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New Versions of Some Results on Fixed Points in *b*-Metric Spaces

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Abstract: The main and the most important objective of this paper is to nominate some new versions of several well-known results about fixed-point theorems such as Caristi's theorem, Pant et al.'s theorem and Karapınar et al.'s theorem in the case of *b*-metric spaces. We use a new technique provided by Miculescu and Mihail in order to prove our theorems. Some illustrative applications and examples are given to strengthen our new findings and the main results.

Keywords: iterative methods; fixed point; *b*-metric; Caristi theorem; orbitally continuous; *k*-continuous

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1. Introduction and Preliminaries

Banach's theorem for fixed point theory is known to be a very useful tool in nonlinear analysis. The Banach result has been generalized in various ways and many applications have been presented. In the past thirty years, a lot of results have been obtained on fixed points of different classes of mappings defined on generalized metric spaces, for example, see [1–27] and references therein. Note that iterative methods and contraction mapping plays a key role in metric fixed-point theory. In addition, fractals can be generated via contraction mappings (Hutchinson's iterated function system) [28]. Some of the topics include *b*-metric space and the corresponding results about fixed point. Bakhtin [3] and Czerwik [6] introduced the notion about *b*-metric space and proved the number of fixed-point theorems in both single-valued and multi-valued mappings upon *b*-metric spaces.

Throughout this manuscript, we use the terms fixed point (FP), metric space (MS), *b*-metric space (bMS), and complete *b*-metric space ($CbMS$).

First, we look back on some background definitions, notations, and results in the bMS setting.

Definition 1. *Suppose $s \geq 1$ and Y is a nonempty set. A function $\mathcal{D} : \mathrm{Y} \times \mathrm{Y} \longrightarrow [0, +\infty)$ denotes a b-metric if $\mathrm{x}, \mathfrak{y}, \mathrm{z} \in \mathrm{Y}$ are valid:*

(1) $\mathcal{D}(\mathrm{x}, \mathfrak{y}) = 0$ *if and only if* $\mathrm{x} = \mathfrak{y}$;
(2) $\mathcal{D}(\mathrm{x}, \mathfrak{y}) = \mathcal{D}(\mathfrak{y}, \mathrm{x})$;
(3) $\mathcal{D}(\mathrm{x}, \mathrm{z}) \leq s[\mathcal{D}(\mathrm{x}, \mathfrak{y}) + \mathcal{D}(\mathfrak{y}, \mathrm{z})]$.

A triplet $(\mathrm{Y}, \mathcal{D}, s)$ is a bMS.

For bMS, the examples are the spaces $l^p(\mathbb{R})$ and $L^p[0,1]$, $p \in (0,1)$.

Recall that the convergence in bMS is defined as in metric spaces as follows.

Definition 2. *Suppose* $(\mathrm{Y}, \mathcal{D}, s)$ *is a bMS,* $\mathrm{x} \in \mathrm{Y}$ *and* $\{\mathrm{x}_n\}$ *is a sequence in* Y.
(a) $\{\mathrm{x}_n\}$ *is convergent in* $(\mathrm{Y}, \mathcal{D}, s)$ *and converges to* x*, if for each* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \mathbb{N}$ *where* $\mathcal{D}(\mathrm{x}_n, \mathrm{x}) < \varepsilon$ *for all* $n > n_\varepsilon$*, we denote this as* $\lim_{n\to\infty} \mathrm{x}_n = \mathrm{x}$ *or* $\mathrm{x}_n \to \mathrm{x}$ *where* $n \to \infty$.
(b) $\{\mathrm{x}_n\}$ *is the Cauchy sequence in* $(\mathrm{Y}, \mathcal{D}, s)$*, if for each* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \mathbb{N}$ *such that* $\mathcal{D}(\mathrm{x}_n, \mathrm{x}_m) < \varepsilon$ *for all* $n, m > n_\varepsilon$.
(c) $(\mathrm{Y}, \mathcal{D}, s)$ *is a CbMS if every Cauchy sequence in* Y *converges to some* $\mathrm{x} \in \mathrm{Y}$.

Next, the lemma for Miculescu and Mihail is a crucial result for achieving our aims.

Lemma 1 (([19], Lemma 2.6))**.** *Suppose* $(\mathrm{Y}, \mathcal{D}, s)$ *is a bMS and* $\{\mathrm{x}_n\}$ *is a sequence in* Y*. If there exists* $\alpha > \log_2 s$ *where the series* $\sum_{n=1}^{+\infty} n^\alpha \mathcal{D}(\mathrm{x}_n, \mathrm{x}_{n+1})$ *converges, then the sequence* $\{\mathrm{x}_n\}$ *is Cauchy.*

Remark 1. *If* $\alpha \geq \log_2 s$*, then Lemma 1 is not valid. Let* $\mathrm{Y} = \mathbb{R}, \mathcal{D}(\mathrm{x}, \mathfrak{y}) = (\mathrm{x} - \mathfrak{y})^2, \mathrm{x}_n = \sum_{k=2}^{n} \frac{1}{k \ln k}, n = 2, 3, \cdots$*. Then,* $s = 2$ *and*

$$\begin{aligned} \sum_{n=2}^{+\infty} n\mathcal{D}(\mathrm{x}_n, \mathrm{x}_{n+1}) &= \sum_{n=2}^{+\infty} \frac{n}{(n+1)^2 \ln^2(n+1)} \\ &\leq \sum_{n=2}^{+\infty} \frac{1}{(n+1)\ln^2(n+1)}. \end{aligned}$$

Therefore, $\sum_{n=2}^{+\infty} n\mathcal{D}(\mathrm{x}_n, \mathrm{x}_{n+1})$ *converges but this sequence* $\{\mathrm{x}_n\}$ *is not Cauchy (using the integral criterion for series convergence, we see that* $\sum_{k=2}^{+\infty} \frac{1}{k \ln^p k}$ *converges for* $p > 1$ *and diverges for* $p \leq 1$*).*

The next two results are the consequences of Lemma 1.

Lemma 2 (([18], Lemma 2.2))**.** *Suppose* $(\mathrm{Y}, \mathcal{D}, s)$ *is a bMS and* $\{\mathrm{x}_n\}$ *is a sequence in* Y*. If there exists* $k \in (0,1)$ *such that*

$$\mathcal{D}(\mathrm{x}_{n+1}, \mathrm{x}_{n+2}) \leq k\mathcal{D}(\mathrm{x}_n, \mathrm{x}_{n+1}), \tag{1}$$

for all $n \in \mathbb{N}$*, this leads to the sequence* $\{\mathrm{x}_n\}$ *being Cauchy.*

Lemma 3 (([19], Corollary 2.8))**.** *Suppose* $(\mathrm{Y}, \mathcal{D}, s)$ *is a bMS and* $\{\mathrm{x}_n\}$ *is a sequence in* Y*. If there exists* $h > 1$ *where the series*

$$\sum_{n=1}^{+\infty} h^n \mathcal{D}(\mathrm{x}_n, \mathrm{x}_{n+1}) \tag{2}$$

converges, then the sequence $\{\mathrm{x}_n\}$ *is Cauchy.*

Remark 2. *Note that if condition (2) is replaced by*

$$\sum_{n=1}^{+\infty} h^{(s-1)n} \mathcal{D}(\mathrm{x}_n, \mathrm{x}_{n+1}), \tag{3}$$

then in this case, we get the appropriate condition for MS as well.

In [4], Caristi presented the next theorem.

Theorem 1 ([4]). *Suppose* $(\mathrm{Y}, \mathcal{D})$ *is a CMS,* $\mathcal{T} : \mathrm{Y} \longrightarrow \mathrm{Y}$ *is a mapping such that*

$$\mathcal{D}(\mathrm{x}, \mathcal{T}\mathrm{x}) \leq \varphi(\mathrm{x}) - \varphi(\mathcal{T}\mathrm{x}), \tag{4}$$

for all $\mathrm{x} \in \mathrm{Y}$*, where* $\varphi : \mathrm{Y} \longrightarrow [0, +\infty)$ *is a lower semicontinuous mapping. This leads to* $\mathcal{T}$ *having FP.*

Dung and Hang [8] showed that Caristi's theorem does not fully extend to *bMS*. It is a negative answer to the latter Kirk-Shahzad's question ([17], Remark 12.6). One year later, Miculescu and Mihail [19] obtained the version of Caristi's theorem in *bMS*. One of the aims of the current work is to improve the mentioned result ([19], Theorem 3.1). Khojasteh et al. [16] gave a light version of Caristi's theorem as follows.

Theorem 2. ([16], Corollary 2.1) *Let* $(\mathrm{Y}, \mathcal{D})$ *be a CMS. Assume that* $\mathcal{T} : \mathrm{Y} \longrightarrow \mathrm{Y}$ *and* $\psi : \mathrm{Y} \times \mathrm{Y} \rightarrow [0, +\infty)$ *are mappings such that* $\mathrm{x} \mapsto \psi(\mathrm{x}, \mathfrak{y})$ *is lower semicontinuous for each* $\mathfrak{y} \in \mathrm{Y}$*. If*

$$\mathcal{D}(\mathrm{x}, \mathfrak{y}) \leq \psi(\mathrm{x}, \mathfrak{y}) - \psi(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}), \tag{5}$$

for all $\mathrm{x}, \mathfrak{y} \in \mathrm{Y}$*, then* $\mathcal{T}$ *has a unique FP.*

The second objective of this paper is to present an alternative of the above theorem in *bMS* (Theorem 5).

Remark 3. *Note that in [16], The partial answers were given by Khojasteh et al. to Reich, Mizoguchi and Takahashi's and Amini-Harandi's conjectures by using a light version of Caristi's FP theorem. In addition, they have shown that some known FP theorems can be obtained from the previously mentioned theorem.*

Definition 3. *Let* $(\mathrm{Y}, \mathcal{D})$ *be an MS and* $\mathcal{T} : \mathrm{Y} \longrightarrow \mathrm{Y}$ *be a mapping.*

(i) *(See [7]) The set* $O(\mathrm{x}, \mathcal{T}) = \{\mathcal{T}^n\mathrm{x} : n = 0, 1, 2, \ldots\}$ *is called the orbit of* $\mathcal{T}$ *at* x*. A map* $\mathcal{T}$ *is said to be orbitally continuous if* $u \in \mathrm{Y}$ *and such that* $u = \lim_{i \to +\infty} \mathcal{T}^{n_i}\mathrm{x}$ *for some* $\mathrm{x} \in \mathrm{Y}$*, then* $\mathcal{T}u = \lim_{i \to +\infty} \mathcal{T}\mathcal{T}^{n_i}\mathrm{x}$*, where* $\{n_i\}$ *is a subsequence of the sequence* $\{n\}$*;*

(ii) *(See [27]) A mapping* $\mathcal{T}$ *is called weakly orbitally continuous if the set* $\{\mathfrak{y} \in \mathrm{Y} : \lim_{i \to +\infty} \mathcal{T}^{n_i}\mathfrak{y} = u$ *implies* $\lim_{i \to +\infty} \mathcal{T}\mathcal{T}^{n_i}\mathfrak{y} = \mathcal{T}u\}$ *is nonempty, whenever the set* $\{\mathrm{x} \in \mathrm{Y} : \lim_{i \to +\infty} \mathcal{T}^{n_i}\mathrm{x} = u\}$ *is nonempty;*

(iii) *(See [26]) A mapping* $\mathcal{T}$ *is called k-continuous,* $k = 1, 2, 3, \ldots$ *if* $\lim_{n \to +\infty} \mathcal{T}^k\mathrm{x} = \mathcal{T}u$ *whenever* $\{\mathrm{x}_n\}$ *is a sequence in* Y *such that* $\lim_{n \to +\infty} \mathcal{T}^{k-1}\mathrm{x}_n = u$*.*

Here, we recall the next theorem of Pant et al. [25].

Theorem 3. ([25], Theorem 2.1) *Let* $(\mathrm{Y}, \mathcal{D})$ *be the CMS and the mappings* $\mathcal{T} : \mathrm{Y} \longrightarrow \mathrm{Y}$, $\varphi : \mathrm{Y} \rightarrow [0, +\infty)$*. If*

$$\mathcal{D}(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}) \leq \varphi(\mathrm{x}) - \varphi(\mathcal{T}\mathrm{x}) + \varphi(\mathfrak{y}) - \varphi(\mathcal{T}\mathfrak{y}). \tag{6}$$

for all $\mathrm{x}, \mathfrak{y} \in \mathrm{Y}$*, then* $\mathcal{T}$ *has a unique fixed point, under one of the following conditions:*

(i) $\mathcal{T}$ *is weakly orbitally continuous;*
(ii) $\mathcal{T}$ *is orbitally continuous;*
(iii) $\mathcal{T}$ *is k-continuous.*

Remark 4. *Note that from condition (6), we obtain*

$$\varphi(\mathcal{T}\mathrm{x}) \leq \varphi(\mathrm{x}). \tag{7}$$

for all $x \in Y$.

The third goal of this paper is to bring a new version of Theorem 3 in *bMS*.

Remark 5. *Pant et al. [25] have shown that Theorem 3 contains results of Banach, Kannan, Chatterjea, Ćirić and Suzuki on fixed points as particular cases. In addition, Theorem 3 is independent of the result of Caristi on fixed point. Note that, Theorem 3 is a new solution to the Rhoades problem about discontinuity at the FP.*

The main and the most important objective of this paper is to nominate some new versions of several well-known results about *FP* such as Caristi's theorem, Pant et al.'s theorem and Karapınar et al.'s theorem in the case of *bMS*. We use a new technique given by Miculescu and Mihail in [19] in order to prove our theorems. Some illustrative applications and examples are given to strengthen our new findings and the main results.

2. Main Results

In this part, we indicate the various known fixed-point theorems in *b*-metric space settings.

2.1. A New Version of the Theorem by Caristi

In this subsection, we afford a new version of Caristi's theorem in *bMS*. The terms orbit, orbitally continuous, weakly orbitally continuous and *k*-continuous in *bMS* are introduced analogously to metric space, see Definition 3.

Lemma 4. *Let* $(Y, \mathcal{D}, s)$ *be a bCMS and* $\mathcal{T} : Y \longrightarrow Y$ *be weakly orbitally continuous mapping. If there exist* $u \in Y$ *and* $x_0 \in Y$ *such that* $u = \lim_{n \to +\infty} \mathcal{T}^n x_0$*, then* $u = \mathcal{T}u$*.*

Proof. Let $u = \lim_{n \to +\infty} \mathcal{T}^n x_0$. Then, $u = \lim_{n \to +\infty} \mathcal{T}\mathcal{T}^n x_0$. The weak orbital continuity of $\mathcal{T}$ leads to $\lim_{n \to +\infty} \mathcal{T}^n x_0 = u = \lim_{n \to +\infty} \mathcal{T}\mathcal{T}^n x_0 = Tu$. So, $u = \mathcal{T}u$. □

Theorem 4. *Let* $(Y, \mathcal{D}, s)$ *be a bCMS and* $\mathcal{T} : Y \longrightarrow Y$ *be weakly orbitally continuous mapping such that*

$$\mathcal{D}(x, \mathcal{T}x) \leq \varphi(x) - h^{s-1}\varphi(\mathcal{T}^r x), \tag{8}$$

for all $x \in Y$*, where* $r \in \mathbb{N}$ *and* $h > 1$ *and* $\varphi : Y \to [0, +\infty)$*. Then,* $\mathcal{T}$ *has at least an FP.*

Proof. Let $x_0 \in$ and $x_n = \mathcal{T}^n x_0, n \in \mathbb{N}$. Put $\lambda = h^{\frac{s-1}{r}}$. From (8), we have

$$\begin{aligned}
\mathcal{D}(x_0, x_1) &\leq \varphi(x_0) - \lambda^r \varphi(x_r) \\
\lambda\mathcal{D}(x_1, x_2) &\leq \lambda\varphi(x_1) - \lambda^{r+1}\varphi(x_{r+1}) \\
&\vdots \\
\lambda^r\mathcal{D}(x_r, x_{r+1}) &\leq \lambda^r\varphi(x_r) - \lambda^{2r}\varphi(x_{2r}) \\
&\vdots \\
\lambda^n\mathcal{D}(x_n, x_{n+1}) &\leq \lambda^n\varphi(x_n) - \lambda^{n+r}\varphi(x_{n+r}).
\end{aligned}$$

The previous inequalities necessitate that

$$\begin{aligned}
\sum_{k=0}^{n} \lambda^k \mathcal{D}(x_k, x_{k+1}) &\leq \varphi(x_0) + \lambda\varphi(x_1) + \cdots + \lambda^{r-1}\varphi(x_{r-1}) \\
&- (\lambda^{r+1}\varphi(x_{r+1}) + \lambda^{r+2}\varphi(x_{r+2}) + \cdots + \lambda^{r+n}\varphi(x_{r+n}) \\
&\leq \varphi(x_0) + \lambda\varphi(x_1) + \cdots + \lambda^{r-1}\varphi(x_{r-1}).
\end{aligned}$$

We now conclude from Lemma 3 that $\{\mathcal{T}^n x_0\}$ is Cauchy. Since Y is complete, this means there is $u \in Y$ where $u = \lim_{n\to+\infty} \mathcal{T}^n x_0$. Therefore, we find that u is an FP of the mapping $\mathcal{T}$ by Lemma 4. $\square$

Remark 6. *One should remember that by putting $r = 1$ in Theorem 4, we obtain Theorem 3.1. from [19]. Moreover, by setting $r = 1$ and $s = 1$, we reach the classical Caristi theorem in MS (refer also to [27], Theorem 2.10).*

Example 1. *Let $Y = [0,1]$ and the functions $\mathcal{T} : Y \longrightarrow Y$, $\varphi : Y \longrightarrow [0,+\infty)$ and $\mathcal{D} : Y \times Y \longrightarrow [0,+\infty)$ defined by $\mathcal{T}x = x^2$, $\varphi(x) = \sqrt{x}$, $\mathcal{D}(x,\mathfrak{y}) = |x - \mathfrak{y}|$. Then, $(Y,\mathcal{D})$ is a metric space and we have*

$$\mathcal{D}(x,\mathcal{T}x) = |x - x^2| = (\sqrt{x} - x)(\sqrt{x} + x) \geq 2x(\sqrt{x} - x) > \sqrt{x} - x = \varphi(x) - \varphi(\mathcal{T}x),$$

for all $x \in (\frac{1}{2},1]$. Therefore, condition (4) is not fulfilled and we cannot apply Theorem 1. On the other hand, by putting $r = 2, s = 1$ in Theorem 4, we arrive at

$$\varphi(x) - \varphi(\mathcal{T}^2x) = \sqrt{x} - x^2 \geq x - x^2 = \mathcal{D}(x,\mathcal{T}x).$$

2.2. Light Version of Caristi's Theorem

Another version from Caristi's theorem in bMS, namely, the *light version of Caristi's theorem*, is the goal of this subsection.

Theorem 5. *Let $(Y,\mathcal{D},s)$ be a CbMS and $\mathcal{T} : Y \longrightarrow Y$, $\psi : Y \times Y \to [0,+\infty)$ be mappings and $\mathcal{T}$ weakly orbitally continuous mapping. If*

$$\mathcal{D}(x,\mathfrak{y}) \leq \psi(x,\mathfrak{y}) - h^{s-1}\psi(\mathcal{T}^r x,\mathcal{T}^r\mathfrak{y}), \tag{9}$$

for every $x,\mathfrak{y} \in Y$, where $r \in \mathbb{N}$ and $h > 1$, then $\mathcal{T}$ has a unique FP.

Proof. Suppose $\mathfrak{y} = Tx$ and $\varphi(x) = \psi(x,\mathcal{T}x)$, for all $x \in Y$. It follows from Theorem 4 that $\mathcal{T}$ has a FP $u \in Y$. If $\mathcal{T}v = v$ where $v \in Y$, then from (9), we attain

$$\mathcal{D}(u,v) \leq \psi(u,v) - h^{s-1}\psi(u,v) \leq 0,$$

which shows that $u = v$. $\square$

Example 2. *Let $Y = \mathbb{R}$, $\mathcal{D} : Y \times Y \longrightarrow [0,+\infty)$ be a b-metric on Y, defined by $\mathcal{D}(x,\mathfrak{y}) = |x - \mathfrak{y}|^2$. $\mathcal{T} : Y \longrightarrow Y$ is a weakly orbitally continuous contraction defined by $\mathcal{T}x = \frac{x}{2}$ and $\psi : Y \times Y \longrightarrow [0,+\infty)$ defined by $\psi(x,\mathfrak{y}) = 2|x - \mathfrak{y}|^2$. Then $(Y,\mathcal{D})$ is a CbMS when $s = 2$. Next, let us consider $h = 2$ and $r \in \mathbb{N}$. Then, we have*

$$\psi(x,\mathfrak{y}) - h^{s-1}\psi(\mathcal{T}^r x,\mathcal{T}^r\mathfrak{y}) = |x - \mathfrak{y}|^2(2 - \frac{1}{2^{2(r-1)}}) \geq \mathcal{D}(x,\mathfrak{y}).$$

Therefore, the conditions of Theorem 5 are fulfilled.

Remark 7. *Note that for the case $s = 1$ and $r = 1$ from Theorem 5, we obtain the results from Khojasteh et al. [16].*

2.3. On the Result of Pant et al. [25]

In this part, we will introduce the next theorem, as a version of Theorem 3 from [25]. We will not state the proof because it has the same proof as Theorem 4.

Theorem 6. *Let* $(\mathrm{Y}, \mathcal{D}, s)$ *be a CbMS. Let* $\mathcal{T} : \mathrm{Y} \longrightarrow \mathrm{Y}$ *and* $\psi : \mathrm{Y} \times \mathrm{Y} \longrightarrow [0, +\infty)$ *be mappings. If*

$$\mathcal{D}(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}) \leq \psi(\mathrm{x}, \mathfrak{y}) - h^{s-1}\psi(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}). \tag{10}$$

for all $\mathrm{x}, \mathfrak{y} \in \mathrm{Y}$, *where* $h > 1$, *this implies* $\mathcal{T}$ *has a unique FP, under one of the following conditions:*

(i) $\mathcal{T}$ *is weakly orbitally continuous;*
(ii) $\mathcal{T}$ *is orbitally continuous;*
(iii) $\mathcal{T}$ *is k-continuous.*

Here, we present a concrete example for the above Theorem, and we show that the conditions for Theorem 6 are satisfied.

Example 3. *Let* $\mathrm{Y} = [0, 1]$ *and* $\mathcal{D} : \mathrm{Y} \times \mathrm{Y} \longrightarrow [0, +\infty)$ *be a b-metric on* Y, *defined by* $\mathcal{D}(\mathrm{x}, \mathfrak{y}) = |\mathrm{x} - \mathfrak{y}|^2$, $\mathcal{T} : \mathrm{Y} \longrightarrow \mathrm{Y}$ *is a contraction, defined by* $\mathcal{T}\mathrm{x} = \frac{\mathrm{x}}{3}$, $\psi : \mathrm{Y} \times \mathrm{Y} \longrightarrow [0, +\infty)$ *is a function defined by* $\psi(\mathrm{x}, \mathfrak{y}) = \frac{(\mathrm{x}+\mathfrak{y})^2}{2}$. *Obviously,* $(\mathcal{X}, \mathcal{D}, 2)$ *is a complete b-metric space. Let* $h = 2$. *We obtain*

$$(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}) = |\mathcal{T}\mathrm{x} - \mathcal{T}\mathfrak{y}|^2 = \frac{|\mathrm{x} - \mathfrak{y}|^2}{9}.$$

On the other side,

$$\psi(\mathrm{x}, \mathfrak{y}) - h^{s-1}\psi(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}) = \frac{7(\mathrm{x} + \mathfrak{y})^2}{18}.$$

When

$$\frac{|\mathrm{x} - \mathfrak{y}|^2}{9} \leq \frac{7(\mathrm{x} + \mathfrak{y})^2}{18},$$

for all $\mathrm{x}, \mathfrak{y} \in [0, 1]$, *we deduce that the conditions for Theorem 6 are met.*

From Theorem 6, we realize following corollary.

Corollary 1. *Let* $(\mathrm{Y}, \mathcal{D}, s)$ *be a complete b-metric space and* $\mathcal{T} : \mathrm{Y} \longrightarrow \mathrm{Y}$, *and let* $\varphi_i : \mathrm{Y} \to [0, +\infty)$, $i = 1, 2$ *be mappings such that* $\mathcal{T}$ *is weakly orbitally continuous. If*

$$\mathcal{D}(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}) \leq \varphi_1(\mathrm{x}) - h^{s-1}\varphi_1(\mathcal{T}\mathrm{x}) + \varphi_2(\mathfrak{y}) - h^{s-1}\varphi_2(\mathcal{T}\mathfrak{y}). \tag{11}$$

for each $\mathrm{x}, \mathfrak{y} \in \mathrm{Y}$, *such that* $h > 1$, *then* $\mathcal{T}$ *has a unique FP.*

Proof. Putting $\psi(\mathrm{x}, \mathfrak{y}) = \varphi_1(\mathrm{x}) + \varphi_2(\mathfrak{y}), \mathrm{x}, \mathfrak{y} \in \mathrm{Y}$ in Theorem 6, we obtain the proof. □

Remark 8. *If* $\varphi_1 = \varphi_2$ *and* $s = 1$ *from Corollary 1, we obtain Theorem 3.*

2.4. On the Result of Karapınar et al. [15]

We first modify Theorem 1 given by Karapınar et al. [15] in the *bMS* setting as follows.

Theorem 7. *Let* $(\mathrm{Y}, \mathcal{D}, s)$ *be a complete bMS,* $\mathcal{T}, \mathcal{I} : \mathrm{Y} \longrightarrow \mathrm{Y}$, *and let* $\psi : \mathrm{Y} \times \mathrm{Y} \longrightarrow \mathbb{R}$ *be mappings such that:*

(a) $\inf_{x,y \in X} \psi(x, y) > -\infty$;
(b) $\mathcal{T}(\mathcal{I}\mathrm{x}) = \mathcal{I}(\mathcal{T}\mathrm{x})$, *for all* $\mathrm{x} \in \mathrm{Y}$;
(c) *the range of* $\mathcal{I}$ *contains the range of* $\mathcal{T}$;
(d) $\mathcal{I}$ *is continuous;*
(e)

$$\mathcal{D}(\mathcal{I}\mathrm{x}, \mathcal{T}\mathrm{x}) > 0 \text{ implies } d(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}) \leq (\psi(\mathcal{I}\mathrm{x}, \mathcal{I}\mathfrak{y}) - \psi(\mathcal{T}\mathrm{x}, \mathcal{T}\mathfrak{y}))\mathcal{D}(\mathcal{I}\mathrm{x}, \mathcal{I}\mathfrak{y}), \tag{12}$$

for all $\mathrm{x}, \mathfrak{y} \in \mathrm{Y}$.

Then, $\mathcal{T}$ and $\mathcal{I}$ have a coincidence point, which means there exists $u \in \mathrm{Y}$ where $\mathcal{T}u = \mathcal{I}u$.

Proof. Let $\mathrm{x}_0 \in \mathrm{Y}$. Since $\mathcal{T}\mathrm{x}_0 \in \mathcal{I}(\mathrm{Y})$, there is an $\mathrm{x}_1 \in \mathrm{Y}$ such that $\mathcal{I}\mathrm{x}_1 = \mathcal{T}\mathrm{x}_0$. Similarly, for any given $\mathrm{x}_n \in \mathrm{Y}$, there is $\mathrm{x}_{n+1} \in \mathrm{Y}$ such that $\mathcal{I}\mathrm{x}_{n+1} = \mathcal{T}\mathrm{x}_n$. If $\mathcal{D}(\mathcal{I}\mathrm{x}_n, \mathcal{T}\mathrm{x}_n) = 0$ for some $n \in \mathbb{N}$, then x_n is a coincidence point. Suppose that

$$\mathcal{D}(\mathcal{I}\mathrm{x}_n, \mathcal{T}\mathrm{x}_n) > 0, \tag{13}$$

for each $n \in \mathbb{N}$. From (12), we obtain

$$\begin{aligned} \mathcal{D}(\mathcal{T}\mathrm{x}_{n+1}, \mathcal{T}\mathrm{x}_n) &\leq (\psi(\mathcal{I}\mathrm{x}_{n+1}, \mathcal{I}\mathrm{x}_n) - \psi(\mathcal{T}\mathrm{x}_{n+1}, \mathcal{T}\mathrm{x}_n))\mathcal{D}(\mathcal{I}\mathrm{x}_{n+1}, \mathcal{I}\mathrm{x}_n) \\ &= (\psi(\mathcal{I}\mathrm{x}_{n+1}, \mathcal{I}\mathrm{x}_n) - \psi(\mathcal{I}\mathrm{x}_{n+2}, \mathcal{I}\mathrm{x}_{n+1}))\mathcal{D}(\mathcal{I}\mathrm{x}_{n+1}, \mathcal{I}\mathrm{x}_n). \end{aligned}$$

Hence,

$$\mathcal{D}(\mathcal{I}\mathrm{x}_{n+2}, \mathcal{I}\mathrm{x}_{n+1}) \leq (\psi(\mathcal{I}\mathrm{x}_{n+1}, \mathcal{I}\mathrm{x}_n) - \psi(\mathcal{I}\mathrm{x}_{n+2}, \mathcal{I}\mathrm{x}_{n+1}))\mathcal{D}(\mathcal{I}\mathrm{x}_{n+1}, \mathcal{I}\mathrm{x}_n), \tag{14}$$

for all $n \in \mathbb{N}$. Recalling condition (13), from (14) we have

$$\frac{\mathcal{D}(\mathcal{I}\mathrm{x}_{n+2}, \mathcal{I}\mathrm{x}_{n+1})}{\mathcal{D}(\mathcal{I}\mathrm{x}_{n+1}, \mathcal{I}\mathrm{x}_n)} \leq \psi(\mathcal{I}\mathrm{x}_{n+1}, \mathcal{I}\mathrm{x}_n) - \psi(\mathcal{I}\mathrm{x}_{n+2}, \mathcal{I}\mathrm{x}_{n+1}), \tag{15}$$

for each $n \in \mathbb{N}$. From inequality (15) and condition (a), we obtain

$$\sum_{j=1}^{n} \frac{\mathcal{D}(\mathcal{I}\mathrm{x}_{j+2}, \mathcal{I}\mathrm{x}_{j+1})}{\mathcal{D}(\mathcal{I}\mathrm{x}_{j+1}, \mathcal{I}\mathrm{x}_j)} < +\infty. \tag{16}$$

Therefore, the series $\sum_{j=1}^{+\infty} \frac{\mathcal{D}(\mathcal{I}\mathrm{x}_{j+2}, \mathcal{I}\mathrm{x}_{j+1})}{\mathcal{D}(\mathcal{I}\mathrm{x}_{j+1}, \mathcal{I}\mathrm{x}_j)}$ converges and

$$\lim_{j \to +\infty} \frac{\mathcal{D}(\mathcal{I}\mathrm{x}_{j+2}, \mathcal{I}\mathrm{x}_{j+1})}{\mathcal{D}(\mathcal{I}\mathrm{x}_{j+1}, \mathcal{I}\mathrm{x}_j)} = 0. \tag{17}$$

From (17), we conclude that for $k \in (0,1)$, there exists $n_0 \in \mathbb{N}$ where

$$\mathcal{D}(\mathcal{I}\mathrm{x}_{j+2}, \mathcal{I}\mathrm{x}_{j+1}) \leq k\mathcal{D}(\mathcal{I}\mathrm{x}_{j+1}, \mathcal{I}\mathrm{x}_j), \tag{18}$$

for all $j \geq n_0$. Now, by applying Lemma 2, the sequence $\mathcal{I}\mathrm{x}_n$ is Cauchy. Let

$$u = \lim_{n \to +\infty} \mathcal{I}\mathrm{x}_n = \lim_{n \to +\infty} \mathcal{T}\mathrm{x}_{n-1}. \tag{19}$$

While $\mathcal{I}$ is continuous, (12) leads to both $\mathcal{I}$ and $\mathcal{T}$ being continuous. On the other hand, $\mathcal{T}$ and $\mathcal{I}$ commute and thus

$$\mathcal{I}u = \mathcal{I}(\lim_{n \to +\infty} \mathcal{T}\mathrm{x}_n) = \lim_{n \to +\infty} \mathcal{I}\mathcal{T}\mathrm{x}_n = \lim_{n \to +\infty} \mathcal{T}\mathcal{I}\mathrm{x}_n = T(\lim_{n \to +\infty} \mathcal{I}\mathrm{x}_n) = \mathcal{T}u. \tag{20}$$

As a result, u is a coincidence point for $\mathcal{T}$ and $\mathcal{I}$. □

Corollary 2. *Suppose $(\mathrm{Y}, \mathcal{D}, s)$ is a CbMS. Let $\mathcal{T} : \mathrm{Y} \longrightarrow \mathrm{Y}$ and $\varphi : \mathrm{Y} \times \mathrm{Y} \longrightarrow \mathbb{R}$ be mappings where $\inf_{x \in \mathrm{Y}} \varphi(x) > -\infty$. If*

$$d(x, Tx) > 0 \textit{ reveals } d(Tx, Ty) \leq (\varphi(x) - \varphi(Tx))d(x, y), \tag{21}$$

this means that T has an FP.

Proof. Put $Ix = x$ and $\psi(x, y) = \varphi(x)$ by Theorem 7. □

Remark 9. *Note that Corollary 2 improves Theorem 1 from [15] to the class of bMS.*

3. Conclusions

The importance of the results obtained here is reflected in the fact that we have improved some known results in the fixed-point theory and demonstrated this validated by the examples presented. On the other hand, the results obtained in metric spaces were obtained in the broad class of spaces in b-metric spaces. A natural question is whether these results can be obtained for some wider classes of spaces such as rectangular b-metric spaces [10], $b_v(s)$-metric spaces [20], orthogonal b-metric-like spaces [29] and modular spaces [30].

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