mathematics

MDPI

Article

Fuzzy Portfolio Selection in the Risk Attitudes of Dimension Analysis under the Adjustable Security Proportions

Kuen-Suan Chen [1,2,3], Yin-Yin Huang [4], Ruey-Chyn Tsaur [5,*] and Nei-Yu Lin [5]

1 Department of Industrial Engineering and Management, National Chin-Yi University of Technology, Taichung 411030, Taiwan
2 Department of Business Administration, Chaoyang University of Technology, Taichung 413310, Taiwan
3 Department of Business Administration, Asia University, Taichung 413305, Taiwan
4 School of Economics and Management, Nanchang Vocational University, 308 Provincial Road, Anyi County, Nanchang 330500, China
5 Department of Management Sciences, Tamkang University, New Taipei City 25137, Taiwan
* Correspondence: rctsaur@yahoo.com.tw

Abstract: Fuzzy portfolio models have received many researchers' focus on the issue of risk preferences. The portfolio based on guaranteed return rates has been developing and considering the dimension of excess investment for the investors in different risk preferences. However, not only excess investment but also shortage investment to the selected portfolio should be considered for risk preferences, including risk-seeking, risk-neutral, and risk-averse, by different degrees of dimensions in excess investment and shortage investment. A comparison to the degree of dimensions for the excess investment and shortage investment indicates that a risk-seeker would like to have excess investment for securities whose return rates are bigger than the guaranteed return rates and shortage investment for securities whose return rates are smaller than the guaranteed return rates. Finally, we present three experiments to illustrate the proposed model. The results show that the different risk preferences derive different fuzzy portfolio selections under s and t dimensions, where a lower value of s is suggested for a risk-seeker as $t > s$, and we suggest the values of s and t to be smaller than or equal to 3. By contrast, for the risk-neutral investor, we suggest $s = t$; $t < s$ is suggested to the investor who is risk-averse.

Keywords: fuzzy portfolio selection; dimension of shortage investment; dimension of excess investment; guaranteed return rate; adjustable security proportion

MSC: 90B50; 90B60

check for updates

Citation: Chen, K.-S.; Huang, Y.-Y.; Tsaur, R.-C.; Lin, N.-Y. Fuzzy Portfolio Selection in the Risk Attitudes of Dimension Analysis under the Adjustable Security Proportions. *Mathematics* **2023**, *11*, 1143. https://doi.org/10.3390/math11051143

Academic Editor: Aleksandar Aleksic

Received: 18 January 2023
Revised: 17 February 2023
Accepted: 22 February 2023
Published: 24 February 2023

1. Introduction

Portfolio selection is used to find the combinations of assets, which are used to optimize the objectives of an investor with respect to maximizing the expected return under the constrained risk. The foundation of portfolio selection was laid by Markowitz [1] who proposed the mean-variance model and considered asset returns as random variables in the multi-variate normal distribution. Most of the researchers have devoted themselves to solve some criticisms of the original portfolio models, and then some of the rigid assumptions of Markowitz's model are relaxed to deal with different investment environments or challenges, including the models in mean-absolute deviation, value at risk, conditional value at risk, or semi-variance, which are with respect to portfolio selections [2–8].

In most of the asset markets, we cannot just assume the factors affecting the market are random variables. In order to solve the portfolio selection, the factors which are other than randomness are usually applied in the possibility theory. Then, fuzzy portfolio selection is proposed to consider the knowledge of experts, investors' subjective opinions, or a quantitative and qualitative analysis in portfolio selection problems. For example, the perceived risk of an investor can be shown in different degrees of linguistic descriptions, and then the constrained risk for the portfolio selection cannot be performed by probability distribution. In addition, the investment behaviors to new economy events cannot be precisely evaluated by the previous return rates for the selected securities, because a lot of factors cannot be considered in the portfolio selection, and thus fuzzy portfolio models are another kind of possible method for solving non-probabilistic portfolio selection. Numerous researchers have the objective of maximization of the fuzzy return rates and constrained the upper investment risk using possibility theory, which were modelled and studied for portfolio selection [9–12]. Thereafter, most researchers have focused on the multi-period fuzzy optimization problems for solving the multi-objective problems by genetic algorithm and neural networks [13–15]. Without the self-dual property in the possibility measures, some researchers extended the credibility measures for the uncertain portfolio selection [16–18].

The major studies in fuzzy portfolio models are summarized as possibility or credibility theories to optimal decisions in a single-period or multi-period fuzzy portfolio selection. With respect to portfolio selection, the habitual behaviors of an investor in the field of risk analysis are also important in the vagueness environment. For example, Mehlawat et al. [19] proposed multi-objective risk measures and evaluate the fuzzy portfolio selection. Yue and Wang [20] used the entropy method to formulate a weighted possibility fuzzy multi-objective and higher order moment portfolio model with the efficiency and effectiveness portfolio selections. Guo et al. [21] considered the capital gain tax to fuzzy portfolio selection and formulated a bi-objective mean-variance model solving by an algorithm in time-varying numerical integral-based particle swarm optimization. Li et al. [22] used a skewness fuzzy variable to formulate a mean-variance-skewness fuzzy portfolio selection, by designing the genetic algorithm and fuzzy simulation technique to show the effective algorithm. Zhou and Xu [23] proposed fuzzy portfolio selection for solving qualitative information represented as hesitant fuzzy elements where both the max-score rule and score-deviation trade-off rule were used to distinguish three types of risk behaviors for the investors. It is important to notice that the risk behavior analysis for an investor is an interesting topic in the research field of fuzzy portfolio selections [24].

In fuzzy portfolio selection, we cannot only use a mean-variance model to reflect the measure of risk; by contrast, we also need to consider the will and behavior of an investor with different risk types in the portfolio selection. Lower returns are equivalent to lower risks, in which investors seldom make significant profits from those securities, and thus most investment behaviors intend to make shortage investment for these securities. By contrast, higher risks are equivalent to higher returns where most investors can realize unexpected returns from those securities, and thus most investors would like to make excess investment to those higher risk securities. Based on the concept of risk behavior of an investor, Tsaur et al. [25] proposed the guaranteed return rate to be the threshold of excess investment for each security in portfolio selection, and then Chen et al. [26] revised the model [25] based on the risk behavior of an investor in a different dimension distance between the guaranteed return rate and return rate for each security. However, models [25,26] just consider the risk behavior of an investor in the excess investment. Not only excess investment but also shortage investment should be considered to the securities. Huang et al. [27] proposed the adjustable security proportion for excess investment and shortage investment based on the selected guaranteed return rates for profitable returns, where the mean-variance model was applied for portfolio selection, whereas the risk behavior of an investor in a different dimension distance for shortage investment and excess investment was still not considered. Therefore, we suppose that if an investor prefers to risk, then his degree of intention in excess investment is higher than the degree of shortage investment; if an investor is averse to risk, then his degree of intention in excess investment is lower than the degree of shortage investment; if an investor is neutral to risk, then his degree of intention in excess investment is equivalent to the degree of shortage investment. The research gap of this study is planned to overcome the degree of risk preference investment in the adjustable security proportion, and then we can consider the dimensions of excess investment and shortage investment by the risk attitudes of an investor.

The organization of this article is as follows. In Section 2, we introduce the definition of fuzzy numbers and their operations. Section 3 proposes the dimensional analysis to the adjustable security proportion in the fuzzy portfolio model. In Section 4, an illustration is presented by the proposed model. Finally, conclusions are discussed in Section 5.

2. Preliminaries

In this section, the fuzzy numbers with fundamental algebraic operations and their defuzzification, fuzzy expected values, and fuzzy variances are introduced and defined, and then Section 3 can be easily understood. A fuzzy set $\widetilde{A}$ is characterized by a membership function defined as $u_{\widetilde{A}}(x) : X \to [0,\ 1]$, which maps the elements of the universe of discourse X to the interval [0, 1]. Therefore, we define a fuzzy number as follows.

Definition 1 ([28]). *Let $\widetilde{A}$ be a fuzzy number as any fuzzy subset of the real line R with a membership function $u_{\widetilde{A}}(x) : R \to [0,\ 1]$ satisfying the following conditions:*

(1) The fuzzy number $\widetilde{A}$ is normal, if there exists an $x \in R$ with $u_{\widetilde{A}}(x) = 1$;
(2) $u_{\widetilde{A}}(x)$ is convex, i.e., $u_{\widetilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{u_{\widetilde{A}}(x), u_{\widetilde{A}}(y)\}$, $\forall\, x, y \in R$ and $\lambda \in [0, 1]$;
(3) $u_{\widetilde{A}}(x)$ is upper semicontinuous, i.e., $\{ x \in R: u_{\widetilde{A}}(x) \geq \alpha\} = \widetilde{A}^{\alpha}$ is a closed subset of U for each $\alpha \in (0, 1]$;
(4) The closure of the set $\{x \in R: u_{\widetilde{A}}(x) \geq 0\}$ is a compact subset of R.

Definition 2 ([28]). *A fuzzy number $\widetilde{A}$ is defined as LR-type fuzzy number as $\widetilde{A} = (a,\ c_1,\ c_2)_{LR}$, then the membership function of $\widetilde{A} = (a,\ c_1,\ c_2)_{LR}$ has the following form:*

$$u_{\widetilde{A}}(x) = \begin{cases} L\left(\frac{a-x}{c_1}\right), & if\ x < a \\ 1, & if\ x = a \\ R\left(\frac{x-a}{c_2}\right), & if\ x > a \end{cases}$$

where a is the central value, and c_1 and c_2 are the left and right spread values.

Let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy numbers of the LR-type defined as $\widetilde{A} = (a,\ c_1,\ c_2)_{LR}$ and $\widetilde{B} = (b,\ d_1,\ d_2)_{LR}$, where a and b are the central values, c_1 and d_1 are the left spread values, and c_2 and d_2 are the right spread values of $\widetilde{A}$ and $\widetilde{B}$, respectively. Then,

$$\widetilde{A} + \widetilde{B} = (a,\ c_1,\ c_2)_{LR} + (b,\ d_1,\ d_2)_{LR} = (a+b,\ c_1+d_1,\ c_2+d_2)_{LR}$$

$$\widetilde{A} - \widetilde{B} = (a,\ c_1,\ c_2)_{LR} - (b,\ d_1,\ d_2)_{LR} = (a-b,\ c_1+d_2,\ c_2+d_1)_{LR}$$

Next, the multiplication of both positive fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ can be derived as

$$\widetilde{A} \bigotimes \widetilde{B} = (a,\ c_1,\ c_2)_{LR} \bigotimes (b,\ d_1,\ d_2)_{LR} = (ab,\ ad_1 + bc_1,\ ad_2 + bc_2)_{LR}$$

Theorem 1 ([29]). *Let $\widetilde{A}$ be a fuzzy number with differentiable membership function with α-level set $\widetilde{A}^{\alpha} = \left\{ x\ \ R:\ u_{\widetilde{A}}(x) \geq \alpha \right\} = [a_1(\alpha),\ a_2(\alpha)]$, $0 \leq \alpha \leq 1$ The lower possibilistic mean value of fuzzy number is defined as $M_*\left(\widetilde{A}\right) = 2\int_0^1 \alpha \cdot a_1(\alpha) d\alpha$, and the upper possibilistic mean value of fuzzy number is defined as $M^*\left(\widetilde{A}\right) = 2\int_0^1 \alpha \cdot a_2(\alpha) d\alpha$. Then, the expected value of a fuzzy number $\widetilde{A}$ is expressed as $M\left(\widetilde{A}\right) = \int_0^1 \alpha \cdot [a_1(\alpha) + a_2(\alpha)] d\alpha$.*

By Theorem 1, the lower possibilistic mean and upper possibilistic mean values for $\widetilde{A}+\widetilde{B}$ can be obtained as Equations (1) and (2) as follows:

$$M_*\left(\widetilde{A}+\widetilde{B}\right)=M_*\left(\widetilde{A}\right)+M_*\left(\widetilde{B}\right) \tag{1}$$

$$M^*\left(\widetilde{A}+\widetilde{B}\right)=M^*\left(\widetilde{A}\right)+M^*\left(\widetilde{B}\right) \tag{2}$$

Then, the sum of possibilistic mean value of $\widetilde{A}$ and $\widetilde{B}$ are obtained as follows:

$$M\left(\widetilde{A}+\widetilde{B}\right)=\frac{M_*\left(\widetilde{A}+\widetilde{B}\right)+M^*\left(\widetilde{A}+\widetilde{B}\right)}{2} \tag{3}$$

Next, the lower and upper possibilistic variances of $\widetilde{A}$ are defined as Equations (4) and (5), respectively [29].

$$Var_*\left(\widetilde{A}\right)=2\int_0^1\alpha\left[M_*\left(\widetilde{A}\right)-a_1(\alpha)\right]^2d\alpha \tag{4}$$

$$Var^*\left(\widetilde{A}\right)=2\int_0^1\alpha\left[M^*\left(\widetilde{A}\right)-a_2(\alpha)\right]^2d\alpha \tag{5}$$

In addition, for ranking the return rate of each security to the guarantee return rate, we use a popular ranking method for fuzzy numbers described as follows:

Theorem 2 ([30])**.** *Let $\widetilde{A}=(a,\ c_1,\ c_2)$ and $\widetilde{B}=(b,\ d_1,\ d_2)$ be triangular fuzzy numbers, the central values be a and b, and the left and right spread values be $c_1,\ c_2$, and $d_1,\ d_2$; then, we define the circumcenter of $\widetilde{A}$ as $S_{\widetilde{A}}=(\overline{x}_0,\ \overline{y}_0)=\left(\frac{6a+(c_2-c_1)}{6},\ \frac{5-c_2c_1}{12}\right)$. The ranking function $R\left(\widetilde{A}\right)$ which maps $\widetilde{A}$ to a real number can be derived as $R\left(\widetilde{A}\right)=\sqrt{(\overline{x}_0)^2+(\overline{y}_0)^2}$. If the ranking value $R\left(\widetilde{A}\right)$ is bigger than $R\left(\widetilde{B}\right)$, then the fuzzy number $\widetilde{A}$ is bigger than fuzzy number $\widetilde{B}$.*

3. The Dimension Risk Analysis in Adjustable Security Proportions

Under the vagueness environment, the fuzzy portfolio model is used to solve the optimal investment proportion for each asset under the maximizing expected return with constrained risk. By considering the s dimension of excess investment and t dimension of shortage investment, we can formulate the fuzzy portfolio model as follows. First, for security j with investment proportion x_j, we define its return rate to be the triangular fuzzy number as $\widetilde{r}_j=\left(r_j,\ c_j,\ d_j\right)$, where r_j is the central value; and c_j, d_j are left and right spreads, $j = 1, \ldots, n$, respectively; and then the expected fuzzy return rate is defined as $\widetilde{R}=\sum_{j=1}^{n}x_j\widetilde{r}_j$. In this study, considering the adjustable security proportion in the fuzzy portfolio selection, the degrees of risk preference for the dimension of excess investment and shortage investment between the selected guaranteed return rates and the security return rates are different to different investors. Second, we rank the n securities to their fuzzy return rates and they are assumed as the ordering of the fuzzy return rates as $\widetilde{r}_1<\widetilde{r}_2<\ldots<\widetilde{r}_n$, in which excess investment for the m securities ($m\leq n$) and the other securities are made a shortage investment based on the selected guaranteed return rate defined as $\widetilde{p}_k=(p_k,\ e_k,\ f_k)$, where p_k is its central value, and $e_k,\ f_k$ are its left and right spread values, respectively. By considering the risk preference of the investor with s dimension of excess investment and t dimension of shortage investment in the expected fuzzy return rate, the following is proposed:

$$\widetilde{R}=\sum_{j=1}^{n}x_j\widetilde{r}_j-\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left|\widetilde{r}_j-\widetilde{p}_k\right|^t+\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left|\widetilde{r}_j-\widetilde{p}_k\right|^s,\ s,t\geq 1 \tag{6}$$

If the fuzzy return rate $\widetilde{r}_j$, $j = 1, \ldots, n$, is larger than $\widetilde{p}_k$ and $R(\widetilde{r}_j) > R(\widetilde{p}_k)$ [30], then we can make an excess investment on security j; otherwise, we can make a shortage investment. The s dimension of excess investment and the t dimension of shortage investment can be formulated as follows:

$$\left|\widetilde{r}_j - \widetilde{p}_k\right|^t = \begin{cases} (\widetilde{p}_k - \widetilde{r}_j)^t & if\ R(\widetilde{p}_k) > R(\widetilde{r}_j) \\ 0 & otherwise \end{cases} \tag{7}$$

$$\left|\widetilde{r}_j - \widetilde{p}_k\right|^s = \begin{cases} (\widetilde{r}_j - \widetilde{p}_k)^s & if\ R(\widetilde{r}_j) > R(\widetilde{p}_k) \\ 0 & otherwise \end{cases} \tag{8}$$

Next, the lower and upper possibilistic mean values for the s dimension excess investment $(\widetilde{r}_j - \widetilde{p}_k)^s$ and t dimension shortage investment $(\widetilde{p}_k - \widetilde{r}_j)^t$ are derived as $M_*(\widetilde{r}_j - \widetilde{p}_k)^s$, $M^*(\widetilde{r}_j - \widetilde{p}_k)^s$, and $M_*(\widetilde{p}_k - \widetilde{r}_j)^t$, $M^*(\widetilde{p}_k - \widetilde{r}_j)^t$, $\forall\, k = 1, 2, \ldots, m$, as follows:

$$M_*(\widetilde{p}_k - \widetilde{r}_j)^t = (p_k - r_j)^t - \frac{1}{3}s(p_k - r_j)^{t-1}(c_j + f_k) \tag{9}$$

$$M^*(\widetilde{p}_k - \widetilde{r}_j)^t = (p_k - r_j)^s + \frac{1}{3}s(p_k - r_j)^{s-1}(d_j + e_k) \tag{10}$$

$$M_*(\widetilde{r}_j - \widetilde{p}_k)^s = (r_j - p_k)^s - \frac{1}{3}s(r_j - p_k)^{s-1}(c_j + f_k) \tag{11}$$

$$M^*(\widetilde{r}_j - \widetilde{p}_k)^s = (r_j - p_k)^s + \frac{1}{3}s(r_j - p_k)^{s-1}(d_j + e_k) \tag{12}$$

where $(\widetilde{p}_k - \widetilde{r}_j)^t = \left[(p_k - r_j)^t, t(p_k - r_j)^{t-1}(c_j + f_k), t(p_k - r_j)^{t-1}(d_j + e_k)\right]$ whose α-level set is defined as $\left[(\widetilde{p}_k - \widetilde{r}_j)^t\right]^\alpha = \left[(\widetilde{p}_k - \widetilde{r}_j)^t_{j1}(\alpha),\ (\widetilde{p}_k - \widetilde{r}_j)^t_{j2}(\alpha)\right]$ for all $\alpha \in [0,\ 1]$. $(\widetilde{r}_j - \widetilde{p}_k)^s = \left[(r_j - p_k)^s, s(r_j - p_k)^{s-1}(c_j + f_k), s(r_j - p_k)^{s-1}(d_j + e_k)\right]$ whose α-level set is defined as $\left[(\widetilde{r}_j - \widetilde{p}_k)^s\right]^\alpha = \left[(\widetilde{r}_j - \widetilde{p}_k)^s_{j1}(\alpha),\ (\widetilde{r}_j - \widetilde{p}_k)^s_{j2}(\alpha)\right]$, $\alpha \in [0,\ 1]$. Then, the expected possibilistic mean values $M(\widetilde{p}_k - \widetilde{r}_j)^t$ and $M(\widetilde{r}_j - \widetilde{p}_k)^s$ are obtained as follows:

$$M(\widetilde{p}_k - \widetilde{r}_j)^t = (p_k - r_j)^t + \frac{1}{6}t(p_k - r_j)^{t-1}\left[(d_j + e_k) - (c_j + f_k)\right] \tag{13}$$

$$M(\widetilde{r}_j - \widetilde{p}_k)^s = (r_j - p_k)^s + \frac{1}{6}s(r_j - p_k)^{s-1}\left[(d_j + e_k) - (c_j + f_k)\right] \tag{14}$$

On the other hand, the expected possibilistic mean value for the proposed fuzzy return rate in Equation (6) can be obtained as follows:

$$\begin{aligned} M&\left[\sum_{j=1}^{n} x_j\widetilde{r}_j - \sum_{j=1}^{n1}\sum_{k=1}^{m} x_j(\widetilde{p}_k - \widetilde{r}_j)^t + \sum_{j=\ n1+1}^{n}\sum_{k=1}^{m} x_j(\widetilde{r}_j - \widetilde{p}_k)^s\right] \\ &= \sum_{j=1}^{n} x_j M(\widetilde{r}_j) - \sum_{j=1}^{n}\sum_{k=1}^{m} x_j M(\widetilde{p}_k - \widetilde{r}_j)^t + \sum_{j=1}^{n}\sum_{k=1}^{m} x_j M(\widetilde{r}_j - \widetilde{p}_k)^s \\ &= \sum_{j=1}^{n} x_j\left[r_j + \tfrac{1}{6}(d_j - c_j)\right] \\ &- \sum_{j=1}^{n1}\sum_{k=1}^{m} x_j\left[(p_k - r_j)^t + \tfrac{1}{6}t(p_k - r_j)^{t-1}\left[(d_j + e_k) - (c_j + f_k)\right]\right] \\ &+ \sum_{j=n1+1}^{n}\sum_{k=1}^{m} x_j\left[(r_j - p_k)^s + \tfrac{1}{6}s(r_j - p_k)^{s-1}\left[(d_j + e_k) - (c_j + f_k)\right]\right] \end{aligned} \tag{15}$$

Then, we can obtain the lower and upper possibilistic variances of the proposed fuzzy return rates shown in Equation (6) as follows:

$$Var_*\left[\sum_{j=1}^{n}x_j\widetilde{r}_j-\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(\widetilde{p}_k-\widetilde{r}_j\right)^t+\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(\widetilde{r}_j-\widetilde{p}_k\right)^s\right]$$
$$=\frac{1}{18}\left[\sum_{j=1}^{n}c_jx_j-t\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(p_k-r_j\right)^{t-1}\left(c_j+f_k\right)+s\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(r_j-p_k\right)^{s-1}\left(c_j+f_k\right)\right]^2 \quad (16)$$

$$Var^*\left[\sum_{j=1}^{n}x_j\widetilde{r}_j-\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(\widetilde{p}_k-\widetilde{r}_j\right)^t+\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(\widetilde{r}_j-\widetilde{p}_k\right)^s\right]$$
$$=\frac{1}{18}\left[\sum_{j=1}^{n}d_jx_j-t\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(p_k-r_j\right)^{t-1}\left(d_j+e_k\right)+s\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(r_j-p_k\right)^{s-1}\left(d_j+e_k\right)\right]^2 \quad (17)$$

The standard deviation of the proposed fuzzy return rates shown in Equation (6) can be obtained as follows:

$$\begin{aligned}
&SD\left[\sum_{j=1}^{n}x_j\widetilde{r}_j-\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(\widetilde{p}_k-\widetilde{r}_j\right)^t+\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(\widetilde{r}_j-\widetilde{p}_k\right)^s\right]\\
&=\frac{1}{2}\left\{\left\{Var_*\left[\sum_{j=1}^{n}x_j\widetilde{r}_j-\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(\widetilde{p}_k-\widetilde{r}_j\right)^t+\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(\widetilde{r}_j-\widetilde{p}_k\right)^s\right]\right\}^{1/2}\right.\\
&\left.+\left\{Var^*\left[\sum_{j=1}^{n}x_j\widetilde{r}_j-\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(\widetilde{p}_k-\widetilde{r}_j\right)^t+\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(\widetilde{r}_j-\widetilde{p}_k\right)^s\right]\right\}^{1/2}\right\}\\
&=\frac{1}{6\sqrt{2}}\left[\sum_{j=1}^{n}\left(c_j+d_j\right)x_j-t\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(p_k-r_j\right)^{t-1}\left(c_j+f_k+d_j+e_k\right)\right.\\
&\left.+s\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(r_j-p_k\right)^{s-1}\left(c_j+f_k+d_j+e_k\right)\right]
\end{aligned} \quad (18)$$

The fuzzy portfolio model with s dimension in the excess investment and t dimension in the shortage investment can be formulated as a linear programming model whose objective function is shown in Equation (15), and the constrained risk by the upper bound of an investor's desired value is shown as in Equation (18). Therefore, the proposed possibilistic mean-standard deviation model of portfolio selection in considering the concept of s dimension excess investment and t dimension in shortage investment is obtained as follows:

$$\begin{aligned}
&Max\ \sum_{j=1}^{n}x_j\left[r_j+\frac{1}{6}\left(d_j-c_j\right)\right]-\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left[\left(p_k-r_j\right)^t+\frac{1}{6}t\left(p_k-r_j\right)^{t-1}\left[\left(d_j+e_k\right)-\left(c_j+f_k\right)\right]\right]\\
&+\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left[\left(r_j-p_k\right)^s+\frac{1}{6}s\left(r_j-p_k\right)^{s-1}\left[\left(d_j+e_k\right)-\left(c_j+f_k\right)\right]\right]\\
&s.t.\ \frac{1}{6\sqrt{2}}\left[\sum_{j=1}^{n}\left(c_j+d_j\right)x_j-t\sum_{j=1}^{n1}\sum_{k=1}^{m}x_j\left(p_k-r_j\right)^{t-1}\left(c_j+f_k+d_j+e_k\right)\right.\\
&\left.+s\sum_{j=n1+1}^{n}\sum_{k=1}^{m}x_j\left(r_j-p_k\right)^{s-1}\left(c_j+f_k+d_j+e_k\right)\right]\leq\sigma\\
&\sum_{j=1}^{n}x_j=1\\
&l_j\leq x_j\leq u_j,\ j=1,\ 2,\ldots,n
\end{aligned} \quad (19)$$

where the lower and upper bounds on the proportion of security j are defined as l_j and u_j, respectively. In addition, $r_j < p_k$ when jth security is the shortage investment; therefore, a bigger dimension of t implies a smaller value of $\left(p_k-r_j\right)^t$. Furthermore, $r_j > p_k$ when jth security is the excess investment; therefore, a bigger dimension of s implies a smaller value of $\left(r_j - p_k\right)^s$. Therefore, the bigger value of s or smaller value of t will derive a smaller objective value of the model (19).

4. Illustrations

4.1. Data Description and Model Explanation

In this study, we use the collected data from April 2002 to January 2004 in Shanghai Stock Exchange, which are the closed prices for each week [31]. By the companies' information offered in the financial reports, there are five securities chosen to formulate the proposed model. The fuzzy return rates for the securities are estimated as $\widetilde{r}_1 = (0.073, 0.054, 0.087)$, $\widetilde{r}_2 = (0.\ 105, 0.075, 0.102)$, $\widetilde{r}_3 = (0.138, 0.096, 0.123)$, $\widetilde{r}_4 = (0.168, 0.126, 0.162)$, and $\widetilde{r}_5 = (0.208, 0.168, 0.213)$, where the first values in the fuzzy return rates are central values, and the second and third values are left and right spread values. In order to range the investment proportion for each security, the lower and upper bounds of investment proportion for security j are derived as $(l_1, l_2, l_3, l_4, l_5)$ = (0.1, 0.1, 0.1, 0.1, 0.1), and $(u_1, u_2, u_3, u_4, u_5)$ = (0.4, 0.4, 0.4, 0.5, 0.6), respectively.

4.2. Results and Discussions

To clearly describe the proposed model, we select the guaranteed return rates to group the fuzzy return rate of the securities. In the first group, we select fuzzy number $\widetilde{p}_1 = (0.1,\ 0.05,\ 0.05)$ which is bigger than $\widetilde{r}_1$; $\widetilde{p}_2 = (0.15,\ 0.1,\ 0.1)$ is bigger than $\widetilde{r}_1$, $\widetilde{r}_2$, and $\widetilde{r}_3$, and $\widetilde{p}_3 = (0.2,\ 0.1,\ 0.15)$ is just smaller than $\widetilde{r}_5$, which are all derived by Theorem 2. Therefore, we can define the first scenario when we select the guaranteed return rate as $\widetilde{p}_1$, where security 1 is set for the shortage investment because its fuzzy return rates are lower than the guaranteed return rate $\widetilde{p}_1$, whereas the other securities 2, 3, 4, and 5 are for excess investment. In the second scenario, securities 1, 2, 3 are the shortage investment because their fuzzy return rates are lower than the guaranteed rate of return $\widetilde{p}_2$; by contrast, the fuzzy return rates of securities 4 and 5 are more $\widetilde{p}_2$ for excess investment. The third scenario shows the securities 1, 2, 3, and 4 to be the shortage investments because their fuzzy return rates are less than the guaranteed return rate $\widetilde{p}_3$, and then we judge security 5 to be the excess investment. In order to clearly state the proposed model, three experiments are conducted for illustration.

4.2.1. Experiment 1

In this experiment, we suppose the risk behavior of an investor is risk-seeking, and he prefers excess investment to shortage investment. The fuzzy portfolio model shown in model (19) assumed the dimensions of shortage to be bigger than excess investments as t > s. The fuzzy portfolio selection is proceeded by the following steps.

Step 1: Formulate a linear programming model for the proposed fuzzy portfolio model

First, the guaranteed return rate $\widetilde{p}_1 = (0.1,\ 0.05,\ 0.05)$ is used to group the securities to be shortage or excess investments. Second, the dimension for the shortage investment is set as t = 2, and the dimension for excess investment is set as s = 1. Third, we formulate model (19) by the collected data, and security 1 is adopted as the shortage investment, and thus the lowermost investment proportion is relaxed from 0.1 to 0. Therefore, the fuzzy portfolio model with t = 2 and s = 1 can be obtained as follows:

$$\begin{gathered} Max\ 0.077474x_1 + 0.119x_2 + 0.185x_3 + 0.248x_4 + 0.331x_5 \\ s.t.\ 0.127986x_1 + 0.454x_2 + 0.538x_3 + 0.676x_4 + 0.862x_5 \leq 6\sqrt{2}\sigma \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ 0 \leq x_1 \leq 0.4\ ,\ 0.1 \leq x_2, x_3 \leq 0.4; 0.1 \leq x_4 \leq 0.5; 0.1 \leq x_5 \leq 0.6 \end{gathered} \tag{20}$$

Step 2: Discussion and analysis

After solving model (20), with the constrained risks from 5% to 9%, we can solve the portfolio selections as shown in Table 1. If the constrained risk is smaller than 5%, then the portfolio is infeasible. On the other hand, if the constrained risk is bigger than 9%, then its optimal portfolio remains the same with the optimal portfolio as x_1 = 0, x_2 = 0.1, x_3 = 0.1, x_4 = 0.2, and x_5 = 0.6, and the expected return rate is 27.86%. With the constrained risk from 5% to 9%, we can find that the investment proportion of security 1 is from its upper bound 0.4 to the shortage investment proportion 0 because the return rate of security 1 is lower than the guaranteed return rate $\widetilde{p}_1$; the investment proportions for securities 2 and 3 are almost the same between the constrained risk from 5% to 9%, and the investment proportion for security 4 finally reaches 0.2 in the increasing process when the proportion of security 5 reaches 0.6. Next, we change the selected guaranteed return rates to $\widetilde{p}_2$ and $\widetilde{p}_3$, respectively. In Table 2, with t = 2, s = 1, and the guaranteed return rate $\widetilde{p}_2$, the risk of the investment is constrained from 2% to 5%. The optimal portfolio is obtained as x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0.4, and x_5 = 0.6 and the expected return rate is 24.78% under the constrained risk of 5%. With the constrained risk from 2% to 5%, we can find that the investment proportion of securities 1, 2, and 3 reach at their lower bounds as 0 in shortage investment because their return rates are less than the guaranteed return rate $\widetilde{p}_2$. In addition, we can find that investment proportions for securities 4 and 5 are increasing between 2% and 5%, because the return rates are higher than the guaranteed return rate $\widetilde{p}_2$. Furthermore, in Table 3, by the constrained risk from 1.5% to 4.5%, we can obtain the optimal portfolio. The optimal portfolio in the maximal expected returns is obtained as x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0.4, and x_5 = 0.6, in which the expected return rate is 20.285% under the constrained risk of 4.5%. By the constrained risk from 1.5% to 4.5%, the investment proportions of securities 1, 2, and 3 reach to their lower bounds as 0 in the shortage investment because their return rates are less than the guaranteed return rate $\widetilde{p}_2$. In addition, the investment proportion for security 4 is also relaxed to the shortage investment but reaches the investment proportion of 0.4, since the expected return rate of security 4 is bigger than securities 1, 2, and 3. By contrast, security 5 is in the increasing process between 1.5% and 4.5%, because its return rate is bigger than the guaranteed return rate $\widetilde{p}_3$.

Next, we solve the proposed model with t = 3, and s = 2, and the results are shown in Tables 4–6 under the guaranteed return rate $\widetilde{p}_1$, $\widetilde{p}_2$, and $\widetilde{p}_3$. Since we add one dimension to the shortage investment and excess investment, by comparing Tables 1–3 to Tables 4–6, we can find that the pattern to obtain the portfolio selection under the constrained risk is similar. However, we can observe two differences in the changed dimension. First, with the increase in the dimension, the maximal expected return rate in the largest constrained risk is lower than the lower dimension results in different guaranteed return rates $\widetilde{p}_1$, $\widetilde{p}_2$, and $\widetilde{p}_3$. Second, the proportion for each security in the shortage investment can be found to be 0 quickly; by contrast, the proportion for each security in the excess investment can be found quickly under the maximum constrained risk.

Table 1. The dimension with t = 2, and s = 1 with a guaranteed return rate $\widetilde{p}_1$ in the proposed model.

Security Proportions \ Constrained Risk	4.5%	5%	5.5%	6%	6.5%	7%	7.5%	8%	8.5%	9%
x_1	Infeasible Solution	0.4	0.4	0.3557	0.2979	0.2401	0.1823	0.1245	0.1245	0
x_2		0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
x_3		0.2096	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
x_4		0.1904	0.1532	0.1	0.1	0.1	0.1	0.1	0.1	0.2
x_5		0.1	0.2468	0.3443	0.4021	0.4599	0.5177	0.5755	0.5755	0.6
Expected Return Rates		0.16198	0.18107	0.19672	0.21138	0.22603	0.4069	0.25534	0.25534	0.27860

Table 2. The dimension with $t = 2$, and $s = 1$ with a guaranteed return rate $\widetilde{p}_2$ in the proposed model.

Constrained Risk / Security Proportions	1.5%	2%	2.5%	3%	3.5%	4%	4.5%	5%
x_1	Infeasible Solution	0.3963	0.2718	0.1474	0.0229	0	0	0
x_2		0.4	0.4	0.4	0.4	0.2093	0	0
x_3		0	0	0	0	0.0907	0.0608	0
x_4		0.1	0.1	0.1	0. 1	0.1	0.3392	0.4
x_5		0.1037	0.2282	0.3526	0.4771	0.6	0.6	0.6
Expected Return Rates		0.12020	0.14624	0.17229	0.19833	0.22371	0.24441	0.24780

Table 3. The dimension with $t = 2$, and $s = 1$ with a guaranteed return rate $\widetilde{p}_3$ in the proposed model.

Constrained Risk / Security Proportions	1%	1.5%	2%	2.5%	3%	3.5%	4%	4.5%
x_1	Infeasible Solution	0.2925	0.008	0	0	0	0	0
x_2		0.4	0.4	0.1417	0	0	0	0
x_3		0.2075	0.4	0.4	0.3264	0.1708	0.0153	0
x_4		0	0.092	0.3583	0.5	0.5	0.5	0.4
x_5		0.1	0.1	0.1	0.1736	0.3292	0.4847	0.6
Expected Return Rates		0.11008	0.13483	0.15429	0.17063	0.18363	0.19662	0.20285

Table 4. The dimension with $t = 3$ and $s = 2$ with a guaranteed return rate $\widetilde{p}_1$ in the proposed model.

Constrained Risk / Security Proportions	2.5%	3%	3.5%	4%	4.5%	5%
x_1	Infeasible Solution	0	0	0	0	0
x_2		0.3827	0.1192	0.1	0.1	0.1
x_3		0.4	0.4	0.1776	0.1	0.1
x_4		0.1173	0.3808	0.5	0.3357	0.2
x_5		0.1	0.1	0.2224	0.4643	0.6
Expected Return Rates		0.14357	0.16199	0.17719	0.19185	0.19854

Table 5. The dimension with $t = 3$ and $s = 2$ with a guaranteed return rate $\widetilde{p}_2$ in the proposed model.

Constrained Risk / Security Proportions	2%	2.5%	3%	3.5%	4%	4.5%	5%
x_1	Infeasible Solution	0.2465	0	0	0	0	0
x_2		0.4	0.2338	0	0	0	0
x_3		0.1535	0.4	0.3485	0.1637	0	0
x_4		0.1	0.2662	0.5	0.5	0.4660	0.4
x_5		0.1	0.1	0.1515	0.3363	0.5340	0.6
Expected Return Rates		0.12427	0.15101	0.17022	0.18450	0.19867	0.20166

4.2.2. Experiment 2

In this experiment, the testing focuses on $t = s$ for risk-neutral. We first solve the proposed model with $t = s = 2$, and the results are shown in Tables 7–9 under the guaranteed return rate $\widetilde{p}_1$, $\widetilde{p}_2$, and $\widetilde{p}_3$. By comparing Tables 1–3 to Tables 7–9, we can find that the expected return rate under the constrained risk is lower when we add one dimension to the excess investment. By increasing the dimension in excess investment, the investment proportions in higher return rate securities 4 and 5 offer more stable results in each constrained risk. Next, by comparing Tables 4–6 to Tables 7–9, we can find that the major difference between (Tables 4 and 5) and (Tables 7 and 8) are Tables 7 and 8 can solve the portfolio under lower constrained risks 2.5% and 2%, respectively. That is, we can solve the portfolio under a lower constrained risk when the dimension of shortage investment is lower. By contrast, compared to Tables 6 and 9, we can find that when we have more securities in shortage investment, a higher dimension in shortage investment enlarges the feasible region of model (19); therefore, we derive portfolio selections under a higher constrained risk than the lower dimension in the shortage investment. On the other hand, we solve the proposed model with $t = s = 3$, and the results are shown in Tables 10–12 under the guaranteed return rate $\widetilde{p}_1$, $\widetilde{p}_2$, and $\widetilde{p}_3$. By comparing Tables 7–9 to Tables 10–12, we can find that higher dimensions to t and s contribute to the narrower feasible region for the portfolio selection, and we can solve the portfolio under smaller constrained risks.

4.2.3. Experiment 3

In this experiment, the testing focuses on $t < s$ for the risk-averse. First, we solve the proposed model with $t = 2$ and $s = 3$, and the results are shown in Tables 13–15 under the guaranteed return rate $\widetilde{p}_1$, $\widetilde{p}_2$, and $\widetilde{p}_3$. It shows that our proposed model can be used to solve model (19) and obtain the efficient portfolio under different constrained risks. Furthermore, by comparing Tables 7–9 to Tables 13–15, we can find three major differences when we add one dimension to the excess investment. First, by increasing the dimension in the excess investment, the maximal expected return rate obtained in the largest constrained risk is lower than the lower dimension results in different guaranteed return rates $\widetilde{p}_1$, $\widetilde{p}_2$, and $\widetilde{p}_3$. From experiments 1 to 3, we can find that the higher dimensions in excess investments derive a lower expected return rate than lower dimensions in excess investments. Second, the lower guaranteed return rate $\widetilde{p}_i$ can obtain a higher expected return rate than the higher guaranteed return rate $\widetilde{p}_j$, $i, j = 1, 2, 3, 4, 5$. Third, through the higher dimension in excess investment, we can find that the higher return rate securities can be quick to reach their maximal investment proportion. On the other hand, we solve the proposed model with $t = 3$ and $s = 5$, and the results are shown in Tables 16–18 under the guaranteed return rate $\widetilde{p}_1$, $\widetilde{p}_2$, and $\widetilde{p}_3$. By comparing Tables 16–18 to Tables 4–6 ($t = 3$, $s = 2$) and Tables 10–12, ($t = 3$, $s = 3$), we can find that too big of an s dimension for excess investment does not make a significant difference to the portfolio selection. That is, too big of a dimension of s in excess investment makes almost no change to the objective function and the constrained risk, and thus, we suggest the values of t and s are, at most, 3.

Table 6. The dimension with $t = 3$ and $s = 2$ with a guaranteed return rate $\widetilde{p}_3$ in the proposed model.

Security Proportions \ Constrained Risk	2%	2.5%	3%	3.5%	4%	4.5%
x_1	Infeasible Solution	0.044	0	0	0	0
x_2		0.4	0.1089	0	0	0
x_3		0.4	0.4	0.2351	0	0
x_4		0.056	0.3911	0.5	0.4934	0.4
x_5		0.1	0.1	0.2649	0.5066	0.6
Expected Return Rates		0.13509	0.15836	0.17754	0.19504	0.19892

Table 7. The dimension with t = 2 and s = 2 with a guaranteed return rate $\widetilde{p}_1$ in the proposed model.

Security Proportions \ Constrained Risk	2%	2.5%	3%	3.5%	4%	4.5%	5%
x_1	Infeasible Solution	0.3978	0	0	0	0	0
x_2		0.3022	0.3827	0.1192	0.1	0.1	0.1
x_3		0.1	0.4	0.4	0.1776	0.1	0.1
x_4		0.1	0.1173	0.3808	0.5	0.3357	0.2
x_5		0.1	0.1	0.1	0.2224	0.4643	0.6
Expected Return Rates		0.11918	0.14357	0.16199	0.17719	0.19185	0.19854

Table 8. The dimension with t = 2 and s = 2 with a guaranteed return rate $\widetilde{p}_2$ in the proposed model.

Security Proportions \ Constrained Risk	1.5%	2%	2.5%	3%	3.5%	4%	4.5%	5%
x_1	Infeasible Solution	0.3228	0.1273	0	0	0	0	0
x_2		0.4	0.4	0.3569	0.2332	0.1095	0	0
x_3		0	0	0.1	0	0	0	0
x_4		0.1772	0.3727	0.5	0.5	0.5	0.466	0.4
x_5		0.1	0.1	0.1431	0.2668	0.3905	0.5340	0.6
Expected Return Rates		0.11889	0.13898	0.15693	0.17086	0.18480	0.19867	0.20166

Table 9. The dimension with t = 2 and s = 2 with a guaranteed return rate $\widetilde{p}_3$ in the proposed model.

Security Proportions \ Constrained Risk	2%	2.5%	3%	3.5%	4%	4.5%	5%
x_1	Infeasible Solution	0.4	0.3924	0.2931	0.1938	0.0944	0
x_2		0.315	0	0	0	0	0
x_3		0	0	0	0	0	0
x_4		0	0.0076	0.1069	0.2062	0.3056	0.4
x_5		0.285	0.6	0.6	0.6	0.6	0.6
Expected Return Rates		0.11854	0.15540	0.16633	0.17726	0.18819	0.19858

Table 10. The dimension with t = 3 and s = 3 with a guaranteed return rate $\widetilde{p}_1$ in the proposed model.

Security Proportions \ Constrained Risk	2%	2.5%	3%	3.5%	4%
x_1	Infeasible Solution	0.1996	0	0	0
x_2		0.4	0.1725	0.1	0.1
x_3		0.2004	0.4	0.1496	0.1
x_4		0.1	0.3275	0.5	0.2
x_5		0.1	0.1	0.2504	0.6
Expected Return Rates		0.127134	0.154601	0.173614	0.190217

Table 11. The dimension with t = 3 and s = 3 with a guaranteed return rate $\widetilde{p}_2$ in the proposed model.

Constrained Risk / Security Proportions	2%	2.5%	3%	3.5%	4%	4.5%
x_1	Infeasible Solution	0.0045	0	0	0	0
x_2		0.3955	0.0525	0	0	0
x_3		0.4	0.4	0.4	0.0867	0
x_4		0.1	0.4475	0.1337	0.3133	0.4
x_5		0.1	0.1	0.4663	0.6	0.6
Expected Return Rates		0.139588	0.162187	0.180877	0.196336	0.199067

Table 12. The dimension with t = 3 and s = 3 with a guaranteed return rate $\widetilde{p}_3$ in the proposed model.

Constrained Risk / Security Proportions	2%	2.5%	3%	3.5%	4%
x_1	Infeasible Solution	0.2001	0	0	0
x_2		0	0	0	0
x_3		0.4	0.3029	0.0783	0
x_4		0.2990	0.5	0.5	0.4
x_5		0.1	0.1971	0.4217	0.6
Expected Return Rates		0.145973	0.172565	0.189003	0.19889

Table 13. The dimension with t = 2 and s = 3 with a guaranteed return rate $\widetilde{p}_1$ in the proposed model.

Constrained Risk / Security Proportions	2%	2.5%	3%	3.5%	4%
x_1	Infeasible Solution	0.3155	0	0	0
x_2		0.1	0.1741	0.1	0.1
x_3		0.3845	0.4	0.1523	0.1
x_4		0.1	0.3259	0.5	0.2
x_5		0.1	0.1	0.2477	0.6
Expected Return Rates		0.12935	0.15463	0.17362	0.19030

Table 14. The dimension with t = 2 and s = 3 with a guaranteed return rate $\widetilde{p}_2$ in the proposed model.

Constrained Risk / Security Proportions	1.5%	2%	2.5%	3%	3.5%	4%	4.5%
x_1	Infeasible Solution	0.2768	0.0647	0	0	0	0
x_2		0.4	0.4	0.4	0.2759	0.0442	0
x_3		0	0	0	0	0	0
x_4		0.2232	0.4353	0.2003	0.1241	0.3558	0.4
x_5		0.1	0.1	0.3997	0.6	0.6	0.6
Expected Return Rates		0.12309	0.14480	0.16393	0.18060	0.19611	0.19907

Table 15. The dimension with t = 2 and s = 3 with a guaranteed return rate $\widetilde{p}_3$ in the proposed model.

Constrained Risk / Security Proportions	2%	2.5%	3%	3.5%	4%	4.5%	5%
x_1	Infeasible Solution	0.4	0.3784	0.2791	0.1798	0.0804	0
x_2		0.2902	0	0	0	0	0
x_3		0	0	0	0	0	0
x_4		0	0.0216	0.12	0.2202	0.3196	0.4
x_5		0.3098	0.6	0.6	0.6	0.6	0.6
Expected Return Rates		0.12137	0.15691	0.16784	0.17877	0.18970	0.19855

Table 16. The dimension with t = 3 and s = 5 with a guaranteed return rate $\widetilde{p}_1$ in the proposed model.

Constrained Risk / Security Proportions	2%	2.5%	3%	3.5%	4%
x_1	Infeasible Solution	0.1682	0	0	0
x_2		0.4	0.1369	0.1	0.1
x_3		0.2318	0.4	0.1064	0.1
x_4		0.1	0.3631	0.5	0.2
x_5		0.1	0.1	0.2936	0.6
Expected Return Rates		0.128984	0.156726	0.176392	0.189312

Table 17. The dimension with t = 3 and s = 5 with a guaranteed return rate $\widetilde{p}_2$ in the proposed model.

Constrained Risk / Security Proportions	2%	2.5%	3%	3.5%	4%	4.5%
x_1	Infeasible Solution	0	0	0	0	0
x_2		0.3960	0.0460	0	0	0
x_3		0.4	0.4	0.4	0.0474	0
x_4		0.104	0.4540	0.1038	0.3526	0.4
x_5		0.1	0.1	0.4962	0.6	0.6
Expected Return Rates		0.139961	0.162579	0.18199	0.197406	0.198901

Table 18. The dimension with t = 3 and s = 5 with a guaranteed return rate $\widetilde{p}_3$ in the proposed model.

Constrained Risk / Security Proportions	2%	2.5%	3%	3.5%	4%
x_1	Infeasible Solution	0.2	0	0	0
x_2		0	0	0	0
x_3		0.4	0.3028	0.0781	0
x_4		0.3	0.5	0.5	0.4
x_5		0.1	0.1972	0.4219	0.6
Expected Return Rates		0.145982	0.172575	0.189023	0.19889

Finally, we list two figures under the guaranteed return rate $\widetilde{p}_2$ in different dimensions of s and t. In Figure 1, we find that when t = 2 is fixed, the increasing dimension of excess investment forms 1 to 3, where s = 1 implies under the same risk, and t = 2, s = 1 have the biggest expected return rate. Therefore, the risk-seeker should select lower dimension s.

Next, in Figure 2, with a guaranteed return rate $\widetilde{p}_2$, we would like to show that when we increase t and fix it to 3, the dimension of excess investment is increasing from 2 to 5, and too big of a value of s cannot offer any useful information for the expected return rate under the same constrained risk, that is, when we adopt $s = 5$ whose results are almost the same as $s = 3$. Therefore, we suggest the values of s and t be smaller than or equal to 3.

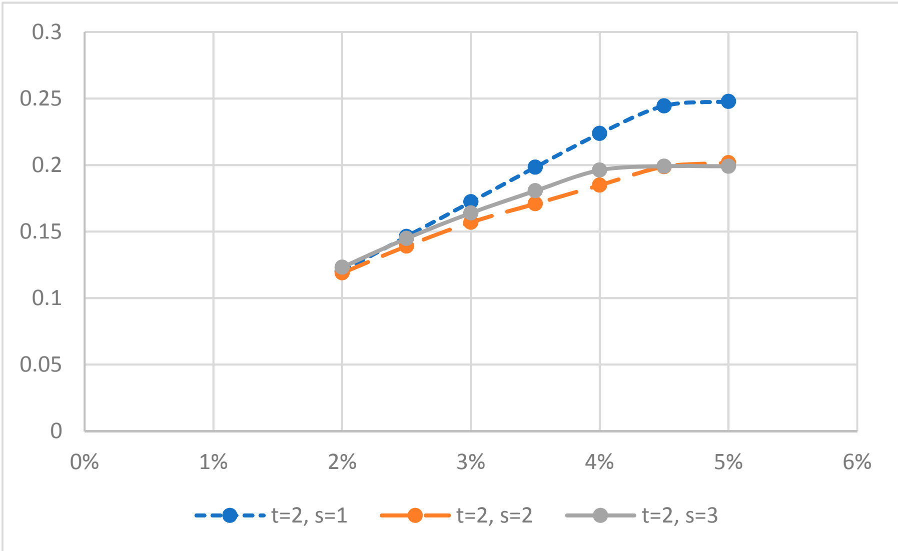

Figure 1. The dimension $t = 2$ with different s under guaranteed return rate $\widetilde{p}_2$.

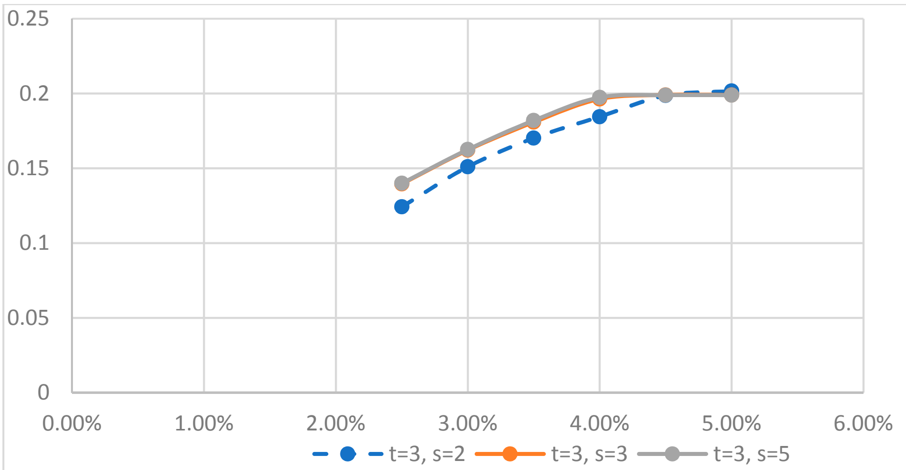

Figure 2. The dimension $t = 3$ with different s under guaranteed return rate $\widetilde{p}_2$.

5. Conclusions

Fuzzy portfolio models have led to a continual increase in the field of single-period or multi-period topics, indirectly resulting in many researchers focusing on the issue of the risk preferences of investors. Some investors might have the challenge of evaluating a better portfolio selection based on the profitable selecting security. Therefore, a method for selecting the most appropriate portfolio based on the guaranteed return rate would be extremely beneficial to these investors, in which any security whose expected return is bigger than the guaranteed return rate will be assumed for excess investment to this security. On the other hand, any security whose expected return is smaller than the guaranteed return rate will be assumed for shortage investment to this security. The present study included different degrees of dimensions to the securities in excess investment or shortage investment that investors expect of maximization of expected return rate and developed a novel decision-making procedure for portfolio selection under the constrained risk. Based on risk preferences, including risk-seeking, risk-neutral, and risk-averse, three kinds of fuzzy portfolio selections comprising different degrees of dimensions in excess investment and shortage investment were established for most of investors. Analysis results indicated that, when using the proposed model, a defuzzy method is required for the ranking between the expected return of each security and the guaranteed return rates. Subsequently, we can decide some securities are for excess investments, and the other securities are for shortage investments. A comparison of the degree of dimensions for the excess investment and shortage investment indicates that a risk-seeker would like to have excess investment for securities whose return rates are bigger than the guaranteed return rates; therefore, a lower value of s is suggested. Then he reduces the security investments whose return rates are lower than the guaranteed return rates; therefore, a bigger value of t is suggested. Next, a risk-seeker will adopt $t > s$, and we suggest the values of s and t to be smaller than or equal to 3. By contrast, for the risk-neutral investor, we suggest $s = t$; and $t < s$ is suggested to the investor who is risk-averse. Lower dimensions in s and t indicate a bigger objective value and feasible region in the linear programming model from the proposed fuzzy portfolio model, and thus we can derive bigger expected return rates from the invested securities. The results suggest that the proposed fuzzy portfolio model can clearly distinguish the relative importance from the ranking results compared to the guaranteed return rate. Finally, using the proposed model, investors could individually select the portfolio for the subjective risk preference, and easily evaluate and analyze the optimized portfolio with ease and convenience, without having to query the experts.

In this study, we expect more investors can be recruited to participate in evaluating and comparing the effects for the dimensions of excess investments and shortage in-vestments. Because risk preferences and standards may differ depending on the perceived risk in the investment, a collaborative discussion involving numerous experts is required to include in the evaluation and selection process for establishing the guaranteed return rate. Therefore, future research should focus on (1) expanding the number of investors, and (2) establishing comprehensive guaranteed return rates according to various experts' opinions on economy trends and business cycling. Furthermore, (3) an investor has a different risk attitude to select a portfolio in a different time period; therefore, multi-period fuzzy portfolio selection in different time periods should be considered with our proposed model.

Author Contributions: Conceptualization, K.-S.C. and R.-C.T.; methodology, R.-C.T. and N.-Y.L.; software, N.-Y.L. and Y.-Y.H.; supervision, K.-S.C. and R.-C.T.; validation, R.-C.T. and Y.-Y.H.; writing—original draft, N.-Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded with the financial support from National Science and Technology council with project No. MOST 110-2221-E-032-033.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees for their helpful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Markowitz, H. Portfolio selection. *J. Financ.* **1952**, *7*, 77–91.
2. Sharpe, W.F. *Portfolio Theory and Capital Markets*; McGraw-Hill: New York, NY, USA, 1970.
3. Merton, R.C. An analytic derivation of the efficient frontier. *J. Financ. Quant. Anal.* **1972**, *10*, 1851–1872. [CrossRef]
4. Pang, J.S. A new efficient algorithm for a class of portfolio selection problems. *Oper. Res.* **1980**, *28*, 754–767. [CrossRef]
5. Perold, A.F. Large-scale portfolio optimization. *Manag. Sci.* **1984**, *30*, 1143–1160. [CrossRef]
6. Vörös, J. Portfolio analysis—An analytic derivation of the efficient portfolio frontier. *Eur. J. Oper. Res.* **1990**, *203*, 294–300. [CrossRef]

7. Best, M.J.; Grauer, R.R. The efficient set mathematics when mean–variance problems are subject to general linear constrains. *J. Econ. Bus.* **1990**, *42*, 105–120. [CrossRef]
8. Best, M.J.; Hlouskova, J. The efficient frontier for bounded assets. *Math. Method Oper. Res.* **2000**, *52*, 195–212. [CrossRef]
9. Gupta, P.; Mehlawat, M.K.; Yadav, S.; Kumar, A. A polynomial goal programming approach for intuitionistic fuzzy portfolio optimization using entropy and higher moments. *Appl. Soft Comput.* **2019**, *85*, 105781. [CrossRef]
10. Wu, Y.; Xu, C.; Ke, Y.; Tao, Y.; Li, X. Portfolio optimization of renewable energy projects under type-2 fuzzy environment with sustainability perspective. *Comput. Ind. Eng.* **2019**, *133*, 69–82. [CrossRef]
11. Mansour, N.; Cherif, M.S.; Abdelfattah, W. Multi-objective imprecise programming for financial portfolio selection with fuzzy returns. *Expert Syst. Appl.* **2019**, *138*, 112810. [CrossRef]
12. Yue, W.; Wang, Y.; Xuan, H. Fuzzy multi-objective portfolio model based on semi-variance–semi-absolute deviation risk measures. *Soft Comput.* **2019**, *23*, 8159–8179. [CrossRef]
13. Liagkouras, K.; Metaxiotis, K. Multi-period mean–variance fuzzy portfolio optimization model with transaction costs. *Eng. Appl. Artif. Intell.* **2018**, *67*, 260–269. [CrossRef]
14. Gupta, P.; Mehlawat, M.K.; Khan, A.Z. Multi-period portfolio optimization using coherent fuzzy numbers in a credibilistic environment. *Expert Syst. Appl.* **2021**, *167*, 114135. [CrossRef]
15. Gupta, P.; Mehlawat, M.K.; Yadav, S.; Kumar, A. Intuitionistic fuzzy optimistic and pessimistic multi-period portfolio optimization models. *Soft Comput.* **2020**, *24*, 11931–11956. [CrossRef]
16. Garcia, F.; González-Bueno, J.; Guijarro, F.; Oliver, J.; Tamošiūnienė, R. Multiobjective approach to portfolio optimization in the light of the credibility theory. *Technol. Econ. Dev. Econ.* **2020**, *26*, 1165–1186. [CrossRef]
17. Gupta, P.; Mehlawat, M.K.; Kumar, A.; Yadav, S.; Aggarwal, A. A credibilistic fuzzy DEA approach for portfolio efficiency evaluation and rebalancing toward benchmark portfolios using positive and negative returns. *Int. J. Fuzzy Syst.* **2020**, *22*, 824–843. [CrossRef]
18. Mehlawat, M.K.; Gupta, P.; Kumar, A.; Yadav, S.; Aggarwal, A. Multi-objective fuzzy portfolio performance evaluation using data envelopment analysis under credibilistic framework. *IEEE Trans. Fuzzy Syst.* **2020**, *11*, 2726–2737. [CrossRef]
19. Mehralizade, R.; Amini, M.; Gildeh, B.S.; Ahmadzade, H. Uncertain random portfolio selection based on risk curve. *Soft Comput.* **2020**, *24*, 13331–13345. [CrossRef]
20. Yue, W.; Wang, Y. A new fuzzy multi-objective higher order moment portfolio selection model for diversified portfolios. *Phys. A Stat. Mech. Its Appl.* **2017**, *465*, 124–140. [CrossRef]
21. Guo, S.; Ching, W.-K.; Li, W.-K.; Siu, T.-K.; Zhang, Z. Fuzzy hidden Markov-switching portfolio selection with capital gain tax. *Expert Syst. Appl.* **2020**, *149*, 113304. [CrossRef]
22. Li, X.; Qin, Z.; Kar, S. Mean-variance-skewness model for portfolio selection with fuzzy returns. *Eur. J. Oper. Res.* **2010**, *202*, 239–247. [CrossRef]
23. Zhou, W.; Xu, Z. Portfolio selection and risk investment under the hesitant fuzzy environment. *Knowl.-Based Syst.* **2018**, *144*, 21–31. [CrossRef]
24. Michalski, G. Portfolio Management Approach in Trade Credit Decision Making. *Romanian J. Econ. Forecast.* **2007**, *3*, 42–53. [CrossRef]
25. Tsaur, R.C.; Chiu, C.-L.; Huang, Y.-Y. Guaranteed rate of return for excess investment in a fuzzy portfolio analysis. *Int. J. Fuzzy Syst.* **2021**, *23*, 94–106. [CrossRef]
26. Chen, K.-S.; Tsaur, R.C.; Lin, N.-C. Dimensions analysis to excess investment in fuzzy portfolio model from the threshold of guaranteed return rates. *Mathematics* **2023**, *11*, 44. [CrossRef]
27. Huang, Y.-Y.; Chen, I.-F.; Chiu, C.-L.; Tsaur, R.-C. Adjustable security proportions in the fuzzy portfolio selection under guaranteed return rates. *Mathematics* **2021**, *9*, 3026. [CrossRef]
28. Zimmermann, H.-J. *Fuzzy Set Theory—And Its Applications*; Springer: Berlin/Heidelberg, Germany, 2011.
29. Carlsson, C.; Fullér, R. On possibilistic mean value and variance of fuzzy numbers. *Fuzzy Sets Syst.* **2001**, *122*, 315–326. [CrossRef]
30. Rao, P.P.B.; Shankar, N.R. Ranking fuzzy numbers with a distance method using circumcenter of centroids and an index of modality. *Adv. Fuzzy Syst.* **2011**, *2011*, 1–7. [CrossRef]
31. Zhang, W.G. Possibilistic mean–standard deviation models to portfolio selection for bounded assets. *Comput. Appl. Math.* **2007**, *189*, 1614–1623. [CrossRef]