

## Article

# New Results Concerning Approximate Controllability of Conformable Fractional Noninstantaneous Impulsive Stochastic Evolution Equations via Poisson Jumps

Yazid Alhojilan <sup>1</sup>  and Hamdy M. Ahmed <sup>2,\*</sup>

<sup>1</sup> Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia

<sup>2</sup> Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El Shorouk Academy, El-Shorouk City 11837, Egypt

\* Correspondence: hamdy\_17eg@yahoo.com

**Abstract:** We introduce the conformable fractional (CF) noninstantaneous impulsive stochastic evolution equations with fractional Brownian motion (fBm) and Poisson jumps. The approximate controllability for the considered problem was investigated. Principles and concepts from fractional calculus, stochastic analysis, and the fixed-point theorem were used to support the main results. An example is applied to show the established results.

**Keywords:** fractional derivative; stochastic system; nonlinear equations; approximate controllability; fractional Brownian motion

**MSC:** 34A08; 34K50; 35R12; 93B05



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## 1. Introduction

The field of fractional calculus is constantly expanding, and applications range from engineering and natural phenomena to financial perspectives (see [1–7]). Recently, there seems to be much enthusiasm for the use of stochastic differential equations to describe a variety of phenomena in population dynamics, physics, electrical engineering, geography, psychology, biochemistry, and other areas of physics and technology (see [8–14]). Stochastic impulsive differential equations arise in a very natural way as mathematical models (see [15–20]). The introduction of drugs into the bloodstream and the consequent absorption into the body are gradual and continuous processes that can be described by noninstantaneous impulsive differential equations (see [21,22]). Now, several authors have discussed different types of controllability for fractional stochastic systems (see [23–29]). To the best of our knowledge, the approximate controllability of CF noninstantaneous impulsive stochastic evolution equations via fBm and Poisson jump mentioned in this study is an area of research that appears to give extra incentive for completing this research.

Assume that the CF noninstantaneous impulsive stochastic evolution equation via fBm, Poisson jump, and the control function has the following form:

$$\begin{cases} D^\ell N(\varphi) + \Delta N(\varphi) = M(\varphi, N(\varphi), N(\ell_1(\varphi)), \dots, N(\ell_m(\varphi))) + Bu(\varphi) \\ + \int_0^\varphi V(\mathbb{k}, N(\mathbb{k}), N(j_1(\mathbb{k})), \dots, N(j_k(\mathbb{k}))) d\omega(\mathbb{k}) \\ + \Omega(\varphi, N(\varphi), N(c_1(\varphi)), \dots, N(c_p(\varphi))) \frac{d\mathfrak{B}_H}{d\varphi} \\ + \int_F \mathfrak{U}(\varphi, N(\varphi), N(\iota_1(\varphi)), \dots, N(\iota_u(\varphi)), f) W(d\varphi, df), \quad \varphi \in (\mathbb{k}_i, \varphi_{i+1}], \quad i \in [0, m] \\ N(\varphi) = \mathfrak{h}_i(\varphi, N(\varphi)), \quad \varphi \in (\varphi_i, \mathbb{k}_i], \quad i \in [1, m] \\ N(0) = N_0, \end{cases} \quad (1)$$

where  $D^\ell$  is the conformable fractional derivative (CFD) of order  $\frac{1}{2} < \ell < 1$  and  $-\Delta$  generates semigroup  $\Theta(\varphi)$ ,  $\varphi \geq 0$ , on  $\mathfrak{H}$ . Here,  $\mathfrak{H}$  and  $G$  are separable Hilbert spaces with

$\|\cdot\|$  and  $u(\cdot) \in L_2(Y, \mathfrak{U})$  is the control function, where  $L_2(Y, \mathfrak{U})$  is the Hilbert space of control functions with  $\mathfrak{U}$  a Hilbert space.  $B$  is a bounded linear operator from  $\mathfrak{U}$  into  $\mathfrak{N}$ , and  $\hbar_i$  is a noninstantaneous impulsive function for all  $i = 1, 2, \dots, m$ . Suppose that the time interval is  $Y = (0, b]$ , where  $\wp_i, \mathbb{k}_i$  are fixed numbers verifying  $0 = \mathbb{k}_0 < \wp_1 \leq \mathbb{k}_1 \leq \wp_2 < \dots < \mathbb{k}_{m-1} < \wp_m \leq \mathbb{k}_m \leq \wp_{m+1} = b$ . Assume  $\{\omega(\wp)\}_{\wp \geq 0}$  is a  $K$ -Wiener process on  $(\mathfrak{F}, \mathfrak{S}, \{\mathfrak{S}_\wp\}_{\wp \geq 0}, \mathfrak{P})$  with values in  $G$  and  $\{\mathfrak{B}_H(\wp)\}_{\wp \geq 0}$  is fBm with Hurst parameter  $H \in (\frac{1}{2}, 1)$  defined on  $(\mathfrak{F}, \mathfrak{S}, \{\mathfrak{S}_\wp\}_{\wp \geq 0}, \mathfrak{P})$  with values in  $Q$ ;  $Q$  is a Hilbert space with  $\|\cdot\|$ . In this paper,  $L(G, \mathfrak{N})$  and  $L(Q, \mathfrak{N})$  are the space of all bounded linear operators from  $G$  into  $\mathfrak{N}$  and from  $Q$  into  $\mathfrak{N}$ , respectively, with  $\|\cdot\|$ .  $M, V, \Omega, \mathfrak{U}$ , and  $\hbar_i$  are defined in Section 2.

The contributions of the present work are as follows:

- The conformable fractional noninstantaneous impulsive stochastic evolution equation with fractional Brownian motion and Poisson jump is presented.
- To the best of the author's knowledge, there has not been any research that has studied the approximate controllability of (1).
- An example is applied to show the established results.

## 2. Preliminaries

Here, we collect the basic concepts, definitions, theorems, and lemmas that are used in the paper.

**Definition 1** (See [7]). The CFD of order  $0 < \mathcal{L} < 1$  of  $z(\wp)$  for  $\wp > 0$  is defined as

$$\frac{d^{\mathcal{L}} z(\wp)}{d\wp^{\mathcal{L}}} = \lim_{\nu \rightarrow 0} \frac{z(\wp + \nu\wp^{1-\mathcal{L}}) - z(\wp)}{\nu}.$$

Furthermore, the conformable integral is defined as

$$I^{\mathcal{L}}(z)(\wp) = \int_0^{\wp} \mathbb{k}^{\mathcal{L}-1} z(\mathbb{k}) d\mathbb{k}.$$

Suppose  $(\mathfrak{F}, \mathfrak{S}, \mathfrak{P})$  is a full probability area connected with a normal filtration  $\mathfrak{S}_{\wp}, \wp \in [0, b]$ , where  $\mathfrak{S}_{\wp}$  is the  $\sigma$ -algebra generated by random variables  $\{\omega(\mathbb{k}), \mathfrak{B}_H(\mathbb{k}), s \in [0, b]\}$  and all  $\mathfrak{P}$ -null sets. Let  $(F, \xi, \zeta(df))$  be a  $\sigma$ -finite measurable space. The stationary Poisson point process  $(p_{\wp})_{\wp \geq 0}$  is defined on  $(\mathfrak{F}, \mathfrak{S}, \mathfrak{P})$  with values in  $F$  and characteristic measure  $\zeta$ . The counting measure of  $p_{\wp}$  is denoted by  $\zeta(\wp, df)$  such that  $W(\wp, \vartheta) := E(\zeta(\wp, \vartheta)) = \wp\zeta(\vartheta)$  for  $\vartheta \in \xi$ . Define  $W(\wp, df) := \zeta(\wp, df) - \wp\lambda(df)$ , the Poisson martingale generated by  $p_{\wp}$ .

Let  $\mathfrak{Z} \in L(Q, Q)$  be an operator defined by  $\mathfrak{Z}\mathfrak{x}_n = \lambda_n \mathfrak{x}_n$  with  $Tr(\mathfrak{Z}) = \sum_{n=1}^{\infty} \lambda_n < \infty$ , where  $\lambda_n \geq 0$  ( $n = 1, 2, \dots$ ) are non-negative real numbers and  $\{\mathfrak{x}_n\}$  ( $n = 1, 2, \dots$ ) is a complete orthonormal basis in  $Q$ .

We introduce the space  $L_2^0 := L_2^0(Q, \mathfrak{N})$  of all  $A$ -Hilbert–Schmidt operators  $\mu : Q \rightarrow \mathfrak{N}$ .  $\mu \in L(Q, \mathfrak{N})$  is called a  $A$ -Hilbert–Schmidt operator, if

$$\|\mu\|_{L_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \mu N_n\|^2 < \infty.$$

**Lemma 1** (see [30]). If  $\mu : [0, b] \rightarrow L_2^0(Q, \mathfrak{N})$  satisfies  $\int_0^b \|\mu(\mathbb{k})\|_{L_2^0}^2 < \infty$  then

$$E \left\| \int_0^{\wp} \mu(\mathbb{k}) d\mathfrak{B}_H(\mathbb{k}) \right\|^2 \leq 2H\wp^{2H-1} \int_0^{\wp} \|\mu(\mathbb{k})\|_{L_2^0}^2 d\mathbb{k}.$$

**Theorem 1** (see [31]). Assume  $(\mathbb{Y}; \mathbb{A})$  is a compact metric space. For a family of functions  $\mathbb{Z} \in C(\mathbb{Y})$ , then the following statements are equivalent:

- (i)  $\mathbb{Z}$  is relatively compact;

(ii)  $\mathbb{Z}$  is equicontinuous and uniformly bounded.

Through this paper, let  $L_2(\mathfrak{F}, \mathfrak{N})$  be a Banach space with

$$\|N(\cdot)\|_{L_2(\mathfrak{F}, \mathfrak{N})}^2 = E \|N(\cdot, \omega)\|^2,$$

where  $E(N) = \int_{\mathfrak{F}} N(\omega) d\mathfrak{P}$ . Assume  $C(Y, L_2(\mathfrak{F}, \mathfrak{N}))$ , from  $Y$  into  $L_2(\mathfrak{F}, \mathfrak{N})$ , is the Banach space of all continuous functions and satisfies  $\sup_{\wp \in Y} E \|N(\wp)\|^2 < \infty$ .

Define  $F = \{\cdot : N(\wp) \in C(Y, L_2(\mathfrak{F}, \mathfrak{N}))\}$ , with

$$\|\cdot\|_F^2 = \sup_{\wp \in Y} E \|N(\wp)\|^2.$$

Obviously,  $F$  is a Banach space.

We require the following hypotheses:

(A1)  $M : Y \times \mathfrak{N}^{m+1} \rightarrow \mathfrak{N}$  verifies the following:

- (i)  $M : Y \times \mathfrak{N}^{m+1} \rightarrow \mathfrak{N}$  is continuous;
- (ii)  $\forall \varepsilon \in N; \varepsilon > 0 \exists e_\varepsilon(\cdot) : Y \rightarrow R^+$  such that

$$\sup_{\|N_0\|^2, \dots, \|N_n\|^2 \leq \varepsilon} E \|M(\wp, N_0, N_1, \dots, N_m)\|^2 \leq e_\varepsilon(\wp),$$

the function  $\mathbb{k} \rightarrow e_\varepsilon(\mathbb{k}) \in L^1((0, b], R^+)$  and  $\exists$  a  $\chi_1 > 0$  such that

$$\liminf_{\varepsilon \rightarrow \infty} \frac{\int_0^\wp e_\varepsilon(\mathbb{k}) d\mathbb{k}}{\varepsilon} = \chi_1 < \infty, \quad \wp \in (0, b].$$

(A2)  $V : Y \times \mathfrak{N}^{k+1} \rightarrow L(G, \mathfrak{N})$  verifies the following:

- (i)  $\forall \wp \in Y$ , the function  $V(\wp, \cdot) : \mathfrak{N}^{k+1} \rightarrow L(G, \mathfrak{N})$  is continuous and  $\forall (N_0, N_1, \dots, N_n) \in \mathfrak{N}^{n+1}$ ; the function  $V(\cdot, N_0, N_1, \dots, N_k) : \mathfrak{J} \rightarrow L(G, \mathfrak{N})$  is  $\mathfrak{S}_\wp$ -measurable;
- (ii)  $\forall \varepsilon \in N; \varepsilon > 0 \exists r_\varepsilon(\cdot) : (0, b] \rightarrow R^+$  such that

$$\sup_{\|N_0\|^2, \dots, \|N_n\|^2 \leq \varepsilon} \int_0^\wp E \|V(\mathbb{k}, N_0, N_1, \dots, N_k)\|_A^2 d\mathbb{k} \leq r_\varepsilon(\wp),$$

the function  $\mathbb{k} \rightarrow r_\varepsilon(\mathbb{k}) \in L^1((0, b], R^+)$  and  $\exists$  a  $\chi_2 > 0$  such that

$$\liminf_{\varepsilon \rightarrow \infty} \frac{\int_0^\wp r_\varepsilon(\mathbb{k}) d\mathbb{k}}{\varepsilon} = \chi_2 < \infty, \quad \wp \in (0, b].$$

(A3)  $\Omega : Y \times \mathfrak{N}^{p+1} \rightarrow L_2^0(Q, \mathfrak{N})$  satisfies the following:

- (i)  $\forall \wp \in Y$ , the function  $\Omega(\wp, \cdot) : \mathfrak{N}^{p+1} \rightarrow L_2^0(Q, \mathfrak{N})$  is continuous and  $\forall (N_0, N_1, \dots, N_p) \in \mathfrak{N}^{p+1}$ ; the function  $\Omega(\cdot, N_0, N_1, \dots, N_p) : J \rightarrow L_2^0(Q, \mathfrak{N})$  is  $\mathfrak{S}_\wp$ -measurable;
- (ii)  $\forall \varepsilon \in N; \varepsilon > 0 \exists \bar{r}_\varepsilon(\cdot) : (0, b] \rightarrow R^+$  such that

$$\sup_{\|N_0\|^2, \dots, \|N_n\|^2 \leq \varepsilon} E \|\Omega(\wp, N_0, N_1, \dots, N_p)\|_{L_2^0}^2 \leq \bar{r}_\varepsilon(\wp),$$

$\mathbb{k} \rightarrow \bar{r}_\varepsilon(\mathbb{k}) \in L^1((0, b], R^+)$  and  $\exists$  a  $\chi_3 > 0$  such that

$$\liminf_{\varepsilon \rightarrow \infty} \frac{\int_0^\wp \bar{r}_\varepsilon(\mathbb{k}) d\mathbb{k}}{\varepsilon} = \chi_3 < \infty, \quad \wp \in (0, b],$$

(A4)  $\mathfrak{U} : Y \times \mathfrak{N}^{u+1} \times F \rightarrow \mathfrak{N}$  satisfies the following:

- (i)  $\mathfrak{U} : Y \times \mathfrak{N}^{u+1} \times F \rightarrow \mathfrak{N}$  is continuous;
- (ii)  $\forall \varepsilon \in N; \varepsilon > 0 \exists q_\varepsilon(\cdot) : Y \rightarrow R^+$  such that

$$\sup_{\|N_0\|^2, \dots, \|N_m\|^2 \leq \varepsilon} \int_F E \|\mathcal{U}(\varphi, N_0, N_1, \dots, N_m, f)\|^2 \lambda(df) \leq q_\varepsilon(\varphi),$$

$\mathbb{k} \rightarrow q_\varepsilon(\mathbb{k}) \in L^1((0, \varphi], R^+)$ , and  $\exists$  a  $\chi_4 > 0$  such that

$$\liminf_{\varepsilon \rightarrow \infty} \frac{\int_0^\varphi q_\varepsilon(\mathbb{k}) d\mathbb{k}}{\varepsilon} = \chi_4 < \infty, \varphi \in (0, b].$$

(A5)  $\mathcal{h}_i : (\varphi_i, \mathbb{k}_i] \times \mathbb{N} \rightarrow \mathbb{N}$  is continuous and verifies the following:

(i)  $\exists \varrho_3 > 0$ , such that

$$E \|\mathcal{h}_i(\varphi, N)\|^2 \leq \varrho_3 E \|N\|^2, \forall N \in \mathbb{N}; \varphi \in (\varphi_i, \mathbb{k}_i], i = 1, 2, \dots, m;$$

(ii)  $\exists \varrho_4 > 0$ , such that

$$E \|\mathcal{h}_i(\varphi, N_1) - \mathcal{h}_i(\varphi, N_2)\|^2 \leq \varrho_4 E \|N_1 - N_2\|^2, \forall N_1, N_2 \in \mathbb{N}; \varphi \in (\varphi_i, \mathbb{k}_i], i = 1, 2, \dots, m.$$

(A6)  $\Delta$  generates a compact semigroup  $\{\Theta(\varphi), \varphi \geq 0\}$  in  $\mathbb{N}$ .

**Definition 2** (see [32]).  $N(\varphi) : Y \rightarrow \mathbb{N}$  is a mild solution of (1) if  $N_0 \in \mathbb{N} \forall s \in [0, b)$  the function  $\mathbb{k}^{\mathbb{L}-1} M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k})))$  is integrable and

$$N(\varphi) = \begin{cases} \Theta\left(\frac{\varphi^\mathbb{L}}{\mathbb{L}}\right) N_0 + \int_0^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k} \\ + \int_0^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) \int_0^\mathbb{k} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\ + \int_0^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \\ + \int_0^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) \int_F \mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) W(d\mathbb{k}, df) \\ + \int_0^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) Bu(\mathbb{k}) d\mathbb{k}, \varphi \in (0, \varphi_1] \\ \mathcal{h}_i(\varphi, N(\varphi)), \varphi \in (\varphi_i, \mathbb{k}_i], i = 1, 2, \dots, m \\ \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}_i^\mathbb{L}}{\mathbb{L}}\right) \mathcal{h}_i(\mathbb{k}_i, N(\mathbb{k}_i)) \\ + \int_{\mathbb{k}_i}^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k} \\ + \int_{\mathbb{k}_i}^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) \int_0^\mathbb{k} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\ + \int_{\mathbb{k}_i}^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \\ + \int_{\mathbb{k}_i}^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) \int_V \mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) W(d\mathbb{k}, df) \\ + \int_{\mathbb{k}_i}^\varphi \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{\varphi^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) Bu(\mathbb{k}) d\mathbb{k}, \varphi \in (\mathbb{k}_i, \varphi_{i+1}], i = 1, 2, \dots, m, \end{cases} \quad (2)$$

is verified.

### 3. Approximate Controllability

Here, we investigate the approximate controllability of (1).

Consider the linear conformable fractional evolution equation in the following form:

$$\begin{cases} D^\mathbb{L} N(\varphi) + \Delta N(\varphi) = Bu(\varphi), \varphi \in (0, b] \\ N(0) = N_0. \end{cases} \quad (3)$$

We present the operators associated with (3) as

$$\Xi_0^b = \int_0^b \mathbb{k}^{\mathbb{L}-1} \Theta\left(\frac{b^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) BB^* \Theta^*\left(\frac{b^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right) d\mathbb{k},$$

and  $\Phi(b, \Xi_0^b) = (bI + \Xi_0^b)^{-1}$ ,  $b > 0$ , where the adjoint of  $B$  and  $\Theta\left(\frac{b^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right)$  are denoted by  $B^*$  and  $\Theta^*\left(\frac{b^\mathbb{L} - \mathbb{k}^\mathbb{L}}{\mathbb{L}}\right)$ , respectively.

The state value of (1) at terminal state  $b$ , corresponding to the control  $u$  and the initial value  $N_0$ , is denoted by  $N(b; N_0, u)$ . Furthermore, the reachable set of (1) at terminal time  $b$  is denoted by  $\Phi(b, N_0) = \{N(b; N_0, u) : u \in L_2(Y, U)\}$ , and its closure in  $\mathfrak{N}$  is  $\overline{\Phi(b, N_0)}$ .

**Definition 3** ([33]). Let (1) be approximately controllable on  $Y$  if  $\overline{\Phi(b, N_0)} = L_2(\mathfrak{F}, \mathfrak{N})$ .

**Lemma 2** ([33]). The linear system (3) is approximately controllable on  $Y$  if and only if  $x(xI + \Phi_0^b)^{-1} \rightarrow 0$  as  $x \rightarrow 0^+$ .

**Lemma 3.**  $\forall \bar{N}_b \in L_2(\mathfrak{F}, \mathfrak{N}) \exists \bar{\psi}$  and  $\bar{\varphi} \in L_2(\mathfrak{F}; L_2(Y; L_2^0))$  such that

$$\bar{N}_b = E\bar{N}_b + \int_0^b \bar{\psi}(\wp) d\omega(\wp) + \int_0^b \bar{\varphi}(\wp) d\mathfrak{B}^H(\wp).$$

We define the control function, for any  $\delta > 0$  and  $\bar{N}_b \in L_2(\mathfrak{F}, \mathfrak{N})$ , in the following form:

$$u^\rho(\wp) = \begin{cases} B^* \Theta^* \left( \frac{b^\ell - \wp^\ell}{\ell} \right) (xI + \Phi_0^b)^{-1} \left[ E\bar{N}_b + \int_0^b \bar{\psi}(\mathbb{k}) d\omega(\mathbb{k}) + \int_0^b \bar{\varphi}(\mathbb{k}) d\mathfrak{B}^H(\mathbb{k}) \right. \\ \quad - \Theta \left( \frac{b^\ell}{\ell} \right) N_0 - \int_0^b \mathbb{k}^{\ell-1} \Theta \left( \frac{b^\ell - \mathbb{k}^\ell}{\ell} \right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k} \\ \quad - \int_0^b \mathbb{k}^{\ell-1} \Theta \left( \frac{b^\ell - \mathbb{k}^\ell}{\ell} \right) \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\ \quad - \int_0^b \mathbb{k}^{\ell-1} \Theta \left( \frac{b^\ell - \mathbb{k}^\ell}{\ell} \right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) W(d\mathbb{k}, df) \\ \quad \left. - \int_0^b \mathbb{k}^{\ell-1} \Theta \left( \frac{b^\ell - \mathbb{k}^\ell}{\ell} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \right], \wp \in (0, \wp_1] \\ \\ B^* \Theta^* \left( \frac{b^\ell - \wp^\ell}{\ell} \right) (xI + \Phi_0^b)^{-1} \left[ E\bar{N}_b + \int_0^b \bar{\psi}(\mathbb{k}) d\omega(\mathbb{k}) + \int_0^b \bar{\varphi}(\mathbb{k}) d\mathfrak{B}^H(\mathbb{k}) \right. \\ \quad - \Theta \left( \frac{\wp^\ell - \mathbb{k}_i^\ell}{\ell} \right) \bar{h}_i(\mathbb{k}_i, N(\mathbb{k}_i)) \\ \quad - \int_{\mathbb{k}_i}^b \mathbb{k}^{\ell-1} \Theta \left( \frac{b^\ell - \mathbb{k}^\ell}{\ell} \right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k} \\ \quad - \int_{\mathbb{k}_i}^b \mathbb{k}^{\ell-1} \Theta \left( \frac{b^\ell - \mathbb{k}^\ell}{\ell} \right) \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\ \quad - \int_{\mathbb{k}_i}^b \mathbb{k}^{\ell-1} \Theta \left( \frac{b^\ell - \mathbb{k}^\ell}{\ell} \right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) W(d\mathbb{k}, df) \\ \quad \left. - \int_{\mathbb{k}_i}^b \mathbb{k}^{\ell-1} \Theta \left( \frac{b^\ell - \mathbb{k}^\ell}{\ell} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \right], \\ \wp \in (\mathbb{k}_i, \wp_{i+1}]. \end{cases} \quad (4)$$

In this paper, we set  $\varrho = \sup_{\cdot \in Y} \|\Theta(\cdot)\|$ ,  $\varrho_B = \|B\|$  and  $\varrho_{B^*} = \|B^*\|$ .

**Theorem 2.** Suppose (A1)–(A6) holds, then (1) has a mild solution on  $Y$ , such that

$$36\varrho^2 \left\{ 1 + \frac{\varrho^2 \varrho_{B^*}^2 \varrho_B^2 b^{2\ell}}{(2\ell-1)x^2} \right\} \left\{ \varrho_3 + \frac{b^{2\ell-1}}{2\ell-1} \chi_1 + \text{Tr}(\mathfrak{Z}) \frac{b^{2\ell-1}}{2\ell-1} \chi_2 + \frac{2Hb^{2(H+\ell-1)}}{2\ell-1} \chi_3 + \frac{b^{2\ell-1}}{2\ell-1} \chi_4 \right\} + \varrho_3 < 1, \quad (5)$$

$$\text{and } \gamma_1 = \left[ \frac{8\varrho^2 b^{2\ell}}{2\ell-1} + \varrho_4 + 4\varrho^2 \varrho_4 \right] < 1.$$

**Proof.** Consider the map  $\Lambda$  on  $\bar{\mathfrak{C}}$  defined by to be verified:

$$\begin{aligned}
(\Lambda N)(\wp) = & \left\{ \begin{aligned} & \Theta\left(\frac{\wp^\ell}{\ell}\right)N_0 + \int_0^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k} \\ & + \int_0^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) Bu(\mathbb{k}) d\mathbb{k} \\ & + \int_0^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) \int_0^\mathbb{k} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\ & + \int_0^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \\ & + \int_0^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) W(d\mathbb{k}, df), \quad \wp \in (0, \wp_1] \\ & h_i(\wp, N(\wp)), \quad \wp \in (\wp_i, \mathbb{k}_i], \quad i = 1, 2, \dots, m \\ & \Theta\left(\frac{\wp^\ell - \mathbb{k}_i^\ell}{\ell}\right) h_i(\mathbb{k}_i, N(\mathbb{k}_i)) + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k} \\ & + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) Bu(\mathbb{k}) d\mathbb{k} \\ & + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) \int_0^\mathbb{k} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\ & + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \\ & + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) \int_V \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) W(d\mathbb{k}, df), \\ & \wp \in (\mathbb{k}_i, \wp_{i+1}], \quad i = 1, 2, \dots, m. \end{aligned} \right.
\end{aligned}$$

□

Next, show that  $\Lambda$  from  $\bar{\mathfrak{C}}$  into itself has a fixed point. Set  $\mathfrak{B}_\varepsilon = \{N \in \bar{\mathfrak{C}}, \quad \|N\|_{\bar{\mathfrak{C}}}^2 \leq \varepsilon\}$ ,  $\varepsilon > 0$ , integer. Therefore,  $\mathfrak{B}_\varepsilon \subset \bar{\mathfrak{C}}$  is a bounded closed convex set in  $\bar{\mathfrak{C}}$ ,  $\forall \varepsilon$ .

From (A1) and Hölder's inequality, we obtain

$$\begin{aligned}
& E \left\| \int_0^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k} \right\|^2 \\
& \leq \frac{\wp^2 b^{2\ell-1}}{2\ell-1} \int_0^\wp \sup_{\|N_0\|^2, \dots, \|N_n\|^2 \leq \varepsilon} E \|M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k})))\|^2 d\mathbb{k} \\
& \leq \frac{\wp^2 b^{2\ell-1}}{2\ell-1} \int_0^\wp e_\varepsilon(\mathbb{k}) d\mathbb{k}.
\end{aligned} \tag{6}$$

It follows that  $\mathbb{k}^{\ell-1} M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k})))$  is integrable on  $Y$ , and by Bochner's theorem,  $\Lambda$  is defined on  $\mathfrak{B}_\varepsilon$ .

From (A2)(ii) with Burkholder–Gundy's inequality, we obtain

$$\begin{aligned}
& E \left\| \int_0^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) \int_0^\mathbb{k} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \right\|^2 \\
& \leq Tr(3) \frac{\wp^2 b^{2\ell-1}}{2\ell-1} \int_0^\wp \left( \sup_{\|N_0\|^2, \dots, \|N_n\|^2 \leq \varepsilon} \int_0^\mathbb{k} E \|V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau)))\|_{\mathcal{H}}^2 d\tau \right) d\mathbb{k} \\
& \leq Tr(3) \frac{\wp^2 b^{2\ell-1}}{2\ell-1} \int_0^\wp r_\varepsilon(\mathbb{k}) d\mathbb{k}.
\end{aligned} \tag{7}$$

From (A3)(ii) with Burkholder–Gundy's inequality, this yields

$$\begin{aligned}
& E \left\| \int_0^\wp \mathbb{k}^{\ell-1} \Theta\left(\frac{\wp^\ell - \mathbb{k}^\ell}{\ell}\right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \right\|^2 \\
& \leq \frac{2H\wp^2 b^{2(H+\ell-1)}}{2\ell-1} \int_0^\wp \sup_{\|N_0\|^2, \dots, \|N_n\|^2 \leq \varepsilon} E \|\Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k})))\|_{\mathcal{H}_0}^2 d\mathbb{k} \\
& \leq \frac{2H\wp^2 b^{2(H+\ell-1)}}{2\ell-1} \int_0^\wp \bar{r}_\varepsilon(\mathbb{k}) d\mathbb{k}.
\end{aligned} \tag{8}$$

From Hölder inequality's and (A4)(ii), we obtain

$$\begin{aligned} E \parallel \int_0^\varphi \mathbb{k}^{\ell-1} \Theta \left( \frac{\varphi^\ell - \mathbb{k}^\ell}{\ell} \right) \int_F \mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df) \parallel^2 \\ \leq \frac{\varrho^2 b^{2\ell-1}}{2\ell-1} \int_0^\varphi \left( \sup_{\|N_0\|^2, \dots, \|N_u\|^2 \leq \varepsilon} \int_F E \parallel \mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) \parallel^2 \lambda df \right) d\mathbb{k} \\ \leq \frac{\varrho^2 b^{2\ell-1}}{2\ell-1} \int_0^\varphi q_\varepsilon(\mathbb{k}) d\mathbb{k}. \end{aligned} \quad (9)$$

From Hölder's inequality and Burkholder–Gundy's inequality with (A1)–(A4), we obtain, for  $\varphi \in (0, \varphi_1]$ ,

$$\begin{aligned} E \parallel \int_0^\varphi \mathbb{k}^{\ell-1} \Theta \left( \frac{\varphi^\ell - \mathbb{k}^\ell}{\ell} \right) Bu(\mathbb{k}) d\mathbb{k} \parallel^2 \leq & \frac{\varrho^4 \varrho_B^{2*} \varrho_B^2 b^{2\ell}}{(2\ell-1)x^2} \left[ E \parallel \bar{N}_b \parallel^2 + Tr(3) \int_0^b E \parallel \bar{\psi}(\mathbb{k}) \parallel_3^2 d\mathbb{k} \right. \\ & + 2Hb^{2H-1} \int_0^b E \parallel \bar{\varphi}(\mathbb{k}) \parallel_{L_2^2}^2 d\mathbb{k} + E \parallel N(0) \parallel^2 + \frac{b^{2\ell-1}}{2\ell-1} \int_0^\varphi e_\varepsilon(\mathbb{k}) d\mathbb{k} \\ & + Tr(3) \frac{b^{2\ell-1}}{2\ell-1} \int_0^\varphi r_\varepsilon(\mathbb{k}) d\mathbb{k} \\ & \left. + \frac{2Hb^{2(H+\ell-1)}}{2\ell-1} \int_0^\varphi \bar{r}_\varepsilon(\mathbb{k}) d\mathbb{k} + \frac{b^{2\ell-1}}{2\ell-1} \int_0^\varphi q_\varepsilon(\mathbb{k}) d\mathbb{k} \right], \end{aligned}$$

and for  $\varphi \in [\mathbb{k}_i, \varphi_{i+1}]$ , we obtain

$$\begin{aligned} E \parallel \int_{\mathbb{k}_i}^\varphi \mathbb{k}^{\ell-1} \Theta \left( \frac{\varphi^\ell - \mathbb{k}^\ell}{\ell} \right) Bu(\mathbb{k}) d\mathbb{k} \parallel^2 \leq & \frac{\varrho^4 \varrho_B^{2*} \varrho_B^2 b^{2\ell}}{(2\ell-1)x^2} \left[ E \parallel \bar{N}_b \parallel^2 + Tr(3) \int_0^b E \parallel \bar{\psi}(\mathbb{k}) \parallel_3^2 d\mathbb{k} \right. \\ & + 2Hb^{2H-1} \int_0^b E \parallel \bar{\varphi}(\mathbb{k}) \parallel_{L_2^2}^2 d\mathbb{k} + \varepsilon \varrho_3 + \frac{b^{2\ell-1}}{2\ell-1} \int_{\mathbb{k}_i}^\varphi e_\varepsilon(\mathbb{k}) d\mathbb{k} \\ & + Tr(3) \frac{b^{2\ell-1}}{2\ell-1} \int_{\mathbb{k}_i}^\varphi r_\varepsilon(\mathbb{k}) d\mathbb{k} \\ & \left. + \frac{2Hb^{2(H+\ell-1)}}{2\ell-1} \int_{\mathbb{k}_i}^\varphi \bar{r}_\varepsilon(\mathbb{k}) d\mathbb{k} + \frac{b^{2\ell-1}}{2\ell-1} \int_{\mathbb{k}_i}^\varphi q_\varepsilon(\mathbb{k}) d\mathbb{k} \right]. \end{aligned}$$

We claim that  $\exists \varepsilon > 0$  such that  $\Lambda(\mathfrak{B}_\varepsilon) \subseteq \mathfrak{B}_\varepsilon$ . If it false, then  $\forall \varepsilon > 0$ ; there is a function  $N_\varepsilon(\cdot) \in \mathfrak{B}_\varepsilon$ , but  $\Lambda(N_\varepsilon) \notin \mathfrak{B}_\varepsilon$ , that is  $\parallel \Lambda(N_\varepsilon)(\varphi) \parallel_{\mathfrak{C}}^2 > \varepsilon$  for some  $\varphi = \varphi(\varepsilon) \in Y$ , where  $\varphi(\varepsilon)$  means that  $\varphi$  is dependent on  $\varepsilon$ .

From (A5) and Equations (6)–(9), we have, for  $\varphi \in (0, \varphi_1]$ ,

$$\begin{aligned} \parallel \Lambda N_\varepsilon \parallel_F^2 \leq & 36 \sup_{\varphi \in Y} \left\{ E \parallel \Theta \left( \frac{\varphi^\ell}{\ell} \right) N_0 \parallel^2 \right. \\ & + E \parallel \int_0^\varphi \mathbb{k}^{\ell-1} \Theta \left( \frac{\varphi^\ell - \mathbb{k}^\ell}{\ell} \right) M(\varphi, N(\varphi), N(\ell_1(\varphi)), \dots, N(\ell_m(\varphi))) d\mathbb{k} \parallel^2 \\ & + E \parallel \int_0^\varphi \mathbb{k}^{\ell-1} \Theta \left( \frac{\varphi^\ell - \mathbb{k}^\ell}{\ell} \right) Bu(\mathbb{k}) d\mathbb{k} \parallel^2 \\ & + E \parallel \int_0^\varphi \mathbb{k}^{\ell-1} \Theta \left( \frac{\varphi^\ell - \mathbb{k}^\ell}{\ell} \right) \int_0^k V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \parallel^2 \\ & + E \parallel \int_0^\varphi \mathbb{k}^{\ell-1} \Theta \left( \frac{\varphi^\ell - \mathbb{k}^\ell}{\ell} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \parallel^2 \\ & \left. + \int_0^\varphi \mathbb{k}^{\ell-1} \Theta \left( \frac{\varphi^\ell - \mathbb{k}^\ell}{\ell} \right) \int_F \mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df) \parallel^2 \right\} \\ \leq & 36\varrho^2 \left\{ 1 + \frac{\varrho^2 \varrho_B^{2*} \varrho_B^2 b^{2\ell}}{(2\ell-1)x^2} \right\} \left\{ E \parallel N(0) \parallel^2 + \frac{b^{2\ell-1}}{2\ell-1} \int_0^b e_\varepsilon(\mathbb{k}) d\mathbb{k} \right. \\ & + Tr(3) \frac{b^{2\ell-1}}{2\ell-1} \int_0^b r_\varepsilon(\mathbb{k}) d\mathbb{k} + \frac{2Hb^{2(H+\ell-1)}}{2\ell-1} \int_0^b \bar{r}_\varepsilon(\mathbb{k}) d\mathbb{k} \\ & + \frac{b^{2\ell-1}}{2\ell-1} \int_0^b q_\varepsilon(\mathbb{k}) d\mathbb{k} \left. \right\} + \frac{\varrho^4 \varrho_B^{2*} \varrho_B^2 b^{2\ell}}{(2\ell-1)x^2} \left[ E \parallel \bar{N}_b \parallel^2 + Tr(3) \int_0^b E \parallel \bar{\psi}(\mathbb{k}) \parallel_3^2 d\mathbb{k} \right. \\ & \left. + 2Hb^{2H-1} \int_0^b E \parallel \bar{\varphi}(\mathbb{k}) \parallel_{L_2^2}^2 d\mathbb{k} \right], \end{aligned} \quad (10)$$

for  $\wp \in (\wp_i, \mathbb{k}_i]$

$$\| \Lambda N_\varepsilon \|_F^2 \leq \sup_{\wp \in J} E \| \mathfrak{h}_i(\wp, N(\wp)) \|^2 \leq \varrho_3 \varepsilon. \quad (11)$$

and for  $\wp \in (\mathbb{k}_i, \wp_{i+1}]$

$$\begin{aligned} \| \Lambda N_\varepsilon \|_F^2 &\leq 36 \sup_{\wp \in J} \left\{ E \left\| \Theta \left( \frac{\wp^\ell - \mathbb{k}_i^\ell}{\varepsilon} \right) \mathfrak{h}_i(\mathbb{k}_i, N(\mathbb{k}_i)) \right\|^2 \right. \\ &\quad + E \left\| \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) M(\wp, N(\wp), N(\ell_1(\wp)), \dots, N(\ell_m(\wp))) d\mathbb{k} \right\|^2 \\ &\quad + E \left\| \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) Bu(\mathbb{k}) d\mathbb{k} \right\|^2 \\ &\quad + E \left\| \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) \int_{\mathbb{k}_i}^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \right\|^2 \\ &\quad + E \left\| \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \right\|^2 \\ &\quad + \left. \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df) \right\} \\ &\leq 36 \varrho^2 \left\{ 1 + \frac{\varrho^2 \varrho_{B^*}^2 \varrho_B^2 b^{2\ell}}{(2\ell-1)x^2} \right\} \left\{ \varepsilon \varrho_3 + \frac{b^{2\ell-1}}{2\ell-1} \int_{\mathbb{k}_i}^b e_\varepsilon(\mathbb{k}) d\mathbb{k} + Tr(3) \frac{b^{2\ell-1}}{2\ell-1} \int_{\mathbb{k}_i}^b r_\varepsilon(\mathbb{k}) d\mathbb{k} \right. \\ &\quad + \frac{2H\varrho^2 b^{2(H+\ell-1)}}{2\ell-1} \int_{\mathbb{k}_i}^b \bar{r}_\varepsilon(\mathbb{k}) d\mathbb{k} + \frac{b^{2\ell-1}}{2\ell-1} \int_{\mathbb{k}_i}^b q_\varepsilon(\mathbb{k}) d\mathbb{k} \left. \right\} \\ &\quad + \frac{\varrho^4 \varrho_{B^*}^2 \varrho_B^2 b^{2\ell}}{(2\ell-1)x^2} \left[ E \| \bar{N}_b \|^2 + Tr(3) \int_0^b E \| \bar{\varphi}(\mathbb{k}) \|^2_3 d\mathbb{k} \right. \\ &\quad + \left. 2Hb^{2H-1} \int_0^b E \| \bar{\varphi}(\mathbb{k}) \|^2_{i_2} d\mathbb{k} \right]. \end{aligned} \quad (12)$$

Adding (10), (11) and (12) in the inequality  $\varepsilon \leq \| (\Lambda N_\varepsilon)(\wp) \|_F^2$ , dividing both sides of the inequality by  $\varepsilon$ , and applying the limit  $\varepsilon \rightarrow +\infty$ , then

$$\begin{aligned} &36 \varrho^2 \left\{ 1 + \frac{\varrho^2 \varrho_{B^*}^2 \varrho_B^2 b^{2\ell}}{(2\ell-1)x^2} \right\} \left\{ \varrho_3 + \frac{b^{2\ell-1}}{2\ell-1} \chi_1 + Tr(3) \frac{b^{2\ell-1}}{2\ell-1} \chi_2 \right. \\ &\quad + \left. \frac{2Hb^{2(H+\ell-1)}}{2\ell-1} \chi_3 + \frac{b^{2\ell-1}}{2\ell-1} \chi_4 \right\} + \varrho_3 \geq 1. \end{aligned}$$

This contradicts (5) Hence, for  $\varepsilon > 0$ ,  $\Lambda(\mathfrak{B}_\varepsilon) \subseteq \mathfrak{B}_\varepsilon$ . Next, we have to demonstrate that  $\Lambda$  has a fixed point on  $\mathfrak{B}_\varepsilon$ . We decompose  $\Lambda$  as  $\Lambda = \Lambda_1 + \Lambda_2$ , where  $\Lambda_1$  and  $\Lambda_2$  are defined on  $\mathfrak{B}_\varepsilon$  by

$$\begin{aligned} (\Lambda_1 N)(\wp) &= \begin{cases} \Theta \left( \frac{\wp^\ell}{\varepsilon} \right) N_0 + \int_0^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k}, & \wp \in (0, \wp_1] \\ \mathfrak{h}_i(\wp, N(\wp)), & \wp \in (\wp_i, \mathbb{k}_i], \quad i = 1, 2, \dots, m \\ \Theta \left( \frac{\wp^\ell - \mathbb{k}_i^\ell}{\varepsilon} \right) \mathfrak{h}_i(\mathbb{k}_i, N(\mathbb{k}_i)) + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k}, & \wp \in (\mathbb{k}_i, \wp_{i+1}], \quad i = 1, 2, \dots, m \end{cases} \\ (\Lambda_2 N)(\wp) &= \begin{cases} \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) Bu(\mathbb{k}) d\mathbb{k} \\ + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\ + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \\ + \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\ell-1} \Theta \left( \frac{\wp^\ell - \mathbb{k}^\ell}{\varepsilon} \right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df), & \wp \in (\mathbb{k}_i, \wp_{i+1}], \quad i = 0, 1, \dots, m \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$



for  $\wp \in Y$ . Next, we show that  $\Lambda_1$  is a contraction and  $\Lambda_2$  is a compact operator. To show that  $\Lambda_1$  is a contraction, let  $N_1, N_2 \in \mathfrak{B}_\varepsilon$ , then for each  $\wp \in Y$  and by (A1) and (A5), we obtain

$$\begin{aligned} E \|(\Lambda_1 N_1)(\wp) - (\Lambda_1 N_2)(\wp)\|^2 &\leq 4E \left\| \int_0^\wp \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \left[ M(\mathbb{k}, N_1(\mathbb{k}), N_1(\ell_1(\mathbb{k})), \dots, N_1(\ell_m(\mathbb{k}))) \right. \right. \\ &\quad \left. \left. - M(\mathbb{k}, N_2(\mathbb{k}), N_2(\ell_1(\mathbb{k})), \dots, N_2(\ell_m(\mathbb{k}))) \right] d\mathbb{k} \right\|^2 \\ &\leq \frac{4\varrho^2 b^{2\mathbb{L}}}{2\mathbb{L}-1} E \|N_1(\wp) - N_2(\wp)\|^2, \quad \wp \in (0, \wp_1] \end{aligned} \quad (13)$$

$$\begin{aligned} E \|(\Lambda_1 N_1)(\wp) - (\Lambda_1 N_2)(\wp)\|^2 &\leq E \|\hbar_i(\wp, N_1(\wp)) - \hbar_i(\wp, N_2(\wp))\|^2 \\ &\leq \varrho_4 E \|N_1(\wp) - N_2(\wp)\|^2, \quad \wp \in (0, \wp_1], \wp \in (\wp_i, \mathbb{k}_i] \end{aligned} \quad (14)$$

$$\begin{aligned} &E \|(\Lambda_1 N_1)(\wp) - (\Lambda_1 N_2)(\wp)\|^2 \\ &\leq 4E \left\| \Theta \left( \frac{(\wp - \mathbb{k}_i)^{\mathbb{L}}}{\mathbb{L}} \right) (\hbar_i(\mathbb{k}_i, N_1(\mathbb{k}_i)) - \hbar_i(\mathbb{k}_i, N_2(\mathbb{k}_i))) \right\|^2 \\ &\quad + 4E \left\| \int_{\mathbb{k}_i}^\wp \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) [M(\wp, N_1(\wp), N_1(\ell_1(\wp)), \dots, N_1(\ell_m(\wp))) \right. \right. \\ &\quad \left. \left. - M(\wp, N_2(\wp), N_2(\ell_1(\wp)), \dots, N_2(\ell_m(\wp))) \right] d\mathbb{k} \right\|^2 \\ &\leq 4 \left[ \varrho^2 \varrho_4 + \frac{\varrho^2 b^{2\mathbb{L}}}{2\mathbb{L}-1} \right] E \|N_1(\wp) - N_2(\wp)\|^2, \quad \wp \in (\mathbb{k}_i, \wp_{i+1}]. \end{aligned} \quad (15)$$

Combining (13), (14) and (15), we obtain

$$\begin{aligned} E \|(\Lambda_1 N_1)(\wp) - (\Lambda_1 N_2)(\wp)\|^2 &\leq \left[ \frac{8\varrho^2 b^{2\mathbb{L}}}{2\mathbb{L}-1} + \varrho_4 + 4\varrho^2 \varrho_4 \right] E \|N_1(\wp) - N_2(\wp)\|^2 \\ &\leq \gamma_1 E \|N_1(\wp) - N_2(\wp)\|^2. \end{aligned}$$

Taking  $\sup_{\wp \in Y}$  for both sides of the inequality, we obtain

$$\sup_{\wp \in J} E \|(\Lambda_1 N_1)(\wp) - (\Lambda_1 N_2)(\wp)\|^2 \leq \gamma_1 \sup_{\wp \in J} E \|N_1(\wp) - N_2(\wp)\|^2$$

Hence,

$$\|\Lambda_1 N_1 - \Lambda_1 N_2\|_F^2 \leq \gamma_1 \|N_1 - N_2\|_F^2.$$

Thus,  $\Lambda_1$  is a contraction.

We show that  $\Lambda_2$  is compact.

First, we prove the continuity of  $\Lambda_2$  on  $\mathfrak{B}_\varepsilon$ .

Let  $\{N_n\} \subseteq \mathfrak{B}_\varepsilon$  with  $N_n \rightarrow y$  in  $\mathfrak{B}_\varepsilon$  and the control function  $u(\wp) = u(\wp, N)$ . Therefore, for each

$\mathbb{k} \in Y$ ,  $N_n(\mathbb{k}) \rightarrow N(\mathbb{k})$  with A2(i), A3(i), and A4(i), we obtain

$V(\mathbb{k}, N_n(\mathbb{k}), N_n(j_1(\mathbb{k})), \dots, N_n(j_k(\mathbb{k}))) \rightarrow V(\mathbb{k}, N(\mathbb{k}), N(j_1(\mathbb{k})), \dots, N(j_k(\mathbb{k})))$ , as  $n \rightarrow \infty$ ,

$\Omega(\mathbb{k}, N_n(\mathbb{k}), N_n(c_1(\mathbb{k})), \dots, N_n(c_p(\mathbb{k}))) \rightarrow \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k})))$ , as  $n \rightarrow \infty$ , and

$\mathcal{U}(\mathbb{k}, N_n(\mathbb{k}), N_n(t_1(\mathbb{k})), \dots, N_n(t_u(\mathbb{k})), f) \rightarrow \mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f)$ , as  $n \rightarrow \infty$ .

From Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
 & \| \Lambda_2 N_n - \Lambda_2 y \|_F^2 = \sup_{\wp \in J} \left\{ E \left\| \int_{\mathbb{k}_i}^{\wp} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \left( Bu(\mathbb{k}, N_n) - Bu(\mathbb{k}, N) \right) d\mathbb{k} \right. \right. \\
 & + \int_{\mathbb{k}_i}^{\wp} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \int_0^{\mathbb{k}} \left( V(\tau, N_n(\tau), N_n(j_1(\tau)), \dots, N_n(j_k(\tau))) \right. \\
 & \left. \left. - V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) \right) d\omega(\tau) d\mathbb{k} \right. \\
 & + \int_{\mathbb{k}_i}^{\wp} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \left( \Omega(\mathbb{k}, N_n(\mathbb{k}), N_n(c_1(\mathbb{k})), \dots, N_n(c_p(\mathbb{k}))) \right. \\
 & \left. \left. - \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) \right) d\mathfrak{B}_H(\mathbb{k}) \right. \\
 & \left. + \int_{\mathbb{k}_i}^{\wp} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \int_F \left( \mathfrak{U}(\mathbb{k}, N_n(\mathbb{k}), N_n(\iota_1(\mathbb{k})), \dots, N_n(\iota_u(\mathbb{k})), f) \right. \right. \\
 & \left. \left. - \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) \right) W(d\mathbb{k}, df) \right\|^2 \Bigg\} \rightarrow 0,
 \end{aligned}$$

as  $n \rightarrow \infty$ , which is continuous.

Next, we show that  $\{\Lambda_2 y : N \in \mathfrak{B}_\epsilon\}$  is an equicontinuous family of functions. Assume  $\epsilon > 0$ , small,  $\mathbb{k}_i < \wp_\alpha < \wp_\beta \leq \wp_{i+1}$ , then

$$\begin{aligned}
 & E \| (\Lambda_2 N)(\wp_\beta) - (\Lambda_2 N)(\wp_\alpha) \|^2 \\
 & \leq E \| \int_{\mathbb{k}_i}^{\wp_\alpha - \epsilon} \mathbb{k}^{\mathbb{L}-1} \left( \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) - \Theta \left( \frac{\wp_\alpha^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \right) Bu(\mathbb{k}) d\mathbb{k} \|^2 \\
 & + E \| \int_{\wp_\alpha - \epsilon}^{\wp_\alpha} \mathbb{k}^{\mathbb{L}-1} \left( \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) - \Theta \left( \frac{\wp_\alpha^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \right) Bu(\mathbb{k}) d\mathbb{k} \|^2 \\
 & + E \| \int_{\wp_\alpha}^{\wp_\beta} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) Bu(\mathbb{k}) d\mathbb{k} \|^2 \\
 & + E \| \int_{\mathbb{k}_i}^{\wp_\alpha - \epsilon} \mathbb{k}^{\mathbb{L}-1} \left( \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) - \Theta \left( \frac{\wp_\alpha^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \right) \\
 & \times \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \|^2 \\
 & + E \| \int_{\wp_\alpha - \epsilon}^{\wp_\alpha} \mathbb{k}^{\mathbb{L}-1} \left( \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) - \Theta \left( \frac{\wp_\alpha^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \right) \\
 & \times \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \|^2 \\
 & + E \| \int_{\wp_\alpha}^{\wp_\beta} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \|^2 \\
 & + E \| \int_{\mathbb{k}_i}^{\wp_\alpha - \epsilon} \mathbb{k}^{\mathbb{L}-1} \left( \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) - \Theta \left( \frac{\wp_\alpha^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \|^2
 \end{aligned}$$

$$\begin{aligned}
& +E \parallel \int_{\wp_\alpha - \epsilon}^{\wp_\alpha} \mathbb{k}^{\mathbb{L}-1} \left( \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) - \Theta \left( \frac{\wp_\alpha^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \parallel^2 \\
& +E \parallel \int_{\wp_\alpha}^{\wp_\beta} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \parallel^2 \\
& +E \parallel \int_{\mathbb{k}_i}^{\wp_\alpha - \epsilon} \mathbb{k}^{\mathbb{L}-1} \left( \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) - \Theta \left( \frac{\wp_\alpha^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \right) \\
& \times \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df) \parallel^2 \\
& +E \parallel \int_{\wp_\alpha - \epsilon}^{\wp_\alpha} \mathbb{k}^{\mathbb{L}-1} \left( \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) - \Theta \left( \frac{\wp_\alpha^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \right) \\
& \times \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df) \parallel^2 \\
& +E \parallel \int_{\wp_\alpha}^{\wp_\beta} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp_\beta^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df) \parallel^2.
\end{aligned}$$

As  $\wp_\beta \rightarrow \wp_\alpha$ , we see that  $E \parallel (\Lambda_2 N)(\wp_\beta) - (\Lambda_2 N)(\wp_\alpha) \parallel^2 \rightarrow 0$  independently of  $N \in \mathfrak{B}_\epsilon$ , with  $\epsilon$  sufficiently small, because the compactness of  $\Theta(\wp)$  for  $\wp > 0$  tends to the continuity in the uniform operator topology. Furthermore, we can show that  $\Lambda_2 N$ ,  $N \in \mathfrak{B}_\epsilon$  are equicontinuous at  $\wp = 0$ . Then,  $\Lambda_2$  maps  $\mathfrak{B}_\epsilon$  into a family of equicontinuous functions. Next, we show that  $T(\wp) = \{(\Lambda_2 N)(\wp) : N \in \mathfrak{B}_\epsilon\}$  is relatively compact in  $\mathfrak{B}_\epsilon$ . Clearly,  $T(0)$  is relatively compact in  $\mathfrak{B}_\epsilon$ .

Assume  $\mathbb{k}_i < \wp \leq \wp_{i+1}$  to be fixed;  $\mathbb{k}_i < \epsilon < \wp$  for  $N \in \mathfrak{B}_\epsilon$ , we define

$$\begin{aligned}
(\Lambda_2^\epsilon N)(\wp) &= \int_{\mathbb{k}_i}^{\wp - \epsilon} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) Bu(\mathbb{k}) d\mathbb{k} \\
&+ \int_{\mathbb{k}_i}^{\wp - \epsilon} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\
&+ \int_{\mathbb{k}_i}^{\wp - \epsilon} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \\
&+ \int_{\mathbb{k}_i}^{\wp - \epsilon} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}} \right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df) \\
&= \Theta \left( \frac{\epsilon^{\mathbb{L}}}{\mathbb{L}} \right) \int_{\mathbb{k}_i}^{\wp - \epsilon} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}} - \epsilon^{\mathbb{L}}}{\mathbb{L}} \right) Bu(\mathbb{k}) d\mathbb{k} \\
&+ \Theta \left( \frac{\epsilon^{\mathbb{L}}}{\mathbb{L}} \right) \int_{\mathbb{k}_i}^{\wp - \epsilon} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}} - \epsilon^{\mathbb{L}}}{\mathbb{L}} \right) \int_0^{\mathbb{k}} \mathfrak{U}(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\
&+ \Theta \left( \frac{\epsilon^{\mathbb{L}}}{\mathbb{L}} \right) \int_{\mathbb{k}_i}^{\wp - \epsilon} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}} - \epsilon^{\mathbb{L}}}{\mathbb{L}} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \\
&+ \Theta \left( \frac{\epsilon^{\mathbb{L}}}{\mathbb{L}} \right) \int_{\mathbb{k}_i}^{\wp - \epsilon} \mathbb{k}^{\mathbb{L}-1} \Theta \left( \frac{\wp^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}} - \epsilon^{\mathbb{L}}}{\mathbb{L}} \right) \\
&\times \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(t_1(\mathbb{k})), \dots, N(t_u(\mathbb{k})), f) W(d\mathbb{k}, df).
\end{aligned}$$

Since  $\Theta \left( \frac{\epsilon^{\mathbb{L}}}{\mathbb{L}} \right)$ ,  $\frac{\epsilon^{\mathbb{L}}}{\mathbb{L}} > 0$  is a compact operator, hence  $T^\epsilon(\wp) = \{(\Lambda_2^\epsilon N)(\wp) : N \in \mathfrak{B}_\epsilon\}$  is relatively compact in  $\mathfrak{N}$  for every  $\epsilon$ ,  $\mathbb{k}_i < \epsilon < \wp$ .

Moreover,  $\forall N \in \mathfrak{B}_\epsilon$ , we have

$$\begin{aligned}
& \| \Lambda_2 y - \Lambda_2^\epsilon y \|_F^2 \leq 16 \sup_{\wp \in J} \left\{ E \left\| \int_{\wp-\epsilon}^{\wp} \mathbb{k}^{\mathcal{L}-1} \Theta \left( \frac{\wp^{\mathcal{L}} - \mathbb{k}^{\mathcal{L}}}{\mathcal{L}} \right) Bu(\mathbb{k}) d\mathbb{k} \right\|^2 \right. \\
& + E \left\| \int_{\wp-\epsilon}^{\wp} \mathbb{k}^{\mathcal{L}-1} \Theta \left( \frac{\wp^{\mathcal{L}} - \mathbb{k}^{\mathcal{L}}}{\mathcal{L}} \right) \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \right\|^2 \\
& + E \left\| \int_{\wp-\epsilon}^{\wp} \mathbb{k}^{\mathcal{L}-1} \Theta \left( \frac{\wp^{\mathcal{L}} - \mathbb{k}^{\mathcal{L}}}{\mathcal{L}} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \right\|^2 \\
& \left. + E \left\| \int_{\wp-\epsilon}^{\wp} \mathbb{k}^{\mathcal{L}-1} \Theta \left( \frac{\wp^{\mathcal{L}} - \mathbb{k}^{\mathcal{L}}}{\mathcal{L}} \right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) W(d\mathbb{k}, df) \right\|^2 \right\} \\
& \leq 16Q^2 \left\{ \frac{(b^{2\mathcal{L}-1} - (b-\epsilon)^{2\mathcal{L}-1}) Q_B^2}{2\mathcal{L}-1} \int_{b-\epsilon}^b E \| u(\mathbb{k}) \|^2 d\mathbb{k} \right. \\
& + Tr(3) \frac{(b^{2\mathcal{L}-1} - (b-\epsilon)^{2\mathcal{L}-1})}{2\mathcal{L}-1} \int_{b-\epsilon}^b \int_0^{\mathbb{k}} E \| V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) \|_A^2 d\tau d\mathbb{k} \\
& + 2H\epsilon^{2H-1} \frac{(b^{2\mathcal{L}-1} - (b-\epsilon)^{2\mathcal{L}-1})}{2\mathcal{L}-1} \int_{b-\epsilon}^b E \| \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) \|_{L_2^0}^2 d\mathbb{k} \\
& \left. + \frac{(b^{2\mathcal{L}-1} - (b-\epsilon)^{2\mathcal{L}-1})}{2\mathcal{L}-1} \int_{b-\epsilon}^b \int_F E \| \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) \|^2 \lambda df d\mathbb{k} \right\}.
\end{aligned}$$

We see that, for each  $N \in \mathfrak{B}_\epsilon$ ,  $\| \Lambda_2 y - \Lambda_2^\epsilon y \|_F^2 \rightarrow 0$  as  $\epsilon \rightarrow 0^+$ . Therefore, there are relative compact sets arbitrarily close to  $T(\wp) = \{(\Lambda_2 N)(\wp) : N \in \mathfrak{B}_\epsilon\}$ ; hence,  $T(\wp)$  is also relatively compact in  $\mathfrak{B}_\epsilon$ .

Thus, by the Arzela–Ascoli theorem  $\Lambda_2$  is a compact operator. Hence,  $\Lambda = \Lambda_1 + \Lambda_2$  is a condensing map on  $\mathfrak{B}_\epsilon$ , and by the fixed-point theorem of Sadovskii, there exists a fixed point  $N(\cdot)$  for  $\Lambda$  on  $\mathfrak{B}_\epsilon$ . Thus, the stochastic system (1) has a mild solution on  $Y$ .

**Theorem 3.** Suppose that Assumptions (A1)–(A6) are satisfied. Moreover, if  $M$ ,  $V$ ,  $\mathfrak{U}$  and  $\Omega$  are uniformly bounded, then (1) be approximately controllable on  $Y$ .

**Proof.** Assume  $N$  is a fixed point of  $\Lambda$ . By the stochastic Fubini theorem, we obtain

$$\begin{aligned}
N(b) &= \bar{N}_b - x(xI + \Xi_0^b)^{-1} \left\{ E\bar{N}_b + \int_0^T \bar{\psi}(s) d\omega(s) + \int_0^T \bar{\varphi}(s) dB^H(s) \right. \\
&- x \int_{\mathbb{k}_m}^b (xI + \Xi_0^b)^{-1} \mathbb{k}^{\mathcal{L}-1} \Theta \left( \frac{b^{\mathcal{L}} - \mathbb{k}^{\mathcal{L}}}{\mathcal{L}} \right) M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) d\mathbb{k} \\
&- x \int_{\mathbb{k}_m}^b (xI + \Xi_0^b)^{-1} \mathbb{k}^{\mathcal{L}-1} \Theta \left( \frac{b^{\mathcal{L}} - \mathbb{k}^{\mathcal{L}}}{\mathcal{L}} \right) \int_0^{\mathbb{k}} V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) d\omega(\tau) d\mathbb{k} \\
&- x \int_{\mathbb{k}_m}^b (xI + \Xi_0^b)^{-1} \mathbb{k}^{\mathcal{L}-1} \Theta \left( \frac{b^{\mathcal{L}} - \mathbb{k}^{\mathcal{L}}}{\mathcal{L}} \right) \int_F \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) W(d\mathbb{k}, df) \\
&\left. - x \int_{\mathbb{k}_m}^b (xI + \Xi_0^b)^{-1} \mathbb{k}^{\mathcal{L}-1} \Theta \left( \frac{b^{\mathcal{L}} - \mathbb{k}^{\mathcal{L}}}{\mathcal{L}} \right) \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) d\mathfrak{B}_H(\mathbb{k}) \right\}.
\end{aligned}$$

□

From the condition on  $M$ ,  $V$ ,  $\mathfrak{U}$  and  $\Omega$ , there exists  $\mathfrak{D} > 0$  such that

$$\begin{aligned}
& \| M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) \|^2 \leq \mathfrak{D}, \quad \| V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) \|^2 \leq \mathfrak{D}, \\
& \| \mathfrak{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) \|^2 \leq \mathfrak{D}, \\
& \| \Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) \|_{L_2^0}^2 \leq \mathfrak{D},
\end{aligned}$$

Consequently, the sequences  
 $\{M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k})))\}, \{\mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f)\},$   
 $\{V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau)))\}, \{\Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k})))\}$  are  
 weakly compact in  
 $L_2(J, \mathbb{N}), L_2(L_3(G, \mathbb{N}))$  and  $L_2(L_2^0(Q, \mathbb{N}))$ ,  
 so there are subsequences  
 $\{M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k})))\}, \{\mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f)\},$   
 $\{V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau)))\}, \{\Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k})))\}$   
 that weakly converge to  $\{M(\mathbb{k})\}, \{\mathcal{U}(\mathbb{k}, f)\}, \{V(\mathbb{k})\}, \{\Omega(\mathbb{k})\}$  in  $L_2(Y, \mathbb{N}), L_2(L_3(G, \mathbb{N})),$   
 and  $L_2(L_2^0(Q, \mathbb{N}))$ .  
 From the above, we have

$$\begin{aligned}
 & E\|N(b) - \bar{N}_b\|^2 \\
 \leq & 36E\|x(xI + \Xi_0^b)^{-1}E\bar{N}_b\|^2 + 36E\left\|\int_0^b x(xI + \Xi_0^b)^{-1}\bar{\psi}(\mathbb{k})d\omega(\mathbb{k})\right\|^2 \\
 + & 36E\left\|\int_0^b x(xI + \Xi_0^b)^{-1}\bar{\varphi}(\mathbb{k})dB^H(\mathbb{k})\right\|^2 \\
 + & 36E\left\|\int_{\mathbb{k}_m}^b x(xI + \Xi_0^b)^{-1}\mathbb{k}^{\mathbb{L}-1}\Theta\left(\frac{b^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}}\right)\left[M(\mathbb{k}, N(\mathbb{k}), N(\ell_1(\mathbb{k})), \dots, N(\ell_m(\mathbb{k}))) - M(\mathbb{k})\right]d\mathbb{k}\right\|^2 \\
 + & 36E\left\|\int_{\mathbb{k}_m}^b x(xI + \Xi_0^b)^{-1}\mathbb{k}^{\mathbb{L}-1}\Theta\left(\frac{b^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}}\right)M(\mathbb{k})d\mathbb{k}\right\|^2 \\
 + & 36E\left\|\int_{\mathbb{k}_m}^b x(xI + \Xi_0^b)^{-1}\mathbb{k}^{\mathbb{L}-1}\Theta\left(\frac{b^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}}\right)\right. \\
 & \quad \times \left.\int_0^{\mathbb{k}} \left[V(\tau, N(\tau), N(j_1(\tau)), \dots, N(j_k(\tau))) - V(\tau)\right]d\omega(\tau)d\mathbb{k}\right\|^2 \\
 + & 36E\left\|\int_{\mathbb{k}_m}^b x(xI + \Xi_0^b)^{-1}\mathbb{k}^{\mathbb{L}-1}\Theta\left(\frac{b^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}}\right)\int_0^{\mathbb{k}} V(\tau)d\omega(\tau)d\mathbb{k}\right\|^2 \\
 + & 36E\left\|\int_{\mathbb{k}_m}^b x(xI + \Xi_0^b)^{-1}\mathbb{k}^{\mathbb{L}-1}\Theta\left(\frac{b^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}}\right)\right. \\
 & \quad \times \left.\int_F \left[\mathcal{U}(\mathbb{k}, N(\mathbb{k}), N(\iota_1(\mathbb{k})), \dots, N(\iota_u(\mathbb{k})), f) - \mathcal{U}(\mathbb{k})\right]W(d\mathbb{k}, df)\right\|^2 \\
 + & 36E\left\|\int_{\mathbb{k}_m}^b x(xI + \Xi_0^b)^{-1}\mathbb{k}^{\mathbb{L}-1}\Theta\left(\frac{b^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}}\right)\int_F \mathcal{U}(\mathbb{k})W(d\mathbb{k}, df)\right\|^2 \\
 + & 36E\left\|\int_{\mathbb{k}_m}^b x(xI + \Xi_0^b)^{-1}\mathbb{k}^{\mathbb{L}-1}\Theta\left(\frac{b^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}}\right)\left[\Omega(\mathbb{k}, N(\mathbb{k}), N(c_1(\mathbb{k})), \dots, N(c_p(\mathbb{k}))) - \Omega(\mathbb{k})\right]d\mathfrak{B}_H(\mathbb{k})\right\|^2 \\
 + & 36E\left\|\int_{\mathbb{k}_m}^b x(xI + \Xi_0^b)^{-1}\mathbb{k}^{\mathbb{L}-1}\Theta\left(\frac{b^{\mathbb{L}} - \mathbb{k}^{\mathbb{L}}}{\mathbb{L}}\right)\Omega(\mathbb{k})d\mathfrak{B}_H(\mathbb{k})\right\|^2.
 \end{aligned}$$

By Lemma 2,  $x(xI + \Xi_0^b)^{-1} \rightarrow 0$  strongly as  $x \rightarrow 0^+$  for all  $\mathbb{k}_m < \mathbb{k} \leq b$ , and furthermore,  $\|x(xI + \Xi_0^b)^{-1}\| \leq 1$ . Thus,  $E\|N(b) - \bar{N}_b\|^2 \rightarrow 0$  as  $x \rightarrow 0^+$  by the Lebesgue-dominated convergence theorem and the compactness of  $\Theta(\mathbb{k})$ . Hence, the system (1) is approximate controllable.

#### 4. Example

Consider the CF noninstantaneous impulsive stochastic partial differential equation with fBm and Poisson jump of the form:

$$\left\{ \begin{array}{l} D_{0+}^{0.6} N(\varphi, z) + \frac{\partial^2}{\partial z^2} N(\varphi, z) = \frac{\cos \varphi}{1 + \cos \varphi} N(\varphi, z) + \mathfrak{w}(\varphi, z) + \int_0^\varphi 3^{-\mathbb{k}} N(\mathbb{k}, z) d\omega(\mathbb{k}) \\ + \frac{e^\varphi}{1 + e^\varphi} N(\varphi, z) \frac{d\mathfrak{B}_H(\varphi)}{d\varphi} + \int_F h(\varphi, N(\varphi, z), f) W(d\varphi, df), \quad \varphi \in (0, \frac{2}{3}] \cup (\frac{5}{4}, 3], \quad 0 \leq z \leq \pi, \\ N(\varphi, 0) = N(\varphi, \pi) = 0, \quad \varphi \in (0, 3], \\ N(\varphi, z) = \frac{2}{7} e^{-(\varphi - \frac{2}{3})} \frac{|N(\varphi, z)|}{1 + |N(\varphi, z)|}, \quad \varphi \in (\frac{2}{3}, \frac{5}{4}], \quad 0 \leq z \leq \pi, \\ (N(0, z)) = N_0(z), \quad 0 \leq z \leq \pi, \end{array} \right. \quad (16)$$

where  $D_{0+}^{0.6}$  is the CFD of order  $\mathcal{L} = 0.6$ ,  $\omega$  is a Wiener process, and  $\mathfrak{B}_H$  is an fBm with  $H \in (\frac{1}{2}, 1)$ .

Assume  $\mathfrak{N} = Q = G = \mathfrak{U} = L_2([0, \pi])$  and  $\Delta$ , where  $\Delta y = -(\frac{\partial^2}{\partial z^2})N$  with domain  $D(\Delta) = \{N \in \mathfrak{N} : N, \frac{dN}{dz} \text{ are absolutely continuous and } (\frac{d^2}{dz^2})N \in \mathfrak{N}, N(0) = N(\pi) = 0\}$ .

Then,  $-\Delta$  generates a strongly continuous semigroup  $\Theta(\cdot)$ , which is compact, analytic, and self-adjoint. Moreover,  $-\Delta$  has a discrete spectrum with eigenvalues  $n^2$ ,  $n \in \mathbb{N}$  and the corresponding normalized eigenfunctions given by

$$\mathfrak{x}_n = \sqrt{\frac{2}{\pi}} \sin nN, \quad n = 1, 2, \dots$$

In addition,  $(\mathfrak{x}_n)_{n \in \mathbb{N}}$  is a complete orthonormal basis in  $\mathfrak{N}$ . Then,

$$-\Delta N = \sum_{n=1}^{\infty} n^2 \langle N, \mathfrak{x}_n \rangle \mathfrak{x}_n, \quad N \in D(\Delta).$$

Moreover,  $-\Delta$  generates an analytic semigroup of the bounded linear operator,  $\{\Theta(\varphi)\}_{\varphi \geq 0}$  on  $\mathfrak{N}$ , and is defined by

$$\Theta(\varphi)N = \sum_{n=1}^{\infty} x^{-n^2 \varphi} \langle N, \mathfrak{x}_n \rangle \mathfrak{x}_n, \quad N \in \mathfrak{N}, \quad \varphi \geq 0.$$

with  $\|\Theta(\varphi)\| \leq x^{-\varphi} \leq 1$ . We define  $B : \mathfrak{U} \rightarrow \mathfrak{N}$  by  $Bu(\varphi)(z) = \mathfrak{w}(\varphi, z)$ ,  $0 \leq z \leq \pi$ ,  $u \in \mathfrak{U}$ . Furthermore,  $M : Y \times \mathfrak{N} \rightarrow \mathfrak{N}$ ,  $V : Y \times \mathfrak{N} \rightarrow L(G, \mathfrak{N})$ ,  $\Omega : Y \times \mathfrak{N} \rightarrow L_2^0(Q, \mathfrak{N})$ ,  $\mathfrak{U} : J \times \mathfrak{N} \times F \rightarrow \mathfrak{N}$ , and  $\mathfrak{h}_i : (\varphi_i, \mathbb{k}_i) \times \mathfrak{N} \rightarrow \mathfrak{N}$  are defined by  $M(\varphi, N)(z) = \frac{\cos \varphi}{1 + \cos \varphi} N(\varphi, z)$ ,  $V(\mathbb{k}, N)(z) = 3^{-\mathbb{k}} N(\mathbb{k}, z)$ ,  $\Omega(\varphi, N)(z) = \frac{e^\varphi}{1 + e^\varphi} N(\varphi, z)$ ,  $\mathfrak{U}(\varphi, N)(z) = \bar{h}(\varphi, N(\varphi, z), f)$ , and  $\mathfrak{h}_1(\varphi, N(\varphi)) = \frac{2}{7} e^{-(\varphi - \frac{2}{3})} \frac{|N(\varphi, \cdot)|}{1 + |N(\varphi, \cdot)|}$ , respectively. Then  $M$ ,  $V$ ,  $\Omega$ ,  $\mathfrak{U}$ , and  $\mathfrak{h}_1$  verify (A1)–(A6).

Let  $B = B^* = I$ . Therefore, all conditions of Theorems 2 and 3 are verified and

$$36\varrho^2 \left\{ 1 + \frac{\varrho^2 \varrho_{B^*}^2 \varrho_B^2 b^{2\mathcal{L}}}{(2\mathcal{L} - 1)x^2} \right\} \left\{ \varrho_3 + \frac{b^{2\mathcal{L}-1}}{2\mathcal{L} - 1} \chi_1 + \text{Tr}(3) \frac{b^{2\mathcal{L}-1}}{2\mathcal{L} - 1} \chi_2 + \frac{2Hb^{2(H+\mathcal{L}-1)}}{2\mathcal{L} - 1} \chi_3 + \frac{b^{2\mathcal{L}-1}}{2\mathcal{L} - 1} \chi_4 \right\} + \varrho_3 < 1,$$

$$\text{and } \gamma_1 = \left[ \frac{8\varrho^2 b^{2\mathcal{L}}}{2\mathcal{L} - 1} + \varrho_4 + 4\varrho^2 \varrho_4 \right] < 1.$$

Thus, (16) is approximately controllable on  $Y$ .

## 5. Conclusions

By using fractional calculus, a compact semigroup, Sadovskii's fixed-point theorem, and stochastic analysis, we investigated the approximate controllability of the given system (1). The obtained theoretical conclusions were illustrated in the later portion with an example. The results can be extended to a fractional stochastic inclusion system.

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