mathematics

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On Some Inequalities Involving Generalized Distance Functions

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Abstract: In this paper, a new class of generalized distance functions with respect to a pair of mappings is introduced. Next, some inequalities involving such distance functions are established. Our obtained results generalize and cover some recent results from the literature. Moreover, new distance inequalities for self-crossing polygons are obtained.

Keywords: distance with respect to a pair of mappings; distance inequalities; sum of distances; self-crossing polygons

MSC: 26D15; 54E35; 51E12

Citation: Jleli, M.; Samet, B. On Some Inequalities Involving Generalized Distance Functions. *Mathematics* **2023**, *11*, 1157. https://doi.org/10.3390/math11051157

Academic Editor: Ana-Maria Acu

Received: 2 February 2023
Revised: 18 February 2023
Accepted: 22 February 2023
Published: 26 February 2023

1. Introduction

In many branches of mathematical analysis, having a metric structure is essential for the study of several problems. For instance, the concept of distance between elements of an abstract set allows us to define many topological properties, such as convergence, Cauchy sequences, continuity and others [1–4]. One of the important properties of a (standard) distance function D on an abstract set M is the triangle inequality, i.e.,

$$D(u,v) \leq D(u,w) + D(w,v) \quad \text{for all } u, v.w \in M.$$

Many generalizations of the concept of a distance function achieved by relaxing the triangle inequality have been introduced in the literature, and examples can be found in [5–10]. For instance, in [5], the triangle inequality was relaxed as

$$D(u,v) \leq k(D(u,w) + D(w,v)) \quad \text{for all } u, v.w \in M,$$

where $k \geq 1$ is a constant.

On the other hand, inequalities involving distance functions are very useful in various areas of mathematics, for instance, in analysis, fixed point theory, operator theory, topology and geometry. Due to this fact, great attention has been paid to the study of inequalities on metric spaces, and examples can be found in [11–18].

Let M be a nonempty set and $D : M \times M \to [0, +\infty)$. We say that D is a distance (or metric) on M, if for all $u, v.w \in M$,

- $D(u,v) = 0 \iff u = v$,
- $D(u,v) = D(v,u)$,
- $D(u,v) \leq D(u,w) + D(w,v)$.

In this case, we say that (M, D) is a metric space.

In [11], Dragomir and Gosa established a polygonal inequality in the setting of metric spaces and provided some applications in normed linear spaces and inner product spaces.

Namely, it was proven that if (M, D) is a metric space, $n \geq 2$ is an integer, $\{u_i\}_{i=1}^n \subset M$ and $\iota_i \geq 0$, with $\iota_1 + \iota_2 + \cdots + \iota_n = 1$, then

$$\sum_{1 \leq i < j \leq n} \iota_i \iota_j D(u_i, u_j) \leq \inf_{u \in M} \sum_{i=1}^{n} \iota_i D(u_i, u). \tag{1}$$

Later, in [15], the above inequality was extended to natural powers of the distance. Namely, it was shown that under the above assumptions, we have

$$\begin{aligned} &\sum_{1 \leq i < j \leq n} \iota_i \iota_j D^m(u_i, u_j) \\ &\leq \frac{1}{2} \inf_{u \in M} \left[2 \sum_{i=1}^{n} \iota_i D^m(u_i, u) + \sum_{k=1}^{m-1} \binom{m}{k} \left(\sum_{i=1}^{n} \iota_i D^k(u_i, u) \right) \left(\sum_{i=1}^{n} \iota_i D^{m-k}(u_i, u) \right) \right] \end{aligned}$$

for all integers $m \geq 2$. In [19], Dragomir studied sums of the form

$$\sum_{1 \leq i < j \leq n} \iota_i \iota_j D^s(u_i, u_j),$$

where $s > 0$. He proved the following:

- If $0 < s \leq 1$, then

$$\sum_{1 \leq i < j \leq n} \iota_i \iota_j D^s(u_i, u_j) \leq \inf_{u \in M} \sum_{i=1}^{n} \iota_i D^s(u_i, u),$$

- If $s > 1$, then

$$\sum_{1 \leq i < j \leq n} \iota_i \iota_j D^s(u_i, u_j) \leq 2^{s-1} \inf_{u \in M} \sum_{i=1}^{n}, \iota_i D^s(u_i, u),$$

$$\left(\frac{2}{1 - \sum_{i=1}^{n} \iota_i^2} \right)^{s-1} \left(\sum_{1 \leq i < j \leq n} \iota_i \iota_j D(u_i, u_j) \right)^s \leq \sum_{1 \leq i < j \leq n} \iota_i \iota_j D^s(u_i, u_j).$$

Some other inequalities of the same type can be found in [12,13]. We also refer to [20], where a continuous version of (1) was obtained.

In this paper, we first introduce the notion of a generalized distance with respect to a pair of mappings and provide some examples of such distance functions (Section 2). Let us provide some motivations for introducing such a notion. Let us observe that some of the above-mentioned inequalities involve the power of a (standard) distance function. Now, if d is a distance function on M, and if we define mapping $D : M \times M \to [0, +\infty)$ as

$$D(u, v) = d^2(u, v), \quad u, v \in M,$$

we obtain, by the triangle inequality, that

$$D(u, v) = d^2(u, v) \leq d^2(u, w) + d^2(w, v) + d(u, w)d(w, v) + d(w, v)d(u, w)$$

for all $u, v, w \in M$, that is,

$$D(u, v) \leq D(u, w) + D(w, v) + f(u, w)g(w, v) + f(w, v)g(u, w), \tag{2}$$

where $f = g = d$. Hence, a natural question is whether inequalities of Dragomir type can be extended to mappings $D : M \times M \to [0, +\infty)$ satisfying (2) for arbitrary $f, g : M \times M \to [0, +\infty)$. A positive answer is obtained in Section 3, where we establish several inequalities of type (1) involving generalized distance functions (Section 3). Finally, in Section 4, some generalized distance inequalities for self-crossing polygons are proved.

2. Generalized Distance Function

Definition 1. *Let M be a nonempty set, and let $f, g : M \times M \to [0, +\infty)$. A mapping*

$$D : M \times M \to [0, +\infty)$$

is said to be a distance with respect to (f, g), if:

(i) $D(u, v) = D(v, u)$ *for all* $u, v \in M$.
(ii) $D(u, u) = 0$ *for all* $u \in M$.
(iii) *There exists* $k > 0$ *such that*

$$D(u, v) \leq k(D(u, w) + D(w, v) + f(u, w)g(w, v) + f(w, v)g(u, w))$$

for all $u, v, w \in M$.

Remark 1. *Let us remark that*

$$D \text{ is a distance with respect to } (f, g) \iff D \text{ is a distance with respect to } (g, f).$$

We provide below some examples of generalized distance functions in the sense of Definition 1.

Example 1. *Let D be a distance on M. Then, for all $f, g : M \times M \to [0, +\infty)$, D is a distance with respect to (f, g). Indeed, for all $u, v, w \in M$, we have*

$$D(u, v) \leq D(u, w) + D(w, v) \leq D(u, w) + D(w, v) + f(u, w)g(w, v) + f(w, v)g(u, w),$$

which shows that (iii) holds with $k = 1$.

Example 2. *Let $M = C([0, 1]; \mathbb{R})$. We consider mapping $D : M \times M \to [0, +\infty)$ defined as*

$$D(u, v) = \max_{0 \leq s \leq 1} |u(s) - v(s)| \int_0^1 |u(t) - v(t)|\, dt, \quad u, v \in M.$$

Clearly, mapping D satisfies properties (i)-(ii) in Definition 1. Moreover, for all $u, v, w \in M$, we have

$$\max_{0 \leq s \leq 1} |u(s) - v(s)| \leq \max_{0 \leq s \leq 1} |u(s) - w(s)| + \max_{0 \leq s \leq 1} |w(s) - v(s)| \tag{3}$$

and

$$\int_0^1 |u(t) - v(t)|\, dt \leq \int_0^1 |u(t) - w(t)|\, dt + \int_0^1 |w(t) - v(t)|\, dt. \tag{4}$$

By multiplying (3) by (4), we obtain

$$\begin{aligned} &D(u, v) \\ &\leq \max_{0 \leq s \leq 1} |u(s) - w(s)| \int_0^1 |u(t) - w(t)|\, dt + \max_{0 \leq s \leq 1} |u(s) - w(s)| \int_0^1 |w(t) - v(t)|\, dt \\ &\quad + \max_{0 \leq s \leq 1} |w(s) - v(s)| \int_0^1 |u(t) - w(t)|\, dt + \max_{0 \leq s \leq 1} |w(s) - v(s)| \int_0^1 |w(t) - v(t)|\, dt \\ &= D(u, w) + D(w, v) + f(u, w)g(w, v) + f(w, v)g(u, w), \end{aligned}$$

where

$$f(u_1, u_2) = \max_{0 \leq s \leq 1} |u_1(s) - u_2(s)|, \quad u_1, u_2 \in M$$

and

$$g(u_1, u_2) = \int_0^1 |u_1(t) - u_2(t)|\,dt, \quad u_1, u_2 \in M.$$

Therefore, (iii) holds with $k = 1$, and D is a distance with respect to (f, g).

Example 3. *Let d_1, d_2 be two distances on M, and let $\alpha_1, \alpha_2 \geq 0$, with $(\alpha_1, \alpha_2) \neq (0,0)$. We consider mapping $D : M \times M \to [0, +\infty)$ defined as*

$$D(u, v) = d_1^{\alpha_1}(u, v) d_2^{\alpha_2}(u, v), \quad u, v \in M.$$

Clearly, mapping D satisfies properties (i)-(ii) in Definition 1. We first consider the following:

- *The case when $\alpha_i > 1$, $i = 1, 2$.*

  Due to the convexity of function $[0, +\infty) \ni t \mapsto t^s$, $s > 1$, for all $u, v, w \in M$ and $i \in \{1, 2\}$, we have

$$\begin{aligned} d_i^{\alpha_i}(u, v) &\leq (d_i(u, w) + d_i(w, v))^{\alpha_i} \\ &= \left(2\, \frac{d_i(u, w) + d_i(w, v)}{2}\right)^{\alpha_i} \\ &= 2^{\alpha_i} \left(\frac{d_i(u, w) + d_i(w, v)}{2}\right)^{\alpha_i} \\ &\leq 2^{\alpha_i - 1}\left(d_i^{\alpha_i}(u, w) + d_i^{\alpha_i}(w, v)\right), \end{aligned}$$

which yields

$$\begin{aligned} \frac{1}{2^{\alpha_1 + \alpha_2 - 2}} D(u, v) &= \frac{1}{2^{\alpha_1 + \alpha_2 - 2}} \prod_{i=1}^{2} d_i^{\alpha_i}(u, v) \\ &\leq d_1^{\alpha_1}(u, w) d_2^{\alpha_2}(u, w) + d_1^{\alpha_1}(u, w) d_2^{\alpha_2}(w, v) \\ &\quad + d_1^{\alpha_1}(w, v) d_2^{\alpha_2}(u, w) + d_1^{\alpha_1}(w, v) d_2^{\alpha_2}(w, v) \\ &= D(u, w) + D(w, v) + f(u, w) g(w, v) + f(w, v) g(u, w), \end{aligned}$$

where

$$f(u_1, u_2) = d_1^{\alpha_1}(u_1, u_2), \quad u_1, u_2 \in M \tag{5}$$

and

$$g(u_1, u_2) = d_2^{\alpha_2}(u_1, u_2), \quad u_1, u_2 \in M. \tag{6}$$

Therefore, (iii) holds with $k = 2^{\alpha_1 + \alpha_2 - 2}$, and D is a distance with respect to (f, g). Next, we consider the following:

- *The case when $0 < \alpha_2 \leq 1 < \alpha_1$.*

  In this case, for all $u, v, w \in M$, we have

$$d_1^{\alpha_1}(u, v) \leq 2^{\alpha_1 - 1}\left(d_1^{\alpha_1}(u, w) + d_1^{\alpha_1}(w, v)\right) \tag{7}$$

and

$$\begin{aligned} d_2^{\alpha_2}(u, v) &\leq (d_2(u, w) + d_2(w, v))^{\alpha_2} \\ &\leq d_2^{\alpha_2}(u, w) + d_2^{\alpha_2}(w, v). \end{aligned} \tag{8}$$

By multiplying (7) by (8), we obtain

$$D(u, v) \leq 2^{\alpha_1 - 1}(D(u, w) + D(w, v) + f(u, w) g(w, v) + f(w, v) g(u, w)),$$

where f and g are defined by (5) and (6). This shows that (iii) holds with $k = 2^{\alpha_1 - 1}$, and D is a distance with respect to (f, g). Now, we consider the following:

- *The case when* $\alpha_2 = 0 < 1 < \alpha_1$.

  In this case, by (7), *we deduce that (iii) holds with* $k = 2^{\alpha_1 - 1}$ *and* $f = g = 0$. *Hence,* D *is a distance with respect to* $(0,0)$.

- *The case when* $0 < \alpha_i \leq 1$, $i = 1,2$.

  In this case, for all $u, v, w \in M$, *we have*

$$D(u,v) \leq D(u,w) + D(w,v) + f(u,w)g(w,v) + f(w,v)g(u,w),$$

where f *and* g *are defined in* (5) *and* (6). *Then, (iii) holds with* $k = 1$, *and* D *is a distance with respect to* (f,g). *Finally, we consider the following:*

- *The case when* $\alpha_1 = 0 < \alpha_2 \leq 1$.

  In this case, by (8), *we deduce that (iii) holds with* $k = 1$ *and* $f = g = 0$. *Hence,* D *is a distance with respect to* $(0,0)$.

3. Inequalities Involving Generalized Distance Functions

The below inequality involving generalized distance functions holds.

Theorem 1. *Let* D *be a distance function on* M *with respect to* (f,g), *in the sense of Definition* 1, *where* $f, g : M \times M \to [0, +\infty)$. *Let* $n \geq 2$ *be an integer,* $\{u_i\}_{i=1}^n \subset M$ *and* $\iota_i \geq 0$, *with* $\iota_1 + \iota_2 + \cdots + \iota_n = 1$. *Then,*

$$\begin{aligned} &\inf_{u\in M}\left[2\sum_{i=1}^n \iota_i D(u_i,u) + \left(\sum_{i=1}^n \iota_i f(u_i,u)\right)\left(\sum_{i=1}^n \iota_i g(u,u_i)\right) + \left(\sum_{i=1}^n \iota_i g(u_i,u)\right)\left(\sum_{i=1}^n \iota_i f(u,u_i)\right)\right] \\ &\geq \frac{2}{k}\sum_{1\leq i<j\leq n} \iota_i\iota_j D(u_i,u_j). \end{aligned} \tag{9}$$

Proof. Let $u \in M$. By property (iii) in Definition 1, we have

$$D(u_i,u_j) \leq k\big(D(u_i,u) + D(u,u_j) + f(u_i,u)g(u,u_j) + f(u,u_j)g(u_i,u)\big), \quad i,j \in \mathbb{I}_n,$$

where $\mathbb{I}_n = \{1, 2, \cdots, n\}$. By multiplying the above inequality by $\iota_i\iota_j$ and summing over i and j, we obtain

$$\begin{aligned} \frac{1}{k}\sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j D(u_i,u_j) \quad \leq \quad & \sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j D(u_i,u) + \sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j D(u,u_j) \\ & + \sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j f(u_i,u)g(u,u_j) + \sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j f(u,u_j)g(u_i,u). \end{aligned} \tag{10}$$

On the other hand, by properties (i)-(ii) in Definition 1, we have

$$\sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j D(u_i,u_j) = 2\sum_{1\leq i<j\leq n} \iota_i\iota_j D(u_i,u_j). \tag{11}$$

Moreover, we have

$$\begin{aligned} \sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j D(u_i,u) \quad &= \quad \sum_{j=1}^n \iota_j \sum_{i=1}^n \iota_i D(u_i,u) \\ &= \quad \sum_{i=1}^n \iota_i D(u_i,u), \end{aligned} \tag{12}$$

$$\sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j D(u,u_j) = \sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j D(u_i,u), \tag{13}$$

$$\sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j f(u_i,u)g(u,u_j) = \left(\sum_{i=1}^n \iota_i f(u_i,u)\right)\left(\sum_{i=1}^n \iota_i g(u,u_i)\right) \tag{14}$$

and

$$\sum_{i,j\in\mathbb{I}_n} \iota_i\iota_j f(u,u_j)g(u_i,u) = \left(\sum_{i=1}^n \iota_i g(u_i,u)\right)\left(\sum_{i=1}^n \iota_i f(u,u_i)\right) \tag{15}$$

Hence, it follows from (10)–(15) that

$$\begin{aligned} &\frac{2}{k}\sum_{1\le i<j\le n} \iota_i\iota_j D(u_i,u_j) \\ &\le 2\sum_{i=1}^n \iota_i D(u_i,u) + \left(\sum_{i=1}^n \iota_i f(u_i,u)\right)\left(\sum_{i=1}^n \iota_i g(u,u_i)\right) + \left(\sum_{i=1}^n \iota_i g(u_i,u)\right)\left(\sum_{i=1}^n \iota_i f(u,u_i)\right). \end{aligned} \tag{16}$$

Finally, by taking the infimum over u in (16), we obtain (9). □

Now, let us study some special cases of Theorem 1. We first consider the case when f and g are symmetric, that is,

$$f(u,v) = f(v,u),\ g(u,v) = g(v,u), \quad u,v \in M.$$

In this case, from Theorem 1, we deduce the below result.

Corollary 1. *Let D be a distance function on M with respect to (f,g), in the sense of Definition 1, where $f,g : M\times M \to [0,+\infty)$ are symmetric. Let $n \ge 2$ be an integer, $\{u_i\}_{i=1}^n \subset M$ and $\iota_i \ge 0$, with $\iota_1+\iota_2+\cdots+\iota_n = 1$. Then,*

$$\sum_{1\le i<j\le n} \iota_i\iota_j D(u_i,u_j) \le k \inf_{u\in M}\left[\sum_{i=1}^n \iota_i D(u_i,u) + \left(\sum_{i=1}^n \iota_i f(u_i,u)\right)\left(\sum_{i=1}^n \iota_i g(u_i,u)\right)\right].$$

By taking $f = g$ in Corollary 1, we deduce the below result.

Corollary 2. *Let D be a distance function on M with respect to (f,f), in the sense of Definition 1, where $f : M\times M \to [0,+\infty)$ is symmetric. Let $n \ge 2$ be an integer, $\{u_i\}_{i=1}^n \subset M$ and $\iota_i \ge 0$, with $\iota_1+\iota_2+\cdots+\iota_n = 1$. Then,*

$$\sum_{1\le i<j\le n} \iota_i\iota_j D(u_i,u_j) \le k \inf_{u\in M}\left[\sum_{i=1}^n \iota_i D(u_i,u) + \left(\sum_{i=1}^n \iota_i f(u_i,u)\right)^2\right].$$

By taking

$$\iota_1 = \iota_2 = \cdots = \iota_n = \frac{1}{n}$$

in Theorem 1, we obtain the below result.

Corollary 3. *Let D be a distance function on M with respect to (f,g), in the sense of Definition 1, where $f,g : M\times M \to [0,+\infty)$. Let $n \ge 2$ be an integer and $\{u_i\}_{i=1}^n \subset M$. Then,*

$$\begin{aligned} &\inf_{u\in M}\left[2n\sum_{i=1}^n D(u_i,u) + \left(\sum_{i=1}^n f(u_i,u)\right)\left(\sum_{i=1}^n g(u,u_i)\right) + \left(\sum_{i=1}^n g(u_i,u)\right)\left(\sum_{i=1}^n f(u,u_i)\right)\right] \\ &\ge \frac{2}{k}\sum_{1\le i<j\le n} D(u_i,u_j). \end{aligned}$$

If f and g are symmetric, we deduce, by Corollary 3, the below result.

Corollary 4. *Let D be a distance function on M with respect to (f,g), in the sense of Definition 1, where $f,g : M \times M \to [0,+\infty)$ are symmetric. Let $n \geq 2$ be an integer and $\{u_i\}_{i=1}^n \subset M$. Then,*

$$\sum_{1\leq i<j\leq n} D(u_i,u_j) \leq k \inf_{u\in M}\left[n\sum_{i=1}^n D(u_i,u) + \left(\sum_{i=1}^n f(u_i,u)\right)\left(\sum_{i=1}^n g(u_i,u)\right)\right].$$

If $f = g$ in Corollary 4, then we deduce the below result.

Corollary 5. *Let D be a distance function on M with respect to (f,f), in the sense of Definition 1, where $f : M \times M \to [0,+\infty)$ is symmetric. Let $n \geq 2$ be an integer and $\{u_i\}_{i=1}^n \subset M$. Then,*

$$\sum_{1\leq i<j\leq n} D(u_i,u_j) \leq k \inf_{u\in M}\left[n\sum_{i=1}^n D(u_i,u) + \left(\sum_{i=1}^n f(u_i,u)\right)^2\right].$$

Next, using the above results, we provide below some upper bounds for the following sum:

$$\sum_{1\leq i<j\leq n} \iota_i\iota_j d_1^{\alpha_1}(u_i,u_j)d_2^{\alpha_2}(u_i,u_j),$$

where $\alpha_1,\alpha_2 \geq 0$ and d_1,d_2 are two distances on M.

We first consider the case when $\alpha_i > 1$, $i = 1,2$.

Corollary 6. *For all $j \in \{1,2\}$, let d_j be a distance on M and $\alpha_j > 1$. Let $n \geq 2$ be an integer, $\{u_i\}_{i=1}^n \subset M$ and $\iota_i \geq 0$, with $\iota_1+\iota_2+\cdots+\iota_n = 1$. Then,*

$$\begin{aligned}&\sum_{1\leq i<j\leq n} \iota_i\iota_j d_1^{\alpha_1}(u_i,u_j)d_2^{\alpha_2}(u_i,u_j)\\ &\leq 2^{\alpha_1+\alpha_2-2}\inf_{u\in M}\left[\sum_{i=1}^n \iota_i d_1^{\alpha_1}(u_i,u)d_2^{\alpha_2}(u_i,u) + \left(\sum_{i=1}^n \iota_i d_1^{\alpha_1}(u_i,u)\right)\left(\sum_{i=1}^n \iota_i d_2^{\alpha_2}(u_i,u)\right)\right].\end{aligned} \quad (17)$$

Proof. By Example 3, since $\alpha_i > 1$, $i = 1,2$, we know that mapping $D : M \times M \to [0,+\infty)$ defined as

$$D(u,v) = d_1^{\alpha_1}(u,v)d_2^{\alpha_2}(u,v), \quad u,v \in M$$

is a distance with respect to $(d_1^{\alpha_1}, d_2^{\alpha_2})$, in the sense of Definition 1, where (iii) holds with constant $k = 2^{\alpha_1+\alpha_2-2}$. Since $d_i^{\alpha_i}$, $i = 1,2$, are symmetric, (17) follows from Corollary 1 by taking $f = d_1^{\alpha_1}$, $g = d_2^{\alpha_2}$ and $k = 2^{\alpha_1+\alpha_2-2}$. □

Next, we consider the case when $0 < \alpha_2 \leq 1 < \alpha_1$.

Corollary 7. *For all $j \in \{1,2\}$, let d_j be a distance on M and $0 < \alpha_2 \leq 1 < \alpha_1$. Let $n \geq 2$ be an integer, $\{u_i\}_{i=1}^n \subset M$ and $\iota_i \geq 0$, with $\iota_1+\iota_2+\cdots+\iota_n = 1$. Then,*

$$\begin{aligned}&\sum_{1\leq i<j\leq n} \iota_i\iota_j d_1^{\alpha_1}(u_i,u_j)d_2^{\alpha_2}(u_i,u_j)\\ &\leq 2^{\alpha_1-1}\inf_{u\in M}\left[\sum_{i=1}^n \iota_i d_1^{\alpha_1}(u_i,u)d_2^{\alpha_2}(u_i,u) + \left(\sum_{i=1}^n \iota_i d_1^{\alpha_1}(u_i,u)\right)\left(\sum_{i=1}^n \iota_i d_2^{\alpha_2}(u_i,u)\right)\right].\end{aligned} \quad (18)$$

Proof. By Example 3, since $0 < \alpha_2 \leq 1 < \alpha_1$, we know that mapping $D = d_1^{\alpha_1}d_2^{\alpha_2}$ is a distance with respect to $(d_1^{\alpha_1}, d_2^{\alpha_2})$, in the sense of Definition 1, where (iii) holds with constant $k = 2^{\alpha_1-1}$. Since $d_i^{\alpha_i}$, $i = 1,2$, are symmetric, (18) follows from Corollary 1 by taking $f = d_1^{\alpha_1}$, $g = d_2^{\alpha_2}$ and $k = 2^{\alpha_1-1}$. □

We now consider the case when $\alpha_2 = 0 < 1 < \alpha_1$. In this case, we deduce the below result obtained in [19].

Corollary 8. *Let d be a metric on M and $\alpha_1 > 1$. Let $n \geq 2$ be an integer, $\{u_i\}_{i=1}^n \subset M$ and $\iota_i \geq 0$, with $\iota_1 + \iota_2 + \cdots + \iota_n = 1$. Then,*

$$\sum_{1 \leq i < j \leq n} \iota_i \iota_j d^{\alpha_1}(u_i, u_j) \leq 2^{\alpha_1 - 1} \inf_{u \in M} \sum_{i=1}^n \iota_i d^{\alpha_1}(u_i, u). \tag{19}$$

Proof. By Example 3, since $\alpha_2 = 0 < 1 < \alpha_1$, we know that $D = d_1^{\alpha_1} d_2^{\alpha_2} = d^{\alpha_1}$ is a distance with respect to $(0,0)$, in the sense of Definition 1, where (iii) holds with constant $k = 2^{\alpha_1 - 1}$. Then, (19) follows from Corollary 1 by taking $f = g = 0$ and $k = 2^{\alpha_1 - 1}$. □

Next, we consider the case when $0 < \alpha_i \leq 1$, $i = 1, 2$.

Corollary 9. *For all $j \in \{1, 2\}$, let d_j be a distance on M and $0 < \alpha_i \leq 1$. Let $n \geq 2$ be an integer, $\{u_i\}_{i=1}^n \subset M$ and $\iota_i \geq 0$, with $\iota_1 + \iota_2 + \cdots + \iota_n = 1$. Then,*

$$\begin{aligned} &\sum_{1 \leq i < j \leq n} \iota_i \iota_j d_1^{\alpha_1}(u_i, u_j) d_2^{\alpha_2}(u_i, u_j) \\ &\leq \inf_{u \in M} \left[\sum_{i=1}^n \iota_i d_1^{\alpha_1}(u_i, u) d_2^{\alpha_2}(u_i, u) + \left(\sum_{i=1}^n \iota_i d_1^{\alpha_1}(u_i, u) \right) \left(\sum_{i=1}^n \iota_i d_2^{\alpha_2}(u_i, u) \right) \right]. \end{aligned} \tag{20}$$

Proof. By Example 3, since $0 < \alpha_i \leq 1$, $i = 1, 2$, we know that mapping $D = d_1^{\alpha_1} d_2^{\alpha_2}$ is a distance with respect to $(d_1^{\alpha_1}, d_2^{\alpha_2})$, in the sense of Definition 1, where (iii) holds with constant $k = 1$. Since $d_i^{\alpha_i}$, $i = 1, 2$, are symmetric, (20) follows from Corollary 1 by taking $f = d_1^{\alpha_1}$, $g = d_2^{\alpha_2}$ and $k = 1$. □

Finally, we consider the case when $\alpha_1 = 0 < \alpha_2 \leq 1$. In this case, we deduce the below result obtained in [19].

Corollary 10. *Let d be a distance on M and $\alpha_1 = 0 < \alpha_2 \leq 1$. Let $n \geq 2$ be an integer, $\{u_i\}_{i=1}^n \subset M$ and $\iota_i \geq 0$, with $\iota_1 + \iota_2 + \cdots + \iota_n = 1$. Then,*

$$\sum_{1 \leq i < j \leq n} \iota_i \iota_j d^{\alpha_2}(u_i, u_j) \leq \inf_{u \in M} \sum_{i=1}^n \iota_i d^{\alpha_2}(u_i, u). \tag{21}$$

Proof. By Example 3, since $\alpha_1 = 0 < \alpha_2 \leq 1$, we know that $D = d_1^{\alpha_1} d_2^{\alpha_2} = d^{\alpha_2}$ is a distance with respect to $(0,0)$, in the sense of Definition 1, where (iii) holds with constant $k = 1$. Then, (21) follows from Corollary 1 by taking $f = g = 0$ and $k = 1$. □

4. Generalized Distance Inequalities for Self-Crossing Polygons

Let D be a distance on M with respect to (f, g), in the sense of Definition 1, where $f, g : M \times M \to [0, +\infty)$. Let $B_1, B_2, \cdots, B_n \in M$, $n \geq 3$, be the vertices of a possibly self-crossing polygon with unit perimeter with respect to D. The perimeter with respect to D is defined as

$$P(B_1, B_2, \cdots, B_n) = \sum_{i=1}^n D(B_i, B_{i+1}), \ B_{n+1} = B_1.$$

Let

$$\rho_n = \inf_{P(B_1, B_2, \cdots, B_n) = 1} \sum_{1 \leq i < j \leq n} D(B_i, B_j), \tag{22}$$

under the assumption of

$$\{\{B_i\}_{i=1}^n \subset M : P(B_1, B_2, \cdots, B_n) = 1\} \neq \emptyset.$$

The below result holds.

Theorem 2. *Let $n \geq 3$. Let D be a distance on M with respect to (f, g), in the sense of Definition 1, where $f, g : M \times M \to [0, +\infty)$. We have*

$$\rho_n \geq \frac{1}{4}\left[\frac{n}{k} - \sum_{1\leq i,j\leq n}\big(f(B_i, B_j)g(B_j, B_{i+1}) + f(B_j, B_{i+1})g(B_i, B_j)\big)\right], \tag{23}$$

where ρ_n is defined in (22).

Proof. Let $B_1, B_2, \cdots, B_n \in M$ be such that

$$P(B_1, B_2, \cdots, B_n) = 1.$$

Let S be the sum of pair-wise distances, that is,

$$S = \sum_{1\leq i<j\leq n} D(B_i, B_j).$$

Then,

$$2S = \sum_{1\leq i,j\leq n} D(B_i, B_j). \tag{24}$$

On the other hand, by property (iii) in Definition 1, we have

$$D(B_i, B_{i+1}) \leq k\big(D(B_i, B_j) + D(B_j, B_{i+1}) + f(B_i, B_j)g(B_j, B_{i+1}) + f(B_j, B_{i+1})g(B_i, B_j)\big).$$

By summing over i, we obtain

$$\begin{aligned} 1 &= P(B_1, B_2, \cdots, B_n) \\ &\leq k\left(\sum_{i=1}^{n} D(B_i, B_j) + \sum_{i=1}^{n} D(B_j, B_{i+1}) + \sum_{i=1}^{n}\big(f(B_i, B_j)g(B_j, B_{i+1}) + f(B_j, B_{i+1})g(B_i, B_j)\big)\right). \end{aligned}$$

On the other hand, we have

$$\sum_{i=1}^{n} D(B_i, B_j) = \sum_{i=1}^{n} D(B_j, B_{i+1}).$$

Hence, the following holds:

$$1 \leq k\left(2\sum_{i=1}^{n} D(B_i, B_j) + \sum_{i=1}^{n}\big(f(B_i, B_j)g(B_j, B_{i+1}) + f(B_j, B_{i+1})g(B_i, B_j)\big)\right).$$

Next, by summing over j and using (24), we obtain

$$n \leq k\left(4S + \sum_{1\leq i,j\leq n}\big(f(B_i, B_j)g(B_j, B_{i+1}) + f(B_j, B_{i+1})g(B_i, B_j)\big)\right),$$

which yields (23). □

Let us consider the special case of Theorem 2 when

$$D(u, v) = d^{\alpha}(u, v), \quad u, v \in M, \tag{25}$$

where $\alpha > 0$ and d is a distance on M. Notice that by Example 3, we know that D is a distance with respect to $(0, 0)$, in the sense of Definition 1, where (iii) holds with

$$k = \begin{cases} 1 & \text{if} \quad 0 < \alpha \leq 1, \\ 2^{\alpha-1} & \text{if} \quad \alpha > 1. \end{cases}$$

Hence, by Theorem 2, we deduce the below result.

Corollary 11. *Let D be the generalized distance defined in (25). Then, for all $n \geq 3$, the following holds:*

$$\rho_n \geq \begin{cases} \frac{n}{4} & \text{if} \quad 0 < \alpha \leq 1, \\ \frac{n}{2^{\alpha+1}} & \text{if} \quad \alpha > 1. \end{cases} \tag{26}$$

In the case when $\alpha > 1$, we have the below additional result.

Theorem 3. *Let D be the generalized distance defined in (25) with $\alpha > 1$. Then, for all $3 \leq n < 2^{\alpha+1}$ and $\{B_i\}_{i=1}^n \subset M$, with $P(B_1, B_2, \cdots, B_n) = 1$, we have*

$$\sum_{1 \leq i < j \leq n} D(B_i, B_j) > \frac{n}{2^{\alpha+1}}. \tag{27}$$

Proof. Let $3 \leq n < 2^{\alpha+1}$ be fixed. Then, by (26), for all $\{B_i\}_{i=1}^n \subset M$, with

$$P(B_1, B_2, \cdots, B_n) = 1,$$

we have

$$\sum_{1 \leq i < j \leq n} D(B_i, B_j) \geq \frac{n}{2^{\alpha+1}}.$$

Let us suppose that (27) is not true. Then, there exist $\{B_i\}_{i=1}^n \subset M$ with

$$P(B_1, B_2, \cdots, B_n) = 1,$$

such that

$$\sum_{1 \leq i < j \leq n} D(B_i, B_j) = \frac{n}{2^{\alpha+1}}. \tag{28}$$

On the other hand, we have

$$\begin{aligned} \sum_{1 \leq i < j \leq n} D(B_i, B_j) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} D(B_i, B_j) \\ &= D(B_{n-1}, B_n) + \sum_{i=1}^{n-2} \left(D(B_i, B_{i+1}) + \sum_{j=i+2}^{n} D(B_i, B_j) \right) \\ &= \sum_{i=1}^{n-1} D(B_i, B_{i+1}) + D(B_n, B_1) + \left(\sum_{i=1}^{n-2} \sum_{j=i+2}^{n} D(B_i, B_j) - D(B_n, B_1) \right) \\ &= P(B_1, B_2, \cdots, B_n) + \left(\sum_{i=1}^{n-2} \sum_{j=i+2}^{n} D(B_i, B_j) - D(B_n, B_1) \right) \\ &= 1 + \left(\sum_{i=1}^{n-2} \sum_{j=i+2}^{n} D(B_i, B_j) - D(B_n, B_1) \right). \end{aligned}$$

Hence, by (28) and due to the assumption on n, we obtain

$$\sum_{i=1}^{n-2} \sum_{j=i+2}^{n} D(B_i, B_j) - D(B_n, B_1) = \frac{n - 2^{\alpha+1}}{2^{\alpha+1}} < 0.$$

On the other hand,

$$\sum_{i=1}^{n-2} \sum_{j=i+2}^{n} D(B_i, B_j) - D(B_n, B_1) \geq 0.$$

Thus, we reach a contradiction. □

We next consider the case when $M = \mathbb{R}^2$ and

$$D(u,v) = \|u - v\|^{\alpha}_{\mathbb{R}^2}, \quad u,v \in \mathbb{R}^2, \tag{29}$$

where $\alpha > 0$ and $\|\cdot\|_{\mathbb{R}^2}$ is the Euclidean norm on $\mathbb{R}^2$. In this case, we obtain the below result.

Theorem 4. *Let D be the generalized distance defined in (29). Then, for all $n \geq 3$:*

(i) *(26) holds,*

(ii) *If n is even and $0 < \alpha \leq 1$, then*

$$\rho_n = \frac{n}{4},$$

(iii) *If n is odd and $0 < \alpha \leq 1$, then*

$$\rho_n \leq \frac{n+1}{4},$$

where ρ_n is defined in (22).

Proof. (i) It immediately follows from Corollary 11 that by taking

$$M = \mathbb{R}^2,\ d(u,v) = \|u - v\|_{\mathbb{R}^2}.$$

(ii) Let n be even and $0 < \alpha \leq 1$. Let us consider the self-crossing polygon, where the vertices are defined as follows:

$$B_i = \begin{cases} (0,0) & \text{if} \quad i \text{ is odd} \\ \left(n^{\frac{-1}{\alpha}}, 0\right) & \text{if} \quad i \text{ is even} \end{cases}, \quad i \in \{1, 2, \cdots, n\}.$$

Then,

$$\begin{aligned} P(B_1, B_2, \cdots, B_n) &= \|B_1 - B_2\|^{\alpha}_{\mathbb{R}^2} + \cdots + \|B_{n-1} - B_n\|^{\alpha}_{\mathbb{R}^2} + \|B_n - B_1\|^{\alpha}_{\mathbb{R}^2} \\ &= n\left(n^{\frac{-1}{\alpha}}\right)^{\alpha} = 1. \end{aligned}$$

Furthermore, by (24), we have

$$\begin{aligned} 2S &= \sum_{1 \leq i,j \leq n} D(B_i, B_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} D(B_i, B_j) \\ &= \sum_{i=1}^{\frac{n}{2}} \sum_{j=1}^{n} D(B_{2i}, B_j) + \sum_{i=0}^{\frac{n-2}{2}} \sum_{j=1}^{n} D(B_{2i+1}, B_j) \\ &= \sum_{i=1}^{\frac{n}{2}} \sum_{j=0}^{\frac{n-2}{2}} D(B_{2i}, B_{2j+1}) + \sum_{i=0}^{\frac{n-2}{2}} \sum_{j=1}^{\frac{n}{2}} D(B_{2i+1}, B_{2j}) \\ &= 2\frac{n}{4}, \end{aligned}$$

which yields $S = \frac{n}{4}$. Then, by (26), we deduce that $\rho_n = \frac{n}{4}$.

(iii) Let n be even and $0 < \alpha \leq 1$. Let us consider the self-crossing polygon, where the vertices are defined as follows:

$$B_i = \begin{cases} (0,0) & \text{if} \quad i \text{ is odd} \\ \left((n-1)^{\frac{-1}{\alpha}}, 0\right) & \text{if} \quad i \text{ is even} \end{cases}, \quad i \in \{1, 2, \cdots, n\}.$$

Then,

$$\begin{aligned} P(B_1, B_2, \cdots, B_n) &= \|B_1 - B_2\|^{\alpha}_{\mathbb{R}^2} + \cdots + \|B_{n-1} - B_n\|^{\alpha}_{\mathbb{R}^2} \\ &= (n-1)\left((n-1)^{\frac{-1}{\alpha}}\right)^{\alpha} = 1. \end{aligned}$$

Furthermore, by (24), we have

$$\begin{aligned} 2S &= \sum_{i=2}^{n-1} D(B_i, B_n) + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} D(B_i, B_j) + \sum_{j=2}^{n-1} D(B_n, B_j) \\ &= 2\sum_{i=2}^{n-1} D(B_i, B_n) + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} D(B_i, B_j) \\ &= 2\sum_{i=1}^{\frac{n-1}{2}} D(B_{2i}, B_n) + \sum_{i=1}^{\frac{n-1}{2}}\sum_{j=1}^{n-1} D(B_{2i}, B_j) + \sum_{i=0}^{\frac{n-3}{2}}\sum_{j=1}^{n-1} D(B_{2i+1}, B_j) \\ &= 2\sum_{i=1}^{\frac{n-1}{2}} D(B_{2i}, B_n) + \sum_{i=1}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-3}{2}} D(B_{2i}, B_{2j+1}) + \sum_{i=0}^{\frac{n-3}{2}}\sum_{j=1}^{\frac{n-1}{2}} D(B_{2i+1}, B_{2j}) \\ &= 2\left(\sum_{i=1}^{\frac{n-1}{2}} D(B_{2i}, B_n) + \sum_{i=1}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-3}{2}} D(B_{2i}, B_{2j+1})\right) \\ &= 2\left(\left((n-1)^{\frac{-1}{\alpha}}\right)^{\alpha}\frac{n-1}{2} + \left((n-1)^{\frac{-1}{\alpha}}\right)^{\alpha}\left(\frac{n-1}{2}\right)^2\right) \\ &= 2\frac{n+1}{4}. \end{aligned}$$

This shows that $S = \frac{n+1}{4}$. Since $\rho_n \leq S$, we obtain $\rho_n \leq \frac{n+1}{4}$. □

5. Conclusions

In this paper, we first introduce the notion of a generalized distance function with respect to a pair of mappings. Namely, given a nonempty set M, we say that

$$D : M \times M \to [0, +\infty)$$

is a distance with respect to (f, g), where $f, g : M \times M \to [0, +\infty)$, if:

(i) $D(u, v) = D(v, u)$ for all $u, v \in M$.
(ii) $D(u, u) = 0$ for all $u \in M$.
(iii) There exists $k > 0$ such that

$$D(u, v) \leq k(D(u, w) + D(w, v) + f(u, w)g(w, v) + f(w, v)g(u, w))$$

for all $u, v, w \in M$.

In Section 2, we provide several examples of generalized distance functions with respect to a pair of mappings. Moreover, motivated by the recent obtained results obtained by Dragomir [19], several inequalities involving sums of the form

$$\sum_{1 \leq i < j \leq n} \iota_i \iota_j D(u_i, u_j),$$

where $\{u_i\}_{i=1}^{n} \subset M$ and $\iota_i \geq 0$, with $\iota_1 + \iota_2 + \cdots + \iota_n = 1$, are established in Section 3. In Section 4, we provide new distance inequalities for self-crossing polygons.

It would be interesting to study the topological properties of distance functions with respect to a pair of mappings, for instance, convergence, Cauchy criterion and completeness.

An interesting problem in this direction is to extend the Banach contraction principle [21] to a set M equipped with a distance function with respect to a pair of mappings.

Author Contributions: All authors equally contributed to this paper. All authors have read and agreed to the published version of the manuscript.

Funding: The second author was supported by Researchers Supporting Project number (RSP2023R4), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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