preceq g$*, then* $s \in \mathcal{G}$.

Remark 2. *If $f_1 \in \mathcal{S}$ and $\mathcal{N}$ and M are positive integers, then the following statements hold:*

1. $\mathcal{N} f_1 = f_1 \cdot \Gamma \cdot f_1 \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot f_1\ (\mathcal{N} - times)$
2. $\mathcal{S} \cdot \Gamma \cdot \mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}\ ((M - times)) \sqsubseteq \mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}\ (M - 1\ times)$

Remark 3. *1. Every M-bi-ideal is an M-LTI.*
2. Every LTI is an M-LTI.

Here is an example showing that the converse does not need to be true, as demonstrated by Example 1.

Example 1.

$$\text{Let } \mathcal{S} = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ 0 & 0 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ 0 & 0 & 0 & 0 & a_{19} & a_{20} & a_{21} & a_{22} \\ 0 & 0 & 0 & 0 & 0 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{26} & a_{27} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{28} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| a_i'^s \in Z^* \right\}$$

and Γ is a unit matrix. Now, we define the partial order relation $\preceq$ on $\mathcal{S}$: for any $A, B \in \mathcal{S}$, $A \preceq_{(S_1,S_2)} B$, if and only if $a_{ij} \preceq b_{ij}$, for all i and j. Then, $\mathcal{S}$ is an ordered Γ-semigroup of matrices over Z^* (non-negative integer) with the partial order relation " $\preceq$ ".

(i) Clearly,

$$B_1 = \left\{ \begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i'^s \in Z^* \right\}.$$

Although B_1 is an M-LTI, it is not an M-bi-ideal of $\mathcal{S}$.

(iii) Clearly,

$$B_2 = \left\{ \begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & b_3 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i'^s \in Z^* \right\}.$$

Hence, B_2 is an M-LTI, but not an LTI of $\mathcal{S}$.

Theorem 1. *1. Let $f_1 \in \mathcal{S}$. The M-LTI generated by an element "f_1" is $\langle f_1 \rangle_{mlt} = \{f_1 \cup \mathcal{N}(f_1 \cdot \Gamma \cdot f_1) \cup f_1 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S} \ \ (M - times)) \cdot \Gamma \cdot f_1 \cdot \Gamma \cdot f_1\}$ and $\mathcal{N} \succeq M$, where $\mathcal{N}$ and M are positive integers.*

2. Let $\mathcal{G}$ be a subset of $\mathcal{S}$. The M-LTI generated by set " $\mathcal{G}$ " is $\langle \mathcal{G} \rangle_{mlt} = \{\mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S} \ \ (M - times)) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G}\}$.

Definition 10. *Let $\mathcal{G}$ be a subset of $\mathcal{S}$, known as an M-left-tri-basis (LTB) of $\mathcal{S}$ if it meets the criteria listed below:*

1. $\mathcal{S} = \langle \mathcal{G} \rangle_{mlt}$.

2. *If $\mathcal{F} \sqsubseteq \mathcal{G}$ such that $\mathcal{S} = \langle \mathcal{F} \rangle_{mlt}$, then $\mathcal{F} = \mathcal{G}$.*

Example 2. *Let $\mathcal{S} = \{k_1, k_2, k_3, k_4, k_5, k_6\}$ and $\Gamma = \{\pi_1, \pi_2\}$, where π_1 and pi_2 are defined on $\mathcal{S}$ with the following table:*

π_1	k_1	k_2	k_3	k_4	k_5	k_6
k_1	k_1	k_1	k_1	k_1	k_1	k_6
k_2	k_1	k_1	k_1	k_2	k_3	k_6
k_3	k_1	k_2	k_3	k_1	k_1	k_6
k_4	k_1	k_1	k_1	k_4	k_5	k_6
k_5	k_1	k_4	k_5	k_1	k_1	k_6
k_6	k_6	k_6	k_6	k_6	k_6	k_6

π_2	k_1	k_2	k_3	k_4	k_5	k_6
k_1	k_1	k_4	k_1	k_4	k_4	k_6
k_2	k_1	k_2	k_1	k_4	k_4	k_6
k_3	k_1	k_4	k_3	k_4	k_5	k_6
k_4	k_1	k_4	k_1	k_4	k_4	k_6
k_5	k_1	k_4	k_3	k_4	k_5	k_6
k_6	k_6	k_6	k_6	k_6	k_6	k_6

$\preceq := \{(k_1,k_1),(k_1,k_6),(k_2,k_2),(k_2,k_6),(k_3,k_3),(k_3,k_6),(k_4,k_4),(k_4,k_6),(k_5,k_5),(k_5,k_6),(k_6,k_6)\}$. *Clearly, $(\mathcal{S}, \Gamma, \preceq)$ is an ordered Γ-semigroup.*

The covering relation $\preceq := \{(k_1,k_6),(k_2,k_6),(k_3,k_6),(k_4,k_6),(k_5,k_6)\}$ is represented by Figure 1, since $\mathcal{G} = \{k_4, k_5\}$ is a M-LTB of $\mathcal{S}$.

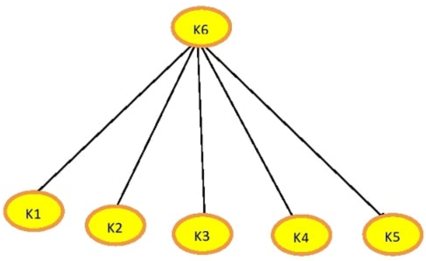

Figure 1. Covering relation.

Theorem 2. *Let $\mathcal{G}$ be the M-LTB of $\mathcal{S}$ and $f_1, f_2 \in \mathcal{G}$. If $f_1 \in (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2]$, then $f_1 = f_2$.*

Proof. Assume that $f_1 \in (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2]$, and suppose that $f_1 \neq f_2$. Let $\mathcal{F} = \mathcal{G} \setminus \{f_1\}$. Obviously, $\mathcal{F} \subset \mathcal{G}$ since $f_1 \neq f_2$, $f_2 \in \mathcal{F}$. To show that $\langle \mathcal{F} \rangle_{mlt} = \mathcal{S}$, clearly, $\langle \mathcal{F} \rangle_{mlt} \sqsubseteq \mathcal{S}$. Still, to prove that $\mathcal{S} \sqsubseteq \langle \mathcal{F} \rangle_{mlt}$, let $s \in \mathcal{S}$. By our hypothesis, $\langle \mathcal{G} \rangle_{mlt} = \mathcal{S}$, and hence, $s \in (\mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G}]$. We have $s \preceq g$ for some $g \in \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G}$. As a result, the following cases will be discussed.
Case-1 : Let $g \in \mathcal{G}$. There are two subcases to examine:
Subcase-1 : Let $g \neq f_1$, then $g \in \mathcal{G} \setminus \{f_1\} = \mathcal{F} \sqsubseteq \langle \mathcal{F} \rangle_{mlt}$.
Subcase-2: Let $g = f_1$. We have $g = f_1 \in (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \sqsubseteq (\mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cup \mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \sqsubseteq \langle \mathcal{F} \rangle_{mlt}$.
Case-2: Let $g \in \mathcal{G} \cdot \Gamma \cdot \mathcal{G}$. Then, $g = g_1 \cdot \pi \cdot g_2$, for some $g_1, g_2 \in \mathcal{G}$ and $\pi \in \Gamma$. In addition, there are four subcases to be considered.
Subcase-1: Let $g_1 = f_1$ and $g_2 = f_1$. Now,

$$\begin{aligned} g &= g_1 \cdot \pi \cdot g_2 \\ &= f_1 \cdot \pi \cdot f_1 \\ &\in (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup \\ &\quad f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \\ &\sqsubseteq ((\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2) \cdot \Gamma \cdot \end{aligned}$$

$$\begin{aligned} &\quad (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2)] \\ &\sqsubseteq (\mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \\ &\sqsubseteq \langle \mathcal{F} \rangle_{mlt}. \end{aligned}$$

Subcase-2: Let $g_1 \neq f_1$ and $g_2 = f_1$. Now,

$$\begin{aligned} g &= g_1 \cdot \pi \cdot g_2 \\ &\in (\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \\ &\sqsubseteq ((\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2)] \\ &\sqsubseteq (\mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \\ &\sqsubseteq \langle \mathcal{F} \rangle_{mlt}. \end{aligned}$$

Subcase-3: Let $g_1 = f_1$ and $g_2 \neq f_1$. Now,

$$\begin{aligned} g &= g_1 \cdot \pi \cdot g_2 \\ &\in (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\}) \\ &\sqsubseteq ((\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2) \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\})] \\ &\sqsubseteq (\mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \\ &\sqsubseteq \langle \mathcal{F} \rangle_{mlt}. \end{aligned}$$

Subcase-4: Let $g_1 \neq f_1$, $g_2 \neq f_1$, and $\mathcal{F} = \mathcal{G} \setminus \{f_1\}$. Now,

$$\begin{aligned} g &= g_1 \cdot \pi \cdot g_2 \\ &\in (\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\}) \\ &\sqsubseteq \langle \mathcal{F} \rangle_{mlt}. \end{aligned}$$

Case-3: Let $g \in \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G}$. Then, $g = g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5$ for some $g_3, g_4, g_5 \in \mathcal{G}$, $s_1, s_2, \ldots, s_n \in \mathcal{S}$ and $\pi, \theta, \theta_1, \pi_1, \pi_2, \ldots, \pi_n \in \Gamma$. We will examine eight subcases.

$g_i \setminus Subcase$	g_3	g_4	g_5
$Subcase - 1$	$f_1 = g_3$	$f_1 = g_4$	$f_1 = g_5$
$Subcase - 2$	$f_1 \neq g_3$	$f_1 = g_4$	$f_1 = g_5$
$Subcase - 3$	$f_1 = g_3$	$f_1 \neq g_4$	$f_1 = g_5$
$Subcase - 4$	$f_1 = g_3$	$f_1 = g_4$	$f_1 \neq g_5$
$Subcase - 5$	$f_1 \neq g_3$	$f_1 \neq g_4$	$f_1 = g_5$
$Subcase - 6$	$f_1 = g_3$	$f_1 \neq g_4$	$f_1 \neq g_5$
$Subcase - 7$	$f_1 \neq g_3$	$f_1 = g_4$	$f_1 \neq g_5$
$Subcase - 8$	$f_1 \neq g_3$	$f_1 \neq g_4$	$f_1 \neq g_5$

Subcase-1: Let $g_3 = f_1, g_4 = f_1$, and $g_5 = f_1$. Now,

$$\begin{aligned} g &= g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \\ &= f_1 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_1 \cdot \theta_1 \cdot f_1 \\ &\in (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \\ &\quad (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup \\ &\quad f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \\ &\sqsubseteq \Big(\Big\{(\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2) \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \\ &\quad (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2)\Big\} \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup \end{aligned}$$

$$\begin{aligned} &\quad f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2\Big] \\ &\sqsubseteq (\mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \\ &\sqsubseteq \langle \mathcal{F} \rangle_{mlt}. \end{aligned}$$

Subcase-2: Let $g_3 \neq f_1, g_4 = f_1$, and $g_5 = f_1$. Now,

$$\begin{aligned} g &= g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \\ &\in (\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \\ &\quad \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \\ &\sqsubseteq \Big((\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \\ &\quad \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2) \cdot \Gamma \cdot (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2\Big] \\ &\sqsubseteq (\mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \\ &\sqsubseteq \langle \mathcal{F} \rangle_{mlt}. \end{aligned}$$

Subcase-6: Let $g_3 = f_1, g_4 \neq f_1$, and $g_5 \neq f_1$. Now,

$$\begin{aligned} g &= g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \\ &\in (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \\ &\quad \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\}) \\ &\sqsubseteq ((\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2) \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \\ &\quad \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\})] \\ &\sqsubseteq (\mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \\ &\sqsubseteq \langle \mathcal{F} \rangle_{mlt}. \end{aligned}$$

Subcase-8: Let $g_3 \neq f_1, g_4 \neq f_1, g_5 \neq f_1$, and $\mathcal{F} = \mathcal{G} \setminus \{f_1\}$. Now,

$$\begin{aligned} g &= g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \\ &\in (\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\}) \cdot \Gamma \cdot (\mathcal{G} \setminus \{f_1\}) \\ &\sqsubseteq (\mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \\ &\sqsubseteq \langle \mathcal{F} \rangle_{mlt}. \end{aligned}$$

It is similar to prove other subcases. Hence, for all the cases, we have $\mathcal{S} \sqsubseteq \langle \mathcal{F} \rangle_{mlt}$. Thus, $\mathcal{S} = \langle \mathcal{F} \rangle_{mlt}$, which is a contradiction. Hence $f_1 = f_2$. □

Lemma 2. *Let $\mathcal{G}$ be the M-LTB of $\mathcal{S}$ and $f_1, f_2, f_3, f_4 \in \mathcal{G}$. If $f_1 \in (\mathcal{N}(f_3 \cdot \Gamma \cdot f_2) \cup f_3 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_4]$, then $f_1 = f_2$ or $f_1 = f_3$ or $f_1 = f_4$.*

Proof. Theorem 2 leads to the proof. □

Definition 11. *For any $s_1, s_2 \in \mathcal{S}$, $s_1 \preceq_{mlt} s_2 \iff \langle s_1 \rangle_{mlt} \sqsubseteq \langle s_2 \rangle_{mlt}$ is called a quasi-order on $\mathcal{S}$.*

Remark 4. *The order $\preceq_{mlt}$ is not a partial order of $\mathcal{S}$.*

Example 3. *By Example 2, $\langle k_5 \rangle_{mlt} \sqsubseteq \langle k_6 \rangle_{mlt}$ and $\langle k_6 \rangle_{mlt} \sqsubseteq \langle k_5 \rangle_{mlt}$ but $k_5 \neq k_6$. Hence, the relation $\preceq_{mlt}$ is not a partial order on $\mathcal{S}$.*

If $\mathcal{F}$ is an M-LTB of $\mathcal{S}$, then $\langle \mathcal{F} \rangle_{mlt} = \mathcal{S}$. Let $s \in \mathcal{S}$. Then, $s \in \langle \mathcal{F} \rangle_{mlt}$, and so, $s \in \langle f_1 \rangle_{mlt}$ for some $f_1 \in \mathcal{F}$. This implies $\langle s \rangle_{mlt} \sqsubseteq \langle f_1 \rangle_{mlt}$. Hence, $s \preceq_{mlt} f_1$.

Remark 5. *If $\mathcal{G}$ is a M-LTB of $\mathcal{S}$ then for any $s \in \mathcal{S}$, there exists $f_1 \in \mathcal{G}$ such that $s \preceq_{mlt} f_1$.*

Lemma 3. *Let $\mathcal{G}$ be an M-LTB of $\mathcal{S}$. If $f_1, f_2 \in \mathcal{G}$ such that $f_1 \neq f_2$, then neither $f_1 \preceq_{mlt} f_2$ nor $f_2 \preceq_{mlt} f_1$.*

Proof. Assume that $f_1, f_2 \in \mathcal{G}$, such that $f_1 \neq f_2$. Suppose that $f_1 \preceq_{mlt} f_2$. Let $\mathcal{F} = \mathcal{G} \setminus \{f_1\}$. Then, $f_2 \in \mathcal{F}$. Let $s \in \mathcal{S}$. By Remark 5, there exists $f_3 \in \mathcal{G}$, such that $s \preceq_{mlt} f_3$. We think about two cases to be discussed. If $f_3 \neq f_1$, then $f_3 \in \mathcal{F}$. Thus, $\langle s\rangle_{mlt} \sqsubseteq \langle f_3\rangle_{mlt} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$. Hence, $\mathcal{S} = \langle \mathcal{F}\rangle_{mlt}$, which is a contradiction. If $f_3 = f_1$, then $s \preceq_{mlt} f_2$. Hence, $s \in \langle \mathcal{F}\rangle_{mlt}$, since $f_2 \in \mathcal{F}$. Hence, $\mathcal{S} = \langle \mathcal{F}\rangle_{mlt}$, which is a contradiction. A similar argument can be made for other cases. □

Lemma 4. *Let $\mathcal{G}$ be the M-LTB of $\mathcal{S}$ and $f_1, f_2, f_3 \in \mathcal{G}$ and $s \in \mathcal{S}$.*

1. *If $f_1 \in (\{f_2 \cdot \pi \cdot f_3\} \cup \mathcal{N}(\{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\}) \cup \{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot$
$\Gamma \cdot \{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\}]$, then $f_1 = f_2$ or $f_1 = f_3$,*
2. *If $f_1 \in (\{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cup \mathcal{N}(\{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\}) \cup \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\}]$, then $f_1 = f_2$ or $f_1 = f_3$ or $f_1 = f_4$.*

Proof. (1) Assume that $f_1 \in (\{f_2 \cdot \pi \cdot f_3\} \cup \mathcal{N}(\{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\}) \cup \{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\}]$ and suppose that $f_1 \neq f_2$ and $f_1 \neq f_3$. Let $\mathcal{F} = \mathcal{G} \setminus \{f_1\}$. Clearly, $\mathcal{F} \subset \mathcal{G}$, since $f_1 \neq f_2$ and $f_1 \neq f_3$ implies $f_2, f_3 \in \mathcal{F}$. To prove that $\langle \mathcal{G}\rangle_{mlt} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$, it suffices to determine that $\mathcal{G} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$. Let $f \in \mathcal{G}$, if $f \neq f_1$ that $f \in \mathcal{F}$, and hence, $f \in \langle \mathcal{F}\rangle_{mlt}$. If $f = f_1$, then

$$\begin{aligned} f = f_1 \quad &\in \quad (\{f_2 \cdot \pi \cdot f_3\} \cup \mathcal{N}(\{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\}) \cup \{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \\ &\qquad \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\}] \\ &\sqsubseteq \quad (\mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cup \mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F} \\ &\sqsubseteq \quad \langle \mathcal{F}\rangle_{mlt}. \end{aligned}$$

Thus, $\mathcal{G} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$. This implies $\langle \mathcal{G}\rangle_{mlt} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$, as $\mathcal{G}$ is an M-LTB of $\mathcal{S}$ and $\mathcal{S} = \langle \mathcal{G}\rangle_{mlt} \sqsubseteq \langle \mathcal{F}\rangle_{mlt} \sqsubseteq \mathcal{S}$. Therefore, $\mathcal{S} = \langle \mathcal{F}\rangle_{mlt}$, which is a contradiction. Hence, $f_1 = f_2$ or $f_1 = f_3$.

(2) Assume that $f_1 \in (\{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cup \mathcal{N}(\{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\}) \cup \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot \mathcal{S} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\}]$ and suppose that $f_1 \neq f_2$ and $f_1 \neq f_3$ and $f_1 \neq f_4$. Let $\mathcal{F} = \mathcal{G} \setminus \{f_1\}$. Clearly, $\mathcal{F} \subset \mathcal{G}$, since $f_1 \neq f_2$, $f_1 \neq f_3$, and $f_1 \neq f_4$ imply that $f_2, f_3, f_4 \in \mathcal{F}$. To prove that $\langle \mathcal{G}\rangle_{mlt} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$, it remains to prove that $\mathcal{G} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$. Let $f \in \mathcal{G}$ if $f \neq f_1$ that $f \in \mathcal{F}$, and so, $f \in \langle \mathcal{F}\rangle_{mlt}$. Hence,

$$\begin{aligned} f \quad &= \quad f_1 \\ &\in \quad (\{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cup \mathcal{N}(\{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \\ &\qquad \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\}) \cup \\ &\qquad \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \\ &\qquad \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \\ &\qquad \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\}] \\ &\sqsubseteq \quad (\mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}] \\ &\sqsubseteq \quad \langle \mathcal{F}\rangle_{mlt}. \end{aligned}$$

Thus, $\mathcal{G} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$. This implies $\langle \mathcal{G}\rangle_{mlt} \sqsubseteq \langle \mathcal{F}\rangle_{mlt}$ as $\mathcal{G}$ is an M-LTB of $\mathcal{S}$ and $\mathcal{S} = \langle \mathcal{G}\rangle_{mlt} \sqsubseteq \langle \mathcal{F}\rangle_{mlt} \sqsubseteq \mathcal{S}$. Therefore, $\mathcal{S} = \langle \mathcal{F}\rangle_{mlt}$, which is a contradiction. Hence, $f_1 = f_2$ or $f_1 = f_3$ or $f_1 = f_4$. □

Lemma 5. *Let $\mathcal{G}$ be an M-LTB of $\mathcal{S}$,*

1. *If $f_1 \neq f_2$ and $f_1 \neq f_3$, then $f_1 \npreceq_{mlt} f_2 \cdot \pi \cdot f_3$.*
2. *If $f_1 \neq f_2, f_1 \neq f_3$ and $f_1 \neq f_4$, then $f_1 \npreceq_{mlt} f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4$, for $f_1, f_2, f_3, f_4 \in \mathcal{G}$, $\pi, \pi_i, \theta, \theta_1 \in \Gamma$ and $s_i \in \mathcal{S}$, $i = 1, 2, \ldots, n$.*

Proof. (1) For any $f_1, f_2, f_3 \in \mathcal{G}$, let $f_1 \neq f_2$ and $f_1 \neq f_3$. Suppose that $f_1 \preceq_{mlt} f_2 \cdot \pi \cdot f_3$ and

$$\begin{aligned} f_1 &\in \langle f_1 \rangle_{mlt} \\ &\sqsubseteq \{(f_2 \cdot \pi \cdot f_3)\}_{mlt} \\ &= (\{(f_2 \cdot \pi \cdot f_3)\} \cup \mathcal{N}(\{(f_2 \cdot \pi \cdot f_3)\} \cdot \Gamma \cdot \{(f_2 \cdot \pi \cdot f_3)\}) \cup \{(f_2 \cdot \pi \cdot f_3)\} \cdot \Gamma \cdot \\ &\quad (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{(f_2 \cdot \pi \cdot f_3)\} \cdot \Gamma \cdot \{(f_2 \cdot \pi \cdot f_3)\}]. \end{aligned}$$

By Lemma 4 (1), it follows that $f_1 = f_2$ or $f_1 = f_3$, which is a contradiction.
(2) For any $f_1, f_2, f_3, f_4 \in \mathcal{G}$, let $f_1 \neq f_2$, $f_1 \neq f_3$, and $f_1 \neq f_4$. Suppose that $f_1 \preceq_{mlt} f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4$, we have

$$\begin{aligned} f_1 &\in \langle f_1 \rangle_{mlt} \\ &\sqsubseteq \{(f_2 \cdot \Gamma \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4)\}_{mlt} \\ &= (\{(f_2 \cdot \Gamma \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4)\} \cup \mathcal{N}(\{(f_2 \cdot \Gamma \cdot (s_1 \cdot \pi_1 \cdot s_2 \\ &\quad \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4)\} \cdot \Gamma \cdot \{(f_2 \cdot \Gamma \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4)\}) \\ &\quad \cup \{(f_2 \cdot \Gamma \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4)\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \\ &\quad \cdot \{(f_2 \cdot \Gamma \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4)\}]. \end{aligned}$$

By Lemma 4 (2), it follows that $f_1 = f_2$, $f_1 = f_3$, or $f_1 = f_4$, which contradicts our assumption. □

Theorem 3. *Let $\mathcal{G}$ be the M-LTB of $\mathcal{S}$, if and only if $\mathcal{G}$ satisfies the following*

1. *For any $s \in \mathcal{S}$,*
   (1.1) there exists $f_2 \in \mathcal{G}$ such that $s \preceq_{mlt} f_2$ (or),
   (1.2) there exists $g_1, g_2 \in \mathcal{G}$ such that $s \preceq_{mlt} g_1 \cdot \pi \cdot g_2$ (or),
   (1.3) there exists $g_3, g_4, g_5 \in \mathcal{G}$ such that $s \preceq_{mlt} g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5$;
2. *If $f_1 \neq f_2$ and $f_1 \neq f_3$ and $f_1 \neq f_4$, then $f_1 \npreceq_{mlt} f_2 \cdot \pi \cdot f_3$, for any $f_1, f_2, f_3 \in \mathcal{G}$;*
3. *If $f_1 \neq f_2$ and $f_1 \neq f_3$ and $f_1 \neq f_4$, then $f_1 \npreceq_{mlt} f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4$, for any $f_1, f_2, f_3, f_4 \in \mathcal{G}, s_i \in \mathcal{S}$ and $\pi_i, \pi, \theta, \theta_1 \in \Gamma$, $i = 1, 2, \ldots, n$.*

Proof. Assume that $\mathcal{G}$ is an M-LTB of $\mathcal{S}$, then $\mathcal{S} = \langle \mathcal{G} \rangle_{mlt}$. To prove that (1), let $s \in \mathcal{S}$, $s \in (\mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G}]$. As $s \in (\mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G}]$, we have $s \preceq g$ for some $g \in \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G}$, we think about the three following cases.
Case-1 : Let $g \in \mathcal{G}$. Then, $g = f_2$ for some $f_2 \in \mathcal{G}$. This implies $\langle g \rangle_{mlt} \sqsubseteq \langle f_2 \rangle_{mlt}$. Hence, $g \preceq_{mlt} f_2$. As $s \preceq g$ for some $g \in \langle f_2 \rangle_{mlt}$. To find out $\langle s \rangle_{mlt} \sqsubseteq \langle f_2 \rangle_{mlt}$. Now, $s \cup \mathcal{N}(s \cdot \Gamma \cdot s) \cup \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot s \cdot \Gamma \cdot s \sqsubseteq \langle f_2 \rangle_{mlt} \cup \mathcal{N}(\langle f_2 \rangle_{mlt} \cdot \Gamma \cdot \langle f_2 \rangle_{mlt}) \cup \langle f_2 \rangle_{mlt} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \langle f_2 \rangle_{mlt} \sqsubseteq f_2 \cup \mathcal{N}(f_2 \cdot \pi \cdot f_2) \cup f_2 \cdot \pi \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2]$. We have $(s \cup \mathcal{N}(s \cdot \Gamma \cdot s) \cup \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot s \cdot \Gamma \cdot s] \sqsubseteq (f_2 \cup \mathcal{N}(f_2 \cdot \pi \cdot f_2) \cup f_2 \cdot \pi \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2]$. Thus, $\langle s \rangle_{mlt} \sqsubseteq \langle f_2 \rangle_{mlt}$, and hence, $s \preceq_{mlt} f_2$.
Case-2 : Let $g \in \mathcal{G} \cdot \Gamma \cdot \mathcal{G}$. Then, $g = g_1 \cdot \pi \cdot g_2$ for some $g_1, g_2 \in \mathcal{G}$ and $\pi \in \Gamma$. This implies $\langle g \rangle_{mlt} \sqsubseteq \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$. Hence, $g \preceq_{mlt} g_1 \cdot \pi \cdot g_2$. As $s \preceq g$ for some $g \in \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$. We have $s \in \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$. We determine that $\langle s \rangle_{mlt} \sqsubseteq \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$. Now, $s \cup \mathcal{N}(s \cdot \Gamma \cdot s) \cup \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot s \cdot \Gamma \cdot s \sqsubseteq (\{g_1 \cdot \pi \cdot g_2\} \cup \mathcal{N}(\{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\}) \cup \{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\}]$. Hence, $(s \cup \mathcal{N}(s \cdot \Gamma \cdot s) \cup \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot s \cdot \Gamma \cdot s] \sqsubseteq (\{g_1 \cdot \pi \cdot g_2\} \cup \mathcal{N}(\{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\}) \cup \{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\}]$. This

implies $\langle s \rangle_{mlt} \sqsubseteq \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$. Hence, $s \preceq_{mlt} g_1 \cdot \pi \cdot g_2$.
Case-3: Let $g \in \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot \mathcal{G}$. Then, $g = g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5$ for some $g_3, g_4 \in \mathcal{G}$. This implies $\langle g \rangle_{mlt} \sqsubseteq \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. Hence, $g \preceq_{mlt} \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. As $s \preceq g$ for some $g \in \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. We have $s \in \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. To prove that $\langle s \rangle_{mlt} \sqsubseteq \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. Now, $s \cup \mathcal{N}(s \cdot \Gamma \cdot s) \cup \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot s \cdot \Gamma \cdot s \sqsubseteq (\{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cup \mathcal{N}(\{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\}) \cup \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\}]$. Hence, $(s \cup \mathcal{N}(s \cdot \Gamma \cdot s) \cup \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot s \cdot \Gamma \cdot s] \sqsubseteq (\{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cup \mathcal{N}(\{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\}) \cup \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\}]$. This implies $\langle s \rangle_{mlt} \sqsubseteq \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. Hence, $s \preceq_{mlt} g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5$. By Lemma 5(1) and Lemma 5(2), we have the proof of (2) and (3), respectively.

Conversely, assume that (1), (2), and (3) hold to prove that $\mathcal{G}$ is an M-LTB of $\mathcal{S}$. To determine that $\mathcal{S} = \langle \mathcal{G} \rangle_{mlt}$, clearly, $\langle \mathcal{G} \rangle_{mlt} \sqsubseteq \mathcal{S}$. By (1), $\mathcal{S} \sqsubseteq \langle \mathcal{G} \rangle_{mlt}$ and $\mathcal{S} = \langle \mathcal{G} \rangle_{mlt}$. It remains to be determined whether $\mathcal{G}$ is a minimal subset of $\mathcal{S}$, $\mathcal{S} = \langle \mathcal{G} \rangle_{mlt}$. Suppose that $\mathcal{S} = \langle \mathcal{F} \rangle_{mlt}$ for some $\mathcal{F} \subset \mathcal{G}$. As $\mathcal{F} \subset \mathcal{G}$, there exists $f_2 \in \mathcal{G} \setminus \mathcal{F}$. As $f_2 \in \mathcal{G} \sqsubseteq \mathcal{S} = \langle \mathcal{F} \rangle_{mlt}$ and $f_2 \notin \mathcal{F}$, it follows that $f_2 \in (\mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cup \mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}]$. As $f_2 \in (\mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cup \mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}]$, this implies $f_2 \preceq g$ for some $g \in \mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cup \mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}$. There are two cases to be observed.
Case-1: Let $g \in \mathcal{F} \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}$. Then, $g = g_1 \cdot \pi \cdot g_2$ for some $g_1, g_2 \in \mathcal{F}$ and $\pi \in \Gamma$. We have $g_1, g_2 \in \mathcal{G}$. As $f_2 \notin \mathcal{F}$, $f_2 \neq g_1$ and $f_2 \neq g_2$. As $g = g_1 \cdot \pi \cdot g_2$, $\langle g \rangle_{mlt} \sqsubseteq \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$. Hence, $g \preceq_{mlt} g_1 \cdot \pi \cdot g_2$. As $f_2 \preceq g$ for some $g \in \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$, we have $f_2 \in \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$ to prove that $\langle f_2 \rangle_{mlt} \sqsubseteq \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$.
Now, $f_2 \cup \mathcal{N}(f_2 \cdot \pi \cdot f_2) \cup f_2 \cdot \pi \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2 \sqsubseteq (\{g_1 \cdot \pi \cdot g_2\} \cup \mathcal{N}(\{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\}) \cup \{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\}]$. Hence, $(f_2 \cup \mathcal{N}(f_2 \cdot \pi \cdot f_2) \cup f_2 \cdot \pi \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \sqsubseteq (\{g_1 \cdot \pi \cdot g_2\} \cup \mathcal{N}(\{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\}) \cup \{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\} \cdot \Gamma \cdot \{g_1 \cdot \pi \cdot g_2\}]$. This implies $\langle f_2 \rangle_{mlt} \sqsubseteq \langle g_1 \cdot \pi \cdot g_2 \rangle_{mlt}$. Hence, $f_2 \preceq_{mlt} g_1 \cdot \pi \cdot g_2$. This contradicts (2).
Case-2: Let $g \in \mathcal{F} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \mathcal{F} \cdot \Gamma \cdot \mathcal{F}$. Then, $g = g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5$ for some $g_3, g_4 \in \mathcal{F}, s_i \in \mathcal{S}$ and $\pi_i, \pi, \theta \in \Gamma$, $i = 1, 2, \ldots, n$. We have $g_3, g_4 \in \mathcal{G}$. As $f_2 \notin \mathcal{F}$, so $f_2 \neq g_3$ and $f_2 \neq g_4$. As $g = g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5$, $\langle g \rangle_{mlt} \sqsubseteq \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. Hence, $g \preceq_{mlt} g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5$. Since $f_2 \preceq g$ for some $g \in \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$, we have $f_2 \in \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. We determine that $\langle f_2 \rangle_{mlt} \sqsubseteq \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. Now, $f_2 \cup \mathcal{N}(f_2 \cdot \pi \cdot f_2) \cup f_2 \cdot \pi \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2 \sqsubseteq (\{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cup \mathcal{N}(\{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\}) \cup \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\}]$. Hence, $(f_2 \cup \mathcal{N}(f_2 \cdot \pi \cdot f_2) \cup f_2 \cdot \pi \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot f_2] \sqsubseteq (\{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cup \mathcal{N}(\{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\}) \cup \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\} \cdot \Gamma \cdot \{g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5\}]$. This implies $\langle f_2 \rangle_{mlt} \sqsubseteq \langle g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5 \rangle_{mlt}$. Hence, $f_2 \preceq_{mlt} g_3 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot g_4 \cdot \theta_1 \cdot g_5$, which is a contradiction to (3). Therefore, $\mathcal{G}$ is an M-LTB of $\mathcal{S}$. □

Theorem 4. *Let $\mathcal{G}$ be an M-LTB of $\mathcal{S}$. Then, $\mathcal{G}$ is an ordered Γ-subsemigroup of $\mathcal{S}$, if and only if $g_1 \cdot \pi \cdot g_2 = g_1$ or $g_1 \cdot \pi \cdot g_2 = g_2$, for any $g_1, g_2 \in \mathcal{G}$ and $\pi \in \Gamma$.*

Proof. If $\mathcal{G}$ is an ordered Γ-subsemigroup of $\mathcal{S}$, then $g_1 \cdot \pi \cdot g_2 \in \mathcal{G}$. As $g_1 \cdot \pi \cdot g_2 \in (\mathcal{N}(g_1 \cdot \Gamma \cdot g_2) \cup g_1 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot g_2 \cdot \Gamma \cdot g_2]$, it follows by Lemma 2 that $g_1 \cdot \pi \cdot g_2 = g_1$ or $g_1 \cdot \pi \cdot g_2 = g_2$. □

4. *M*-RTB Generator

We present some results on the M-right-tri-ideal (RTI) generator based on an ordered Γ-semigroup.

Definition 12. *Let $\mathcal{S}$ be an ordered Γ- semigroup. $\mathcal{G} \sqsubseteq \mathcal{S}$ is said to be an M-RTI of $\mathcal{S}$ if it meets the criteria listed below:*

1. *$\mathcal{G}$ is a Γ-subsemigroup,*
2. *$\mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S} \ \ (M - times)) \cdot \Gamma \cdot \mathcal{G} \sqsubseteq \mathcal{G}$,*
3. *If $g \in \mathcal{G}$ and $s \in \mathcal{S}$, such that $s \preceq g$, then $s \in \mathcal{G}$.*

Theorem 5.
1. *For $f_1 \in \mathcal{S}$, the M-RTI generated by "f_1" is $\langle f_1 \rangle_{mrt} = \{f_1 \cup \mathcal{N}(f_1 \cdot \Gamma \cdot f_1) \cup f_1 \cdot \Gamma \cdot f_1 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S} \ \ (M - times)) \cdot \Gamma \cdot f_1\}$ and $\mathcal{N} \succeq M$, where $\mathcal{N}$ and M are positive integers;*
2. *For $\mathcal{G} \sqsubseteq \mathcal{S}$, the M-RTI generated by "$\mathcal{G}$" is $\langle \mathcal{G} \rangle_{mrt} = \{\mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cup \mathcal{G} \cdot \Gamma \cdot \mathcal{G} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S} \ \ (M - times)) \cdot \Gamma \cdot \mathcal{G}\}$.*

Definition 13. *Let $\mathcal{G}$ be a subset $\mathcal{S}$ called a M-right tri-basis (RTB) of $\mathcal{S}$ if it satisfies the following conditions:*

1. *$\mathcal{S} = \langle \mathcal{G} \rangle_{mrt}$.*
2. *If $\mathcal{F} \sqsubseteq \mathcal{G}$ such that $\mathcal{S} = \langle \mathcal{F} \rangle_{mrt}$, then $\mathcal{F} = \mathcal{G}$.*

Theorem 6. *Let $\mathcal{G}$ be an M-RTB of $\mathcal{S}$ and $f_1, f_2 \in \mathcal{G}$. If $f_1 \in (\mathcal{N}(f_2 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot f_2 \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_2]$, then $f_1 = f_2$.*

Proof. The proof is the same as in Theorem 2. □

Lemma 6. *Let $\mathcal{G}$ be an M-RTB of $\mathcal{S}$ and $f_1, f_2, f_3, f_4 \in \mathcal{G}$. If $f_1 \in (\mathcal{N}(f_3 \cdot \Gamma \cdot f_2) \cup f_2 \cdot \Gamma \cdot f_4 \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot f_3]$, then $f_1 = f_2$ or $f_1 = f_3$ or $f_1 = f_4$.*

Proof. Theorem 2 leads to the proof. □

Definition 14. *For any $s_1, s_2 \in \mathcal{S}$, $s_1 \preceq_{mrt} s_2 \iff \langle s_1 \rangle_{mrt} \sqsubseteq \langle s_2 \rangle_{mrt}$ is called a quasi-order on $\mathcal{S}$.*

Remark 6. *The order $\preceq_{mrt}$ is not a partial order of $\mathcal{S}$.*

Example 4. *By Example 2, $\langle k_4 \rangle_{mrt} \sqsubseteq \langle k_6 \rangle_{mrt}$ and $\langle k_6 \rangle_{mrt} \sqsubseteq \langle k_4 \rangle_{mrt}$ but $k_4 \neq k_6$. Hence, the relation $\preceq_{mrt}$ is not a partial order on $\mathcal{S}$.*

If $\mathcal{F}$ is an M-RTB of $\mathcal{S}$, then $\langle \mathcal{F} \rangle_{mrt} = \mathcal{S}$. Let $s \in \mathcal{S}$. Then, $s \in \langle \mathcal{F} \rangle_{mrt}$ and so $s \in \langle f_1 \rangle_{mrt}$ for some $f_1 \in \mathcal{F}$. This implies $\langle s \rangle_{mrt} \sqsubseteq \langle f_1 \rangle_{mrt}$. Hence, $s \preceq_{mrt} f_1$.

Remark 7. *If $\mathcal{G}$ is an M-RTB of $\mathcal{S}$, then for any $s \in \mathcal{S}$, there exists $f_1 \in \mathcal{G}$ such that $s \preceq_{mrt} f_1$.*

Lemma 7. *Let $\mathcal{G}$ be an M-RTB of $\mathcal{S}$. If $f_1, f_2 \in \mathcal{G}$ such that $f_1 \neq f_2$, then neither $f_1 \preceq_{mrt} f_2$ nor $f_2 \preceq_{mrt} f_1$.*

Proof. The proof follows from Lemma 3. □

Lemma 8. *Let $\mathcal{G}$ be the M-RTB of $\mathcal{S}$ and $f_1, f_2, f_3 \in \mathcal{G}$ and $s \in \mathcal{S}$.*

1. *If $f_1 \in (\{f_2 \cdot \pi \cdot f_3\} \cup \mathcal{N}(\{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\}) \cup \{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot f_3\}]$, then $f_1 = f_2$ or $f_1 = f_3$;*
2. *If $f_1 \in (\{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cup \mathcal{N}(\{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\}) \cup \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\} \cdot \Gamma \cdot (\mathcal{S} \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot \mathcal{S}) \cdot \Gamma \cdot \{f_2 \cdot \pi \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \theta \cdot f_3 \cdot \theta_1 \cdot f_4\}]$, then $f_1 = f_2$ or $f_1 = f_3$ or $f_1 = f_4$.*

Proof. The proof follows from Lemma 4. □

Lemma 9. *Let $\mathcal{G}$ be the M-RTB of $\mathcal{S}$,*

1. *If $f_1 \neq f_2$ and $f_1 \neq f_3$, then $f_1 \not\preceq_{mrt} f_2 \cdot \pi \cdot f_3$.*
2. *If $f_1 \neq f_2, f_1 \neq f_3$ and $f_1 \neq f_4$, then $f_1 \not\preceq_{mrt} f_3 \cdot \theta \cdot f_4 \cdot \theta_1 \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \pi \cdot f_2$, for $f_1, f_2, f_3, f_4 \in \mathcal{G}$, $\pi, \pi_i, \theta, \theta_1 \in \Gamma$ and $s_i \in \mathcal{S}$, $i = 1, 2, \ldots, n$.*

Proof. The proof follows from Lemma 5. □

Theorem 7. *Let $\mathcal{G}$ be the M-RTB of $\mathcal{S}$, if and only if the following conditions are met by $\mathcal{G}$.*

1. *For any $s \in \mathcal{S}$,*
   (1.1) there exists $f_2 \in \mathcal{G}$, such that $s \preceq_{mrt} f_2$ (or),
   (1.2) there exists $g_1, g_2 \in \mathcal{G}$, such that $s \preceq_{mrt} g_1 \cdot \pi \cdot g_2$ (or),
   (1.3) there exists $g_3, g_4, g_5 \in \mathcal{G}$, such that $s \preceq_{mrt} g_4 \cdot \theta \cdot g_5 \cdot \theta_1 \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \pi \cdot g_3$;
2. *If $f_1 \neq f_2$, $f_1 \neq f_3$, and $f_1 \neq f_4$, then $f_1 \not\preceq_{mrt} f_2 \cdot \pi \cdot f_3$, for any $f_1, f_2, f_3 \in \mathcal{G}$,*
3. *If $f_1 \neq f_2$, $f_1 \neq f_3$, and $f_1 \neq f_4$, then $f_1 \not\preceq_{mrt} f_3 \cdot \theta \cdot f_4 \cdot \theta_1 \cdot (s_1 \cdot \pi_1 \cdot s_2 \cdot \ldots \cdot \pi_n \cdot s_n) \cdot \pi \cdot f_2$, for any $f_1, f_2, f_3, f_4 \in \mathcal{G}, s_i \in \mathcal{S}$ and $\pi_i, \pi, \theta, \theta_1 \in \Gamma$, $i = 1, 2, \ldots, n$.*

Proof. Theorem 3 leads to the proof. □

Theorem 8. *Let $\mathcal{G}$ be an M-RTB of $\mathcal{S}$. Then, $\mathcal{G}$ is an ordered Γ-subsemigroup of $\mathcal{S}$, if and only if $g_1 \cdot \pi \cdot g_2 = g_1$ or $g_1 \cdot \pi \cdot g_2 = g_2$, for any $g_1, g_2 \in \mathcal{G}$ and $\pi \in \Gamma$.*

Proof. The proof is the same as Theorem 4. □

5. Conclusions

Several characterizations of the M-LTB (RTB) of an ordered Γ-semigroup are described in this article. Our discussion has focused on some of their fundamental characteristics and has also examined some of them using the M-tri-ideal generator. We presented the M-LTB (RTB) of an ordered Γ-semigroup, which was constructed from an ordered Γ-semigroup element and subset. At the end of our discussion, we explored the relationship between partial order and the M-LTB (RTB). In the future, we plan to explore a few more types of tri-basis and tri-M-basis. Our study will examine their research on Γ-hyper semigroups using bi-basis and M-bi-basis.

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