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# Exponential Stability of Nonlinear Time-Varying Delay Differential Equations via Lyapunov–Razumikhin Technique

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**Abstract:** In this article, some new sufficient conditions for the exponential stability of nonlinear time-varying delay differential equations are given. An extension of the classical asymptotical stability theorem in terms of a Lyapunov–Razumikhin function is obtained. The condition of non-positivity of the time derivative of a Razumikhin function is weakened. Additionally, the resulting sufficient asymptotic stability conditions allow us to guarantee uniform exponential stability and evaluate the exponential convergence rate of the system solutions. The effectiveness of the results is demonstrated by some examples.

**Keywords:** time-varying system; delay differential system; Razumikhin condition; exponential stability

**MSC:** 34K20

## 1. Introduction

There have been a number of developments in searching for the stability criteria for nonlinear delay differential systems, but most of these have been restricted to finding asymptotic stability conditions.

As noted in [1], this may be caused by the fact that 'a "geometric" view of nonlinear dynamics leads one to adopt the view that notions of stability should be invariant under (nonlinear) changes of variables'. For this reason, 'exponential stability is not a natural mathematical notion when nonlinear coordinate changes are allowed. This is why the notion of asymptotic stability is important' [1].

However, in some cases it is necessary not only to establish the fact of asymptotic stability, but also to know the rate of convergence in the asymptotic stability property—for example, in order to guarantee some asymptotic properties of solutions to a cascaded system based on an analysis of the properties of its isolated subsystems [1]. In practical applications, it is also important to guarantee a certain rate of convergence of some characteristics of the system.

At the same time, the exponential stability of nonlinear systems is not easily verified. Moreover, for systems with delay, even in the linear case, there are difficulties in estimating the rate of convergence of solutions (see, for example, references in [2]).

Therefore, researchers are attracted by the problem of obtaining requirements for the Lyapunov function, which can guarantee estimates for the convergence of solutions.

The specificity of using finite-dimensional functions to analyze the stability properties of systems with delay is that the derivative of the Lyapunov function is estimated not in the entire neighborhood of the origin, but only within its part.

The idea of such a modification of the requirements for the derivative was simultaneously proposed by N.N. Krasovskii and B.S. Razumikhin [3,4]. Subsequently, the conditions

that determine a part of the neighborhood of zero for estimating the derivative were called Razumikhin conditions.

On one hand, these conditions simplify the practical verification of the relevant requirements that guarantee one type of stability or another. On the other hand, the absence of an estimate for the derivative of the Lyapunov–Razumikhin function along the entire solution makes it difficult to estimate the rate of decrease of the function itself, and, consequently, the estimate of the rate of convergence of solutions.

Therefore, for equations with delay, such estimates are mainly obtained using the Lyapunov–Krasovskii functionals [5,6]. Similar results using the functions instead the functionals have occurred from time to time and have become more frequent in the last decade; in this case, mainly autonomous or linear equations are considered (see, e.g., references in [7]).

The purpose of this paper is to establish sufficient conditions for the exponential stability of nonlinear time-varying delay differential systems, under less restrictive assumptions on the Lyapunov–Razumikhin function (the function $V(t, x)$ below) than those made in the literature so far.

Namely, the time-derivative of the constructed Razumikhin function is allowed to be neither sign-definite (as in the classical asymptotic stability theorem) nor semidefinite (as in some more general theorems, see e.g., [8]). Moreover, a new upper estimate on the rate of decreasing of the function is obtained.

It is important to note that, in this paper, consideration is given only to deterministic systems. Nowadays, practical control systems need to meet several important requirements, for example, time delays, stochastic perturbations, time-varying parameters, and impulse effects. The research problem of stability analysis for related systems has received a great deal of attention.

The Lyapunov–Razumikhin method was originally developed to study the stability of deterministic systems with delay. However, the Razumikhin technique has also been a powerful and effective method for investigating the stability of other types of delay systems. In 1996, X. Mao extended the technique to stochastic functional differential equations to investigate $p$-th moment exponential stability [9]. In the past several decades, a large number of works on this topic have been reported in the literature (e.g., [10–12]).

In particular, several novel criteria of the moment exponential stability are derived for the related systems; see, for example, [13–16] and references therein.

An analysis of these results shows that Razumikhin-type theorems for deterministic systems can be appropriately developed and extended to stochastic delay systems with additional features and effects. Taking into account the behavior of a specific system leads to using special tools and adding certain conditions to ensure stability. Therefore, when studying a system of a certain type, it is easier to rely on results for deterministic systems than to derive stability conditions from the available results for systems of another type. Therefore, the development of the Razumikhin method for deterministic continuous systems plays an important role both in theory and applications.

The paper is organized as follows. In Section 2, we introduce notation, definitions, and other preliminaries. Section 3 gives new sufficient conditions for uniform asymptotic stability and uniform exponential stability in terms of the Lyapunov–Razumikhin functions. Section 4 contains a remark on necessary conditions for uniform asymptotic stability, which are relevant to the results obtained. In Section 5, we consider scalar linear time-varying equations and show the effectiveness of the exponential stability theorem using some particular examples. Some more examples of nonlinear equations are given in Section 6. Section 7 discusses some known related results. Concluding remarks are given in the final section.

## 2. Preliminaries

The following notation will be used throughout the paper: $R^+$ is the set of all non-negative real numbers; $R^n$ is the $n$-dimensional vector space and $|x|$ is a norm of a vector

$x \in R^n$; time delay is denoted by $r$ ($r > 0$), $R^+ = [0, +\infty)$; $R^n$ denotes the $n$-dimensional space of vectors $x = (x_1, \ldots, x_n)^\top$ with the norm $|x|$; $C = C([-r, 0], R^n)$ is the Banach space with the supremum-norm $\| \cdot \|$. For a continuous function $x(t) \in C([t_0 - r, t_0 + \bar{t}), R^n)$ ($t_0 \in R^+$, $\bar{t} > 0$) an element $x_t \in C$ is defined for any $t \in [t_0, t_0 + \bar{t})$ by $x_t(s) = x(t + s)$, $-r \le s \le 0$, and $\dot{x}(t)$ stands for the right-hand derivative.

Consider the nonlinear system described by a time-varying delay differential equation of the form:

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where $x(t) \in R^n$, $f(t, x) : R^+ \times C \to R^n$ is a given nonlinear function satisfying $f(t, 0) = 0$ for all $t \in R^+$. Then (1) admits the zero solution.

In the following, we assume that the Carathéodory-type conditions from [17] are imposed on Equation (1). We denote the solution of (1) with initial conditions $t_0 \in R^+$ and $\varphi_0 \in C$ by $x(t; t_0 \varphi_0)$.

In addition, we use the following standard definition:

**Definition 1.** *The zero solution of* (1) *is said to be uniformly exponentially stable (UES) if there exist constants* $\Delta$, $\delta$, *and* $r$ *such that*

$$\|\varphi_0\| < r, \ t \ge t_0 \ \Rightarrow \ |x(t; t_0 \varphi_0)| \le \Delta \|\varphi_0\| e^{-\delta(t - t_0)}.$$

*We say that the zero solution is globally uniformly exponentially stable (GUES) if* $r = \infty$.

*From now on, to shorten expressions, instead of saying the zero solution of the equation is GUES, we say that the equation is GUES.*

We also use the well-known concepts of uniform asymptotic stability (UAS) and global uniform asymptotic stability (GUAS).

Let $V(t, x) : R^+ \times R^n \to R$ be a given function. Now, let $x(t)$ be a solution of (1) starting from $(t, \varphi)$, and denote by $V'(t, \varphi)$ the upper right-hand derivative of $V(t, x(t))$ [18]:

$$\varlimsup_{h \to 0^+} \frac{V(t + h, x(t + h; t, \varphi)) - V(t, \varphi(0))}{h}. \tag{2}$$

Suppose that $V(t, x)$ is locally Lipschitzian in $x$ (uniformly in $t \in R^+$). Then, the derivative (2) is bounded and is equal to the upper Dini derivative of $V(.)$ (along the trajectory of (1)):

$$\varlimsup_{h \to 0^+} \frac{V(t + h, \varphi(0) + h f(t, \varphi)) - V(t, \varphi(0))}{h},$$

where $f(.)$ is the right-hand side function of (1).

Obviously, a continuously differentiable function satisfies the equality

$$V'(t, \varphi) = \frac{\partial V(t, \varphi(0))}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, \varphi(0))}{\partial x_i} X_i(t, \varphi).$$

In the sequel, we assume that $V(t, x)$ is continuous in $t$ and locally Lipschitzian in $x$ (uniformly in $t$).

We also define the class of functions $\mathcal{K} = \{a \in C(R^+, R^+), \sigma(u)$ is continuous, strictly increasing, and $a(0) = 0\}$. A *class-$\mathcal{K}_\infty$ function* is a function $a \in \mathcal{K}$ which is unbounded.

As is known, the idea that made it possible to obtain constructive results on stability in terms of functions for systems with delays was that it is sufficient to check the sign of the derivative $V'$ at each time not on the whole set $C$, but only on its subset [3,4]. As a rule, one of two sets is used:

$$\Omega_t(V) = \{\varphi \in C : V(t + s, \varphi(s)) \le V(t, \varphi(0)), \ -r \le s \le 0\},$$

$$\Omega_t(V, \eta) = \{\varphi \in C : V(t + s, \varphi(s)) \le \eta(V(t, \varphi(0))), \ -r \le s \le 0\},$$

where $\eta \in \mathcal{K}$ is such that $\eta(u) > u$ for $u > 0$.

The following Razumikhin-type result of N.N. Krasovskii became the basis for further studies of the asymptotic stability of delay systems in terms of functions.

**Theorem 1.** *Suppose that there is a function $V : R^+ \times R^n \to R^+$ such that:*

1.  $a_1(|x|) \leqslant V(t, x) \leqslant a_2(|x|)$;
2.  *if $t \in R^+$ and $\varphi \in \Omega_t(V, \eta)$ then $V'(t, \varphi) \leqslant -a_3(|x|)$;*

*for some functions $a_i \in \mathcal{K}$.*

*Then Equation (1) is UAS.*

It can be proved in the standard way that if $a_i \in \mathcal{K}_\infty$, then the uniform asymptotic stability will be global (GUAS).

## 3. Main Result

First, we prove the following lemma, which provides an upper bound for the behavior of the function $V(t, x)$ along solutions of Equation (1).

**Lemma 1.** *Suppose that there exist numbers $\bar{a} > 0$, $\underline{a} \in R$, and the functions $\eta \in \mathcal{K}$, $V \in C^1(R^+ \times R^n, R^+)$, and $p \in C([-r, +\infty), R)$, such that the following conditions are met:*

1.  *for all $t \in [-r, +\infty)$ and $s \in [0, r]$ the following inequalities $\underline{a} \leqslant \int_t^{t+s} p(\theta)d\theta \leqslant \bar{a}$ hold;*
2.  $\eta(u) \geqslant e^{\bar{a}} u$ *for $u > 0$:*
3.  $V'(t, \varphi) \leqslant -p(t)V(t, \varphi(0))$ *whenever $t \in R^+$ and $\varphi \in \Omega_t(V, \eta)$.*

*Then, for any solution $x(t) = x(t; t_0, \varphi_0)$ of Equation (1) and for all $t \geqslant t_0$ the estimation*

$$V(t, x(t)) \leqslant \max_{-r \leqslant s \leqslant 0} V(t_0 + s, \varphi_0(s)) e^{-(\int_{t_0}^t p(\theta)d\theta + a)}$$

*is valid with $a = \min\{0, \underline{a}\}$.*

**Proof.** We define the function $v(t) = V(t, x(t))$ along an arbitrary solution $x(t) = x(t; t_0, \varphi_0)$, and the function $u(t) = v(t) - \max_{-r \leqslant s \leqslant 0} v(t_0 + s) e^{-\left[\int_{t_0}^t p(\theta)d\theta + a\right]}$. It is clear that $u(t_0) \leqslant 0$. We define $t^* = \inf\{t \geqslant t_0 : u(t) > 0\} \geqslant t_0$. Then, $u(t^*) = 0$, while $u(t) \leqslant 0$ for $t \in [t_0 - r, t^*)$, and $u(t) > 0$ for $t \in (t^*, t^* + \varepsilon)$. So we have $\dot{u}(t^*) > 0$. Notice that $\dot{u}(t^*) = \dot{v}(t^*) + p(t^*) \max_{-r \leqslant s \leqslant 0} v(t_0 + s) e^{-\left[\int_{t_0}^{t^*} p(\theta)d\theta + a\right]} \leqslant \dot{v}(t^*) + p(t^*)v(t^*)$. On the other hand,

$$v(t^*) = \max_{-r \leqslant s \leqslant 0} v(t_0 + s) e^{-\left[\int_{t_0}^{t^*} p(\theta)d\theta + a\right]},$$

and for all $s \in [-r, 0]$ we have

$$v(t^* + s) \leqslant \max_{-r \leqslant s \leqslant 0} v(t_0 + s) e^{-\left[\int_{t_0}^{t^*+s} p(\theta)d\theta + a\right]}$$

$$= \max_{-r \leqslant s \leqslant 0} v(t_0 + s) e^{-\left[\int_{t_0}^{t^*} p(\theta)d\theta + a\right]} e^{\int_{t^*+s}^{t^*} p(\theta)d\theta + a}$$

$$\leqslant v(t^*) e^{\bar{a}} \leqslant \eta(v(t^*)).$$

Then, item 3 of the theorem implies $\dot{u}(t^*) \leqslant 0$. The obtained contradiction proves that $u(t) \leqslant 0$, which was to be proved. $\square$

Based on the lemma just proved, it is easy to obtain the following result:

**Theorem 2.** *In addition to the conditions of Lemma 1, suppose that for some $T > 0$, $A > 0$ and for all $t \in R^+$, we have the estimate*

$$\int_t^{t+T} p(\theta)d\theta \geqslant A.$$

*Then, for an arbitrary solution $x(t) = x(t; t_0, \varphi_0)$ of Equation (1) and for all $t \geqslant t_0$, there holds*

$$V(t, x(t)) \leqslant M \max_{-r \leqslant s \leqslant 0} V(t_0 + s, \varphi_0(s)) e^{-\frac{A}{T}(t-t_0)}$$

*with $M = e^{A - \underline{a} - a}$.*
*In addition:*

1. *if $a_1(|x|) \leqslant V(t, x)$ ($a_1 \in \mathcal{K}$) then Equation (1) is asymptotically stable;*
2. *if $a_1(|x|) \leqslant V(t, x) \leqslant a_2(|x|)$ ($a_1, a_2 \in \mathcal{K}$) then Equation (1) is UAS;*
3. *if $a_1(|x|) \leqslant V(t, x) \leqslant a_2(|x|)$ ($a_1, a_2 \in \mathcal{K}$), and there exist constants $c$, $c_1 > 0$, and $c_2 \in (0, 1)$ such that $a_1^{-1}(Ma_2(s)) \leqslant cs$ and $a_1^{-1}(c_1 a_2(s)) \leqslant c_2$, then Equation (1) is UES.*

**Proof.** Indeed, it follows from the additional condition that for $t \geqslant T$ the inequality $\int_{t_0}^{t_0+t} p(\theta)d\theta \geqslant A(t/T - 1) + \underline{a}$ is valid. Thus, we can immediately get the required estimate for $V(t, x(t))$.

Then, item 1 implies $a_1(|x(t)|) \leqslant V(t, x(t)) \leqslant M \max_{-r \leqslant s \leqslant 0} V(t_0 + s, \varphi_0(s)) e^{-\frac{A}{T}(t-t_0)}$, and item 2 implies $a_1(|x(t)|) \leqslant V(t, x(t)) \leqslant M \max_{-r \leqslant s \leqslant 0} V(t_0 + s, \varphi_0(s)) e^{-\frac{A}{T}(t-t_0)} \leqslant Ma_2(\|\varphi_0\|)e^{-\frac{A}{T}(t-t_0)}$. The assertions of items 1 and 2 follow directly from these estimates.

It remains to prove item 3. We define the function $v(t) = \max_{-r \leqslant s \leqslant 0} V(t + s, x(t + s))$ along an arbitrary solution $x(t) = x(t; t_0, \varphi_0)$. Note that the resulting estimate for $V$ implies $v(t + t_1) \leqslant Me^{-\frac{A}{T}(t_1 - r)}v(t) \leqslant c_1 v(t)$ for every $t \geqslant t_0$ and a sufficiently large value $t_1$. Therefore, $a_1(\|x_{t+t_1}\|) \leqslant c_1 a_2(\|x_t\|)$ and $\|x_{t+t_1}\| \leqslant c_2\|x_t\|$. Further, take an arbitrary $t > t_0$ and, following the idea of the proof from [19], denote by $N$ the lower integer part of $(t - t_0)/t_1$. Then $t - Nt_1 \in [t_0, t_0 + t_1)$, and $\|x_t\| \leqslant c_2\|x_{t-t_1}\| \leqslant c_2^2\|x_{t-2t_1}\| \leqslant \cdots \leqslant c_2^N\|x_{t-Nt_1}\|$. Moreover, the inequality $v(t) \leqslant Mv(t_0)$ implies $\|x_{t-Nt_1}\| \leqslant a_1^{-1}(Ma_2(\|\varphi_0\|) \leqslant c\|\varphi_0\|$. Thus, $\|x_t\| \leqslant c_2^N c\|\varphi_0\| \leqslant c_2^{\frac{t-t_0}{t_1}} c\|\varphi_0\| \leqslant c\|\varphi_0\|e^{-\delta(t-t_0)}$, where $\delta = \delta(c_2) = \frac{1}{t_1}\log\frac{1}{c_2} > 0$. This implies the last assertion of the theorem. $\square$

**Remark 1.** *If we replace $\mathcal{K}$ by $\mathcal{K}_\infty$ in the conditions of items 1–3, then we obtain the conditions for global uniform asymptotic (exponential) stability.*

**Remark 2.** *It is easy to see that the condition $\alpha_1|x|^m \leqslant V(t, x) \leqslant \alpha_2|x|^m$ implies*

$$|x(t)| \leqslant \Delta_1\|x_{t_0}\|e^{-\delta_1(t-t_0)}$$

*with $\Delta_1 = (\alpha_2 e^{A-\underline{a}-a}/\alpha_1)^{1/m}$, $\delta_1 = \frac{A}{Tm}$. In this case, $a_1^{-1}(c_1 a_2(s)) = (c_1\alpha_2/\alpha_1)^{1/m}$, hence the second inequality in item 3 holds for an arbitrary $c_2 \in (0, 1)$ (it suffices to put $c_1 = c_2^m(\alpha_1/\alpha_2)$). Moreover, $\delta(c_2) \to \gamma/m = \delta_1$ for $c_2 \to 0$.*

**Remark 3.** *Recall the definition of GUAS($a_0$) [1], which qualifies the speed of convergence in the GUAS property, and serves to relax exponential stability (here we modify it according to the type of equation considered). We say that the zero solution to the Equation (1) is GUAS($a_0$) for a given $a_0 \in \mathcal{K}_\infty$ if there exists a class-$\mathcal{K}_\infty$ function $a$ and a positive constant $\delta > 0$ so that $|x(t; t_0, \varphi_0)| \leqslant a_0(e^{-\delta(t-t_0)}a(\|\varphi_0\|))$ holds for all $\varphi_0 \in C$ (recall that GUAS is always equivalent to the existence of some $a$ and $a_0$ like this.) We see that due to the exponential estimate for $V(t, x(t))$, item 2 of Theorem 2 actually implies GUAS($a_1^{-1}$).*

**Corollary 1.** *Let the conditions of Theorem 2 hold for $t \geqslant \bar{t} > 0$. Then all statements remain true.*

**Proof.** For the proof, we use the assumptions on the right-hand side of Equation (1) and results from [17].

By Assumption 2 from [17], for each compact $K \subset C$ there exists a locally Lebesgue-integrable function $L_K(t)$ and $N(K) > 0$ such that for all $\varphi \in K$ and $t \in R^+$ the estimate $|f(t, \varphi)| \leqslant L_K(t) \|\varphi\|$ holds, and $\int_t^{t+1} L_K(s) ds \leqslant N(K)$. In addition, for an arbitrary $a > 0$ every solution $x(t)$ of Equation (1) is contained in some compact $K(q) \subset C$ as long as $|x(t)| \leqslant a$ [17].

Now, let $x(t) = x(t; t_0 \varphi_0)$ be an arbitrary solution of (1). It is clear that it suffices to check the case $t_0 < \bar{t}$. If

$$\|x_t\| \leqslant a \text{ for all } t \in [t_0, \bar{t}] \tag{$*$}$$

then

$$\|x_{\bar{t}}\| \leqslant \|\varphi_0\| + \max_{\bar{t} - r \leqslant s \leqslant \bar{t}} \int_{t_0}^s |f(\theta, x_\theta| d\theta \leqslant$$

$$\|\varphi_0\| + \int_{t_0}^{\bar{t}} L_a(\theta) \|x_\theta\| d\theta.$$

The Grönwall–Bellman inequality now implies

$$\|x_{\bar{t}}\| \leqslant \|\varphi_0\| e^{N_a \bar{t}}.$$

It remains to note that inequality $(*)$ will be satisfied whenever $\|\varphi_0\| \leqslant a e^{-N_a \bar{t}}$. $\square$

**Remark 4.** *Note that if $V(t, x) \leqslant a_2(|x|)$ ($a_2 \in \mathcal{K}$), then the estimate for the derivative in Theorem 2 can be replaced by the following one: $V'(t, \varphi) \leqslant -p(t)a_3(|\varphi(0)|)$ for all $t \in R^+$ and $\varphi \in \Omega_t(V, \eta)$, where $a_3 \in \mathcal{K}$. Indeed, in this case we set $U(t, x) = \Lambda(V(t, x))$, where $\Lambda$ is some $C^1$ positive definite proper function such that $\Lambda'(s)a_3(a_2^{-1}(s)) \geqslant \Lambda(s)$. Then, the function $U$ satisfies the conditions of Lemma 1.*

### 4. A Remark on Necessary Conditions for Uniform Asymptotic Stability

In this section, the discussion will center on the integral conditions for the function $p(t)$ that ensure the exponential convergence of the function $V(t, x(t))$ to zero (Theorem 2). In fact, these conditions are equivalent to a restriction on the antiderivative of $p(t)$, which naturally arises in the analysis of linear ordinary differential equations with time-varying coefficients. Here, we give a statement showing that, in the nonlinear case, a similar estimate for the right-hand side is a necessary condition for uniform asymptotic stability.

Note that the requirements imposed on integrals of $p(t)$ to be bounded from below, which are specified in Theorem 2, are equivalent to the following condition (see [20]):

$$\exists a > 0 \text{ and } b \text{ such that } \int_{t_0}^t p(\theta) d\theta \geqslant a(t - t_0) + b \text{ for every } t_0, \ t \geqslant t_0, \tag{3}$$

and (3) is equivalent to GUES for the scalar ordinary differential equation $\dot{y}(t) = -p(t)y(t)$.

Moreover, for GUES of the linear system $\dot{y}(t) = -P(t)y(t)$ with a symmetric positive semidefinite matrix $P(t)$, it is necessary and sufficient that (3) be satisfied for $p(t) = |P(t)x|$ and every unit vector $x \in R^n$ [21].

If the inequality $x^\top P(t)x \geqslant 0$ is violated, then the estimate (3) provides sufficient GUES conditions via a quadratic time-varying Lyapunov function [20]. Note that for a linear system, GUAS and GUES are equivalent.

Z. Artstein ([22] Theorem 6.2) showed that the growth condition (3) is necessary in a very general situation for ordinary differential equations. There holds a similar statement for Equation (1):

**Theorem 3.** *Suppose that Equation (1) is uniformly stable. Then, the equation is uniformly asymptotically stable only if there exists a compact neighborhood of zero $W \subset R^n$ such that for every $\delta > 0$ there are numbers $a > 0$ and $b$ such that*

$$\int_{t_0}^{t} |f(\theta, \varphi)| d\theta \geqslant a(t - t_0) + b$$

*for every $t_0 \in R^+$, $t \geqslant t_0$ and every $\varphi \in C$ such that $\varphi(s) \equiv c$, $|c| \geqslant \delta$, and $c \in W$.*

**Proof.** Assumptions about the right-hand side of Equation (1) ensure the positive precompactness of (1) in an appropriate function space and the existence of the so-called limiting equations; in addition, for every limiting equation and every pair $(t_0, \varphi_0)$ the initial-value problem has a unique solution [23].

Using the relationships of uniform asymptotic stability of Equation (1) and stability properties of the corresponding limiting equations ([23] Theorem 6), we can now repeat the proof of Theorem 6.2 from [22] with slight modifications. □

It should be emphasized that the necessary condition for UAS from Theorem 3 without additional assumptions is not sufficient, even for ordinary differential equations ([22] Theorem 6.3) (see also Section 5).

## 5. Application: Scalar Linear Time-Varying Equations

In this section, we consider the scalar linear differential equation

$$\dot{x}(t) = -a(t)x(t) + b(t)x(t - \tau(t)), \tag{4}$$

where $a(t)$, $b(t)$, and $\tau(t)$ are piecewise continuous, $0 \leqslant \tau(t) \leqslant r$.

Note that by Theorem 3, the uniform asymptotic stability of the Equation (4) implies the condition

$$\int_{t_0}^{t} |-a(\theta) + b(\theta)| d\theta \geqslant a_0(t - t_0) + b_0 \tag{5}$$

for every $t_0 \in R^+$, $t \geqslant t_0$, and some numbers $a_0 > 0$ and $b_0$.

There have been a lot of investigations concerning sufficient stability conditions for Equation (4).

Most of the results obtained in this line of study impose restrictions on the sign of coefficients. For example, by Theorem 1, an Equation (4) is uniformly asymptotically stable if $p_l(t) = a(t) - q|b(t)| \geqslant p_0 > 0$ for some $q > 1$ and $p_0 > 0$; it is also clear that for GUES of Equation (4), it is sufficient for the function $p_l(t)$ to satisfy the conditions of Theorem 2. In both cases, an arbitrary (bounded) delay is allowed. Moreover, if no restrictions are imposed on the delay, then for constant coefficients, the condition is the best possible.

For Equation (4) with time-varying $a$ and $b$ as well as with a time-varying delay, sufficient GUES conditions can be refined through information about the nature of the delay variation. However, in this case, the conditions known so far usually also imply that some function depending on $a$, $b$, and $\tau(t)$ must retain its sign (at least for large values of $t$); see [24,25] and references therein.

Note that the effect of delay variation can have a decisive influence on the stability even for Equation (4) with constant coefficients: one can give an example such that for each fixed member of the range of the delay function, the associated autonomous equation is exponentially stable, yet the time-varying equation considered is unstable [26]. However, the exact form of the relation "delay–time" is usually unknown; thus, in practice, it is more convenient to check the conditions that are valid for all delay values from a given range.

Equation (4) with $a(t) \equiv 0$ has a separate and even richer history of study going back to the work of Myshkis (see e.g., [27]), and this study is still ongoing. Following A.D. Myshkis, various forms of the so-called "3/2-criterion" were obtained, which establish an upper bound either for the coefficient $b(t)$ or for the integral of it; see some discussion and references in [28,29]. However, in those results, the lower bound for $b(t)$ (at least for the limit as $t$ tends to infinity) remains zero.

In [30], the assertion is proved that if the equation $\dot{x}(t) = -b(t)x(t)$ is GUES and a certain additional integral condition for $b(t)$ is satisfied, then Equation (7) is GUES.

Note that the equation $\dot{x}(t) = -b(t)x(t)$ is GUES if and only if (3) hold for $b(t)$ (see Section 4), while for the equation $\dot{x}(t) = -b(t)x(t - \tau(t))$ this condition is not sufficient and additional upper bounds are required. For example, even for constant values $b(t) \equiv b > 0$ and $\tau(t) \equiv r$ the equation $\dot{x}(t) = -bx(t - r)$ is GUAS only if $br < \pi/2$; for time-varying non-negative functions $b(t)$ and $\tau(t)$, this condition turns into restrictions imposed on their supremum or some integral (see, e.g., [27]).

But even such conditions stop working as soon as the coefficient $b(t)$ is allowed to change sign. The authors of [25] investigate whether sufficiently small bounds $r$ for the delay and $b_0$ for the integral $\int_{t-\tau(t)}^{t} |b(s)| ds \leqslant b_0$ and exponential stability of $\dot{x}(t) = -b(t)x(t)$ cause exponential stability of the equation $\dot{x}(t) = -b(t)x(t - \tau(t))$ with an oscillating coefficient $b(t)$. They provide an example showing that, unlike the case of the equation with $b(t) \geq 0$, such $r$ and $b_0$ cannot be found in the general case.

Thus, the study of the stability of linear delay equations with sign-changing coefficients is a rather challenging problem. Even for the scalar Equation (4) there are only a few results of this kind; see [25] and references therein.

Next, we deduce sufficient GUES conditions for several examples of Equation (4) using Theorem 2, and compare the conditions with those obtained by some other methods.

**Example 1.** *Consider the scalar equation*

$$\dot{x}(t) = -x(t) + b(t)x(t - \tau(t)) \tag{6}$$

*with $0 \leqslant \tau(t) \leqslant r$.*

*The classical theorem (Theorem 1) implies GUAS of (1) for $|b(t)| < 1$; the same inequality follows from other results (see, e.g., [24] and references therein).*

*Let $b(t)$ be a periodic function of period 1 given by*

$$b(t) = \begin{cases} 0, & t \in [0, c), \\ d, & t \in [c, 1), \end{cases}$$

*where $c \in (0, 1)$ and $d > 1$.*

*For the derivative of the function $V(x) = |x|$ by virtue of the Equation (1) for $\eta(s) = qs$ ($q > 1$), we obtain the estimate from Lemma 1 with $p(t) = 1 - q|b(t)|$. If $r \leqslant 1$, then $\bar{a} = \min\{r, c\}$, and if $n - 1 < r \leqslant n$ with integer $n > 1$, then $\bar{a} = nc$. For $T = 1$, we get $A = c + (1 - c)(1 - qd) = 1 - qd(1 - c)$. Then, by Theorem 2 the conditions $q \geqslant e^{\bar{a}}$, $A > 0$ are sufficient for GUES of Equation (1).*

*In Table 1 we compare the results for Equation (1) obtained using various assertions.*

**Table 1.** Comparison of some sufficient conditions for GUAS (GUES) of Equation (1).

| Method | Stability Conditions | Convergence Rate |
|---|---|---|
| [31] | $(1-c)d < \frac{1-e^{-1}}{2} \approx 0.3161$<br>GUAS | – |
| [32] | $(1-c)d < 1, r < -\ln[(1-c)d]$,<br>GUAS | – |
| [33] | $(1-c)d < e^{-c}$,<br>$e^{-c} > e^{-1} \approx 0.3679$<br>GUES | $A/(1+r)$,<br>$(A = 1 - e^c d(1-c))$ |
| Theorem 2 | $(1-c)d < e^{-\bar{a}}$<br>$\bar{a} = $<br>$\begin{cases} nc \text{ if } \max\{c, n-1\} < r \leqslant n \\ (n \in \{1, 2, \dots, \}) \\ r \text{ if } r < c \end{cases}$<br>GUES | $A$<br>$(A = 1 - e^{\bar{a}} d(1-c))$ |

*Note that the conditions of GUAS from [32] follow from the estimates obtained using Theorem 2. Thus, the number d can be large enough, provided that c and r are small enough. A comparison of the conditions for r, c, and d shows that for large values of r, Theorem 2 gives more restrictive conditions than [33], and vice versa for small ones.*

**Example 2.** *Consider the scalar equation*

$$\dot{x}(t) = -b(t)x(t - \tau(t)) \tag{7}$$

*with a bounded $b(t)$ and a time-varying delay $\tau(t) \in [0, r]$.*

*Using the function $V(x) = |x|$, we find that the estimation in condition 3 of Lemma 1 holds for $p(t) = b(t) - |b(t)|q \int_{t-r}^{t} |b(s)| ds$, where $\eta(u) = qu$ and $q \geqslant e^{\bar{a}}$. We denote $\hat{b} = \sup_{t \in R^+} |b(t)|$, then $p(t) \geqslant b(t) - qr\hat{b}^2$ and $\int_{t}^{t+\tilde{T}} p(\theta) d\theta \geqslant A = \tilde{b} - qr\tilde{T}\hat{b}^2$. Moreover, in condition 1 of Lemma 1, the number $\bar{a}$ can be chosen to be equal to $r\hat{b}$. Thus, if the zero solution of the equation $\dot{x}(t) = -b(t)x(t)$ is GUES, then the GUES of Equation (7) is guaranteed for sufficiently small values of $r$ (at least for $0 \leqslant r < \tilde{b}/(e^{r\hat{b}}\hat{b}^2\tilde{T})$).*

*To compare known GUES conditions with the conditions of Theorem 2, consider a special case of Equation (7) (see ([30] Example 2) and ([25] Example 4)) with the function*

$$b(t) = \begin{cases} \alpha, & 2n \leqslant t \leqslant 2n + 1, \\ -\beta, & 2n + 1 \leqslant t \leqslant 2n + 2, \end{cases} \tag{8}$$

*where $0 < r \leqslant 1$, $\alpha > \beta > 0$, and $n = 0, 1, 2, \ldots$.*

*For (7), (8) with a constant delay, the condition of GUES is obtained in the form*

$$\frac{(2e^{\beta} - e^{-(\alpha - \beta)} - 1)\alpha r}{1 - e^{-(\alpha - \beta)}} < 1 \tag{9}$$

*in [30], and in the form*

$$r\alpha \max\left\{\frac{\alpha}{\alpha - \lambda}, \frac{\beta}{\lambda - \beta}\right\}e^{\lambda} < 1 \text{ for some } \lambda \in (\beta, \alpha) \tag{10}$$

*in [25].*

*From Theorem 2, we get the conditions $A\frac{qr^2}{2}(\alpha - \beta)^2 + (\alpha - \beta) - qr(\alpha^2 + \beta^2) > 0$ and $q \geqslant e^{r(\alpha - \beta^2 qr)}$. Table 2 lists some numerical values of the parameters $\alpha$, $\beta$, and $r$ and indicates whether the conditions of Theorem 2 or Equations (9) or (10) are satisfied for these sets.*

**Table 2.** Fulfillment of sufficient GUES conditions for (7), (8) with different values of the parameters $\alpha$, $\beta$, and $r$.

| Values of ($\alpha$, $\beta$, $r$) | Theorem 2 | (9) [30] | (10) [25] |
|---|---|---|---|
| (0.6, 0.2, 0.6) | + | + | + |
| (0.6, 0.3, 0.5) | + | − | − |
| (0.6, 0.3, 0.4) | + | + | − |
| (0.6, 0.3, 0.6) | − | − | − |

*Comparing the results for Equation (7) with $b(t)$ given by (8), we observe that the conditions do not follow from each other and may or may not be satisfied depending on the parameter values. In addition, the conditions obtained using Theorem 2 remain applicable in the case of variable delay, including a state-dependent one.*

Note that all the mentioned stability conditions are sufficient. Not every pair of conditions can be compared; one or the other may win, depending on the particular equation. However, it is worth noting that the previously obtained results discussed in this

section were obtained using methods that make significant use of linearity of the equation under study. The results of this paper are obtained on the basis of Lyapunov–Razumikhin functions and are applicable to more general time-varying equations (see Section 6 for some examples of nonlinear equations).

**Example 3.** *Consider the system*

$$\dot{x}(t) = -\sin^2 t \cdot x(t) + bx(t - \tau(t)), \ \tau(t) \in [0, r].\tag{11}$$

*We study the stability conditions for this equation with arbitrary values of $r > 0$ and $b > 0$. For $b \geqslant 1/2$, the necessary condition GUES of the Equation (11) does not hold for $r = 0$. Numerical experiments show that the solutions of Equation (11) for $b \geqslant 1/2$ increase indefinitely for all $r > 0$. Using the function $V(x) = |x|$, we find that the estimation in condition 3 of Lemma 1 holds for $p(t) = \sin^2 t - q|b|$, and the conditions of Theorem 2 are satisfied for Equation (11) whenever $\eta = 1/2 - q|b| > 0$ and $q \geqslant e^{1/2 + r\eta}$. In this case, for all solutions of the equation, the estimate $|x(t)| \leqslant M\|x_{t_0}\|e^{-\eta(t - t_0)}$ is valid with $M = e^{1 + \pi \eta}$. The given GUES conditions are satisfied, for example, for $b = 0.2$, $r = 6$, $q \geqslant 2.243$, $\eta \leqslant 0.0513$, and $M \geqslant 3.2$. It can also be noted that none of the GUES conditions given in [24,25] are not applicable here.*

*From Figure 1, we find that Equation (11) is GUES with the decay rate guaranteed by our conditions: for the selected parameter values, the norm $|x(t)|$ is less than $M\|x_{t_0}\|e^{-\eta(t - t_0)}$. The above GUES conditions remain valid for the equation $\dot{x}(t) = -\sin^2 t \cdot x(t) + \sum_{i=1}^{N} b_i(t)x(t - \tau_i(t))$ with $\sum_{i=1}^{N} |b_i(t)| \leqslant b$ and $0 \leqslant \tau_i(t) \leqslant r$ for all $i = 1, \dots, N$ (see an example with $N = 2$ in Figure 1a). For comparison, the time evolutions of other functions that estimate the decay rate of solutions are also shown. These functions are calculated using formulas from ([33] Theorem 1). It is clearly seen that our estimate of M is very conservative and much worse than the estimate by [33], but the decay rate estimate gives a better result (0.0513 versus 0.0176). We also see that for $b \geqslant 0.5$ the solutions of Equation (11) diverge (Figure 1b).*
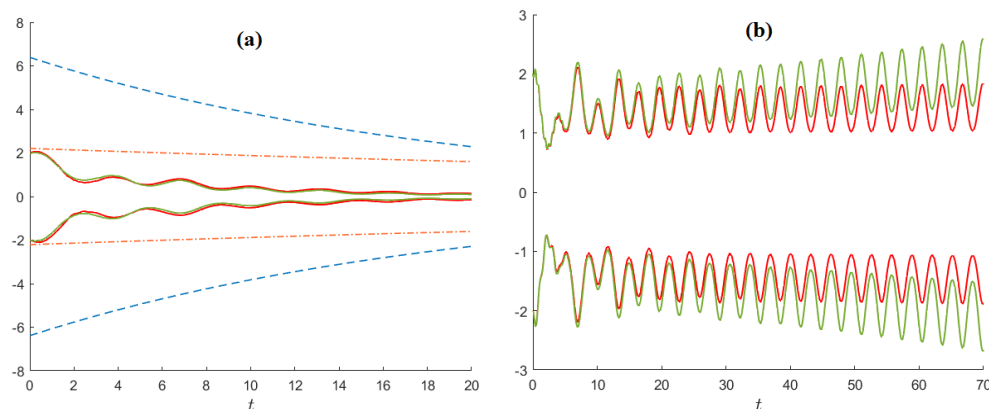


**Figure 1.** Evolution of the states with $t_0 = 0$, $\varphi_1(s) = -2(1 + \frac{1}{3}s\cos^2(3s))$, and $\varphi_2(s) = 2(1 + \frac{1}{3}s\sin^2(6s))$: (**a**) for Equation (7) with $b = 0.2$ and $\tau(t) = 6(1 - 0.2\cos^2 t)$ (red line); for the equation $\dot{x}(t) = -\sin^2 t \cdot x(t) + b_1(t)x(t - \tau_1(t)) + b_2(t)x(t - \tau_2(t))$ with $b_1(t) = 0.1(1 - 0.2\sin^2 t)$, $b_2(t) = 0.1(1 - 0.1\cos^2 t)$, $\tau_1(t) = 4(1 - 0.5\sin t)$, and $\tau_2(t) = 6/(1 + \sin^2 t)$ (green line); the blue dashed lines indicate the time evolution of the functions $\pm M\|x_{t_0}\|e^{-\eta(t - t_0)}$, the orange dash-dotted lines indicate the time evolution of functions estimating the decay rate of solutions by ([33] Theorem 1). (**b**) for Equation (7) with $\tau(t) \equiv r = 6$ and $b = 0.5$ (red line) or $b = 0.52$ (green line).

## 6. Nonlinear Examples

**Example 4.** *Consider the system*

$$\dot{x}(t) = f(t, x_t) + g(t, x_t),\tag{12}$$

*where $x(t) \in R^n$, and suppose that for some $\delta > 0$ the inequality $\|\varphi\| < \delta$ implies $f_i(t,\varphi)\varphi_i(0) \leqslant -a_i(t)\varphi_i^2(0)$ and $|g_i(t,\varphi)| \leqslant b_i(t)|\varphi_i(0)|$ for all $i = 1,\ldots,n$. Using the function $V(x) = \max_{i \in \{1,\ldots,n\}} |x_i|$, we obtain that system (12) is UES whenever $p(t) = \min_{1 \leqslant i \leqslant n}(a_i(t) - qb_i(t))$ satisfies the conditions of Theorem 2 for some $q > 1$.*

It should be pointed out that the idea of generalizing "linear" results to a certain class of nonlinear systems using the direct Lyapunov method is used quite often; see e.g., [29].

**Example 5.** *Let $\mu_i \geqslant 0$, $f_i : R^+ \times C \to R$ and $g_i : R^+ \times C \to R$ be continuous and locally Lipschitz uniformly in t and take non-negative values for non-negative $\varphi$ ($i = 1,\ldots,n$). Define $f(t,\varphi) = (f_1(t,\varphi),\ldots,f_n(t,\varphi))^\top$, $g(t,\varphi) = diag(g_1(t,\varphi),\ldots,g_n(t,\varphi))$, and $\mu = diag(\mu_1,\ldots,\mu_n)$, and consider the system*

$$\dot{x}(t) = f(t,x_t) - (\mu + g(t,x_t))x(t), \tag{13}$$

*arising in models of population dynamics [34].*

*It is also assumed that $x(t) \in R^n$ takes non-negative values and there exist matrices $L^k$ ($k = 0,1,\ldots,m+1$) such that $f(t,\varphi) \leqslant \sum_{k=0}^{m} L^k\varphi(-r_k) + \int_{-r_{m+1}}^{0} L^{m+1}(s)\varphi(s)ds$ with $r_0 = 0$, $0 < r_i \leqslant r$ for $k = 1,\ldots,m$ [34]. Let us denote $L = \sum_{k=0}^{m} L^k + \int_{-r_{m+1}}^{0} L^{m+1}(s)ds$. In [34], the exponential convergence of positive solutions of (13) is proved under the condition that $\mu - L$ is a nonsingular M-matrix. Let us use the function $V(x) = \max_{i \in \{1,\ldots,n\}} |x_i|$. Estimating the derivative of this function by virtue of system (13) and applying Theorem 2 with a constant $p(t)$ (see also Theorem 4 below), we obtain a sufficient condition for the exponential convergence of positive solutions of (13) in the form:*

$$\min_{i \in \{1,\ldots,n\}} \left( \mu_i - \sum_{j=1}^{n} L_{ij} \right) > 0. \tag{14}$$

*Notice that whenever all $L_{ii} = 0$, which corresponds to most specific models, condition (14) means that the matrix $\mu - L$ is strictly diagonally dominant. This implies that $\mu - L$ is a nonsingular M-matrix [35]. It is easy to see how, for system (13) with time-varying parameters, condition (14) can be generalized using Theorem 2.*

## 7. Discussion

On one hand, Razumikhin's conditions simplify the practical checking of the corresponding requirements that guarantee one type of stability or another, since the resulting estimate of the derivative does not depend on the history of the system; this circumstance, in turn, gives more "freedom" for delay in comparison with the method of functionals. On the other hand, the absence of an estimate for the derivative of the Lyapunov function along the entire solution makes it difficult to estimate the rate of convergence of solutions.

To obtain an estimate for the convergence of solutions based on Theorem 1, one can use a special structure of the time-dependent Lyapunov function, so that the convergence to zero of the function itself implies a certain rate of convergence of solutions (see, for example, [36]).

Another approach is to modify the requirements for the function $V(t,x)$. A significant number of modifications of Razumikhin-type theorems are based on ideas representing the so-called comparison principle. Two types of comparison theorems [37] give rise to two types of sufficient stability conditions.

In the first case, the estimate of the derivative is constructed over the entire domain, and therefore, it has the form of a functional. Accordingly, the comparison equation is a delay equation. In the second case, the derivative of the function $V(t,x)$ is estimated over some set in the spirit of Razumikhin's conditions, but to determine this set, it is necessary to know the solutions of the comparison equation [37]. In the first case, to apply the result, it is necessary to obtain information about the properties of solutions of some

delay equation; in the second one, it is necessary to find a solution to a (nonlinear) ordinary differential equation.

Therefore, it is natural to try to obtain more specific forms both for estimates of the derivative of the function $V(t, x)$ and for Razumikhin-type conditions. The particular form of the proposed comparison equations makes it possible to obtain more constructive requirements that provide suitable properties for the solutions of these equations and, at the same time, do not lead to the "trivial" stability conditions that follow from the classical theorems. In this case, the requirements for the Lyapunov function are easier to check in specific applications, and sometimes it is also possible to obtain information about the convergence of solutions.

The results which do not make use of any Razumikhin conditions, but use a functional as an estimate of the derivative, mainly go back to the following statement:

**Lemma 2** (Halanay's lemma, [38])**.** *Let $v(t)$ be a piecewise $C^1$ non-negative valued function that admits constants $r > 0$, $0 < \beta < \alpha$ such that $\dot{v}(t) \leqslant -\alpha v(t) + \beta \max_{t-r \leqslant s \leqslant t} v(s)$ holds for all $t \geqslant t_0$. Then, the inequality $v(t) \leqslant \max_{-r \leqslant s \leqslant 0} v(t_0 + s)e^{-\gamma(t-t_0)}$ holds for all $t \geqslant t_0$ with $\gamma \in (0, \alpha - \beta)$ be the root of the equation $\gamma = \alpha - \beta e^{\gamma r}$.*

There are a number of extensions of Halanay's lemma to the time-varying case; they are formulated in different forms and under different assumptions, depending, among other things, on the purpose of use (e.g., [31,32,39–41]). Moreover, in a number of cases, the proposed conditions imply classical requirements for the Lyapunov–Razumikhin function, which ensure (asymptotic) stability.

The requirements on the derivative in such results are relaxed due to the fact that the proposed conditions do not impose restrictions on the values of the system parameters themselves, such as delay and norms of the system coefficients, but instead involve integration of the time-varying parameters.

The second approach, which is based on the use of Razumikhin's conditions, uses the following result in most cases:

**Theorem 4.** *Suppose that there exists a function $V : R^+ \times R^n \to R^+$, numbers $\gamma > 0$ and $q > 1$, and functions $a_1$, $a_2 \in \mathcal{K}$ such that $a_1(|x|) \leqslant V(t, x) \leqslant a_2(|x|)$, and $V'(t, \varphi) \leqslant -\gamma V(t, \varphi(0))$ whenever $t \in R^+$ and $\max_{-r \leqslant s \leqslant 0} V(t + s, \varphi(s)) \leqslant qV(t, \varphi(0))$. Then, along any solution $x(t) = x(t; t_0, \varphi_0)$ of (1), the following inequality*

$$V(t, x(t)) \leqslant \max_{-r \leqslant s \leqslant 0} V(t_0 + s, \varphi_0(s))e^{-\min(\gamma, \ln q/r)(t-t_0)}$$

*holds for all $t \geqslant t_0$. If, in addition, $a_1|x|^m \leqslant V(t, x) \leqslant a_2|x|^m$ for some integer $m > 0$, then Equation (1) is GUES.*

This statement is proved in various ways and used in a number of papers (see [7] and references therein). Some estimates for the convergence of the function $V(t, x)$ along solutions use other Razumikhin-type conditions. For discussion of a number of such results, see [8]. Note that, in such statements, the estimate of the derivative of a (non-negative) function $V(t, x)$ is usually non-positive on the entire time interval. The question arises whether it is possible to allow the sign of the estimate of the derivative to change, and at the same time to ensure the (asymptotic) stability of the zero solution of this equation.

One of the main ideas to combine these requirements is to use integral constraints on the parameters of the system studied (including the delay). As a result, it turns out that a delay system can be asymptotically stable even if the system is described by an equation with an unstable zero solution on some (short) time intervals. Therefore, a "classical" function for such a system cannot exist.

The first result allowing usage of an indefinite time-derivative of the function $V(t, x)$ under a Razumikhin-type condition was probably given in [42]: sufficient conditions for uniform stability are justified therein, provided that on the set $\Omega_t(V)$ the estimate

$$V'(t, \varphi) \leqslant \psi(t) V(t, \varphi(0)), \tag{15}$$

is valid, where $\int_0^\infty \max\{\psi(s), 0\} ds < \infty$, and for uniform asymptotic stability, the estimate $\int_{t_0}^t \max\{-\psi(s), 0\} ds \geqslant \varepsilon(t - t_0)$ is additionally required for some $\varepsilon > 0$ and sufficiently large values of $t$. It is clear that such a function $\psi(t)$ does not have to be negative all the time. On the other hand, the conditions of [42] cannot be satisfied if, for example, $\psi(t)$ is periodic and $\psi(t) > 0$ for all $t$ from $[a, b] \subset [0, T]$ where $T$ is the period and $b > a$. For such functions $\psi(t)$ in (15), sufficient conditions for uniform asymptotic stability and exponential stability were obtained in [33]: inequality (15) should be verified on the set $\Omega_t(V, \eta)$, wherein the functions $\psi$ and $\eta$ are linked by a certain relation. Note that Theorem 2 implies Theorem 4 for $p(t) \equiv \gamma$.

The results obtained here are close to the Razumikhin-type asymptotic stability theorem from [33], but another approach is used and other exponential stability conditions are obtained. It should be mentioned at this point that in [33] the estimate of the decay rate of solutions decreases with increasing delay, while in the estimate of Theorem 2 an increase in delay can lead to an increase in the number $M$. Thus, the results of [33] and Theorem 2 complement each other.

## 8. Conclusions

This paper has dealt with the problem of uniform exponential stability for nonlinear time-varying delay systems. We apply the Razumikhin method to establish sufficient conditions for exponential stability, and propose an extension of the Razumikhin approach for time-varying systems. Namely, the time-derivative of the constructed Razumikhin function is allowed to be indefinite, and a new upper estimate on the rate of decreasing of the function is also obtained.

The statements deduced can be seen as extensions of those in some earlier work discussed in the previous section, as well as some known sufficient conditions for the exponential stability of linear delay equations. The application of the exponential stability conditions derived here to specific examples shows that the previous related results are either inapplicable or give some other exponential estimates for the solutions.

We note that the use of the Razumikhin method gives rise to stability conditions that are sufficient but not necessary. Therefore, such stability conditions may be overly conservative. Various approaches have been taken to reduce the conservatism of the method, and numerous results have been obtained in this area. However, no universally good stability criterion exists; which criterion is better depends on what a specific system is.

The simple examples given in the paper show that Theorem 2 can be less conservative than other methods.

The method used in this paper for estimating the function $V$ along solutions of a differential equation can be extended to wider classes of differential equations. It can also be used to solve other problems. For example, in [39], similar estimates are used for sufficient conditions for practical stability (exponential stability is a special case; see ([39] Remark 1)). Parenthetically, in [39] the coefficient of $V$ in the estimate of the derivative is negative.

In [16], methods from [33] are extended to stochastic systems, as well as to studying other types of stability, in particular input-to-state stability and integral input-to-state stability. In [14], the ideas of [33] were developed for impulsive stochastic systems with delay. For such systems, it is necessary not only to construct estimates for a function $V$ on continuity intervals, but also to take into account the change in this function at impulse

moments. The most important feature of the results from [14] is that time-derivatives of Razumikhin functions are allowed to be unbounded.

We also note the recent work [43], in which the reaction–diffusion epidemic model with a delayed impulse is studied. For this model, a new synchronization criterion is obtained. In this case, the Lyapunov–Krasovskii functional is used, but the estimate for the derivative also has the form $V' \leqslant -\gamma V$ with $\gamma > 0$.

Therefore, the results obtained in this paper can be extended both to other types of equations and the study of other qualitative properties. In the future, the estimates of Razumikhin functions will be refined and used to study other types of stability.

## Abbreviations

The following abbreviations are used in this manuscript:

GUAS     Global uniform asymptotic stability
GUES     Global uniform exponential stability

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