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Flows in Infinite-Dimensional Phase Space Equipped with a Finitely-Additive Invariant Measure

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Abstract: Finitely-additive measures invariant to the action of some groups on a separable infinite-dimensional real Hilbert space are constructed. The invariantness of a measure is studied with respect to the group of shifts on a vector of Hilbert space, the orthogonal group and some groups of symplectomorphisms of the Hilbert space equipped with the shift-invariant symplectic form. A considered invariant measure is locally finite, σ finite, but it is not countably additive. The analog of the ergodic decomposition of invariant finitely additivemeasures with respect to some groups are obtained. The set of measures that are invariant with respect to a group is parametrized using the obtained decomposition. The paper describes the spaces of complex-valued functions which are quadratically integrable with respect to constructed invariant measures. This space is used to define the Koopman unitary representation of the group of transformations of the Hilbert space. To define the strong continuity subspaces of a Koopman group, we analyze the spectral properties of its generator.

Keywords: A. Weil theorem; finitely-additive measure; shift-invariant measure on an infinite-dimensional space; isometry-invariant measure on a Hilbert space; Koopman representation of a Hamiltonian flow

MSC: 28C20; 28D05; 37A05

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1. Introduction

1.1. Motivation

Shift-invariant finitely additive measures on a Hilbert space present the realization of invariant measures on a topological group without the locally compactness property. According to the A. Weil theorem, there is no Lebesgue measure on an infinite dimensional Euclidean space. For this reason, the paper [1] states that there is no function of a set on an infinite dimensional space such that this function defines the notion of a volume on this space. We analyze different approaches to the extension of the volume notion for an infinitely dimensional space and study the invariantness with respect to the action of groups of finitely additive measures on a Hilbert space.

A studying of invariant measures on the phase space of an infinite-dimensional Hamiltonian system is important for the statistical mechanics of infinite-dimensional systems. An isometry-invariant measure is the base for the constructing of the Koopman representation of a group of shifts along a vector field in a separable Hilbert space (the group isometries in a Hilbert space are defined as the group generated by the group of shift and the orthogonal group). The isometry invariance of a measure on a separable Hilbert space allows the opportunity to analyze a random Hamiltonian and the asymptotic behavior of compositions of independent random Hamiltonian flows with values in the group of isometries of a separable Hilbert space.

Gibbs measures of infinite-dimensional Hamiltonian equations (including a nonlinear Schrödinger equation, a nonlinear wave equation, a Kleyn–Gordon equation, and a Korteveg–de Vrize equation) are studied in the works [2–8]. The Gibbs measure of every considered Hamiltonian system is invariant with respect to the flow generated by this system. However, the Gibbs measures of a Hamiltonian system can be singular with respect to another one even in the class of Gaussian measures of quadratic Hamiltonian systems.

An invariant measure of a Hamiltonian flow different from the Gibbs measure can be suggested on the basis of the complete integrability of a Hamiltonian system admitting action-angle coordinates [9,10]. In papers [11,12], on the contrary, a countably additive invariant measure is used for the constructing of action-angle coordinates for the Koopman presentation of a Hamiltonian system. Invariant measures are useful for constructing a hydrodynamical approach to classical and quantum integrable systems [13].

In this paper, we describe a finitely additive measure on a real separable Hilbert space such that this measure is invariant with respect to a family of Hamiltonian flows including one-parametric groups of shifts along a vector of a Hilbert space.

The considered group of self-mappings of a Hilbert space has the unitary representation in the space of quadratic integrables with respect to an invariant-measure complex-valued function. Properties of continuity in the strong operator topology of the unitary representation are studied.

To describe the family of shift-invariant measures, the notions of the ring-ergodicity and ring-decomposibility of a measure with respect to a group are introduced (see Definition 1 in Section 2.3). Using a ring-ergodic component of a measure which is invariant with respect to a group, we obtain the separable space of functions that are quadratically integrable with respect to ring-ergodic invariant measures.

A unitary representation of a group of self-mappings of a Hilbert space is discontinuous in general. We describe subgroups admitting the continuity of its representation in the strong operator topology. For the Koopman unitary representation of a Hamiltonian flow of an infinite system of oscillators, the subspaces of continuity are described in terms of the spectrum of the Koopman group generator.

1.2. Historical Background

A nontrivial countably additive σ-finite locally finite Borel left-invariant measure on a topological group G does not exist, according to the A. Weil theorem, if the group G is not locally compact. Hence, there is no nontrivial countably additive σ-finite locally finite Borel shift-invariant measure on an infinite-dimensional normalized linear space. Therefore, the studying of shift-invariant measures on a Hilbert space deals with an additive function of a set without some properties of the Lebesgue measure. We study finitely additive measures on an infinite-dimensional separable real Hilbert space so that these measures are invariant under shifts on a vector and orthogonal transformations. The focus of our research is the space of functions on a Hilbert space that are quadratically integrable with respect to an isometry-invariant measure. Unitary groups acting by means of isometric transformations of the space of arguments in the above space of quadratically integrable functions are investigated.

Thus, a shift-invariant measure on a topological group without the local compactness property is considered as an additive non-negative function, which is defined on a ring of subsets of the space. However, this function of a set would not have at least one of properties of the Lebesgue measure listed in the A. Weil theorem [14–20].

One approach is based on the construction of a countably additive measure without the σ-finiteness property. Countably additive measures on topological vector spaces of numerical sequences are introduced in [14–16,19]. However, the introduced measures are not σ-finite nor locally finite.

As for the other approach, shift-invariant σ-finite locally finite measures on a separable Banach space are introduced in [17,20]. However, every constructed measure, at first, is not countably additive and, secondly, is not defined on the ring of bounded Borel subsets. The paper [20] describes the construction of a shift-invariant measure on a Hilbert space as a finitely additive function of a set defined on some ring of subsets of a Hilbert space. This ring of subsets (the domain of a finitely additive function of a set) is not invariant with respect to every orthogonal transformation since this ring depends on the choice of the orthonormal basis (ONB) in the Hilbert space.

The studying of a finitely additive invariant with respect to shifts and rotations measured on a Hilbert space is the continuation of the investigation of the same problem in a finite-dimensional Euclidean space. The problem of the existence of an invariant with respect to an isometric transformation measure on a finite-dimensional Euclidean space was investigated during the last century in the form of the following question. *Does the measure λ on the d-dimensional Euclidean space exist so that this measure is*

(1) *Defined on a bounded subset of the Euclidean space;*
(2) *Invariant with respect to a shift and a rotation;*
(3) *Normalized by the condition $\lambda([0,1]^d) = 1$?*

There is no a countably additive measure with these properties for every natural number d according to the article by F. Hausdorff [21]. In 1923, S. Banach proved the existence of a finitely additive measure which is defined on the σ-ring of all bounded subsets of Euclidean spaces $\mathbb{R}^d$, $d = 1,2$, so that this measure is invariant with respect to any isometry ([22], p. 81). Hence, finitely additive measures can admit invariance with respect to a wider group than a countably additive one.

The paradox of Hausdorff–Banach–Tarskii includes some restrictions on the properties of a finitely additive measure on the Euclidean space, which are discussed in [23]. In particular, there is no finitely additive measure on the Euclidean space $\mathbb{R}^d$ with dimension $d \geq 3$, which is defined on the ring of all bounded subsets of this space and invariant with respect to shifts and rotations. Nevertheless, according to the work by S. Banach, in the case $d = 1,2$, there is a finitely additive non-negative measure ν_d on the space $\mathbb{R}^d$ such that this measure is shift and rotation invariant, is defined on every bounded subset of the space $\mathbb{R}^d$ and is normalized by the condition $\nu_d([0,1)^d) = 1$.

In the class of countably additive measures, there is the unique normalized shift-invariant complete Borel measure on the space $\mathbb{R}^d$ satisfying the normalization condition 3. This is the Lebesgue measure. In addition, this measure is invariant with respect to the orthogonal group.

In 1923, the following S. Ruziewicz problem was posed. Let $\mathcal{B}(\mathbb{R}^n)$ be the ring of bounded Lebesgue measurable sets in the n-dimensional real space $\mathbb{R}^n$. Let λ_n be the Lebesgue measure on $\mathcal{B}(\mathbb{R}^n)$ normalized by $\lambda_n([0,1)^n) = 1$. The following question was posed by Rusiewicz: *Is λ_n, up to proportionality, the unique finitely additive isometry-invariant positive measure mapping the ring $\mathcal{B}(\mathbb{R}^n)$ into the semiaxis $[0,+\infty)$?*

In 1923, Banach gave a negative answer to this question for $\mathbb{R}^1$ and $\mathbb{R}^2$. For $\mathbb{R}^n$ with $n \geq 3$, the positive answer to the Ruziewicz question was provided in [24,25].

The Lebesgue measure can be defined as the complete countably additive shift-invariant extension of the measure defined on the ring $r(\mathbb{R}^d)$ and normalized by condition 3). Here, $r(\mathbb{R}^d)$ is the ring generated by the collection of bounded d-dimensional rectangles. Hence, the Ruziewicz problem can be reformulated in the following way: *what finitely additive isometry-invariant positive measure is the extension of the measure defined on the ring $r(\mathbb{R}^d)$ and normalized by condition 3)?*

However, in the separable infinite-dimensional real Hilbert space E, the Ruziewicz question should be reformulated since there is no Lebesgue measure on space E or on the sphere in space E. In the present paper, we study the following question.

What measure on space E exists so that this measure satisfies the conditions:

(1) *It is invariant to any bijective isometric transformation of this space;*
(2) *The domain of this measure is the ring $\mathcal{R}$ of subsets of space E which contains any measurable rectangle (a measurable rectangle in the space E is an infinite-dimensional parallelepiped, such that the product of lengths of its edges converges unconditionally) of space E;*
(3) *The normalized condition $\lambda(\{x \in E: \ (x,e_k) \in [0,1) \ \forall\, k \in \mathbb{N}\}) = 1$ holds for some ONB $\mathcal{E} = \{e_k\}$?*

The constructing of a finitely additive shift-invariant measure and study of its properties was started in the work [17]. Results of Sections 3–7 on the rotation-invariant measure on a Hilbert space were announced in [26]. The construction of a symplectic-invariant measure was presented in [27]. The present article is the review of results on finitely additive invariant measures. The notion of ring ergodicity is introduced in this paper and applications of this notion are new results. The new result of this article are the analysis of the continuity of Koopman unitary representation together with the spectral properties of a Koopman generator.

1.3. The Main Result and Comparison with Similar Approaches

The purpose of this paper is to introduce a finitely additive measure on an infinite-dimensional real separable Hilbert space so that this measure is invariant with respect to shifts and orthogonal mappings (i.e., the measure is shift- and rotation-invariant). Moreover, the introduced measure is locally finite and σ-finite. However, it is nether countably additive nor a Borel measure.

In the separable real Euclidean space, there is no normalized shift-invariant countably additive σ-finite and locally finite measure. There are different shift-invariant finitely additive σ-finite and locally finite measures, which are normalized by the condition $\lambda([0,1)^{\mathbb{N}} \bigcap \ell_2) = 1$ (see [17,20]). The existence of a shift and rotation-invariant measure on a real separable Hilbert space was proven in [17] using the transfinite induction procedure. In the present paper, the extension of one of these measures up to the isometry-invariant normalized measure is introduced.

The construction of a shift and rotation-invariant measure on the real separable Hilbert space E is based on the analysis of the deformation under the action of orthogonal mappings on a shift-invariant measure $\lambda_{\mathcal{E}}$ on the space E. We obtain the criterion of an absolute continuity of the measure $\lambda_{\mathcal{E}}$ with respect to the image $\lambda_{\mathcal{E}} \circ \mathbf{U}$ of a measure $\lambda_{\mathcal{E}}$ under the action of an orthogonal mapping $\mathbf{U}$. We prove that if the measure $\lambda_{\mathcal{E}} \circ \mathbf{U}$ is absolutely continuous with respect to the measure $\lambda_{\mathcal{E}}$, then measures $\lambda_{\mathcal{E}} \circ \mathbf{U}$ and $\lambda_{\mathcal{E}}$ coincide. In the opposite case, measures $\lambda_{\mathcal{E}} \circ \mathbf{U}$ and $\lambda_{\mathcal{E}}$ are defined on the different rings $\mathcal{R}_{\mathbf{U}\mathcal{E}}$, $\mathcal{R}_{\mathcal{E}}$ and $\mathcal{R}_{\mathbf{U}\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{E}} = \{A \subset E : \ \lambda_{\mathbf{U}\mathcal{E}}(A) \subset \lambda_{\mathcal{E}}(A) = 0\}$.

The equivalence relation $\sim$ on the set of the ONB of space E is introduced by the following way. Two orthonormal bases, $\mathcal{E}$ and $\mathcal{F}$, are equivalent to each other if and only if $\lambda_{\mathcal{F}} = \lambda_{\mathcal{E}}$. We prove that if two bases, $\mathcal{E}$ and $\mathcal{F}$, are not equivalent, then the restrictions of measures $\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{F}}$ on the intersection of their domains are equal to zero. This property allows gluing of the measures $\lambda_{\mathcal{E}}$, which are defined on the subset rings $\mathcal{R}_{\mathcal{E}}$ into the unique measure λ, which is defined on the unique ring $\mathcal{R}$. This analysis of measures $\lambda_{\mathcal{E}} : \ \mathcal{R}_{\mathcal{E}} \to \infty$, corresponding to different ONB $\mathcal{E}$, gives the rule for defining an isometry-invariant measure $\lambda : \ \mathcal{R} \to [0, +\infty)$, where $\mathcal{R}$ is the ring of subsets generated by the collection of sets $\bigcup\limits_{\mathcal{E}} \mathcal{R}_{\mathcal{E}}$.

The paper [18] is devoted to the properties of a measure on a topological vector space, which is invariant with respect to a shift in a vector from some admissible subspace of the topological vector space. A constructed measure has all the properties of Lebesgue measures listed in the Weil Theorem except invariance with respect to a shift in an arbitrary vector.

The problem of the existence of translation-invariant measures on an Abelian topological group G can be considered as the description of a shift-invariant linear functional on the space of proper functions on group G. The description of the translation invariant functionals on space $L^q(G)$, $1 < q < +\infty$ is given in [28]. Invariant means on an infinite product of measured spaces with an infinite measure are defined and studied in [29] using a limit of normalized finite-dimensional approximation. A generalized shift-invariant measure is investigated in [30] as a shift-invariant functional on the space of test functions of the Schwartz type. Constructed functionals have properties of invariance with respect to the group of orthogonal mappings. However, the problem of the existence of a measure as the additive set function on some rings of the subsets is currently unresolved.

Ergodic properties of countable-additive measures on a topological group with respect to the action of a group of automorphisms are important for the analysis of problems of dynamical systems and the dynamical properties of solutions of evolution PDE's [31–34]. Now we extend this approach to the study of ergodic properties of finitely-additive measures on topological groups without the locally compactness property.

Let G be a group of mappings of a space E into itself. Let $\mathcal{R}$ be a ring of subsets of a space E which is invariant with respect to the group G. An invariant with respect to the group-G measure $\mu : \mathcal{R} \to [0, +\infty)$ is called *ring-ergodic* with respect to group G if for any two G-invariant subrings r_1, r_2 of the ring $\mathcal{R}$ the following two conditions

(i) the ring $\mathcal{R}$ is completion with respect to the measure μ of the ring, which is generated by the collection of sets $r_1 \bigcup r_2$,
(ii) $\mu|_{r_1} \neq 0,\ \mu|_{r_2} \neq 0$,

imply that there is a set $A \in r_1 \bigcap r_2$ such that $\mu(A) > 0$ (conversely, the measure μ is called *ring-decomposible*).

Roughly speaking, the definition of ring ergodicity changes the condition of an invariant subset to the condition of an invariant subring in the definition of the ergodicity of a measure with respect to a group.

The decomposition of an G-invariant measure μ to the sum of ring-ergodic mutually singular measures is called the *ring-ergodic decomposition of the measure μ*.

The properties of ergodicity or the decomposability of a G-invariant measure are important to the study of the uniqueness of a G-invariant measure. We use the notion of the ring ergodicity of a shift-invariant measure to parametrize the collection of mutually singular shift-invariant measures. The ring-ergodic decomposition of a shift-invariant measure is obtained. Thus, the obtained ergodic decompositions describe the collection of measures satisfying the condition of invariantness with respect to the considered group and the normalization condition from Section 1.2.

The Lebesgue measure on the Euclidean space $\mathbb{R}^{2N}$ is invariant not only to the group of isometries but with respect to symplectomorphisms of the space $\mathbb{R}^{2N}$ equipped with a shift-invariant symplectic form. This property is important to applications in statistical mechanics. Let us equip the Hilbert space E with a shift-invariant symplectic form. Then, the measures considered above have neither invariantness with respect to the group of simplectomorphism nor with respect to the subgroup of linear symplectomorphisms. We consider the measures that are invariant to the subgroup of symplectomorphisms preserving two-dimensional symplectic subspaces. In addition, we prove that there is no measure on a Hilbert space equipped with the shift-invariant symplectic form such that this measure is invariant with respect to the above subgroup of symplectomorphisms and the orthogonal group.

The invariance of a measure on the space of complex matrices with respect to a group of unitary transformations is studied in [35]. In this paper, the Pickerell measures on the space of infinite complex matrices and on the Grassman manifold of infinite-dimensional Hilbert space are constructed. Pickerell measures are the two-parametric family of probability measures on the space of complex matrices such that each of these measures is invariant with respect to a infinite subgroup of a unitary-operators group acting on the space of complex matrices by means of conjugation [36].

The measures of the algebras of operators are studied in [37]. Some of these measures were defined by means of operator intervals [38,39]. The invariance of the introduced measures on algebras with respect to the action of some groups was obtained.

1.4. Organization of the Paper

The structure of the present article is the following.

Section 2 introduces the family of shift-invariant measures $\{\lambda_{\mathcal{E}}\}$, where a measure $\lambda_{\mathcal{E}} :\ \mathcal{R}_{\mathcal{E}} \to [0, +\infty)$ is defined on the ring $\mathcal{R}_{\mathcal{E}}$ of subsets of the space E. The ring $\mathcal{R}_{\mathcal{E}}$ and the measure $\lambda_{\mathcal{E}}$ depend on the choice of ONB $\mathcal{E}$ in the space E [17,40]. For the constructed measure, we obtain its decomposition onto the sum of pairwise singular measures which are ring ergodic with respect to the group of shifts or to the subgroup of shifts with a continuity property.

Section 3 contains the description of the mutual position for two ONB in space E in terms of an infinite orthogonal transition matrix. The condition of the proximity of one ONB to another in terms of the transition matrix is introduced. Section 4 shows that if ONB $\mathcal{E}$ and $\mathcal{F}$ satisfy the proximity condition, then the measure $\lambda_{\mathcal{E}}$ coincides with the measure $\lambda_{\mathcal{F}}$. Section 5 demonstrates that if ONB $\mathcal{E}$ and $\mathcal{F}$ do not satisfy the proximity condition, then $\lambda_{\mathcal{E}}(A) = 0 = \lambda_{\mathcal{F}}(A) \ \forall \ A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$.

To solve the problems of Sections 4 and 5, properties of the intersection of a measurable rectangle with its image under the action of a shift or an orthogonal mapping are studied. This geometric problem is interesting as the infinite-dimensional generalization of the theory of k-dimensional sections of n-dimensional cubes [41,42]. The solving of this problem gives the opportunity to introduce the equivalence relation on the set of ONB of the space E in terms of the proximity condition from Section 3.

The proof of the existence of an isometry-invariant analog of a Lebesgue measure λ on a Hilbert space is given in Section 6 using the introduced equivalence relation on the set of ONB. The decomposition of the measure λ into the sum of mutually singular shift-invariant measures is obtained. In Section 6, we study the space $\mathbb{H} = L_2(E, \mathcal{R}, \lambda, \mathbb{C})$ of complex valued functions which are quadratically integrable with respect to an isometry-invariant measure. The orthogonal decomposition of space $\mathbb{H}$ corresponding to the mutually singular decomposition of the measure λ is obtained. Any component of the orthogonal decomposition is invariant with respect to a shift in any vector of the space E. The whole space $\mathbb{H}$ is invariant with respect to a shift and to an orthogonal transformation.

In Section 7, we study the unitary group in the Hilbert space $\mathbb{H}$ which is generated by the orthogonal mapping of arguments of the functions from the space $\mathbb{H}$. The Koopman representation of the orthogonal group in space E by means of the unitary group in space $\mathbb{H}$ is obtained. The condition of strong continuity in space $\mathbb{H}$ and the description of continuity subspaces for these unitary groups are obtained in Section 7. These results are important for extending the procedure of the averaging of random orthogonal mappings to the infinite-dimensional case, and for obtaining the differential equation describing the mean values of the compositions of independent random orthogonal mappings [43].

To study the symplectic-invariant measure, in Section 8 we equip a Hilbert space with a shift-invariant symplectic form. We introduce a measure which is the continuation of a shift-invariant measure. A continued measure is invariant with respect to a group of symplectomorphisms (namely, the group of symplectomorphisms, preserving two-dimensional symplectic subspaces). The unitary Koopman representation of the above group of symplectomorphisms is obtained in the space of functions that are quadratically integrable with respect to a symplectic-invariant measure. The continuity of a Koopman group and its spectral properties are studied. In addition, we prove that a considered symplectic-invariant measure has no continuation that is invariant with respect to orthogonal groups.

Section 9 is the conclusion of the main results of the article.

2. Shift-Invariant Measures on a Hilbert Space

Let E be a real separable Hilbert space. Let $\mathcal{S}$ be a set of ONB in space E.

Here, we introduce a family of shift-invariant measures on the Hilbert space E. This family of measures, $\{\lambda_{\mathcal{E}}, \ \mathcal{E} \in \mathcal{S}\}$, is parametrized by the choice of ONB $\mathcal{E} = \{e_i\}$ ([17,20]).

A set $\Pi \subset E$ is said to be a rectangle if there is an ONB $\mathcal{E} = \{e_i\}$ and elements $a, b \in \ell_\infty$ such that

$$\Pi = \{x \in E : \ (x, e_j) \in [a_j, b_j) \ \ \forall \, j \in \mathbb{N}\}. \tag{1}$$

A rectangle (1) is called measurable if either $\Pi = \varnothing$ or the following condition holds

$$\sum_{j=1}^{\infty} \max\{0, \ln(b_j - a_j)\} < \infty. \tag{2}$$

Let $\mathcal{K}$ be a collection of measurable rectangles in the space E. Let $\mathcal{E}$ be an ONB in the space E. Let $\mathcal{K}_{\mathcal{E}}$ be a set of measurable rectangles in E such that the edges of any rectangle

$\Pi \in \mathcal{K}_{\mathcal{E}}$ are collinear to vectors of ONB $\mathcal{E}$. In other words, if $\Delta_j \subset \mathbb{R}$ is the projection of a set Π onto the axis Oe_j for any $j \in \mathbb{N}$, then $\Pi = \{x \in E: \ (x, e_j) \in \Delta_j \ \forall\, j \in \mathbb{N}\}$. Let $\mathrm{r}_{\mathcal{E}}$ be a ring of subsets of the space E which is generated by the set $\mathcal{K}_{\mathcal{E}}$. According to [20], the ring $r_{\mathcal{E}}$ is generated by the following semiring $s_{\mathcal{E}}$ of subsets of space E

$$s_{\mathcal{E}} = \{A_0 \backslash (\bigcup_{j=1}^{m} A_j), \quad m \in \mathbb{N}, A_0, \dots, A_m \in \mathcal{K}_{\mathcal{E}}\}. \tag{3}$$

Let r be a ring of subsets of space E generated by the collection $\mathcal{K}$.

Let λ be a function of a set such that the function λ is defined on the collection of sets $\mathcal{K}$ by the equality

$$\lambda(\Pi) = \exp\Big[\sum_{j=1}^{\infty} \ln(b_j - a_j)\Big] \tag{4}$$

for any non-empty measurable rectangle (1), and $\lambda(\varnothing) = 0$. According to the condition (2), we have $\lambda(\Pi) \in [0, +\infty)$ for any $\Pi \in \mathcal{K}$. Let $\lambda_{\mathcal{E}}$ be the restriction of the function of a set λ to the collection of sets $\mathcal{K}_{\mathcal{E}}$.

According to papers [17,20], the function $\lambda_{\mathcal{E}}$ is an additive function on the collection of sets $\mathcal{K}_{\mathcal{E}}$ and it has the unique extension to the measure $\lambda_{\mathcal{E}}: \ \mathrm{r}_{\mathcal{E}} \to \mathbb{R}$. This measure, $\lambda_{\mathcal{E}}: \ \mathrm{r}_{\mathcal{E}} \to [0, +\infty)$, is invariant with respect to a shift on a vector of space E.

A set $A \subset E$ is said to be $\lambda_{\mathcal{E}}$-measurable if for any $\epsilon > 0$ there are sets $A_*, A^* \in \mathrm{r}_{\mathcal{E}}$ such that $A_* \subset A \subset A^*$ and $\lambda(A^* \backslash A_*) < \epsilon$. Then, the collection $\mathcal{R}_{\mathcal{E}}$ of $\lambda_{\mathcal{E}}$-measurable subsets of the space E is the ring. The measure $\lambda_{\mathcal{E}}: \ \mathrm{r}_{\mathcal{E}} \to [0, +\infty)$ has the unique extension to the ring $\mathcal{R}_{\mathcal{E}}$ by the equality $\lambda_{\mathcal{E}}(A) = \inf\limits_{A^* \in \mathrm{r}_{\mathcal{E}}, A^* \supset A} \lambda(A^*) \ \ \forall\, A \in \mathcal{R}_{\mathcal{E}}$.

The function of a set $\lambda_{\mathcal{E}} : \mathcal{R}_{\mathcal{E}} \to [0, +\infty)$ is the finitely additive measure which is invariant with respect to a shift on any vector of the space E [17]. This measure is locally finite, σ-finite, complete. However, this measure is not σ-additive and it is not defined on the σ ring of bounded Borel subsets. In particular, the ring $\mathcal{R}_{\mathcal{E}}$ does not contain a ball in space E with a sufficiently large radius [44].

Thus, for a given ONB $\mathcal{E} \in \mathcal{S}$, there is the ring of subsets $\mathcal{R}_{\mathcal{E}}$ and there is the shift-invariant finitely additive locally finite and σ-finite measure $\lambda_{\mathcal{E}}: \ \mathcal{R}_{\mathcal{E}} \to [0, +\infty)$.

2.1. Dependence of the Measure $\lambda_{\mathcal{E}}$ on ONB

The paper [17] describes the procedure of extending the family of measures $\{\lambda_{\mathcal{F}}: \ \mathcal{R}_{\mathcal{F}} \to [0, +\infty); \ \mathcal{F} \in \mathcal{S}\}$ to the measure $\lambda: \mathcal{R} \to [0, +\infty)$, where $\mathcal{R}$ is the ring generated by the collection of sets $\bigcup\limits_{\mathcal{F} \in \mathcal{S}} \mathcal{R}_{\mathcal{F}}$ (or generated by the collection of sets $\mathcal{K}$).

The existence of the measure $\lambda: \ \mathcal{R} \to [0, +\infty)$ such that $\lambda|_{\mathcal{R}_{\mathcal{F}}} = \lambda_{\mathcal{F}} \ \forall\, \mathcal{F} \in \mathcal{S}$ is proven in the work [17] by using some total ordering $\prec$ on the set $\mathcal{S}$ of ONB in the space E and by applying transfinite induction procedure. It is proven that the measure λ:

(1) Is invariant with respect to any orthogonal mapping and to a shift on a vector $\mathbf{h} \in E$;
(2) Does not depend on the choice of total ordering $\prec$ on the set $\mathcal{S}$.

The dependence of the properties of the ring $\mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ on the mutual position of two ONB $\mathcal{E}$ and $\mathcal{F}$ is not considered in the paper [17]. In the present article, we describe the dependence of the ring $\mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ on the mutual position of two ONB, $\mathcal{E}$ and $\mathcal{F}$.

2.2. Representation of the Group of Shifts in the Space $\mathcal{H}_{\mathcal{E}} = L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ and the Subgroup of Strong Continuity

The shift-invariant measure $\lambda_{\mathcal{E}}$ defines the space $\mathcal{H}_{\mathcal{E}} = L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ of quadratically integrable functions. In order to construct the space $\mathcal{H}_{\mathcal{E}}$, one should consider the space $S(\mathcal{R}_{\mathcal{E}})$ of finite linear combinations over the field $\mathbb{C}$ of the indicator functions of sets from the ring $\mathcal{R}_{\mathcal{E}}$. Let us introduce a non-negative hermitian sesquilinear form: for any

$A,\ B \in \mathcal{R}_{\mathcal{E}}$, one can pose $(\chi_A,\, \chi_B) = \lambda(A \cap B)$. For arbitrary functions $f,\, g \in S(\mathcal{R}_{\mathcal{E}})$, where

$$f(x) = \sum_{k=1}^{s} \alpha_k \chi_{A_k}(x) \quad \text{and} \quad g(x) = \sum_{l=1}^{p} \beta_l \chi_{B_l}(x)$$

we have

$$(f,\, g) = \sum_{k=1}^{s} \sum_{l=1}^{p} \alpha_k \overline{\beta}_l (\chi_{A_k},\, \chi_{B_l}).$$

We call functions $f,\, g \in S(\mathcal{R}_{\mathcal{E}})$ *equivalent* if $(f-g,\, f-g) = 0$. Thus, the linear space of classes of the equivalence of functions from $S(\mathcal{R}_E)$ is pre-Hilbert, and after the procedure of completion, $\mathcal{H}_{\mathcal{E}}$ is obtained. The same construction of the space $\mathcal{H} = L_2(E, \mathcal{R}, \lambda, \mathbb{C})$ will be used for other choices of a measure λ on a ring $\mathcal{R}$ of subsets of the space E.

The space E, as the group with respect to the summation operation, is represented in the space $\mathcal{H}_{\mathcal{E}}$ by the Abelian unitary group of shift operators $\mathcal{S} = \{\mathbf{S}_h,\ h \in E\}$ acting by the rule $\mathbf{S}_h u(x) = u(x-h),\ x \in E$. The subgroup $\mathbf{S}_{th},\ t \in \mathbb{R}$ of the group $\mathcal{S}$ is a one-parameter unitary group in the space $\mathcal{H}_{\mathcal{E}}$ for every vector $h \in E$.

In paper [17], the criterion of the strong continuity of the one-parameter unitary group $\mathbf{S}_{th},\ t \in \mathbb{R}$ in the space $\mathcal{H}_{\mathcal{E}}$ is obtained.

Theorem 1 ([17]). *Let $\mathcal{E}$ be an ONB in the space E and $h \in E$. Then, the one-parameter unitary group $S_{th},\ t \in \mathbb{R}$ is continuous in the strong operator topology of the space $\mathcal{H}_{\mathcal{E}}$ if and only if $\{(h, e_k)\} \in l_1$.*

If $\mathcal{E}$ is ONB in the space E, then $L_1(\mathcal{E}) = \{h \in E :\ \{(h, e_k)\} \in l_1\}$ is the linear subspace of the space E (hence, $L_1(\mathcal{E})$ is the subgroup of the group E). The subgroup $L_1(\mathcal{E})$ equipped with the norm $\|x\|_{L_1(\mathcal{E})} = \sum\limits_{k=1}^{\infty} |(x, e_k)|$ is the Banach topological group.

Corollary 1. *Let $\mathcal{E}$ be an ONB in space E. If $\mathcal{S}_1(\mathcal{E}) = \{\mathbf{S}_h,\ h \in L_1(\mathcal{E})\}$ when equipped with the strong operator topology τ_{sot} of the space $B(\mathcal{H}_{\mathcal{E}})$, then the topological group of linear operators $(\mathcal{S}_1(\mathcal{E}),\ \tau_{sot})$ is the continuous unitary representation in the space $\mathcal{H}_{\mathcal{E}}$ of the Abelian topological group $(L_1(\mathcal{E}),\ \|\cdot\|_{L_1(\mathcal{E})})$.*

Corollary 2. *Let $\mathcal{E}$ be an ONB in space E. If the group $\mathcal{S}$ is equipped with the strong operator topology τ_{sot} of the space $B(\mathcal{H}_{\mathcal{E}})$, then the topological group of linear operators $(\mathcal{S},\ \tau_{sot})$ is the unitary representation of the Abelian topological group $(E,\ \|\cdot\|_E)$ in the space $\mathcal{H}_{\mathcal{E}}$. However, this representation is not continuous.*

2.3. Decomposition of a Shift-Invariant Measure $\lambda_{\mathcal{E}} :\ \mathcal{R}_{\mathcal{E}} \to [0, +\infty)$ Ring-Ergodic with Respect to a Strongly Continuous Group of Shifts

The ergodic properties of countable-additive measures on a topological group with respect to the action of a group of automorphisms are important for the analysis of the problems of dynamical systems and the dynamical properties of evolution PDE [31–34]. Now, we extend the approach of ergodic theory to the study of the properties of finitely additive measures on a topological group without the locally compactness property.

Let G be a group of mappings of a space E into itself. Let $\mathcal{A}$ be an invariant with respect to the group-G algebra of subsets of a space E. A G-invariant measure $\mu : \mathcal{A} \to [0, +\infty)$ is called ergodic with respect to the group G if for every G-invariant set $A \in \mathcal{A}$ either $\mu(A) = 0$ or $\mu(E \backslash A) = 0$.

Now we consider a decomposition of the ring which is the domain of the measure onto invariant subrings instead of the decomposition of the space onto invariant subspaces.

Let G be a group of mapping of a space E into itself. Let $\mathcal{R}$ be an invariant with respect to the group-G ring of subsets of a space E.

Let us note that the ring $\mathcal{R}$ is called the completion of the ring r with respect to a measure $\mu: r \to [0,+\infty)$ if

$$\mathcal{R} = \{A \in E: \forall \epsilon > 0 \, \exists B_1, B_2 \in r: \lambda(B_2 \backslash B_1) < \epsilon\}.$$

Definition 1. *An invariant with respect to the group-G measure $\mu : \mathcal{R} \to [0,+\infty)$ is called*

(1) *Ring-decomposible with respect to group G if there are two G-invariant subrings r_1, r_2 of ring $\mathcal{R}$ satisfying conditions (i) and (ii) such that $\mu(A) = 0 \, \forall \, A \in r_1 \bigcap r_2$;*
(2) *Ring-ergodic with respect to group G if, for any two G-invariant subrings r_1, r_2 of ring $\mathcal{R}$, conditions (i) and (ii) imply that there is a set $A \in r_1 \bigcap r_2$ such that $\mu(A) > 0$.*

Here, (i) and (ii) are the following conditions:

(i) *Ring $\mathcal{R}$ is the completion with respect to the measure μ of the ring which is generated by the collection of sets $r_1 \bigcup r_2$, $\mu(A) = 0 \, \forall \, A \in r_1 \bigcap r_2$;*
(ii) $\mu_{r_1} \neq 0, \; \mu|_{r_2} \neq 0,$

If an invariant with respect to the group-G measure $\mu : \mathcal{R} \to [0,+\infty)$ is ring-decomposible with respect to group G, then this measure admits the decomposition $\mu = \nu_1 + \nu_2$ into the sum of two mutually singular G-invariant measures $\nu_i(A) = \sup\limits_{B \in r_j, B \subset A} \mu(B)$, $i = 1,2$.

The decomposition of a G-invariant measure μ to the sum of ring-ergodic mutually singular measures is called the *ring-ergodic decomposition of the G-invariant measure μ*.

Now we present the example of ring-ergodic decomposition for the measure $\lambda_{\mathcal{E}}$. Let $\mathcal{E}$ be an ONB in space E. Let us consider the measurable space $(E, \mathcal{E})$ equipped with the measure $\lambda_{\mathcal{E}}$.

Let us consider the representation of the topological group $(E, \|\cdot\|_E)$ by the group of unitary operators $\mathcal{S}$ in the space $\mathcal{H}_{\mathcal{E}}$.

We study the following questions. Is the measure $\lambda_{\mathcal{E}}$ ring-ergodic with respect to group E? What ring-ergodic components with respect to group E does the measure $\lambda_{\mathcal{E}}$ admit?

A non-empty rectangle $\Pi \in \mathcal{K}_{\mathcal{E}}$ is called E-equivalent to a rectangle $Q \in \mathcal{K}_{\mathcal{E}}$ ($\Pi \sim_E Q$) if there is a vector $h \in E$ such that $Q = \Pi + h$. For a rectangle $\Pi = \Pi_{a,b}, \; a, b \in l_\infty$, the point $c(\Pi) \in l_\infty$ is called the center of the rectangle $\Pi_{a,b}$ if $c(\Pi) = \frac{1}{2}(a+b)$. A rectangle $\Pi = \Pi_{a,b}, \; a, b \in l_\infty$ is non-empty if and only if $a_j < b_j \, \forall \, j \in \mathbb{N}$ and $\max\{0, |c_j| - \frac{1}{2}\} \in l_2$. For every vector $c \in l_\infty$, the symbol $\mathcal{K}_{\mathcal{E}}(c)$ denotes the collection of non-empty rectangles $\Pi \in \mathcal{K}_{\mathcal{E}}$ such that $c(\Pi) - c \in l_2$ (any rectangle $\Pi \in \mathcal{K}_{\mathcal{E}}(c)$ is E-equivalent to the rectangle with the center c).

Let $\mathcal{C}_\infty$ be a set of vectors $c \in l_\infty$ such that $\mathcal{K}_{\mathcal{E}}(c) \neq \emptyset$. Let $S_\infty = \mathcal{C}_\infty / l_2$ be the set of classes of E-equivalent vectors of the set $\mathcal{C}_\infty$.

Lemma 1. *Let $c_1, c_2 \in \mathcal{S}_\infty$ and $c_1 \neq c_2$. If $\Pi' \in \mathcal{K}_{\mathcal{E}}(c_1)$, $\Pi'' \in \mathcal{K}_{\mathcal{E}}(c_2)$, then $\lambda_{\mathcal{E}}(\Pi' \bigcap \Pi'') = 0$.*

Proof. If $\lambda_{\mathcal{E}}(\Pi') = 0$ or $\lambda_{\mathcal{E}}(\Pi'') = 0$, then the statement is true. If $\lambda_{\mathcal{E}}(\Pi') \neq 0$ and $\lambda_{\mathcal{E}}(\Pi'') \neq 0$, then for any $k \in \mathbb{N}$ we have $a'_k = c_{1,k} - \frac{1}{2} - \alpha'_k$, $b'_k = c_{1,k} + \frac{1}{2} + \alpha'_k$ where $\alpha' \in l_1$. Analogously, for any $k \in \mathbb{N}$, we have $a''_k = c_{2,k} - \frac{1}{2} - \alpha''_k$, $b''_k = c_{2,k} + \frac{1}{2} + \alpha''_k$ where $\alpha'' \in l_1$.

We have $\Delta c = c_2 - c_1 \notin l_2$ since $c_1 \neq c_2$ in $\mathcal{S}_\infty$. Let $\Pi = \Pi' \bigcap \Pi''$ and $[a_k, b_k)$ be the projection of the rectangle Π to the k-th coordinate axis span(e_k). Therefore, $b_k - a_k \leq \max\{0, \; \min\{b''_k - a''_k, \, b'_k - a'_k\} - |\Delta c_k|\}$ for any $k \in \mathbb{N}$. Hence, $b_k - a_k \leq 1 + \beta_k, \; k \in \mathbb{N}$ where $\beta_k = \alpha_k - |\Delta c_k|, \; k \in \mathbb{N}$. Thus, $\prod\limits_{k=1}^{\infty}(b_k - a_k) \leq \prod\limits_{k=1}^{\infty}(1 + \beta_k) = 0$, since $\beta \notin l_1$ and $\beta_k < 0$ for every sufficiently large k. □

Let $r_{c,\mathcal{E}}$ be the ring generated by the family of sets $\mathcal{K}_{\mathcal{E}}(c)$ for every $c \in \mathcal{S}_\infty$ (the construction of this ring is described in papers [20]).

In the paper [20], it is proven that the function of a set $\lambda_{\mathcal{E}}:\ \mathcal{K}_{\mathcal{E}} \to [0,+\infty)$ is additive and has the unique additive extension to the measure $\lambda_{\mathcal{E}}$ whose domain is the ring $r_{\mathcal{E}}$. The same arguments prove the following statement.

Lemma 2. *The function of a set $\lambda_{c,\mathcal{E}}$ is additive and has the unique additive extension to the measure $\lambda_{c,\mathcal{E}}$ whose domain is the ring $r_{c,\mathcal{E}}$. Moreover, $\lambda_{c,\mathcal{E}} = \lambda_{\mathcal{E}}|_{\mathcal{K}_{\mathcal{E}}(c)}$.*

Proof. The ring $r_{c,\mathcal{E}}$ is generated by the semiring

$$\Lambda_{c,\mathcal{E}} = \{A_0 \backslash (\bigcup_{j=1}^{n} A_j),\ n \in \mathbb{N},\ A_0,\dots,A_n \in \mathcal{K}_{c,\mathcal{E}}\}.$$

For a given $n \in \mathbb{N}$, let us introduce families of sets

$$\Lambda^{(n)}_{c,\mathcal{E}} = \{A_0 \backslash (\bigcup_{j=1}^{n} A_j),\ A_0,\dots,A_n \in \mathcal{K}_{c,\mathcal{E}}\}, \qquad V^{(n)}_{c,\mathcal{E}} = \{\bigcup_{j=1}^{n} A_j,\ A_1,\dots,A_n \in \mathcal{K}_{c,\mathcal{E}}\}.$$

Let us note that $V^{(1)}_{c,\mathcal{E}} = \mathcal{K}_{c,\mathcal{E}}$. Since the function of a set $\lambda_{\mathcal{E}}:\ \mathcal{K}_{\mathcal{E}} \to [0,+\infty)$ is additive ([20]), its restriction $\lambda_{c,\mathcal{E}}:\ \mathcal{K}_{c,\mathcal{E}} \to [0,+\infty)$ is additive too. Using the induction procedure, we can prove that the function $\lambda_{c,\mathcal{E}}$ has the unique additive extension on classes $V^{(n)}_{c,\mathcal{E}}$, $\Lambda^{(n)}_{c,\mathcal{E}}$ for every $n \in \mathbb{N}$ (see also [27], Theorem 3.1). Therefore, the function $\lambda_{c,\mathcal{E}}$ has the unique extension onto the semi-ring $\Lambda_{c,\mathcal{E}}$, and, hence, to the ring $r_{c,\mathcal{E}}$.

Let $\mathcal{R}_{c,\mathcal{E}}$ be the completion of the ring $r_{c,\mathcal{E}}$ by the measure $\lambda_{c,\mathcal{E}}$. The ring $r_{c,\mathcal{E}}$ is the subring of the ring $r_{\mathcal{E}}$ since $\mathcal{K}_{\mathcal{E}}(c) \subset \mathcal{K}_{\mathcal{E}}$. Therefore, $\lambda_{c,\mathcal{E}} = \lambda_{\mathcal{E}}|_{r_{c,\mathcal{E}}}$. □

Lemma 3. *Let $c_1, c_2 \in \mathcal{S}_\infty$ and $c_1 \neq c_2$. If $A \in \mathcal{R}_{c_1,\mathcal{E}} \bigcap \mathcal{R}_{c_2,\mathcal{E}}$, then $\lambda_{\mathcal{E}}(A) = 0$.*

Proof. For every $i = 1,2$, the indicator function of a set from the ring $\mathcal{R}_{c_i,\mathcal{E}}$ can be approximated in $\mathcal{H}_{\mathcal{E}}$-norm by the linear combination of the indicator functions of rectangles from the family $\mathcal{K}_{\mathcal{E}}(c_i)$. Thus, the statement of Lemma 3 is the consequence of Lemma 1. □

Let $\rho_{\mathcal{E}}$ be the ring which is generated by the collection of sets $\bigcup\limits_{c \in \mathcal{S}_\infty} \mathcal{R}_{c,\mathcal{E}}$.

Hence, the ring $\rho_{\mathcal{E}}$ is generated by the semi-ring

$$s_{\mathcal{E}} = \{A_0 \backslash \bigcup_{j=1}^{N} A_j,\ N \in \mathbb{N},\ A_0 \in \mathcal{R}_{c_0,\mathcal{E}},\ A_j \in \mathcal{R}_{c_j,\mathcal{E}},\ c_0, c_1, \dots, c_N \in \mathcal{S}_\infty\}. \tag{5}$$

Since the systems of sets $\mathcal{R}_{c_j,\mathcal{E}}$ are rings, we can assume that $c_j \neq c_i$ for every different $i,j = 0,1,\dots,N$. Hence, $\lambda_{\mathcal{E}}(A_0 \bigcap (\bigcup\limits_{j=1}^{N} A_j)) = 0$ according to the Lemma 3. Thus, we should define $\lambda(A_0 \backslash \bigcup\limits_{j=1}^{N} A_j) = \lambda_{\mathcal{E}}(A_0)$ for any set $A_0 \backslash \bigcup\limits_{j=1}^{N} A_j$, $j \in \mathbb{N}$, $A_0 \in \mathcal{R}_{c_0,\mathcal{E}}$, $A_j \in \mathcal{R}_{c_j,\mathcal{E}}$ from the semi-ring (5). Then, the function $\lambda:\ s_{\mathcal{E}} \to [0,+\infty)$ is additive on the semiring (5). Moreover, this additive function satisfies the condition $\lambda_{\mathcal{E}}(A) = \|\chi_A\|^2_{\mathbb{H}_{\mathcal{E}}}\ \forall\ A \in s$. Additive function $\lambda:\ s_{\mathcal{E}} \to [0,+\infty)$ on the semi-ring (5) admits the unique additive extension to the additive function of a set $\mu_{\mathcal{E}}:\ \rho_{\mathcal{E}} \to [0,+\infty)$ on the ring $\rho_{\mathcal{E}}$. Moreover, the measure $\mu_{\mathcal{E}}$ satisfy the condition $\lambda(A) = \|\chi_A\|^2_{\mathbb{H}}\ \forall\ A \in \rho_{\mathcal{E}}$.

The semi-ring $s_{\mathcal{E}}$ and the generated by this semi-ring ring $\rho_{\mathcal{E}}$ are invariant with respect to a shift on a vector of the space E. The measure $\mu_{\mathcal{E}}:\ \rho_{\mathcal{E}} \to [0,+\infty)$ is shift-invariant measure on the space E by its construction.

Then, according to Lemmas 1–3 we obtain the following statement.

Theorem 2. *Let $\mathcal{E}$ be an ONB in the space E. Then*

$$\mathcal{H}_{\mathcal{E}} = \oplus_{c\in\mathcal{S}_\infty}\mathcal{H}_{c,\mathcal{E}},$$

where $\mathcal{H}_{c,\mathcal{E}} = L_2(E,\mathcal{R}_{c,\mathcal{E}},\lambda_{c,\mathcal{E}},\mathbb{C})$.

Proof. Let $c \in \mathcal{S}_\infty$ and $\mathcal{H}_{c,\mathcal{E}} = L_2(E,\mathcal{R}_{c,\mathcal{E}},\lambda_{c,\mathcal{E}},\mathbb{C})$. Then, the space $\mathcal{H}_{c,\mathcal{E}}$ is the subspace of the space $\mathcal{H}_{\mathcal{E}}$ since the measure $\lambda_{c,\mathcal{E}}$ is the restriction of the measure $\lambda_{\mathcal{E}}$. Hence, according to Lemma 3 subspaces $\mathcal{H}_{c_1,\mathcal{E}}$ and $\mathcal{H}_{c_2,\mathcal{E}}$ are orthogonal subspaces of the space $\mathcal{H}_{\mathcal{E}}$ if $c_1 \neq c_2$.

The linear space $\mathrm{span}(\chi_\Pi,\ \Pi \in \mathcal{K}_{\mathcal{E}})$ is dense in the space $\mathcal{H}_{\mathcal{E}}$ according to Lemma 3.4 [45]. Hence, the family of functions

$$\left\{\sum_{j=1}^{N} a_j\chi_{\Pi_j},\quad N\in\mathbb{N},\, a_j\in\mathbb{C},\, \Pi_j\in\mathcal{K}_{\mathcal{E}}\right\} \tag{6}$$

is dense in the space $\mathcal{H}_{\mathcal{E}}$. Since every element of the family (6) belongs to the linear space $\mathrm{span}(\mathcal{H}_{c,\mathcal{E}},\ c\in\mathcal{S}_\infty)$, the linear space $\mathrm{span}(\mathcal{H}_{c,\mathcal{E}},\ c\in\mathcal{S}_\infty)$ is dense in the space $\mathcal{H}_{\mathcal{E}}$.

Then, according to Lemma 3 we have

$$\overline{\mathrm{span}(\mathcal{H}_{c,\mathcal{E}},\ c\in\mathcal{S}_\infty)}^{\mathcal{H}_{\mathcal{E}}} = \oplus_{c\in\mathcal{S}_\infty}\mathcal{H}_{c,\mathcal{E}}.$$

Hence, the statement is proved. □

Corollary 3. *The ring $\mathcal{R}_{\mathcal{E}}$ is the completion of the ring $\rho_{\mathcal{E}}$ with respect to the measure $\mu_{\mathcal{E}}$.*

Proof. According to the Theorem 2 for any set $A \in \mathcal{R}_{\mathcal{E}}$ there is the sequence of finite collection of rectangles $\Pi_1^{(n)},\dots\Pi_{m_n}^{(n)} \in \mathcal{K}_{\mathcal{E}},\ n\in\mathbb{N}$, such that the sequence of linear combinations $\{\sum\limits_{j=1}^{m_n}\sigma_j\chi_{\Pi_j}\}$ converges to χ_A in the space $\mathcal{H}_{\mathcal{E}}$. It means that the ring $\mathcal{R}_{\mathcal{E}}$ is the completion or rings $\rho_{\mathcal{E}}$ with respect to the measure $\mu_{\mathcal{E}}$. □

Corollary 4. *Let $c\in\mathcal{S}_\infty$ and $\mathcal{S}_\infty^c = \mathcal{S}_\infty\backslash\{c\}$. Then*

$$\mathcal{H}_{\mathcal{E}} = \mathcal{H}_{c,\mathcal{E}}\oplus\mathcal{H}_{\mathcal{E}}^c$$

where $\mathcal{H}_{\mathcal{E}}^c = \bigoplus\limits_{b\in\mathcal{S}_\infty^c}\mathcal{H}_{b,\mathcal{E}}$.

Corollary 5. *Let $c\in\mathcal{S}_\infty$ and $\mathcal{S}_\infty^c = \mathcal{S}_\infty\backslash\{c\}$. Let ${\mathcal{R}_{\mathcal{E}}}^c$ be the completion with respect to measure $\mu_{\mathcal{E}}$ of the ring which is generated by the set $\bigcup\limits_{b\in{\mathcal{S}_{\mathcal{E}}}^c}\mathcal{R}_{b,\mathcal{E}}$. Then, the ring $\mathcal{R}_{\mathcal{E}}$ is the completion with respect to measure $\mu_{\mathcal{E}}$ of the ring generated by the collection of sets $\mathcal{R}_{c,\mathcal{E}}\bigcup{\mathcal{R}_{\mathcal{E}}}^c$.*

The domain of a measure $\lambda_{c,\mathcal{E}}$ depends on a class of vectors $c\in\mathcal{S}_\infty$. Let us introduce following extensions $\nu_{c,\mathcal{E}}:\ \mathcal{R}_{\mathcal{E}}\to[0,+\infty),\ c\in\mathcal{S}_\infty$, of measures $\lambda_{c,\mathcal{E}}$:

$$\nu_{c,\mathcal{E}}(A) = \left\{\begin{array}{l}\lambda_{c,\mathcal{E}}(A)\ if\ A\in\mathcal{R}_{c,\mathcal{E}};\\ 0,\ if\ A\in\mathcal{R}_{c',\mathcal{E}},\ c'\in\mathcal{S}_\infty,\ c'\neq c.\end{array}\right\} \tag{7}$$

Let $c,c'\in\mathcal{S}_\infty,\ c'\neq c$. Then the following statement take place. If $\nu_{c',\mathcal{E}}(A) > 0$ for some $A\in\mathcal{R}_{\mathcal{E}}$, then $\nu_{c,\mathcal{E}}(A) = 0$ and vice versa according to (7). On the contrary, if $A\in\mathcal{R}_{c,\mathcal{E}}$, $B\in\mathcal{R}_{c,\mathcal{E}}$, $\lambda_{\mathcal{E}}(A) > 0$, $\lambda_{\mathcal{E}}(B) > 0$, then there is a vector $h\in E$ such that $\lambda_{\mathcal{E}}(A\bigcap\mathbf{S}_h(B)) > 0$.

Theorem 3. *For any $c \in \mathcal{S}_\infty$ the measure $\nu_{c,\mathcal{S}}$ is invariant under the action of the Abelian unitary group $\mathbb{S} = \{\mathbf{S}_h,\ h \in E\}$. The $\mathbb{S}$-invariant measure $\lambda_\mathcal{E} :\ \mathcal{R}_\mathcal{E} \to [0, +\infty)$ admits the ring-ergodic decomposition*

$$\lambda_\mathcal{E} = \sum_{c \in \mathcal{S}_\infty} \nu_{c,\mathcal{E}} \tag{8}$$

into the sum of mutually singular components.

Proof. For every $c \in \mathcal{S}_\infty$ the ring $\mathcal{R}_{c,\mathcal{E}}$ is invariant with respect to the group $\mathbb{S}$ by construction. Invariant subrings $\mathcal{R}_{c,\mathcal{E}},\ c \in \mathcal{S}_\infty$ of the ring $\mathcal{R}_\mathcal{E}$ are independent in the following sense. If $c, c' \in \mathcal{S}_\infty,\ c' \neq c$, then $\lambda_\mathcal{E}(A \bigcap \mathbf{S}_h(B)) = 0$ for every $A \in \mathcal{R}_{c,\mathcal{E}}$, $B \in \mathcal{R}_{c',\mathcal{E}}$, $h \in E$. According to Lemma 3 the equality

$$\lambda_\mathcal{E}(A) = \sum_{c \in \mathcal{S}_\infty} \nu_{c,\mathcal{E}}(A) \tag{9}$$

holds for every set $A \in r_\mathcal{E}$ since every set $A \in r_\mathcal{E}$ is the finite union of sets from the semi-rings (3).

Since the ring $\mathcal{R}_\mathcal{E}$ is the completion of the ring $r_\mathcal{E}$ with respect to the measure $\lambda_\mathcal{E}$, the equality (9) is valid. In fact, if $A \in \mathcal{R}_\mathcal{E}$ then there are sequences $\{C_i\}, \{D_i\} :\ \mathbb{N} \to r_\mathcal{E}$ such that $C_i \subset A \subset D_i$ and $\lambda_\mathcal{E}(D_i \backslash C_i)$. For every $i \in \mathbb{N}$ the equality (9) is valid for sets C_i, D_i since $C_i, D_i \in r_\mathcal{E}$. Since $0 \leq \lambda_{c,\mathcal{E}}(D_i \backslash C_i) \leq \lambda_\mathcal{E}(D_i \backslash C_i)\ \forall\ i$ then the equality (9) holds for every $A \in \mathcal{R}_\mathcal{E}$.

The measure $\lambda_{c,\mathcal{E}}$ is invariant with respect to the group $\mathbb{S}$ by the same property of the measure $\lambda_\mathcal{E}$. Thus, the measures in the decomposition (8) are mutually singular and $\mathbb{S}$-invariant.

Let us prove that the measure $\nu_{c,\mathcal{E}}$ is ring-ergodic with respect to the group $\mathbb{S}$. Let us assume the contrary that the measure $\nu_{c,\mathcal{E}}$ is not ring-ergodic measure of the group $\mathbb{S}$.

Hence, there are subrings $r_1, r_2 \subset \mathcal{R}_{c,\mathcal{E}}$ such that r_1, r_2 are invariant with respect to the group $\mathbb{S}$, the ring $\mathcal{R}_{c,\mathcal{E}}$ is the completion with respect to the measure $\nu_{c,\mathcal{E}}$ of the collection of sets $r_1 \bigcup r_2$, $\nu_{c,\mathcal{E}}|_{r_i}$ is nontrivial measure for $i = 1, 2$ and $\nu_{c,\mathcal{E}}(A) = 0\ \forall\ A \in r_1 \bigcap r_2$. Therefore, there are sets $A \in r_1,\ B \in r_2$ such that $\nu_{c,\mathcal{E}}(A) > 0$, $\nu_{c,\mathcal{E}}(B) > 0$ and

$$\nu_{c,\mathcal{E}}(A \bigcap \mathbf{S}_h(B)) = 0\ \forall\ h \in E \tag{10}$$

since $A \in r_1$ and $\mathbf{S}_h(B) \in r_2\ \forall\ h \in E$.

Since $\nu_{c,\mathcal{E}}(A) > 0$, $\nu_{c,\mathcal{E}}(B) > 0$, there are sets $\Pi, Q \in \mathcal{K}_\mathcal{E}(c)$ such that $\Pi \subset A,\ Q \subset B$ and $\nu_{c,\mathcal{E}}(\Pi) > 0$, $\nu_{c,\mathcal{E}}(Q) > 0$. Since $\Pi, Q \in \mathcal{K}_\mathcal{E}(c)$, there is a vector $h = c(\Pi) - c(Q) \in E$ such that $\nu_{c,\mathcal{E}}(\Pi \bigcap \mathbf{S}_h(Q)) > 0$. It is the contradiction with condition (10). □

2.4. Decomposition of a Shift-Invariant Measure Ring-Ergodic with Respect to the Strongly Continuous Subgroup of Shifts

The action of the group E transforms a ring $\mathcal{R}_{c,\mathcal{E}}$ into itself for every $c \in S_\infty$. But the representation $\{\mathbf{S}_h,\ h \in E$ of the group E is not strongly continuous in spaces $\mathcal{H}_\mathcal{E}$ and $\mathcal{H}_{c,\mathcal{E}},\ c \in S_\infty$.

The Abelian group $\mathbb{S}_1 = \{\mathbf{S}_h,\ h \in L_1(\mathcal{E})$ equipped with the strong operator topology τ_{sot} is the continuous unitary representation in the spaces $\mathcal{H}_{c,\mathcal{E}},\ c \in S_\infty$ of the subgroup $L_1(\mathcal{E})$ of the group E equipped with l_1-norm on the coordinates with respect to ONB $\mathcal{E}$. The proof of last statement is based on the estimate $\forall\ \epsilon > 0\ \exists\ v \in \mathcal{H}_\mathcal{E} :\ \|S_h u - u\|_{\mathcal{H}_\mathcal{E}} \leq \epsilon + \|h\|_{L_1(\mathcal{E})} \|v\|_{\mathcal{H}_\mathcal{E}}$ from [20].

A nonempty rectangle $\Pi \in \mathcal{K}_\mathcal{E}$ is called $L_1(\mathcal{E})$-equivalent to a rectangle $Q \in \mathcal{K}_\mathcal{E}$ ($\Pi \sim_{L_1(\mathcal{E})} Q$) if there is a vector $h \in L_1(\mathcal{E})$ such that $Q = \mathbf{S}_h(\Pi)$. Hence, if $\Pi \sim_{L_1(\mathcal{E})} Q$, then $\Pi \sim_E Q$.

Let $\mathcal{K}^1_\mathcal{E}(0)$ be a collection of non-empty rectangles $\Pi \in \mathcal{K}_\mathcal{E}(0)$ such that $c(\Pi) \in l_1$ (any rectangle $\Pi \in \mathcal{K}^1_\mathcal{E}(0)$ is $L_1(\mathcal{E})$-equivalent to a centered rectangle). For every vector $c \in S_\infty$

and every vector $d \in l_2$ the symbol $\mathcal{K}^1_{\mathcal{E}}(c,d)$ denotes the collection of rectangles $\Pi \in \mathcal{K}_{\mathcal{E}}$ such that $c(\Pi) - c - d \in l_1$.

Let $c \in \mathcal{S}_\infty$, $\Pi \in \mathcal{K}_{\mathcal{E}}(c)$ and $d \in l_2$. Then the rectangle $Q = \Pi + d \in \mathcal{K}_{\mathcal{E}}(c)$ is non-empty. Let $D_2 = l_2/l_1$ be the set of classes of l_1-equivalent vectors of the space l_2. The following statement has the same proof as the Lemma 1.

Lemma 4. *Let $c \in \mathcal{S}_\infty$, $d_1, d_2 \in D_2$ and $d_1 \neq d_2$. If $\Pi' \in \mathcal{K}_{\mathcal{E}}(c,d_1)$, $\Pi'' \in \mathcal{K}_{\mathcal{E}}(c,d_2)$, then $\lambda_{\mathcal{E}}(\Pi' \bigcap \Pi'') = 0$.*

For any $c \in S_\infty$ we obtain the decomposition of the space $\mathcal{H}_{c,\mathcal{E}}$ to the orthogonal sum of subspaces such that any of this subspaces is invariant with respect to the group $\mathbb{S}_1$.

Let $c \in \mathcal{S}_\infty$. Let $r_{\mathcal{E}}(c,d)$ be the ring generated by the family of sets $\mathcal{K}_{\mathcal{E}}(c,d)$ for every $d \in l_2$ (the construction of this ring is described in papers [20,27]). The following statement can be obtained as Lemma 2.

Lemma 5. *The function of a set $\lambda_{\mathcal{E}} : \mathcal{K}_{\mathcal{E}}(c,d) \to [0,+\infty)$ is additive and has the unique additive extension to the measure $\lambda_{c,d,\mathcal{E}}$ whose domain is the ring $r_{c,d,\mathcal{E}}$.*

Lemma 6. *Let $c \in \mathcal{S}_\infty$. Let $d_1, d_2 \in l_2$ and $d_1 \neq d_2$. If $A \in \mathcal{R}_{c,d_1,\mathcal{E}} \bigcap \mathcal{R}_{c,d_2,\mathcal{E}}$, then $\lambda_{\mathcal{E}}(A) = 0$.*

Let $\mathcal{R}_{c,d,\mathcal{E}}$ be the completion of the ring $r_{c,d,\mathcal{E}}$ by the measure $\lambda_{\mathcal{E}}|_{r_{c,d,\mathcal{E}}}$. The symbol $\lambda_{c,d,\mathcal{E}}$ denotes the completion of the measure $\lambda_{\mathcal{E}}|_{r_{c,d,\mathcal{E}}}$.

Theorem 4. *Let $\mathcal{E}$ be on ONB in the space E and $c \in \mathcal{S}_\infty$. Then*

$$\mathcal{H}_{c,\mathcal{E}} = \oplus_{d \in \mathcal{D}_2} \mathcal{H}_{c,d,\mathcal{E}},$$

where $\mathcal{H}_{c,d,\mathcal{E}} = L_2(E, \mathcal{R}_{c,d,\mathcal{E}}, \lambda_{c,d,\mathcal{E}}, \mathbb{C})$. The ring $\mathcal{R}_{c,\mathcal{E}}$ is the completion with respect to the measure $\lambda_{c,\mathcal{E}}$ of the ring which is generated by the collection of sets $\bigcup\limits_{d \in \mathcal{D}_2} \mathcal{R}_{c,d,\mathcal{E}}$.

Let us introduce the following extensions $\nu_{c,d,\mathcal{E}} : \mathcal{R}_{c,\mathcal{E}} \to [0,+\infty)$, $d \in \mathcal{D}_2$, of measures $\lambda_{c,d,\mathcal{E}}$:

$$\nu_{c,d,\mathcal{E}}(A) = \left\{ \begin{array}{l} \lambda_{c,d,\mathcal{E}}(A) \; if \; A \in \mathcal{R}_{c,d,\mathcal{E}}; \\ 0, \; if \; A \in \mathcal{R}_{c,d',\mathcal{E}}, \; d' \in L_1(\mathcal{E}), \; d' \neq d. \end{array} \right\} \tag{11}$$

Then the following statement take place.

Corollary 6. *For any $d \in \mathcal{D}_2$ the measure $\nu_{c,d,\mathcal{S}}$ is invariant under the action of the Abelian unitary group $\mathbb{S}_1 = \{\mathbf{S}_h, \; h \in L_1(\mathcal{E})\}$. The measure $\lambda_{c,\mathcal{E}} : \mathcal{R}_{c,\mathcal{E}} \to [0,+\infty)$ admits the decomposition*

$$\lambda_{c,\mathcal{E}} = \sum_{d \in \mathcal{D}_2} \nu_{c,d,\mathcal{E}} \tag{12}$$

into the sum of mutually singular components invariant with respect to the group $\mathbb{S}_1$.

Invariant components of the measure $\lambda_{c,\mathcal{E}}$ in decomposition (12) are independent in the following sense. If $d, d' \in \mathcal{S}_\infty$, $d' \neq d$ then $\lambda_{\mathcal{E}}(A \bigcap \mathbf{S}_h(B)) = 0$ for every $A \in \mathcal{R}_{c,d,\mathcal{E}}$, $B \in \mathcal{R}_{c,d',\mathcal{E}}$, $h \in E$. Therefore, if $\nu_{c,d',\mathcal{E}}(A) > 0$ for some $A \in \mathcal{R}_{c,\mathcal{E}}$, then $\nu_{c,d,\mathcal{E}}(A) = 0$ and vice versa according to (11).

On the contrary, the following statement takes place.

Lemma 7. *If $A \in \mathcal{R}_{c,d,\mathcal{E}}$, $B \in \mathcal{R}_{c,d,\mathcal{E}}$, $\lambda_{c,d,\mathcal{E}}(A) > 0$, $\lambda_{c,d,\mathcal{E}}(B) > 0$, then there is a vector $h \in L_1(\mathcal{E})$ such that $\lambda_{c,d,\mathcal{E}}(A \bigcap \mathbf{S}_h(B)) > 0$.*

Proof. Since $\lambda_{c,d,\mathcal{E}}(A) > 0$, $\lambda_{c,d,\mathcal{E}}(B) > 0$, there is the rectangles $\Pi, Q \in \mathcal{K}_{\mathcal{E}}(c,d)$ such that $\Pi \subset A$, $Q \subset B$ and $\nu_{c,d,\mathcal{E}}(\Pi) > 0$, $\nu_{c,d,\mathcal{E}}(Q) > 0$.

Since $\Pi, Q \in \mathcal{K}_{\mathcal{E}}(c,d)$, $c(\Pi) - c - d$, $c(Q) - c - d \in L_1(\mathcal{E})$. Therefore, there is a vector $h \in L_1(\mathcal{E})$ such that $c(Q) = c(\Pi) + h$. Hence, the rectangles Π and $Q' = Q - h$ has the common center $C(\Pi)$. Let $P = \Pi \bigcap Q'$. Then, $c(P) = c(\Pi)$ and $p_i = \min\{\pi_i, q_i\}$ $\forall\, i \in \mathbb{N}$ where p_i, q_i, π_i are the length of projections of rectangles P, Q, Π on the axis e_i respectively. Since $\nu_{c,d,\mathcal{E}}(\Pi) > 0$, $\nu_{c,d,\mathcal{E}}(Q) > 0$, $\{\ln(\pi_i)\} \in l_1$, $\{\ln(q_i)\} \in l_1$. Hence, $\{\ln(p_i)\} \in l_1$. Therefore, $\lambda(P) > 0$. $\square$

Theorem 5. *Let $\mathcal{E}$ be on ONB in the space E, $c \in \mathcal{S}_\infty$ and $d \in D_2$. Then the decomposition (12) of $\mathbb{S}_1$-invariant measure $\nu_{c,\mathcal{E}}$ is ring-ergodic.*

The proof of the Theorem 5 has the same scheme as the proof of the Theorem 3.

Similar decomposition will be obtained for the measure that is invariant with respect to the group of orthogonal mappings in Section 6, Theorem 13.

Now we prove that the space $\mathcal{H}_{0,\mathcal{E}}$ is separable.

Let $\mathcal{E}$ be an ONB in the space E and $\Pi_{-\frac{1}{2},\frac{1}{2}} \equiv \Pi \in \mathcal{K}_{\mathcal{E}}(0)$. Let $\mathcal{H}_k(\Pi)$ be linear the space of functions $\Pi_{-\frac{1}{2},\frac{1}{2}} \to \mathbb{C}$ of the form $\{\phi_k(x_1,\ldots,x_k)1(x_{k+1},\ldots),\ \phi_k \in L_2([-\frac{1}{2},\frac{1}{2}])^k\}$ for every $k \in \mathbb{N}$. The space $H_k(\Pi)$ equipped with the norm $\|\phi_k(x_1,\ldots,x_k)1(x_{k+1},\ldots)\|_{\mathcal{H}_k} = \|\phi_k\|_{L_2([-\frac{1}{2},\frac{1}{2}])^k}$ is the Hilbert space. Let $S_{\mathcal{E}}(\Pi_{-\frac{1}{2},\frac{1}{2}})$ be a linear hull of the set $\bigcup\limits_{k\in\mathbb{N}} \mathcal{H}_k(\Pi)$ equipped with the norm $\|\cdot\|_{\mathcal{H}(\Pi)}$ of inductive limit of the sequence of Hilbert spaces $\{\mathcal{H}_k(\Pi)\}$. The completion of the normed linear space $(S_{\mathcal{E}}(\Pi_{-\frac{1}{2},\frac{1}{2}}), \|\cdot\|_{\mathcal{H}(\Pi)})$ is the Hilbert space $\mathcal{H}(\Pi)$ which is the inductive limit of the sequence of Hilbert spaces $\{\mathcal{H}_k(\Pi)\}$.

Lemma 8. *The system of functions $\mathcal{P} = \{\mathbf{S}_h\phi\chi_\Pi,\ \phi \in \mathcal{H}(\Pi),\ h \in L_1(\mathcal{E})\}$ is the total system in the space $\mathcal{H}_{0,\mathcal{E}}$.*

Proof. By the definition of the space $\mathcal{H}_{0,\mathcal{E}}$ the set

$$\{\mathbf{S}_h\chi_Q,\ h \in L_1(\mathcal{E}),\ Q \in \mathcal{K}_{\mathcal{E}}(0)\} \tag{13}$$

is total in the space $\mathcal{H}_{0,\mathcal{E}}$. $\square$

Therefore, to prove the Lemma it is sufficient to show that for every $Q \in \mathcal{K}_{\mathcal{E}}(0)$ the function χ_Q is the limit of a sequence of linear combination of functions from the system $\mathcal{P}$. As the consequence we obtain that for every $Q \in \mathcal{K}_{\mathcal{E}}(d)$, $d \in L_1(\mathcal{E})$, the function χ_Q is also the limit of a sequence of linear combination of functions from the system $\mathcal{P}$.

For any $Q \subset \Pi_{-\frac{1}{2},\frac{1}{2}}$, $Q \in \mathcal{K}_{\mathcal{E}}(0)$ and for any $\epsilon > 0$ there is a number $m \in \mathbb{N}$ such that $\lambda_{\mathcal{E}}(P_m \backslash Q) < \epsilon$ where the rectangle Π_m is defined by the following rule. If $Q = \Pi_{-q,q}$, then $P_m = \Pi_{-p,p}$, where

$$p_1 = q_1, \ldots, p_m = q_m,\ p_{m+1} = p_{m+2} = \ldots = \frac{1}{2}.$$

Therefore, $\chi_{P_m} \in \mathcal{H}_m(\Pi) \subset \mathcal{H}(\Pi)$. Thus, for every $Q \in \mathcal{K}_{\mathcal{E}}(0):\ Q \subset \Pi_{-\frac{1}{2},\frac{1}{2}}$ there is a sequence $\{u_k\}:\ \mathbb{N} \to \mathcal{H}(\Pi)$ such that $\|\chi_Q - u_m\|_{\mathcal{H}_{\mathcal{E}}} \to 0$ as $m \to \infty$.

According to Lemma 3.3 [27] for any $\epsilon > 0$ and every $P, Q \in \mathcal{K}_{\mathcal{E}}(0)$ there are mutually disjoint rectangles $\Pi_1, \ldots, \Pi_m \in \mathcal{K}_{\mathcal{E}}$ such that $\lambda_{\mathcal{E}}((P\backslash Q)\backslash(\bigcup\limits_{j=1}^{m}\Pi_j)) < \epsilon$. Therefore, for every $Q \in \mathcal{K}_{\mathcal{E}}(0)$ and every $\epsilon > 0$ there are a system of vectors $h_1, \ldots, h_N \in L_1(\mathcal{E})$ and a system of rectangles $\Pi_1, \ldots, \Pi_N \in \mathcal{K}_{\mathcal{E}}(0)$ such that $\|\chi_Q - \sum\limits_{j=1}^{N} \mathbf{S}_{h_j}\chi_{\Pi_j}\|_{\mathcal{H}_{0,E}}$. $\square$

Corollary 7. *The space $\mathcal{H}_{0,\mathcal{E}}$ is separable.*

Proof. The space $\mathcal{H}(\Pi)$ is separable since it contains the ONB

$$\{\psi_{k_1,\ldots,k_n}(x) = \phi_{k_1}(x_1)\ldots\phi_{k_n}(x_n)\} \prod_{j=n+1}^{\infty} 1(x_j), \quad n \in \mathbb{N},\ k_1,\ldots,k_n \in \mathbb{N}\},$$

where $\{\phi_k\}$ is an ONB in the space $L_2([-\frac{1}{2},\frac{1}{2}])$.

The space $L_1(\mathcal{E})$ is separable since in isomorphic to the Banach space l_1.

The system of vectors 13 is total in the space $\mathcal{H}_{0,\mathcal{E}}$ by the definition of this space. Moreover, $\|\mathbf{S}_h\chi_Q - \chi_Q\|^2_{\mathcal{H}_{0,\mathcal{E}}} \leq \|h\|_{L_1(\mathcal{E})}\|\chi_Q\|^2_{\mathcal{H}_{0,\mathcal{E}}}$ according to Lemma 7 in [17]. Hence, if $\{h_m\}$ is dense system of elements in the space $L_1(\mathcal{E})$ then the countable system of elements

$$\{\mathbf{S}_{h_m}\psi_{k_1,\ldots,k_n}, \quad m,n,k_1,\ldots,k_n \in \mathbb{N}\}$$

is total in the space $\mathcal{H}_{0,\mathcal{E}}$. □

3. Proximity for Two ONB and Orthogonal Transition Matrix

Let us study the dependence of the ring $\mathcal{R}_{\mathcal{E}\mathcal{F}} = \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{E}}$ on the mutual position of two ONB $\mathcal{E},\mathcal{F} \in \mathcal{S}$ in the space E. The description of this dependence gives the opportunity to define the procedure of the extension of the family of measures $\lambda_{\mathcal{F}}$, $\mathcal{F} \in \mathcal{S}$, to the rotation invariant measure.

Let $\mathbf{U}$ be an orthogonal operator in the space E. Let $\mathcal{E},\mathcal{F}$ be a pair of ONB in the space E such that $\mathcal{F} = \mathbf{U}(\mathcal{E})$. Let us consider two measures $\lambda_{\mathcal{E}}: \mathcal{R}_{\mathcal{E}} \to [0,+\infty)$ and $\lambda_{\mathcal{F}}: \mathcal{R}_{\mathcal{F}} \to [0,+\infty)$. We study measures $\lambda_{\mathcal{E}}|_{\mathcal{R}_{\mathcal{E}\mathcal{F}}}$ and $\lambda_{\mathcal{F}}|_{\mathcal{R}_{\mathcal{E}\mathcal{F}}}$ where $\mathcal{R}_{\mathcal{E}\mathcal{F}} = \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$.

Let $\|C\| = \|c_{i,j}\|$ be the matrix of transition of the basis $\mathcal{E}$ into the basis $\mathcal{F}$. Hence, matrix elements are $c_{i,j} = (e_i, f_j) = (e_i, \mathbf{U}e_j)$, $i,j \in \mathbb{N}$. Therefore, $(e_k, \mathbf{U}^{-1}e_l) = c_{l,k}$, $k,l \in \mathbb{N}$ and $\sum\limits_{k=1}^{\infty} c_{k,i}c_{k,j} = \delta_{ij}$, $i,j \in \mathbb{N}$ where $\delta_{i,j}$ is the Kronecker symbol.

The sequence $\{c_{\cdot,j}\}$ is the unit vector of Hilbert space ℓ_2 since $\|c_{\cdot,j}\|$ are coordinates of the unit vector f_j, $j \in \mathbb{N}$, with respect to the basis $\mathcal{E}$. But the sequence $\{c_{\cdot,j}\}$ can be not belong to the space ℓ_1. We will show that if $\{c_{\cdot,j}\} \notin \ell_1$ then $\lambda_{\mathcal{E}}(A) = 0 = \lambda_{\mathcal{F}}(A)$ for any set $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$. For a given ONB $\mathcal{E}$ the symbol $L_1(\mathcal{E})$ denotes the linear subspace $L_1(\mathcal{E}) = \{x \in E: \{(x,e_j)\} \in \ell_1\}$. If conditions $f_j \in L_1(\mathcal{E})\ \forall\, j \in \mathbb{N}$ and $e_j \in L_1(\mathcal{F})\ \forall\, j \in \mathbb{N}$ hold then the property of absolute continuity of measures $\lambda_{\mathcal{E}}$, $\lambda_{\mathcal{F}}$ with respect to each other is controlled by following conditions on the pair of bases $\mathcal{E}$, $\mathcal{F}$

$$\prod_{j=1}^{\infty} \|c_{\cdot,j}\|_{\ell_1} < +\infty; \tag{14}$$

$$\prod_{j=1}^{\infty} \|c_{j,\cdot}\|_{\ell_1} < +\infty. \tag{15}$$

We prove that the condition (14) is equivalent to the condition (15). If the conditions (14) and (15) are satisfied then $\mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}} = \mathcal{R}_{\mathcal{E}} = \mathcal{R}_{\mathcal{F}}$ and the equality $\lambda_{\mathcal{E}}(A) = \lambda_{\mathcal{F}}(A)$ holds for any $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$. In the opposite case the measures $\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{F}}$ take only zero values on an arbitrary set of the ring $\mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$.

The next Lemma defines the geometrical sense of values of products in left hand side of inequalities (14) and (15).

Lemma 9. *Let $\Pi_{0,1}$ be a unit rectangle from the collection of sets $\mathcal{K}_{\mathcal{E}}$. Then*

$$\inf_{Q \in \mathcal{K}_{\mathcal{F}}:\ \Pi_{0,1} \subset Q} \lambda_{\mathcal{F}}(Q) = \prod_{j=1}^{\infty} \|c_{\cdot,j}\|_{\ell_1}.$$

Proof. According to the conditions $Q \in \mathcal{K}_{\mathcal{F}} : \Pi_{0,1} \subset Q$ the length l_j of the orthogonal projection of the rectangle Q on the line $Of_j = \{x = tf_j, t \in \mathbb{R}\}$, $j \in \mathbb{N}$ (i.e., the length of j-th edge of the rectangle Q) no less than the sum of lengths of orthogonal projections of edges of the rectangle $\Pi_{0,1}$ onto the line Of_j: $l_j \geq \sum\limits_{i=1}^{\infty} \mid (e_i, f_j)) \mid \ \forall j \in \mathbb{N}$. Therefore, we obtain the statement of Lemma 9. □

Lemma 10. *Let* $\{c_k\} \in \ell_2$ *and* $\|\{c_k\}\|_{\ell_2} = 1$. *If* $\max\limits_{j \in \mathbf{N}} |c_j| = \alpha$, *then* $\|c_j\|_{\ell_1} \geq \alpha + \sqrt{1-\alpha^2}$. *In particular,* $\|\{c_k\}\|_{\ell_1} \geq 1$.

The statement is the consequence of the inequality

$$\mid c_1 \mid + \ldots + \mid c_m \mid \geq \sqrt{c_1^2 + \ldots + c_m^2}$$

which holds for any $m \in \mathbb{N}$ and for any collection of complex numbers $c_1, \ldots, c_m$. □

Corollary 8. *Let* $c^{(m)}$ *be a sequence of vectors of the space* ℓ_2 *with coordinates* $c_k^{(m)}$, $k \in \mathbb{N}$ *such that* $\lim\limits_{m \to \infty} \|\{c_k^{(m)}\}\|_{\ell_2} = 1$. *If* $\lim\limits_{m \to \infty} \|c^{(m)}\|_{\ell_1} = 1$ *then* $\lim\limits_{m \to \infty} \alpha_m = 1$ *where* $\alpha_m = \max\limits_{k \in \mathbb{N}} \mid c_k^{(m)}) \mid$ *for every* $m \in \mathbb{N}$.

Corollary 9. *Let conditions of the Corollary 8 be hold. Then there is the number* $m_0 \in \mathbb{N}$ *such that the maximum* $\max\limits_{k \in \mathbb{N}} \mid c_k^{(n)} \mid$ *is reached on the only one number* $k = i_n$ *for any* $n \geq m_0$.

Lemma 11. *Let* $\mathcal{F}, \mathcal{E}$ *be a pair of ONB in the space E. Let* $\{m_j\}$ *be a sequence of natural numbers such that* $\max\limits_{i \in \mathbb{N}} \mid c_{i,j} \mid = c_{m_j,j}$ *for every* $j \in \mathbb{N}$. *Then the condition*

$$\sum_{j=1}^{\infty} \sum_{i \neq m_j} \mid c_{i,j} \mid < +\infty \tag{16}$$

is equivalent to the condition (14).

Proof. Let the condition (14) be hold. Let $l_j = \sum\limits_{i=1}^{\infty} \mid c_{i,j} \mid$. Then the series $\sum\limits_{j=1}^{\infty} \ln(l_j)$ converges. Hence, the series $\sum\limits_{j=1}^{\infty} (l_j - 1)$ converges and $\lim\limits_{j \to \infty} (l_j - 1) = 0$.

Let $\alpha_j = \max\limits_i \mid c_{i,j} \mid$. According to Lemma 10 inequalities $l_j \geq \alpha_j + \sqrt{1-\alpha_j^2} \geq 1$ hold for any $j \in \mathbb{N}$. Hence, $l_j \geq 1 - \beta_j + \sqrt{2\beta_j - \beta_j^2}$ for any $j \in \mathbb{N}$. Here $\beta_j = 1 - \alpha_j$, hence $\beta_j \in [0,1)$. Then, $\lim\limits_{j \to \infty} \beta_j = 0$ according to Corollary 8. Hence, $l_j - 1 \sim \sqrt{2\beta_j}$ as $j \to \infty$ and the series

$$\sum_{j=1}^{\infty} \sqrt{\beta_j} \tag{17}$$

converges as well as the series $\sum\limits_{j=1}^{\infty} (l_j - 1)$. Since $\sum\limits_{i \neq m_j} \mid c_{ij} \mid = l_j - 1 + \beta_j$ for any $j \in \mathbb{N}$, the condition (16) is the consequence of the convergence of the series $\sum\limits_{j=1}^{\infty} (l_j - 1)$ and (17).

Let the condition (16) be hold. Let $\gamma_j = \sum\limits_{i \neq m_j} \mid c_{ij} \mid = l_j - \alpha_j$, $j \in \mathbb{N}$. Then, $\lim\limits_{j\to\infty} \gamma_j = 0$. Since $l_j = \alpha_j + \gamma_j \leq 1 + \gamma_j$, the convergence of the series $\sum\limits_{j=1}^{\infty} \gamma_j$ is the consequence of the condition (16). Therefore, the series $\sum\limits_{j=1}^{\infty} (l_j - 1)$ converges. Hence, the condition (14) holds. □

Theorem 6. *Conditions (14) and (15) are equivalent.*

Proof. Let the condition (14) be hold. Hence, $\lim\limits_{j\to\infty} l_j = 1$. Therefore, $\lim\limits_{j\to\infty} \alpha_j = 1$ according to the Corollary 8. Hence, there is the number $j_0 \in \mathbb{N}$ such that $\alpha_j > \frac{1}{\sqrt{2}}$ for any $j > j_0$.

There is the sequence of natural numbers m_j, $j \in \mathbb{N}$ such that

$$\max_k \mid c_{k,j} \mid = c_{m_j,j} = \alpha_j.$$

Moreover, the number m_j is uniquely defined for every $j > j_0$. On the other hand there is the sequence of natural numbers M_k, $k \in \mathbb{N}$ such that $\max\limits_j \mid c_{k,j} \mid = c_{k,M_k}$. Since $\mid c_{m_j,j} \mid > \frac{1}{\sqrt{2}}$ for all $j > j_0$, $\max\limits_i \mid c_{m_j,i} \mid = \mid c_{m_j,j} \mid$ for all $j > j_0$. I.e., the maximal element $\mid c_{m_j,j} \mid$ of j-th column is the maximal element of m_j-th row in matrix $\|C\|$ for any $j > j_0$. Hence, $M_{m_j} = j$, $\forall\, j > j_0$.

Thus, $\max\limits_{i\in\mathbb{N}} \mid c_{m_j,i} \mid = \mid c_{m_j,M_{m_j}} \mid = \mid c_{m_j,j} \mid$ for any $j > j_0$.

The condition (14) implies $\sum\limits_{j\in\mathbb{N}} \sum\limits_{k\in\mathbb{N}\setminus\{m_j\}} \mid c_{k,j} \mid < +\infty$ according to the Lemma 11. Therefore,

$$\sum_{j=1}^{j_0} \sum_{k\in\mathbb{N}} \mid c_{k,j} \mid + \sum_{j=j_0+1}^{+\infty} \sum_{k\in\mathbb{N}\setminus\{m_j\}} \mid c_{k,j} \mid < +\infty. \tag{18}$$

Let $\mathbb{N}_1$ be the set of values of the sequence m_j, $j > j_0$. Let $\mathbb{N}_0$ be a set $\mathbb{N}\setminus\mathbb{N}_1$. In the condition (18) we can rearrange the order of summation of the series of non-negative terms:

$$\sum_{j=1}^{j_0} \sum_{k\in\mathbb{N}} \mid c_{k,j} \mid + \sum_{j=j_0+1}^{+\infty} \sum_{k\in\mathbb{N}\setminus\{m_j\}} \mid c_{k,j} \mid =$$

$$= \left(\sum_{k\in\mathbb{N}_0} \sum_{j=1}^{j_0} + \sum_{k\in\mathbb{N}_1} \sum_{j=1}^{j_0} + \sum_{k\in\mathbb{N}_0} \sum_{j>j_0} + \sum_{k\in\mathbb{N}_1} \sum_{j>j_0;\, j\neq M_k} \right) \mid c_{k,j} \mid < +\infty. \tag{19}$$

In the last equality we use the following presentation of the set of summation indexes

$$\{\{(k,j),\ k \in \mathbb{N}\setminus\{m_j\}\},\ j > j_0\} =$$

$$= \{(k,j),\ j > j_0,\ k \in \mathbb{N}_0\} \bigcup \{\{(k,j),\ j > j_0,\ j \neq M_k\},\ k \in \mathbb{N}_1\}.$$

Therefore, according to (19) we obtain the following condition

$$\sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}\setminus\{M_k\}} \mid c_{k,j} \mid \leq \sum_{k\in\mathbb{N}_0} \sum_{j\in\mathbb{N}} \mid c_{k,j} \mid + \sum_{k\in\mathbb{N}_1} \sum_{j\in\mathbb{N}\setminus\{M_k\}} \mid c_{k,j} \mid < +\infty.$$

Since $\sum\limits_{k\in\mathbb{N}} \sum\limits_{j\in\mathbb{N}\setminus\{M_k\}} \mid c_{k,j} \mid < +\infty$, according to the Lemma 11 the condition (15) holds.

If we swap the bases $\mathcal{E}$ and $\mathcal{F}$, then we obtain that the condition (15) implies the condition (14). □

Corollary 10. *Let $\|C\|$ be the transition matrix from one ONB to another. Then the product $\prod\limits_{j=1}^{\infty}\|c_{\cdot,j}\|_{\ell_1}$ converges if and only if the product $\prod\limits_{i=1}^{\infty}\|c_{i,\cdot}\|_{\ell_1}$ converges.*

ONB $\mathcal{E}$ is called near to the ONB $\mathcal{F}$ if $\mathcal{E}$ and $\mathcal{F}$ satisfy the condition (14). If the condition (14) is not satisfy for two ONB $\mathcal{E}$ and $\mathcal{F}$, then ONB $\mathcal{E}$ is called distant from the ONB $\mathcal{F}$.

In the Section 4 we show that if ONB $\mathcal{E}$ is near to ONB $\mathcal{F}$, then measures $\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{F}}$ coincide. In the Section 5 we show that if ONB $\mathcal{E}$ is distant from ONB $\mathcal{F}$, then measures $\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{F}}$ are defined on different rings such that both measures $\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{F}}$ take zero value on an arbitrary set from the ring $\mathcal{R}_{\mathcal{E}}\bigcap\mathcal{R}_{\mathcal{F}}$. Results of Sections 4 and 5 are obtained by the analysis of mutual position of rectangles with edges collinear to vectors of different ONB. Results of Sections 4 and 5 give tools for the proof of the existence of shift- and rotation-invariant measures on the Hilbert space.

4. Ring $\mathcal{R}_{\mathcal{E}}\bigcap\mathcal{R}_{\mathcal{F}}$ in the Case of Nearness of Bases $\mathcal{E}$ and $\mathcal{F}$

Let us prove that if the condition (14) holds (as well as the equivalent condition (15)), then rectangles Π and $Q=\mathbf{U}(\Pi)$ belong to the ring $\mathcal{R}_{\mathcal{E}}$ and $\lambda_{\mathcal{E}}(Q)=\lambda_{\mathcal{E}}(\Pi)$. At the first step to this goal we apply orthogonal mappings $\mathbf{V}$ of the space E which only change the order of vectors in the basis $\mathcal{E}$. Also we use the following property of the measure $\lambda_{\mathcal{E}}$ to be invariant with respect to permutation of vectors of basis $\mathcal{E}$.

Lemma 12. *Let $\mathbf{V}$ be an orthogonal transformation of the space E changing the order of vectors of the basis $\mathcal{E}$ only. Let $\mathcal{E}'=\mathbf{V}(\mathcal{E})$. Then, $\mathcal{R}_{\mathcal{E}'}=\mathcal{R}_{\mathcal{E}}$ and $\lambda_{\mathcal{E}'}(A)=\lambda_{\mathcal{E}}(A)$ for all $A\in\mathcal{R}_{\mathcal{E}}$.*

Proof. The collection of absolutely measurable rectangles $\mathcal{K}_{\mathcal{E}}$ coincides with the collection $\mathcal{K}_{\mathcal{E}'}$. In fact, for any rectangle $\Pi\in\mathcal{K}_{\mathcal{E}}$ its edges are collinear to vectors of ONB $\mathcal{E}'$ and the product of lengths of edges converges unconditional. Therefore, $\Pi\in\mathcal{K}_{\mathcal{E}'}$. The opposite is also true. Moreover, $\lambda_{\mathcal{E}'}(\Pi)=\lambda_{\mathcal{E}}(\Pi)$ for any $\Pi\in\mathcal{K}_{\mathcal{E}}$ according to unconditional measurability. We have $r_{\mathcal{E}}=r_{\mathcal{E}'}$ since $\mathcal{K}_{\mathcal{E}'}=\mathcal{K}_{\mathcal{E}}$. Since the finitely additive function $\lambda_{\mathcal{E}'}$ coincides with $\lambda_{\mathcal{E}}$ on the collection of sets $\mathcal{K}_{\mathcal{E}'}=\mathcal{K}_{\mathcal{E}}$, additive functions $\lambda_{\mathcal{E}'}$ and $\lambda_{\mathcal{E}}$ has the unique (only one) additive extension to the ring $r_{\mathcal{E}}=r_{\mathcal{E}'}$. Therefore the completion of the measures $\lambda_{\mathcal{E}'}$ and $\lambda_{\mathcal{E}}$ coincides with each other. Thus, we prove the statement on the transformation $\mathbf{V}$. □

Let $\mathcal{E}=\{e_j\}$ and $\mathcal{F}=\{f_k\}$ be a pair of ONB in the space E. Let us define subspaces $E_n=\text{span}(e_1,\dots,e_n)$ and $E^n=E_n^{\perp}$; $F_n=\text{span}(f_1,\dots,f_n)$ and $F^n=F_n^{\perp}$ for any $n\in\mathbb{N}$.

Theorem 7. *Let the condition (14) for the pair of ONB $\mathcal{E}$ and $\mathcal{F}$ be hold. Then there is the permutation of vectors of ONB $\mathcal{F}$ such that this permutation transform ONB $\mathcal{F}$ into ONB $\mathcal{F}'$ satisfying the condition*

$$\exists\, m_0\in\mathbb{N}:\ \forall\, j>m_0\quad |\, c'_{j,j}\,|=\max_{k\in\mathbb{N}}|\, c'_{j,k}\,|=\max_{i\in\mathbb{N}}|\, c'_{i,j}\,|, \tag{20}$$

where $c'_{k,j}=(f'_k,e_j)$, $k,j\in\mathbb{N}$.

Proof. According to condition (14) we have $\lim\limits_{j\to\infty}\alpha_j=1$. Therefor the set

$$M'=\{j\in\mathbb{N}:\ \alpha_j\le\frac{1}{\sqrt{2}}\}$$

is finite. Since the condition (14) implies the condition (15), the set $M'' = \{k \in \mathbb{N} : \max_{j\in\mathbb{N}} | c_{k,j} | \leq \frac{1}{\sqrt{2}}\}$ is finite analogously. Let us denote by m', m'' numbers of elements in finite sets M', M'' respectively. Let us prove that $m' = m''$.

Let us assume that $m' > m''$ (the case $m' < m''$ can be considered analogously). Then $| c_{k,j} | \leq \frac{1}{\sqrt{2}}$ for any $k \in M''$, $j \in M'$.

We will done finite number of permutations of rows and columns of the matrix $\|c_{ij}\|$. For each permutation of two rows (of two columns) we done the permutation of corresponding vectors in the basis $\mathcal{F}$ (in the basis $\mathcal{E}$).

Step 1. Let us permute the vectors of ONB $\mathcal{F}$ with the numbers from the set M'' onto first m'' positions. The natural order of numbers in the set M'' and in its complement are preserved. Analogously, let us permute the vectors of ONB $\mathcal{E}$ with the numbers from the set M' onto first m' positions. The natural order of numbers in the set M' and in its complement are preserved.

After the above permutation of bases $\mathcal{E}$ and $\mathcal{F}$ we obtain the matrix of transition with following properties. Each row with number greater than m'' contains the only one element with the modulus greater than $\frac{1}{\sqrt{2}}$. According to the permutation in step 1 this element belongs to the column with the number greater than m'. Conversely, each column with number greater than m' contains the only one element with modulus greater than $\frac{1}{\sqrt{2}}$. This element belongs to the row with the number greater than m''.

Hence, there is the permutation of columns with the numbers $m'+1, m'+2, \ldots$ such that each row with a number $k > m''$ contains the only one element with the modulus greater than $\frac{1}{\sqrt{2}}$ and this element belong to the column with the number $m' + (k - m'')$. After this permutation of the vectors of the basis $\mathcal{F}$ the matrix $\|c_{i,j}\|$ satisfies the condition: for any $j > m'$ (and for any $i > m''$) an element with the maximal modulus in the j-th column (in i-th row) is the element $c_{i,j}$ with $i - j = m'' - m'$.

Step 2. We transform the ONB $\mathcal{F}$ by the following rule. We change the vector f_k, $k > m''$, onto the vector $-f_k$ under the condition $c_{k,k+m'-m''} < 0$. In opposite case we remain f_k.

The ring of subsets $\mathcal{R}_{\mathcal{F}}$ (and $\mathcal{R}_{\mathcal{E}}$) and the measure $\lambda_{\mathcal{F}}$ (and $\lambda_{\mathcal{E}}$) so not change under the transformation in steps (1) and (2) according to Lemma 12. The matrix $\|c_{i,j}\|$ under the above transformation satisfies the conditions $c_{k,k+m'-m''} > \frac{1}{\sqrt{2}}\ \ \forall\, k > m''$.

In the proof of the Lemma 11 we should prove that the condition (14) implies estimates

$$\sum_{j>m''} (1 - c_{j,j+m'-m''}) < \sum_{k=1}^{\infty} (1- | c_{k,M_k} |) < \sum_{k=1}^{\infty} \sqrt{1- | c_{k,M_k} |} < +\infty.$$

Therefore, there is the number N such that

$$\sum_{j=N+1}^{\infty} \sqrt{1 - c_{j,j+m'-m''}} < \frac{1}{2}; \qquad \sum_{k=N+1}^{\infty} (l_k^T - | c_{k,k+m'-m''} |) < \frac{1}{2}, \tag{21}$$

here $l_k^T = \sum_{j=1}^{\infty} | c_{kj} |$.

Let $E^{N+m'-m''}$ ($E_{N+m'-m''}$) be the subspace of the space E such that the orthonormal system of vectors $\{e_{N+m'-m''+1}, \ldots\}$ ($\{e_1, \ldots, e_{N+m'-m''}\}$) forms the ONB in the space $E^{N+m'-m''}$ ($E_{N+m'-m''}$). Let $\mathbf{P}_{E_a}$ be an orthogonal projector in the space E onto a subspace E_a of the space E. Let us consider the system of vectors $\tilde{f}_{N+1}, \tilde{f}_{N+2}, \ldots$. Here $\tilde{f}_k = \mathbf{P}_{E^{N+m'-m''}} f_k$ for any $k > N$. Therefore, $\|f_k - \tilde{f}_k\|_E \leq l_k^T - | c_{k,k+m'-m''} |$ and for any $k > N$ we have

$$\|e_{k+m'-m''} - \tilde{f}_k\| \leq \left\| \sum_{i\neq k+m'-m''} c_{k,i} e_i + (1 - c_{k,k+m'-m''}) e_{k+m'-m''} \right\| =$$

$$= \sqrt{2(1 - c_{k,k+m'-m''})} < 1.$$

Hence, the system of vectors $\{\tilde{f}_k,\ k > N\}$ of the space $E^{N+m'-m''}$ is the perturbation of ONB $\{e_k,\ k > N + m' - m''\}$ which is small in the following sense $\sum\limits_{j=N+1}^{\infty} \|e_{j+m'-m''} - \tilde{f}_j\|_E < 1$ according to (21). Therefore, the system of vectors $\{\tilde{f}_k,\ k > N\}$ is the Riesz basis in the space $E^{N+m'-m''}$ (see [46], chapter 1.6). The system of vectors $\{f_1, \dots, f_N, \tilde{f}_{N+1}, \dots\}$ is the Riesz basis in the space E since it is nearby to the ONB $\mathcal{F} = \{f_j\}$ in the following sense $\sum\limits_{j=N+1}^{\infty} \|f_j - \tilde{f}_j\|_E \leq \sum\limits_{j=N+1}^{\infty} (l_j^T - \mid c_{j,j+m'-m''} \mid) < 1$.

The subsystem $\{\tilde{f}_{N+1}, \dots\}$ of Riesz basis $\{f_1, \dots, f_N, \tilde{f}_{N+1}, \dots\}$ belongs to the subspace $E^{N+m'-m''}$ and forms Riesz basis in the subspace $E^{N+m'-m''}$ of codimension $N + m' - m''$. Therefore, the system of vectors $\{\hat{f}_1, \dots, \hat{f}_N, \tilde{f}_{N+1}, \dots\}$ forms Riesz basis in the space E (here $\hat{f}_i = \mathbf{P}_{E_{N+m'-m''}} f_i,\ i = 1, \dots, N$). Hence, the system of vectors $\{\hat{f}_1, \dots, \hat{f}_N\}$ forms the basis in the space $E_{N+m'-m''}$. It is impossible in the case $m' > m''$. The contradiction proves that $m'' = m'$. □

Remark 1. *The conditions (14)–(16) are invariant with respect to changing of numbering of vectors of bases $\mathcal{E}$, $\mathcal{F}$.*

Remark 2. *The theory of determinants of linear operators [47] forms different approaches to a definition of a determinant and to study conditions of its existence.*

The Poincare Theorem gives the condition on the infinite matrix of a linear operator in some basis sufficient to the existence of determinant. Poincare Theorem (see [48], p. 400) states that the following two conditions (A) and (B) are sufficient for the existence of the determinant of an infinite matrix $\|C\|$ (the determinant of the infinite matrix $\|C\|$ is defined as the limit of n-th order main angular minor of matrix $\|C\|$ as $n \to \infty$). Here (A) is the condition of unconditional convergence of products of diagonal matrix elements; (B) is the condition of absolute convergence of the series of non-diagonal elements of matrix $\|C\|$. If the matrix $\|C\|$ is orthogonal, then the condition (A) is the consequence of the condition (B). In this case the condition (16) on the pair of ONB $\mathcal{E}$ and $\mathcal{F}$ is equivalent to the condition B) of the Theorem 25 [48] for the matrix which is connected with the matrix $\|C\| = \|(f_k, e_j)\|$ by means of permutations of rows and columns. Thus, the deformation of the measure under the action of linear mapping is connected with the determinant of the mapping.

Let us introduce some notations. Let $\mathcal{E}$, $\mathcal{F}$ be a pair of ONB. Finite-dimensional subspaces $E_n = \mathrm{span}(e_1, \dots, e_n)$, $F_n = \mathrm{span}(f_1, \dots, f_n)$ and their orthogonal completions $E^n = (E_n)^\perp$, $F^n = (F_n)^\perp$ are defined for any $n \in \mathbb{N}$. Operators $\mathbf{P}_{E_n}$ and $\mathbf{P}_{E^n}$ in the space E are operators of orthogonal projections onto subspaces E_n and E^n respectively. For any $n \in \mathbb{N}$ the matrix $C_n = \|(e_i, f_j)\|,\ i, j \in \{1, \dots, n\}$, is the matrix of orthogonal projector $\mathbf{P}_{F_n, E_n} :\ F_n \to E_n$ from the subspace F_n into the subspace E_n in pair of bases $\{f_1, \dots, f_n\}$ and $\{e_1, \dots, e_n\}$. Let λ_n be the Lebesgue measure in an n-dimensional Euclidean space.

Lemma 13. *Let $\mathcal{E}$, $\mathcal{F}$ be a pair of bases which satisfies conditions (14) and (20). Then for any $\epsilon > 0$ there is a number $N_\epsilon \in \mathbb{N}$ such that $\mathrm{Tr}(C_n^T C_n) \geq n - \epsilon$ for any $n \geq N_\epsilon$.*

Proof. To prove Lemma 13 we firstly obtain some asymptotic estimates for the spectrum of the matrix $C_n = \|(e_i, f_j)\|,\ i, j \in \{1, \dots, n\}$.

Norms of projections of vectors $f_j,\ j = 1, \dots, n$ onto a subspace E_n have following expressions $\|\mathbf{P}_{E_n} f_j\|^2 = \sum\limits_{k=1}^{n} c_{kj}^2$. Therefore,

$$1 \geq \|\mathbf{P}_{E_n} f_j\|^2 \geq c_{jj}^2 = \alpha_j^2 \quad \forall\, j \in \{1, \dots, n\}. \tag{22}$$

Let a number $\epsilon > 0$ be fixed. Since the series $\sum_{k=1}^{\infty}(1-\alpha_k)$ converges (Lemma 11), there is a number $m_\epsilon \in \mathbb{N}$ such that

$$\sum_{k=m_\epsilon+1}^{n}(1-\alpha_k) \leq \frac{\epsilon}{2} \tag{23}$$

for any $n \geq m_\epsilon$. Then, according to (22) and (23) we have the estimate

$$\sum_{k=m_\epsilon+1}^{n}(1-\|\mathbf{P}_{E_n}f_k)\|) \leq \sum_{k=m_\epsilon+1}^{\infty}(1-\alpha_k) \leq \frac{\epsilon}{2} \quad \forall\, n > m_\epsilon.$$

Since the sequence of operators $\{\mathbf{P}_{E_n}\}$ converges to unit operator in the strong operator topology, $\lim\limits_{n\to\infty}\|\mathbf{P}_{E_n}f_j\| = 1$ for any $j \in \{1,\ldots,m_\epsilon\}$. Therefore, there is a number $N_\epsilon > m$ such that the inequality

$$\sum_{j=1}^{m_\epsilon}(1-\|\mathbf{P}_{E_n}f_j\|) < \frac{\epsilon}{2} \tag{24}$$

holds for any $n \geq N_\epsilon$. Thus, for any $\epsilon > 0$ there are numbers $m_\epsilon \in \mathbb{N}$ and $N_\epsilon > m_\epsilon$ such that the condition

$$\sum_{j=1}^{n}(1-\|\mathbf{P}_{E_n}f_j\|) = \sum_{j=1}^{m_\epsilon}(1-\|\mathbf{P}_{E_n}f_j\|) + \sum_{j=m_\epsilon+1}^{n}(1-\|\mathbf{P}_{E_n}f_j\|) < \epsilon \tag{25}$$

hods for all $n \geq N_\epsilon$.

Columns of the matrix $\|C_n\|$ are coordinate columns of vectors $\mathbf{P}_{E_n}f_1,\ldots,\mathbf{P}_{E_n}f_n$ with respect to ONB $\{e_1,\ldots,e_n\}$ of the space E_n. Hence, the equality $(C_n^T C_n)_{jj} = \|\mathbf{P}_{E_n}f_j\|^2$ holds for any $j \in \{1,\ldots,n\}$. Therefore,

$$\mathrm{Tr}(C_n^T C_n) = \sum_{j=1}^{n}\|\mathbf{P}_{E_n}f\|_j^2 = \sum_{j=1}^{n}(1-(1-\|\mathbf{P}_{E_n}f\|_j))^2 \geq n - 2\sum_{j=1}^{n}(1-\|\mathbf{P}_{E_n}f\|_j)$$

for any $n \in \mathbb{N}$. Thus, according to (25) there is a number $N_\epsilon \in \mathbb{N}$ such that $\mathrm{Tr}(C_n^T C_n) \geq n - 2\epsilon$ for any $n \geq N_\epsilon$. $\square$

Corollary 11. *Let the assumption of Lemma 13 be hold. Then for any $\epsilon \in (0,\frac{1}{2})$ there is a number $N_\epsilon \in \mathbb{N}$ such that $\det(C_n^T C_n) \geq 1-2\epsilon$ for any $n \geq N_\epsilon$.*

Proof. The matrix $C_n^T C_n$ is positive $n \times n$ matrix. Hence, it has the ONB of eigenvectors and the collection of n positive eigenvalues $\mu_1,\ldots,\mu_n$. The matrix C_n is the matrix of the operator of orthogonal projection $\mathbf{P}_{F_n,E_n} :\ F_n \to E_n$ (and C_n^T is the matrix of operator of orthogonal projection $\mathbf{P}_{E_n,F_n}$ from the subspace E_n onto the subspace F_n). Therefore, the eigenvalues of the matrix $C_n^T C_n$ are no greater than 1 since $\|\mathbf{P}_{F_n,E_n}x\|_E \leq \|x\|_F\ \forall\, x \in F_n$ and $\|\mathbf{P}_{E_n,F_n}y\|_F \leq \|y\|_E\ \forall\, y \in E_n$. Thus, $0 < \mu_k \leq 1\ \forall\, k \in \{1,\ldots,n\}$.

Let us fix some $\epsilon \in (0,\frac{1}{2})$. Then, according to Lemma 13 there is the number $N_\epsilon \in \mathbb{N}$ such that the inequality $\mathrm{Tr}(C_n^T C_n) = \sum_{k=1}^{n}(1-(1-\mu_k)) = n - \sum_{k=1}^{n}(1-\mu_k) \geq n-\epsilon$ holds for any $n \geq N_\epsilon$. Hence, $\sum_{k=1}^{n} t_k < \epsilon$ where $t_k = 1-\mu_k \in [0,\epsilon]$. Therefore, $\det(C_n^T C_n) = \prod_{k=1}^{n}\mu_k = \prod_{k=1}^{n}(1-t_k) \geq 1 + \sum_{k=1}^{n}\ln(1-t_k)$. According to Lagrange Theorem we have $\ln(1-t) > -\frac{t}{1-\epsilon}\ \ \forall\, t \in [0,\epsilon]$. Therefore, the inequality $\det(C_n^T C_n) \geq 1-2\epsilon$ holds for any $n \geq N_\epsilon$. $\square$

Lemma 14. *Let $\mathcal{E}$, $\mathcal{F}$ be a pair of bases which satisfy the conditions (14) and (20). Then $\lim\limits_{n\to\infty}\lambda_n(\mathbf{P}_{F_n,E_n}(Q_n)) = \lambda_{\mathcal{F}}(Q)$ for any $Q \in \mathcal{K}_{\mathcal{F}}$.*

Proof. Since the rectangle $Q \in \mathcal{K}_{\mathcal{F}}$ is measurable, the equality $\lambda_{\mathcal{F}}(Q) = \lim\limits_{n\to\infty} \lambda_n(Q_n)$ holds. Here Q_n is the n-dimensional section of the rectangle Q by the hyperplane F_n for each $n \in \mathbb{N}$.

Let $p_n = \mathbf{P}_{F_n,E_n}(Q_n)$ be the orthogonal projection of the n-dimensional rectangle $Q_n = F_n \bigcap Q$ from the subspace F_n onto the subspace E_n. The matrix $\|c^{(n)})_{i,j}\|$, $i, j \in \overline{1, n}$ is the Jacobi matrix of the linear mapping of orthogonal projection $\mathbf{P}_{F_n,E_n}$ in the bases $f_1, \dots, f_n$ and $e_1, \dots, e_n$ in subspaces F_n and E_n. Therefore, $\lambda_n(p_n) =\mid \det(\|C_n\|)|\lambda_n(Q_n)$.

According to Corollary 11 for any $\epsilon \in (0, \frac{1}{2})$ there is the number $N_\epsilon \in \mathbb{N}$ such that the condition $\det(C_n^T C_n) = (\det(C_n))^2 \geq 1 - \epsilon$ holds for any $n \geq N_\epsilon$. Moreover, $| \det(C_n) | \leq 1$. Therefore,

$$\lim_{n\to\infty} \lambda_n(\mathbf{P}_{F_n,E_n}(Q_n)) = \lim_{n\to\infty} (| \det(C_n) | \lambda_n(Q_n)) = \lambda_{\mathcal{F}}(Q).$$

□

Remark 3. *The proof of the Lemma 14 is based on the existence of the determinant* $\det(C) = \lim\limits_{n\to\infty} \det(C_n)$ *of the matrix* $C = \|(f_k, e_j)\|$ *(see [47–50]).*

Theorem 8. *Let the condition (14) be hold. If* $Q \in \mathcal{K}_{\mathcal{F}}$, *then* $Q \in \mathcal{R}_{\mathcal{E}}$ *and* $\lambda_{\mathcal{E}}(Q) = \lambda_{\mathcal{F}}(Q)$. *On the contrary, if* $P \in \mathcal{K}_{\mathcal{E}}$, *then* $P \in \mathcal{R}_{\mathcal{F}}$ *and* $\lambda_{\mathcal{F}}(P) = \lambda_{\mathcal{E}}(P)$.

Proof. According to the Theorem 7 we can count that the condition (20) holds. (In opposite case we can change the numbering of basis vectors to obtain (20) but the rings $\mathcal{R}_{\mathcal{E}}$, $\mathcal{R}_{\mathcal{F}}$ and measures $\lambda_{\mathcal{E}}$, $\lambda_{\mathcal{F}}$ do not change). If we prove the first statement, then the second one follows from the first statement and the Theorem 6. To prove the first statement of the Theorem 8 it is sufficient to prove the statement for a rectangle $Q \in \mathcal{K}_{\mathcal{F}}$ such that $\lambda_{\mathcal{F}}(Q) > 0$.

In fact, let the first statement of the Theorem 8 is proved for a rectangle $Q' \in \mathcal{K}_{\mathcal{F}}$ such that $\lambda_{\mathcal{F}}(Q') > 0$. Let $Q \in \mathcal{K}_{\mathcal{F}}$ and $\lambda_{\mathcal{F}}(Q) = 0$. Then, for any $\epsilon > 0$ the rectangle $Q \in \mathcal{K}_{\mathcal{F}}$ can be inscribed into the measurable rectangle $Q' \in \mathcal{K}_{\mathcal{F}}$ with positive measure $\lambda_{\mathcal{F}}(Q') < \epsilon$. According to the above assumption $Q' \in \mathcal{R}_{\mathcal{E}}$ and $\lambda_{\mathcal{E}}(Q') = \lambda_{\mathcal{F}}(Q')$. Therefore, the external measure $\bar{\lambda}_{\mathcal{E}}$ admits estimates $\bar{\lambda}_{\mathcal{E}}(Q) \leq \lambda_{\mathcal{E}}(Q') = \lambda_{\mathcal{F}}(Q') < \epsilon$. Since $\epsilon > 0$ is arbitrary, $\lambda_{\mathcal{E}}(Q) = 0 = \lambda_{\mathcal{F}}(Q)$.

Let us show that if $Q \in \mathcal{K}_{\mathcal{F}}$ and $\lambda_{\mathcal{F}}(Q) > 0$, then for any $\epsilon > 0$ there are

(1) a set $S \in \mathcal{R}_{\mathcal{E}}$ such that $Q \subset S$ and $\lambda_{\mathcal{E}}(S) < (1 + \epsilon)\lambda_{\mathcal{F}}(Q)$ (the upper estimate);
(2) a set $s \in \mathcal{R}_{\mathcal{E}}$ such that $s \subset Q$ and $\lambda_{\mathcal{E}}(s) > (1 - \epsilon)\lambda_{\mathcal{F}}(Q)$ (the low estimate).

I. The upper estimate. Let $Q \in \mathcal{K}_{\mathcal{F}}$ and $\lambda_{\mathcal{F}}(Q) > 0$. Let $\{d_k\}$ be a sequence of lengths of edges of the rectangle Q. Then, $\sum\limits_{k=1}^{\infty} \max\{0, d_k - 1\} < \infty$ since rectangle Q is measurable. Moreover, the series $\sum\limits_{k=1}^{\infty} (d_k - 1)$ converges absolutely since $\lambda_{\mathcal{F}}(Q) > 0$. Hence, $0 < d_0 \leq 1 \leq D_0 < +\infty$ where $d_0 = \inf\{d_k\}$, $D_0 = \sup\{d_k\}$. Since measures $\lambda_{\mathcal{E}}$, $\lambda_{\mathcal{F}}$ are invariant with respect to a shift, we can count that the rectangle Q is centered and it can be parametrized by the following way

$$Q = \{x \in E : \ (x, f_k) \in [-\frac{1}{2}d_k, \frac{1}{2}d_k], \, k \in \mathbb{N}\}.$$

Since the length of the rectangle Q projection onto the axis Oe_j, $j = 1, 2, \dots$ is equal to $b_j = \sum\limits_{k=1}^{\infty} d_k \mid c_{k,j} \mid$, $j \in \mathbb{N}$, the rectangle Q can be inscribed into the rectangle Π such that edges of the rectangle Π are collinear to the vectors of ONB $\mathcal{E}$ and lengths of edges form the sequence $\{b_j\}$.

Let us prove that the rectangle $\Pi \in \mathcal{K}(\mathcal{E})$ and $\lambda_{\mathcal{E}}(\Pi) > 0$. It is sufficient to prove that the series $\sum\limits_{j=1}^{\infty} \mid b_j - 1 \mid$ converges. For every $j \in \mathbb{N}$ we have

$$b_j = \sum_{k=1}^{\infty} d_k \mid c_{k,j} \mid = \sum_{k=1}^{\infty} \mid c_{k,j} \mid + \sum_{k=1}^{\infty} (d_k - 1) \mid c_{k,j} \mid = l_j + \sum_{k=1}^{\infty} (d_k - 1) \mid c_{k,j} \mid .$$

Hence, we have estimates

$$\sum_{j=1}^{\infty} \mid b_j - 1 \mid \leq \sum_{j=1}^{\infty} (l_j - 1) + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mid d_k - 1 \mid \mid c_{k,j} \mid =$$

$$= \sum_{j=1}^{\infty} (l_j - 1) + \sum_{k=1}^{\infty} \mid d_k - 1 \mid \hat{l}_k < +\infty. \tag{26}$$

In fact, $l_j = \sum\limits_{k=1}^{\infty} \mid c_{k,j} \mid > 1 \ \forall \ j$ and the series $\sum\limits_{j=1}^{\infty} (l_j - 1)$ converges according to condition (14). Analogously, $\hat{l}_k = \sum\limits_{j=1}^{\infty} \mid c_{k,j} \mid > 1 \ \forall \ k$ and the series $\sum\limits_{k=1}^{\infty} (\hat{l}_k - 1)$ converges according to the Theorem 6. Hence, the sequence $\{\hat{l}_k\}$ is bounded. Therefore, the estimate (26) holds.

Let us construct the centered rectangle $\tilde{\Pi} \in \mathcal{K}_{\mathcal{E}}$ such that lengths of edges of this rectangle form the sequence $\{\tilde{b}_j\}$, $\tilde{b}_j = \max\{1, b_j\}$, $j \in \mathbb{N}$. Then, the rectangle $\tilde{\Pi}$ is absolutely measurable according to the estimate (26).

Since the rectangle $Q \in \mathcal{K}_{\mathcal{F}}$ is measurable, the equality $\lambda_{\mathcal{F}}(Q) = \lim\limits_{n\to\infty} \lambda_n(Q_n)$ holds where Q_n is the n-dimensional section of the rectangle Q by the hyperplane F_n for each $n \in \mathbb{N}$.

Let $p_n = \mathbf{P}_{F_n,E_n}(Q_n)$ be the orthogonal projection of the n-dimensional rectangle $Q_n = F_n \bigcap Q$ from the subspace F_n onto the subspace E_n. In the proof of Lemma 14 the equality $\lambda_n(p_n) = \mid \det(\|C_n\|) \mid \lambda_n(Q_n)$ is obtained. Here the matrix $C_n = \|c^{(n)}{}_{i,j}\|$, $i, j \in \overline{1, n}$ is the Jacobi matrix of the linear mapping of orthogonal projection $\mathbf{P}_{F_n,E_n}$ in the bases $f_1, \ldots, f_n$ and $e_1, \ldots, e_n$ in subspaces F_n and E_n.

Let $P_n = \mathbf{P}_{E_n}(Q)$ be the orthogonal projection onto the subspace E_n of the unit rectangle Q. Let $\tilde{\Pi}_n$ be the projection of the rectangle $\tilde{\Pi}$ onto n-dimensional hyperplane E_n. Then, $p_n \subset P_n \subset \tilde{\Pi}_n$ for any $n \in \mathbb{N}$ and P_n be the convex subset of the space E_n. Let

$$\prod_{j=1}^{\infty} \tilde{b}_j = B < \infty. \tag{27}$$

Hence, $\lambda_n(\tilde{\Pi}_n) = \tilde{b}_1 \cdots \tilde{b}_n \leq B \ \forall \ n \in \mathbb{N}$.

The rectangle Q can be parametrized by shifts of n-dimensional rectangle Q_n along the vectors $f_{n+1}, \ldots$ by the equality $Q = \left\{ Q_n + \sum\limits_{k=n+1}^{\infty} d_k t_k f_k,\ t_k \in [-\frac{1}{2}, \frac{1}{2}],\ \sum\limits_{k=n+1}^{\infty} d_k^2 t_k^2 < \infty \right\}$. Therefore, the set P_n admits the parametrization

$$P_n = \left\{ p_n + \sum_{k=n+1}^{\infty} d_k t_k \mathbf{P}_{E_n}(f_k),\ t_k \in \left[-\frac{1}{2}, \frac{1}{2}\right],\ \sum_{k=n+1}^{\infty} d_k^2 t_k^2 < \infty \right\} =$$

$$= \left\{ p_n + \sum_{j=1}^{n} t_j \beta_{n,j} e_j,\ t_j \in [-\frac{1}{2}, \frac{1}{2}] \right\} \tag{28}$$

where $\beta_{n,j} = \sum\limits_{k=n+1}^{\infty} d_k \mid c_{k,j} \mid$. For every $n \in \mathbb{N}$ we define the numbers

$$\Delta_{j,n} = \sum_{k=n+1}^{\infty} \mid c_{k,j} \mid, \; j = 1, \ldots, n. \tag{29}$$

Hence, the set P_n lies in the convex hull S_n of shifts of the set p_n along the axes Oe_j onto the vectors $tD_0\Delta_{j,n}e_j$, $t \in [-\frac{1}{2}, \frac{1}{2}]$, $j = 1, \ldots, n$:

$$P_n \subset \bigcup_{(t_1,\ldots,t_n)\in[-1,1]^n} (p_n + \sum_{j=1}^{n} \frac{1}{2} t_j D_0 \Delta_{j,n} e_j) \equiv S_n \subset \tilde{\Pi}_n.$$

For every $n \in \mathbb{N}$ the set S_n is the result of sequentially for $j = 1, \ldots, n$ elongations of the set p_n along the axis Oe_j onto the value $\frac{1}{2}D_0\Delta_{j,n}$ in directions of vectors e_j and $-e_j$. We prove that the set S_n is Jordan measurable and obtain the estimate

$$\lambda_n(S_n) \leq \lambda(p_n) + \sum_{j=1}^{n} BD_0\Delta_{n,j} \tag{30}$$

for its Jordan measure by using the induction with respect to sequentially elongation.

Let $S_{n,0} = p_n$, $S_{n,k} = \bigcup\limits_{(t_1,\ldots,t_k)\in[-1,1]^k} (p_n + \frac{1}{2} \sum\limits_{j=1}^{k} t_j D_0 \Delta_{j,n} e_j)$ for every $k = 1, \ldots, n$. Then, $S_n = S_{n,n}$. We prove that sets $S_{n,k}$, $k = 1, \ldots, n$ are Jordan measurable and obtain estimates (30) by the induction with respect to indexes $k \in \{0, 1, \ldots, n\}$. The set $S_{n,0}$ is Jordan measurable and $\lambda_n(S_{n,0}) = \lambda(p_n)$. Let $k \in \{0, \ldots, n-1\}$. Let the set $S_{n,k}$ is Jordan measurable and the estimate

$$\lambda_n(S_{n,k}) \leq \lambda)p_n + \sum_{j=1}^{k} BD_0\Delta_{n,j} \tag{31}$$

holds. Then, $S_{n,k+1} = S_{n,k} \bigcup \Delta S_{n,k}$ where

$$\Delta S_{n,k} = S_{n,k+1} \backslash S_{n,k} = \bigcup_{t_{k+1}\in[-1,1]} (S_{n,k} + \frac{1}{2} t_{k+1} D_0 \Delta_{n,k+1} e_{k+1}).$$

Therefore, the set $\Delta S_{n,k}$ is Jordan measurable.

Inequalities $\lambda_{n-1}(\mathbf{P}_{\Gamma_{k+1}}(S_{n,k})) \leq \lambda_{n-1}(\mathbf{P}_{\Gamma_{k+1}})(S_{n,k}) \leq \tilde{b}_1 \ldots \tilde{b}_k \tilde{b}_{k+2} \ldots \tilde{b}_n \leq B$ take place since the orthogonal projection $\mathbf{P}_{\Gamma_{k+1}}$ of the set $S_{n,k}$ onto the $(n-1)$-dimensional hyperplane $\Gamma_{k+1} = \text{span}(e_1, \ldots, e_k, e_{k+2}, \ldots, e_n)$ belongs to the projection of the rectangle $\tilde{\Pi}_n$ onto this hyperplane. We have $\lambda_n(\Delta S_{n,k}) \leq BD_0\Delta_{n,k+1}$ since the elongation of the set $S_{n,k}$ to the set $S_{n,k+1}$ along the axis Oe_{k+1} is equal to $D_0\Delta_{n,k+1}$. Therefore, sets $S_{n,k}$ are Jordan measurable and estimates (31) are hold for any $k = 0, 1, \ldots, n$. Thus, for every $n \in \mathbb{N}$ the set S_n is measurable and the estimate (30) takes place.

Lemma 15. *Let the condition (14) be hold. Then* $\lim\limits_{n\to\infty} \left(\sum\limits_{j=1}^{n} \Delta_{j,n} \right) = 0$.

Proof. Let us fix a number $\epsilon > 0$. Since the series $\sum\limits_{j=1}^{\infty} (1 - \alpha_j)$ converges, there is the number $m_\epsilon \in \mathbb{N}$ such that $\sum\limits_{j=1+m}^{\infty} (1 - \alpha_j) < \frac{\epsilon}{2}$ for every $m \geq m_\epsilon$. According to the condition (14) the inequality $\sum\limits_{k=1}^{\infty} \mid c_{k,j} \mid < +\infty$ holds for any $j \in \mathbb{N}$. Hence, there is the number $N > m_\epsilon$ such that $\sum\limits_{j=1}^{m_\epsilon} \sum\limits_{k=N+1}^{\infty} \mid c_{k,j} \mid < \frac{\epsilon}{2BD_0}$. Thus, for every $n > N$ we have estimates

$$\sum_{j=1}^{n}\Delta_{j,n}=\sum_{j=1}^{m_\epsilon}\Delta_{j,n}+\sum_{j=m_\epsilon+1}^{n}\Delta_{j,n}=\sum_{j=1}^{m_\epsilon}\sum_{k=n+1}^{\infty}\mid c_{k,j}\mid+\sum_{j=m_\epsilon+1}^{n}\sum_{k=n+1}^{\infty}\mid c_{k,j}\mid<$$

$$<\sum_{j=1}^{m_\epsilon}\sum_{k=N+1}^{\infty}\mid c_{k,j}\mid+\sum_{j=m_\epsilon+1}^{n}(1-\alpha_j)<\epsilon.$$

□

Corollary 12. *Let the condition (14) be hold. Then for every $n\in\mathbb{N}$ there is the measurable set $S_n\subset E_n$, such that $P_n\subset S_n$ and the sequence $\{S_n\}$ satisfy the condition $\lim\limits_{n\to\infty}(S_n)=\lambda_{\mathcal{F}}(Q)$.*

The statement of the Corollary 12 is the consequence of the Lemmas 14, 15 and the estimates (30).

Let B_j be the projection of the rectangle Q onto the axis Oe_j. Then, B_j is the segment with the length $b_j=\sum\limits_{k=1}^{\infty}d_k\mid(e_j,f_k)\mid$. Since P_n is the projection of the rectangle Q onto the hyperplane E_n and $P_n\subset S_n$,

$$Q\subset S_n\times B_{n+1}\times B_{n+2}\times\cdots\quad\forall\,n\in\mathbb{N}.$$

According to (26) the series $\sum\limits_{k=1}^{\infty}(b_k-1)$ converges absolutely.

Let us fix a number $\epsilon>0$. Then, there is the number $n_0\in\mathbb{N}$ such that, at first, $\prod\limits_{j=n+1}^{\infty}b_j\in[1,1+\frac{\epsilon}{2})$ for any $n\geq n_0$; at second, $\lambda_n(S_n)\leq(1+\frac{\epsilon}{2})\lambda_{\mathcal{F}}(Q)$ for any $n\geq n_0$ according to the Corollary 12. Therefore, there is the set $S=S_{n_0}\times(B_{n_0+1}\times B_{n_0+2}\times\cdots)$ such that $\lambda_{\mathcal{E}}(S)<(1+\frac{\epsilon}{2})^2\lambda_{\mathcal{F}}(Q)$ and $S\supset Q$, $S\in\mathcal{R}_{\mathcal{E}}$. Hence, the estimation from above for $\bar{\lambda}_{\mathcal{E}}(Q)$ is obtained.

II. The low estimate. Let

$$Q=\{x\in E:\ (x,f_k)\in[-\frac{d_k}{2},\frac{d_k}{2}]\}$$

and

$$Q_n=\{x\in E:\ (x,f_k)\in[-\frac{d_k}{2},\frac{d_k}{2}]\,\forall\,k=1,\dots,n;\ (x,f_i)=0\,\forall\,i>n\}.$$

For every $n\in\mathbb{N}$ the set $p_n=\mathbf{P}_{E_n}(Q_n)$ is the orthogonal projection of the set Q_n onto the hyperplane E_n.

Hence, $p_n=\left\{\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{n}d_kt_kc_{j,k}e_j,\ t_1,\dots,t_n\in[-\frac{1}{2},\frac{1}{2}]\right\}$ for any $n\in\mathbb{N}$. Then, for any $x\in p_n$ and any $l=1,\dots,n$ the equality $(x,f_l)=\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{n}d_kt_kc_{j,k}c_{j,l}$ holds. Since $\sum\limits_{j=1}^{\infty}c_{j,k}c_{j,l}=\delta_{kl}$, $(x,f_l)=d_lt_l-\sum\limits_{k=1}^{n}\sum\limits_{j=n+1}^{\infty}d_kt_kc_{j,k}c_{j,l}$, $l=1,\dots,n$. Therefore,

$$\sup_{x\in p_n}\mid(x,f_l)-d_lt_l\mid\leq\frac{1}{2}\sum_{k=1}^{n}d_k\sum_{j=n+1}^{\infty}(\mid c_{j,k}\mid\mid c_{j,l}\mid)\leq$$

$$\leq\frac{1}{2}D_0\left(\sum_{k=1}^{n}\sum_{j=n+1}^{\infty}(\mid c_{j,k}\mid)\right)\sum_{j=n+1}^{\infty}\mid c_{j,l}\mid.\tag{32}$$

Let us consider numerical sequences $\Delta_{l,n}$, $n\in\mathbb{N}$, $l\in\{1,\dots,n\}$ (see (29)) and

$$\gamma_n=\sum_{k=1}^{n}\sum_{j=n+1}^{\infty}\mid c_{j,k}\mid=\sum_{k=1}^{n}\Delta_{k,n}.\tag{33}$$

According to (32)

$$\sup_{x\in p_n} \mid (x,f_l) \mid\leq d_l t_l + \frac{1}{2} D_0 \gamma_n \Delta_{l,n} \quad \forall\, l = 1,\dots,n. \tag{34}$$

Consider the set $\sigma_n \subset p_n$ where

$$\sigma_n = \Big\{ \sum_{k,j=1}^{n} d_k t_k c_{j,k} e_j,\ \mid t_k \mid\leq \frac{1}{2}\Big(1 - \frac{2}{d_0}\Delta_{k,n}\Big),\ k = 1,\dots,n \Big\}. \tag{35}$$

The set σ_n is the convex polyhedron in the space E_n. It is the image of n-dimensional rectangle $\Big\{ \sum\limits_{k,j=1}^{n} d_k t_k f_k;\ \mid t_k \mid\leq \frac{1}{2}(1 - \frac{2}{d_0}\Delta_{k,n}),\ k = 1,\dots,n \Big\}$ from the space F_n under the action of projector $\mathbf{P}_{F_n,E_n}$. Therefore,

$$\lambda_n(\sigma_n) = \det(C_n) \prod_{j=1}^{n} \Big(1 - \frac{2}{d_0}\Delta_{j,n}\Big) \lambda_n(Q_n). \tag{36}$$

The n-dimensional set σ_n admits the extension s_n along the directions $e_{n+1}, e_{n+2}, \dots$ such that $s_n \subset Q$. Consider the set

$$s_n = \Big\{ \sigma_n + \sum_{j=n+1}^{\infty} t_j e_j,\ t_j \in [-a_j, a_j],\ j \geq n+1 \Big\}. \tag{37}$$

Here for every $j \geq n+1$ the number a_j is chosen from the interval $(0, \frac{1}{2})$ such that the condition $s_n \subset Q$ holds. The condition $s_n \subset Q$ is equivalent to the system of inequalities $\sup\limits_{x\in s_n} \mid (x,f_k) \mid\leq \frac{d_k}{2},\ k \in \mathbb{N}$.

Since $\mid t_l \mid\leq \frac{1}{2}(1 - \frac{2}{d_0}\Delta_{l,n})$ for every $l = 1,\dots,n$ in the parametrization of the set (35), according to (34) we obtain estimates

$$\sup_{x\in \sigma_n} \mid (x,f_l) \mid\leq \frac{1}{2} d_l (1 - \Delta_{l,n}(\frac{2}{d_0} - \frac{D_0}{d_0}\gamma_n)).$$

Since $a_j \leq \frac{1}{2}\ \forall\, j \geq n+1$, according to (35) we have

$$\sup_{x\in s_n} \mid (x,f_k) \mid\leq \sup_{y\in \sigma_n} \mid (y,f_k) \mid + \sum_{j=n+1}^{\infty} a_j \mid c_{jk} \mid\leq$$

$$\leq \frac{d_k}{2}(1 - \Delta_{k,n}\frac{2 - D_0}{d_0}\gamma_n) + \frac{\Delta_{k,n}}{2} \leq \frac{d_k}{2}(1 - \Delta_{k,n}\frac{1 - D_0}{d_0}\gamma_n)$$

for any $k \in \{1,\dots,n\}$. Therefore, the inequality $\sup\limits_{x\in s_n} \mid (x,f_k) \mid\leq \frac{1}{2} d_k,\ k = 1,\dots,n$ holds for any sufficiently large n.

For every $k \geq 1+n$ we have

$$\sup_{x\in s_n} \mid (x,f_k) \mid\leq a_k c_{k,k} + \frac{1}{2}\sum_{j\neq k} c_{jk} = a_k \alpha_k + \frac{1}{2}(l_k - \alpha_k).$$

Hence, conditions $\sup\limits_{x\in s_n} \mid (x,f_k) \mid < \frac{1}{2} d_k,\ k > n+1$, are satisfied if the values a_k are defined by equations

$$a_k = \frac{1}{2\alpha_k}(d_k - l_k + \alpha_k) = \frac{1}{2} - \frac{l_k - d_k}{2\alpha_k},\ k = n+1,\dots.$$

Thus, if the number $n \in \mathbb{N}$ is sufficiently large, then there is the set $s_n \in \mathcal{R}_{\mathcal{E}}$ of type (37) such that $Q \supset s_n$ and its $\lambda_{\mathcal{E}}$-measure has the estimation

$$\lambda_{\mathcal{E}}(s_n) = \lambda_n(\sigma_n) \prod_{j=n+1}^{\infty} 2a_j \geq$$

$$\geq [\det(C_n)\lambda_n(Q_n)\prod_{j=1}^{n}(1 - \frac{2}{d_0}\Delta_{l,n})] \prod_{j=n+1}^{\infty} \Big(1 - \frac{l_k - d_k}{\alpha_k}\Big). \tag{38}$$

According to Corollary 11 we have $\lim\limits_{n\to\infty} \det C_n = 1$; $\lim\limits_{n\to\infty} \lambda_n(Q_n) = \lambda_{\mathcal{F}}(Q)$ since $Q \in \mathcal{K}_{\mathcal{F}}$. According to Lemma 11 and Theorem 7 we have

$$\lim_{n\to\infty} \Big(\sum_{k=n+1}^{\infty} \sum_{j=k+1}^{\infty} \mid c_{k,j} \mid + \sum_{j=n+1}^{\infty} \sum_{k=j+1}^{\infty} \mid c_{k,j} \mid \Big) = 0$$

and $\lim\limits_{n\to\infty} (\sum\limits_{l=1}^{n} \sum\limits_{j=n+1}^{\infty} \mid c_{l,j} \mid) = 0$. Hence, $\lim\limits_{n\to\infty} (\sum\limits_{l=1}^{n} \Delta_{l,n}) = 0$. Therefore,

$$\lim_{n\to\infty} \prod_{l=1}^{n} \Big(1 - \frac{2}{d_0}\Delta_{l,n}\Big) = 1.$$

Since $\alpha_k \in (0,1]$ and $\lim\limits_{k\to\infty} \alpha_k = 1$, the series $\sum\limits_{k=1}^{\infty} \frac{l_k - 1}{\alpha_k}$ converges according to the condition (15). According to the condition $\lambda_{\mathcal{F}}(Q) \in (0,+\infty)$ the series $\sum\limits_{k=1}^{\infty} \frac{1-d_k}{\alpha_k}$ converges. Hence, $\lim\limits_{n\to\infty} \prod\limits_{j=n+1}^{\infty} \Big(1 - \frac{l_k - d_k}{\alpha_k}\Big) = 1$. Therefore, $\lim\limits_{n\to\infty} \lambda_{\mathcal{E}}(s_n) = \lambda_{\mathcal{F}}(Q)$ according to (38) and we obtain the low estimate. The proof of the Theorem 8 is finished. □

5. The Ring $\mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{E}}$ in the Case of Distant Bases $\mathcal{F}$ and $\mathcal{E}$

Theorem 9. *If the condition (14) is not satisfied, then $\lambda_{\mathcal{F}}(A) = 0$ and $\lambda_{\mathcal{E}}(A) = 0$ for any set $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$.*

Lemma 16. *Let $s_{\mathcal{E}} = \Pi^0 \backslash (\bigcup\limits_{i=1}^{n} \Pi^i)$, $\Pi^k \in \mathcal{K}_{\mathcal{E}}\ \forall\, k = 0, \dots, n$. Let $\lambda_{\mathcal{E}}(s_{\mathcal{E}}) > 0$. Then there is the rectangle $P_{\mathcal{E}} \in \mathcal{K}_{\mathcal{E}}$ such that $P_{\mathcal{E}} \subset s_{\mathcal{E}}$ and $\lambda_{\mathcal{E}}(P_{\mathcal{E}}) > 0$.*

Proof. Firstly, we consider the case $s_{\mathcal{E}} = \Pi \backslash \Pi'$ where $\Pi, \Pi' \in \mathcal{K}_{\mathcal{E}}$ (we can assume that $\Pi' \subset \Pi$). Let $\Pi = \{x \in E: \ (x, e_j) \in [a_j, b_j),\ j \in \mathbb{N}\}$ and $\Pi' = \{x \in E: \ (x, e_j) \in [a'_j, b'_j),\ j \in \mathbb{N}\}$. Then, $[a_j, b_j) \supset [a'_j, b'_j)\ \forall\, j \in \mathbb{N}$. Since $\lambda_{\mathcal{E}}(\Pi \backslash \Pi') > 0$, there is the number $k \in \mathbb{N}$ such that $b''_k - a''_k > 0$ and $[a_k, b_k) \backslash [a'_k, b'_k) \supset [a''_k, b''_k)$. Let Π'' be the rectangle such that k-th edge of Π'' is the segment $[a''_k, b''_k)$ and its j-th edge coincides with j-th edges of the rectangle Π for every $j \neq k$. Then, the rectangle Π'' satisfies the following conditions: $\Pi'' \subset \mathcal{K}_{\mathcal{E}}$, $\Pi'' \subset \Pi$, $\Pi'' \bigcap \Pi' = \emptyset$, $\Pi'' \subset s_{\mathcal{E}}$ and $\lambda_{\mathcal{E}}(\Pi'') > 0$.

In general case $s_{\mathcal{E}} = \Pi^0 \backslash (\bigcup\limits_{i=1}^{n} \Pi^i)$, $\Pi^i \in \mathcal{K}_{\mathcal{E}}$, $i = 0, 1, \dots, n$, the statement of the Lemma 16 can be obtained by the applying of the induction method with respect to $n \in \mathbb{N}$. □

Let $l_j = \sum\limits_{k=1}^{\infty} \mid c_{k,j} \mid$ be the length of the projection of the unit rectangle $Q \in \mathcal{K}_{\mathcal{F}}$ onto the axis Oe_j for every $j \in \mathbb{N}$. Let $l^T_k = \sum\limits_{j=1}^{\infty} \mid c_{k,j} \mid$ be the length of the projection of the unit rectangle $\Pi \in \mathcal{K}_{\mathcal{E}}$ onto the axis Of_k for every $k \in \mathbb{N}$. Then, $l_j \geq 1$, $l_k^T \geq 1$ according to Lemma 10 Since the condition (14) is not satisfied, following two conditions hold

$$\prod_{j=1}^{\infty} l_j = +\infty; \quad \prod_{k=1}^{\infty} l_k^T = +\infty.$$

These two conditions are equivalent to following two equalities

$$\sum_{j=1}^{\infty}(l_j - 1) = +\infty; \quad \sum_{k=1}^{\infty}(l_k^T - 1) = +\infty. \tag{39}$$

Let $L^T = \sup\limits_{k \in \mathbb{N}} l_k^T$ and $L = \sup\limits_{j \in \mathbb{N}} l_j$. There are three possible cases for sequences $\{l_k^T\}$ and $\{l_j\}$.

(1) $L^T = +\infty$ and $L = +\infty$.
(2) $L^T < +\infty$ snd $L < +\infty$.
(3) either $L^T = +\infty$ and $L < +\infty$, or vice versa.

Let us prove the statement of the Theorem 9 for every of these three cases.
Let us study the case (1). Consider the case $L^T = +\infty$ and $L = +\infty$.

Lemma 17. *Let $L^T = +\infty$. If $\Pi \in \mathcal{K}_{\mathcal{E}}$ and $\lambda_{\mathcal{E}}(\Pi) > 0$, then $\overline{\lambda_{\mathcal{F}}}(\Pi) = +\infty$.*
Let $L = +\infty$. If $Q \in \mathcal{K}_{\mathcal{F}}$ and $\lambda_{\mathcal{F}}(Q) > 0$, then $\overline{\lambda_{\mathcal{E}}}(Q) = +\infty$.

Proof. Since $L^T = +\infty$, there are two possible cases: either

(i) $\exists\, k_0 \in \mathbb{N} : \; l_{k_0}^T = +\infty$;

or

(ii) $\forall\, k \in \mathbb{N} : \; l_k^T < +\infty$.

Let us prove that for every of these cases conditions $\Pi \in \mathcal{K}_{\mathcal{E}} \, : \, \lambda_{\mathcal{E}}(\Pi) > 0$ imply $\overline{\lambda_{\mathcal{F}}}(\Pi) = +\infty$. Hence, $\overline{\lambda_{\mathcal{F}}}(\Pi) = +\infty$.

(i) Let the condition $\exists\, k_0 \in \mathbb{N} : \; l_{k_0}^T = +\infty$ be satisfied. Then the projection of the rectangle Π onto the axis Of_{k_0} is unbounded segment. However, the projection of a set S onto the axis Of_{k_0} is the finite union of bounded segments. Therefore, it is impossible to cover a rectangle $\Pi \in \mathcal{K}_{\mathcal{E}} \, : \; \lambda_{\mathcal{E}}(\Pi) > 0$ by the finite union S of rectangles from the collection $\mathcal{K}_{\mathcal{F}}$.
(ii) Let the condition $l_k^T < +\infty \; \forall\, k \in \mathbb{N}$ be satisfied. If $\Pi \in \mathcal{K}_{\mathcal{E}} \, : \; \lambda_{\mathcal{E}}(\Pi) > 0$, then the sequence of lengths of projections of the rectangle Π onto the axes Of_k, $k \in \mathbb{N}$, is unbounded. If $\{r_k\}$ is the sequence of lengths of edges of the rectangle Π, then $\inf\limits_j r_j = r_0 > 0$ according to the condition $\lambda_{\mathcal{E}}(\Pi) > 0$. Therefore, the length of the projection of the rectangle Π onto the axis Of_k no less than $r_0 l_k^T$. The sequence l_k^T is unbounded according to (ii). Hence, the sequence of lengths of projections of the rectangle Π onto the axes Of_k, $k \in \mathbb{N}$ is unbounded.

On the other hand, if $S = \bigcup\limits_{k=1}^{m} Q_k$ is the finite union of rectangles from the collection $\mathcal{K}_{\mathcal{F}}$, then the sequence of lengths of projections of the set S onto an axis Of_j is bounded. Hence, in the case (ii) it is impossible to cover the rectangle $\Pi \in \mathcal{K}_{\mathcal{E}} \, : \; \lambda_{\mathcal{E}}(\Pi) > 0$ by the finite union S of rectangles from the collection $\mathcal{K}_{\mathcal{F}}$. Therefore, $\overline{\lambda_{\mathcal{F}}}(\Pi) = +\infty$. □

Corollary 13. *If $L = +\infty$, then the condition $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ implies $\lambda_{\mathcal{F}}(A) = 0$. If $L^T = +\infty$, then the condition $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ implies $\lambda_{\mathcal{E}}(A) = 0$. If the condition 1) is satisfied, then for any set $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ the equality $\lambda_{\mathcal{F}}(A) = \lambda_{\mathcal{E}}(A) = 0$ holds.*

Proof. Let $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$. Let us assume the opposite that $\lambda_{\mathcal{E}}(A) > 0$. Then, according to the Lemma 16 there is a rectangle $p \in \mathcal{K}_{\mathcal{E}}$ such that $p \subset A$ and $\lambda_{\mathcal{E}}(p) > 0$. According to the Lemma 17 the condition $L^T = +\infty$ implies that $\overline{\mu_{\mathcal{F}}}(A) \geq \overline{\mu_{\mathcal{F}}}(p) = +\infty$. This is the contradiction to the condition $A \in \mathcal{R}_{\mathcal{F}}$. Therefore, $\lambda_{\mathcal{E}}(A) = 0$. Analogously, the condition $L = +\infty$ implies that $\lambda_{\mathcal{F}}(A) = 0$. □

Let us study the case (2). Consider the case $L^T < +\infty$ and $L < +\infty$.

Lemma 18. *Let* $\Pi \supset Q$ *where* $\Pi \in \mathcal{K}_{\mathcal{E}}$, $Q \in \mathcal{K}_{\mathcal{F}}$. *Let the condition (2) be satisfied. Then* $\lambda_{\mathcal{F}}(Q) = 0$.

Proof. Let us assume the opposite that $\lambda_{\mathcal{F}}(Q) > 0$. Let $\{d_k\}$ be the sequence of lengths of the edges of the rectangle Q. We can assume that $d_k \leq 1$. In fact, in opposite case we can change the rectangle Q onto the smaller inscribed rectangle $Q' :\ Q' \subset Q \subset \Pi$ such that lengths of edges of the rectangle Q' no greater than 1. Then, $\delta_k = 1 - d_k \geq 0\ \forall\ k$, $\lim\limits_{k\to\infty} d_k = 1$ and

$$\sum_{k=1}^{\infty} \delta_k < +\infty \tag{40}$$

according to the condition $\lambda_{\mathcal{F}}(Q) \in (0, +\infty)$.

Therefore, if $\Pi \in \mathcal{K}_{\mathcal{E}}$, $\Pi \supset Q$ and $\{D_j\}$ is the sequence of lengths of edges of the rectangle Π, then

$$D_j \geq \sum_{k=1}^{\infty} d_k \mid (f_k, e_j) \mid = \sum_{k=1}^{\infty} (1 - \delta_k) \mid c_{j,k} \mid = l_j - \sum_{k=1}^{\infty} \delta_k \mid c_{j,k} \mid .$$

Hence,

$$\sum_{j=1}^{\infty} (D_j - 1) \geq \sum_{j=1}^{\infty} (l_j - 1 - \sum_{k=1}^{\infty} \mid c_{j,k} \mid \delta_k) = \sum_{j=1}^{\infty} (l_j - 1) - \sum_{k=1}^{\infty} (\sum_{j=1}^{\infty} \mid c_{j,k} \mid) \delta_k =$$

$$= \sum_{j=1}^{\infty} (l_j - 1) - \sum_{k=1}^{\infty} l_k^T \delta_k \geq \sum_{j=1}^{\infty} (l_j - 1) - L^T \sum_{k=1}^{\infty} \delta_k = +\infty$$

since the first series diverges according to the assumption of violation of the condition (14) and the second series is converges according to the condition (40) with finiteness of the value $L^T = \sup\limits_k l_k^T$. But the condition $\Pi \in \mathcal{K}_{\mathcal{E}}$ implies that $\sum\limits_{j=1}^{\infty} (D_j - 1) < +\infty$. The obtained contradiction proves the statement of Lemma 18. □

Corollary 14. *Let the condition (2) be satisfied. Then for any rectangle* $\Pi \in \mathcal{K}_{\mathcal{E}}$ *the equality* $\underline{\lambda_{\mathcal{F}}}(\Pi) = 0$ *holds.*

Proof. Let us assume the opposite, that there is a rectangle $\Pi \in \mathcal{K}_{\mathcal{E}}$ such that $\underline{\mu_{\mathcal{F}}}(\Pi) > 0$. Then, there is the set $s \in r_{\mathcal{F}}$ such that $\lambda_{\mathcal{F}}(s) > 0$ and $s \subset \Pi$. Hence, according to Lemma 16 there is a rectangle $q \in \mathcal{K}_{\mathcal{F}}$ such that $q \subset s \subset \Pi$ and $\lambda_{\mathcal{F}}(q) > 0$. Therefore, $\lambda_{\mathcal{E}}(\Pi) = +\infty$ according to Lemma 18. This is the contradiction with the condition $\Pi \in \mathcal{K}_{\mathcal{E}}$. □

Let us show that $\lambda_{\mathcal{F}}(A) = 0$ for any $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$. For this goal we prove that if the rectangle $q \in \mathcal{K}_{\mathcal{F}}$ and $\lambda_{\mathcal{F}}(q) > 0$, then it is impossible to cover the rectangle $q \in \mathcal{K}_{\mathcal{F}}$ by the finite union $S \in r_{\mathcal{E}}$ of rectangles $\Pi_1, \dots, \Pi_m \in \mathcal{K}_{\mathcal{E}}$.

Lemma 19. *Let* $S = \bigcup\limits_{i=1}^{m} \Pi_i$ *where* $\Pi_i \in \mathcal{K}_{\mathcal{E}}$, $i = 1, \dots, m$ *and* $\Pi_i \neq \varnothing\ \forall\ i = 1, \dots, m$. *Then there is a hyperplane* Γ *of finite codimension* $k \leq m - 1$ *of type*

$$\Gamma^{j_1,\dots,j_k}_{c_1,\dots,c_k} = \{x \in E :\ (x, e_j) = c_j,\ j = 1, \dots, k\} \tag{41}$$

such that $\Gamma \bigcap S = \Gamma \bigcap \Pi_{j_*}$ *for some* $j_* \in \{1, \ldots, m\}$.

Proof. Let us prove the statement by the induction. Firstly we note that the statement is true for $m = 1$ since in this case $S = \Pi_1$ and $\Gamma = E$.

Let $\mathbf{P}_{e_j}^{\perp}$ be the orthogonal projection onto the subspace $(x, e_j) = 0$ in the space E for every $j \in \mathbb{N}$. For every $j \in \mathbb{N}$ the intersection $\Gamma_c^j \bigcap \Pi_1$ is either empty set or rectangle in the hyperplane Γ_c^j.

Let $m \in \mathbb{N}$.

The intersection $\Gamma_c^j \bigcap S = \bigcup\limits_{i=1}^{m} (\Gamma_c^j \bigcap \Pi_i)$ is the union of no more than m nonempty rectangles $\Gamma_c^j \bigcap \Pi_i$, $i = 1, \ldots, m$ in the hyperplane Γ_c^j. Therefore, for every $j \in \mathbb{N}$ the set-valued function $\mathbf{P}_{e_j}^{\perp}(\Gamma_c^j \bigcap S)$, $c \in \mathbb{R}$, can has only finite number of values and these values lies in a set of subsets of the subspace $E \ominus \mathrm{span}(e_j)$.

If the set-valued function $\mathbf{P}_{e_j}^{\perp}(\Gamma_c^j \bigcap S)$, $c \in \mathbb{R}$, has only one nonempty value B_j for any $j \in \mathbb{N}$, then the set S is the rectangle $(\times)_{j=1}^{\infty} \mathbf{P}_{Oe_j}(S)$ and the statement of the Lemma is true for hyperplane Γ_c^j for any j and for any $c \in \mathbf{P}_{Oe_j}(S)$.

In opposite case there is a number $j \in \mathbb{N}$ and there are $c', c'' \in \mathbf{P}_{Oe_j}(S)$ such that the sets $\Gamma_{c'}^j \bigcap S$ and $\Gamma_{c''}^j \bigcap S$ are nonempty sets and $\mathbf{P}_{e_j}^{\perp}(\Gamma_{c'}^j \bigcap S) \neq \mathbf{P}_{e_j}^{\perp}(\Gamma_{c''}^j \bigcap S)$. Hence, for at least one of two numbers c', c'' (for certainty, for the number c') the set $\mathbf{P}_{e_j}^{\perp}(\Gamma_{c'}^j \bigcap S)$ can't be the union of m nonempty rectangles $\mathbf{P}_{e_j}^{\perp}(\Gamma_{c'}^j \bigcap \Pi_i)$, $i = 1, \ldots, m$ in the space $E \ominus \mathrm{span}(e_j)$. Therefore, there are numbers $j \in \mathbb{N}$ and $c' \in \mathbb{R}$ such that the set $\Gamma_{c'}^j$ is the union of no more than $m - 1$ rectangles in the space $E \ominus \mathrm{span}(e_j)$. Then, we can apply the assumption of induction to the set $\Gamma_{c'}^j(S)$ in the space $E \ominus \mathrm{span}(e_j)$. Thus, we obtain the statement of the Lemma. □

Corollary 15. *Let* $q \in \mathcal{K}_{\mathcal{F}}$ *and* $q \subset S$ *where* $S = \bigcup\limits_{i=1}^{m} \Pi_i$, $\Pi_i \in \mathcal{K}_{\mathcal{E}}$ $\forall\, i = 1, \ldots, m$. *Then there are a hyperplane* Γ *of type (41) and a number* $i^* \in \{1, \ldots, m\}$ *such that* $\Gamma \bigcap q \subset \Gamma \bigcap \Pi_{i_*}$.

Lemma 20. *Let* $\mathcal{E}$, $\mathcal{F}$ *be a pair of ONB such that the condition (2) is satisfied. If* $q \in \mathcal{K}_{\mathcal{F}}$ *and* $\lambda_{\mathcal{F}}(q) > 0$, *then* $\overline{\lambda_{\mathcal{E}}}(q) = +\infty$. *If* $p \in \mathcal{K}_{\mathcal{E}}$ *and* $\lambda_{\mathcal{E}}(p) > 0$, *then* $\overline{\lambda_{\mathcal{F}}}(p) = +\infty$.

Proof. We can assume without loss of generality that q is a centered rectangle such that the sequence of lengths of its edges $\{d_k\}$ satisfies the condition $d_k \leq 1 \,\forall\, k$ (see proof of the Lemma 18). Since $\lambda_{\mathcal{F}}(q) > 0$, $d_0 = \inf\limits_k d_k > 0$ and $D_0 = \sup\limits_k d_k < +\infty$. Lengths of the projections of the rectangle q onto axes Oe_j form the numerical sequence $\{D_j\}$. Then, $D_j \leq D_0 L$. On the other hand, the condition

$$\prod_{j=1}^{\infty} D_j = \infty \tag{42}$$

holds according to Lemma 18. For every $j \in \mathbb{N}$ the projection of the rectangle q onto the axis Oe_j is the segment containing the interval $\left(-\frac{D_j}{2}, \frac{D_j}{2}\right)$.

The rectangle q admits the parametrization

$$q = \left\{x = \sum_{i=1}^{\infty} d_i t_i f_i,\ t_i \in [-\frac{1}{2}, \frac{1}{2}],\ \sum_{i=1}^{\infty} t_i^2 < +\infty\right\}.$$

Let us fix a numbers $j \in \mathbb{N}$ and $b \in (-\frac{D_j}{2}, \frac{D_j}{2})$. Let us consider the intersection $\Gamma_b^j \bigcap q = \{x \in q : \ x_j = b\}$. The set $\Gamma_b^j \bigcap q$ admits the parametrization

$$\Gamma_b^j \bigcap q = \Big\{ x = \sum_{i=1}^{\infty} d_i t_i f_i, \ t_i \in \Big[-\frac{1}{2}, \frac{1}{2}\Big], \ \sum_{i=1}^{\infty} t_i^2 < +\infty; \ \sum_{i=1}^{\infty} d_i c_{i,j} t_i = b \Big\}. \tag{43}$$

Since $|b| < \frac{D_j}{2}$, $\Gamma_b^j \bigcap q \neq \emptyset$. Hence, there is a vector $\{t_i^0\} \in \ell_2$ such that $\sum\limits_{i=1}^{\infty} d_i c_{i,j} t_i^0 = b$ and $| \ t_i^0 \ | < \frac{1}{2} \ \forall \ i \in \mathbb{N}$.

Since $\alpha_j = \max\limits_{i \in \mathbb{N}} | \ c_{i,j} \ | > 0$ for given j, we can define the positive number

$$\delta_j = \min\{ \frac{1}{2\alpha_j d_j} (\frac{D_j}{2} - |b|), \ \frac{1}{2} - | \ t_j^0 \ | \} > 0.$$

Hence, for given number j there is the number $i_0 > j$ such that $\sum\limits_{i=i_0}^{\infty} d_i \ | \ c_{i,j} \ | < \frac{1}{2} \alpha_j d_j \delta_j$. Therefore, for every collection of numbers $\hat{t}_j = (t_1, \ldots, t_{j-1}, t_{j+1}, \ldots)$ such that

$$t_i \in \Big[-\frac{d_j \delta_j}{4 i_0 D_0}, \frac{d_j \delta_j}{4 i_0 D_0} \Big], \ i = 1, \ldots, i_0, \ i \neq j; \quad t_i \in \Big[-\frac{1}{2}, \frac{1}{2}\Big], \ i > i_0$$

there is the number $t_j = t_j(\hat{t}_j) \in (t_j^0 - \delta_j, t_j^0 + \delta_j)$ such that $d_j c_{j,j} t_j + \sum\limits_{i \neq j} d_i c_{i,j} t_i = b$.

Thus, points with parameters t_i, $i \in \mathbb{N}$, in the parametrization (43) belong to the set $\Gamma_b^j \bigcap q$ if the following conditions hold:

$$t_i \in [-\frac{1}{2}, \frac{1}{2}], \ i > i_0;$$

$$| \ t_i - t_i^0 \ | < \frac{d_j \delta_j}{4 i_0 D_0}, \ i = 1, \ldots, i_0, \ i \neq j; \tag{44}$$

$$t_j = t_j(\hat{t}_j) \in (t_j^0 - \delta_j, t_j^0 + \delta_j).$$

For any $x \in \Gamma_b^j \bigcap q$ which is given by parametrization (43) with the parameters from the set (44) we have $(x, e_k) = \sum\limits_{i=1}^{\infty} d_i c_{i,k} t_i$. Hence,

$$\sup_{x \in \Gamma_b^j \bigcap q} | \ (x, e_k) \ | \geq \sum_{i=1}^{\infty} \frac{1}{2} d_i \ | \ c_{i,k} \ | - \sum_{i=1}^{i_0} \frac{1}{2} d_i \ | \ c_{i,k} \ |$$

for any $k \in \mathbb{N}$. Therefore, for every $k > i_0$ we obtain

$$\{(x, e_k), \ x \in \Gamma_b^j \bigcap q\} \supset \Big(-\frac{1}{2} D_k + \frac{1}{2} \sum_{i=1}^{i_0} d_i \ | \ c_{i,k} \ |, \ \frac{1}{2} D_k - \frac{1}{2} \sum_{i=1}^{i_0} d_i \ | \ c_{i,k} \ | \Big).$$

According to (42) and the condition $L^T < \infty$ we have

$$\sum_{k=k_0+1}^{\infty} (D_k - 1) - \sum_{k=k_0+1}^{\infty} \sum_{i=1}^{i_0} d_i \ | \ c_{i,k} \ | \geq \sum_{k=k_0+1}^{\infty} (D_k - 1) - D_0 \sum_{i=1}^{i_0} l_i^T = +\infty.$$

Then, we obtain

$$\prod_{k=k_0+1}^{\infty} (D_k - \sum_{i=1}^{i_0} d_i \ | \ c_{i,k} \ |) = +\infty. \tag{45}$$

Therefore, every intersection of the rectangle q by the one-codimensional hyperplane of type $\Gamma_c^j = \{x : (x, e_j) = c, j \in \mathbb{N}, c \in (-\frac{D_j}{2}, \frac{D_j}{2})\}$ can not be covered by some measurable rectangle $\Pi^* \in \mathcal{K}_{\mathcal{E}}$. Because if $\Pi^* \supset (\Gamma_b^j \bigcap q)$, then for every $k > i_0$ the k-th edge of rectangle Π^* should has the length no less than $D_k - \sum\limits_{i=1}^{i_0} d_i \mid c_{i.k} \mid$. Thus, the condition $\Pi^* \in \mathcal{K}_{\mathcal{E}}$ is violated according to (45).

The same reasoning allows us to show that every intersection of rectangle q by the m-codimensional hyperplane $\Gamma_{c_1,\dots,c_m}^{j_1,\dots,j_m}$ of type

$$\{x : (x, e_{j_1}) = c_1, \dots, (x, e_{j_m}) = c_m, j_1, \dots, j_m \in \mathbb{N},$$

$$c_k \in (-\frac{D_k}{2}, \frac{D_k}{2}) \; \forall \, k \in \{j_1, \dots, j_m\}\}$$

with some $m \in \mathbb{N}$ can not be covered by some measurable rectangle $\Pi^* \in \mathcal{K}_{\mathcal{E}}$.

Let us assume that $\overline{\lambda}_{\mathcal{E}}(q) < +\infty$. Then there is the set

$$S = \bigcup_{k=1}^{N} \Pi_k, \; N \in \mathbb{N}, \; \Pi_k \in \mathcal{K}_{\mathcal{E}} \; \forall \, k \in \{1, \dots, N\},$$

such that $q \subset S$. Then, $\{x \in S : (x, e_j) = c\} \neq \emptyset$ for every $j \in \mathbb{N}$ and for every $c \in (-\frac{1}{2}D_j, \frac{1}{2}D_j)$. Therefore, according to the Corollary 15 the condition $q \subset S = \bigcup\limits_{k=1}^{N} \Pi_k$ implies that there are numbers $j^* \in \{1, \dots, N\}$ and $c_1, \dots, c_N$ such that $c_i \in (-\frac{1}{2}D_i, \frac{1}{2}D_i)$ and $\Gamma_{c_1,\dots,c_m}^{j_1,\dots,j_m} \bigcap q \subset \Pi_{j^*}$. Hence, the intersection $q \bigcap \Gamma_{c_1,\dots,c_m}^{j_1,\dots,j_m}$ is covered by one rectangle Π_{j^*}.

The obtained contradiction prove that the rectangle $q \in \mathcal{K}_{\mathcal{F}} : \lambda_{\mathcal{F}}(q) > 0$ can't be covered by the finite union of rectangles from the collection $\mathcal{K}_{\mathcal{E}}$. Hence, $\overline{\lambda}_{\mathcal{E}}(q) = +\infty$. □

Corollary 16. *Let the pair of bases satisfy the condition (2). Then for any* $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ *the equality* $\lambda_{\mathcal{E}}(A) = \lambda_{\mathcal{E}}(A) = 0$ *holds. (The proof is the same as the proof for the Corollary 13).*

Let us study the case (3). Consider the case $L^T = +\infty$, $L < +\infty$, or vice versa.

Let $L^T = +\infty$, $L < +\infty$. Then the sequence l_k^T, $k \in \mathbb{N}$, either is an unbounded real valued sequence or takes values $+\infty$. The sequence l_j, $j \in \mathbb{N}$ is bounded against, but $\sum\limits_{j=1}^{\infty} (l_j - 1) = +\infty$. Since $L^T = +\infty$, according to the corollary 13 $\lambda_{\mathcal{E}}(A) = 0$ for any set $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$.

Theorem 10. *Let the condition (3) by satisfied. Then* $\lambda_{\mathcal{F}}(A) = 0$ *for any set* $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$.

Proof. Let us assume the opposite, that there is a set $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ such that $\lambda_{\mathcal{F}}(A) > 0$. Then, according to the Lemma 16 there is the rectangle $q \in \mathcal{K}_{\mathcal{F}}$ such that $q \subset A$ and $\lambda_{\mathcal{F}}(q) > 0$. Let $\{d_k\}$ be the sequence of lengths of edges of the rectangle $q \in \mathcal{K}_{\mathcal{F}}$. Then, we can assume without loss of generality that $d_k \leq 1 \, \forall \, k \in \mathbb{N}$. Since $\lambda_{\mathcal{F}}(q) > 0$, $\sum\limits_{k=1}^{\infty} \mid d_k - 1 \mid < +\infty$. In particular

$$\lim_{k \to \infty} d_k = 1. \tag{46}$$

The lengths of projections of the rectangle q onto the axes Oe_j, $j \in \mathbb{N}$ are

$$D_j = \sum_{k=1}^{\infty} \mid c_{j,k} \mid d_k, \; j \in \mathbb{N}.$$

Therefore, if $\Pi \in \mathcal{K}_{\mathcal{E}}$ and $\Pi \supset q$, then

$$\lambda_{\mathcal{E}}(\Pi) \geq \exp(\sum_{j=1}^{\infty} \ln(D_j)).$$

To prove the Theorem 10 we firstly obtain following five statements.

Lemma 21. *If the condition*

$$\overline{\lim_{j\to\infty}} l_j = L_0 > 1, \tag{47}$$

is satisfied, then $\sum\limits_{n=1}^{\infty} \max\{0, \ln(D_j)\} = +\infty$.

Proof. Let us fix $\epsilon > 0$. According to (46) there is the number K_ϵ such that $d_k > 1 - \epsilon$ for any $k > K_\epsilon$.

Since $\lim\limits_{j\to\infty} c_{k,j} = 0$ for any k, we have $\lim\limits_{j\to\infty} \left(\max\limits_{k\in 1,\ldots,K_\epsilon} | c_{k,j} | \right) = 0$. Hence, there is the number J_ϵ such that $\sum\limits_{k=1}^{K_\epsilon} | c_{k,j} | < \epsilon$ for every $j > J_\epsilon$. Therefore, for every $j > J_\epsilon$ we have the estimate

$$D_j = \sum_{k=1}^{\infty} | c_{k,j} | d_k \geq \sum_{k=K_\epsilon}^{\infty} | c_{k,j} | (1-\epsilon) \geq (1-\epsilon)(l_j - \epsilon). \tag{48}$$

Since $\epsilon > 0$ is arbitrary, according to (47) we can choose the value $\epsilon > 0$ in (48) such that there is a strictly monotone sequence of numbers $\{j_n\}$ satisfying following condition: $D_{j_n} > \frac{1}{2}(1 + L_0) > 1$ for any $n \in \mathbb{N}$. Hence, $\sum\limits_{n=1}^{\infty} (D_{j_n} - 1) = +\infty$ and Lemma 21 is proved. □

Corollary 17. *If the condition (47) is satisfied, then a rectangle* $q \in \mathcal{K}_{\mathcal{F}}$ *with positive measure* $\lambda_{\mathcal{F}}(q) > 0$ *can't be covered by a rectangle* $\Pi \in \mathcal{K}_{\mathcal{E}}$.

Let us introduce the notation $\delta_k = 1 - d_k$, $k \in \mathbb{N}$.

Lemma 22. *Let the condition (47) be violated. If* $\|\delta\|_2 = \sqrt{\delta_1^2 + \delta_2^2 + \ldots} < 1$*, then a rectangle* q *with positive measure* $\lambda_{\mathcal{F}}(q) > 0$ *can't be inscribe into any rectangle* $\Pi \in \mathcal{K}_{\mathcal{E}}$.

Proof. Since $l_j \geq 1 \ \ \forall\, j \in \mathbb{N}$, the negation of the condition (47) implies that there is the limit $\lim\limits_{j\to\infty} l_j = 1$. According to Lemma 10 the condition $\alpha_j < \frac{1}{\sqrt{2}}$ implies the estimate $l_j \geq \sqrt{2}$; the condition $\alpha_j > \frac{1}{\sqrt{2}}$ implies the inequality $l_j \geq \alpha_j + \sqrt{1-\alpha_j^2}$ (remember that $\alpha_j = \max\limits_{k\in\mathbb{N}} | c_{k,j} |$, $j \in \mathbb{N}$). Hence, the equality $\lim\limits_{j\to\infty} l_j = 1$ implies that $\lim\limits_{j\to\infty} \alpha_j = 1$. If $\alpha_j = 1 - \beta_j$ (see the notation used in the Lemma 11), then $l_j \geq 1 - \beta + \sqrt{2\beta_j - \beta_j^2}$. Therefore, the asymptotic equality

$$\beta_j = \frac{1}{2}(l_j - 1)^2(1 + o(1)) \tag{49}$$

as $j \to \infty$ holds. For every $j \in \mathbb{N}$ we have the estimate

$$D_j - 1 = \sum_{k=1}^{\infty} | c_{k,j} | d_k - 1 = l_j - 1 - \sum_{k=1}^{\infty} | c_{k,j} | \ \delta_k =$$

$$= l_j - 1 - \sum_{k\neq j}^{\infty} | c_{k,j} | \delta_k - | c_{j,j} | \delta_j \geq l_j - 1 - \delta_j - \sqrt{1-\alpha_j^2}\|\delta\|_2. \tag{50}$$

According to (49) we have $\sqrt{1-\alpha_j^2}\|\delta\|_2 = \sqrt{\beta_j(1+\beta_j)}\|\delta\|_2 \sim (l_j-1)\|\delta_j\|_2$ as $j \to \infty$. Therefore, there are numbers $J \in \mathbb{N}$ and $\sigma \in (0, 1-\|\delta\|_2)$ such that

$$\sqrt{1-\alpha_j^2}\|\delta\|_2 \leq (1-\sigma)(l_j-1)$$

for every $j > J$. Hence,

$$D_j - 1 \geq \sigma(l_j-1) - \delta_j \ \ \forall \ j > J$$

according to (50). Since the condition (14) is not satisfied, the nonnegative series $\sum\limits_{j=1}^{\infty}(l_j-1)$ diverges. Therefore, $\sum\limits_{j=1}^{\infty}(D_j-1) = +\infty$, hence $\lambda(\Pi) = \prod\limits_{j=1}^{\infty} D_j = +\infty$. $\square$

Lemma 23. *Let $q \in \mathcal{K}_{\mathcal{F}}$ and the sequence $\{d_k\}$ of lengths of its edges satisfy the condition of the Lemma 22. Then $\overline{\lambda_{\mathcal{E}}}(q) = +\infty$.*

The proof of the Lemma 23 repeats the proof of the Lemma 20.

Lemma 24. *Let $q \in \mathcal{F}$ and $\lambda_{\mathcal{F}}(q) > 0$. Then $\overline{\lambda_{\mathcal{E}}}(q) = +\infty$.*

Proof. Let us assume the opposite, that $\overline{\lambda_{\mathcal{E}}}(q) < +\infty$. Hence, there is a set $S = \bigcup\limits_{s=1}^{m} \Pi_s$ such that $\Pi_1, \dots, \Pi_m \in \mathcal{K}_{\mathcal{E}}$ and $q \subset S$.

Since $\lambda_{\mathcal{F}}(q) > 0$, there is a number $N \in \mathbb{N}$ such that $\sum\limits_{k=N+1}^{\infty} \delta_k < 1$.

We can assume that the rectangle q is centered. Consider the orthogonal projections $q_N = \mathbf{P}_{F_N}(q)$, $q^N = \mathbf{P}_{F^N}(q)$ of the rectangle q onto N-dimensional subspace $F_N = \text{span}(f_1, \dots, f_N)$ and onto its orthogonal complement respectively. Let Q_N^1 be the centered unit rectangle in the subspace F_N. Let $Q = Q_N^1 \times q^N$. Then the rectangle Q satisfies conditions of Lemma 22. Therefore,

$$\overline{\lambda_{\mathcal{E}}}(Q) = +\infty. \tag{51}$$

Lengths of edges of the rectangle q satisfy conditions $d_k \leq 1 \; \forall \, k \in \mathbb{N}$. Hence, $q_N \subset Q_N^1$ and $q \subset Q$. Since $\lambda_N(q_N) > 0$, there is the collection of vectors $h_1, \dots, h_M \in F_N$ such that $\bigcup\limits_{j=1}^{M} \mathbf{S}_{h_j}(q_N) \supset Q_N^1$ (here $\mathbf{S}_{h_j}(q_N) = q_N + h_j$). Since $\bigcup\limits_{j=1}^{M} \mathbf{S}_{h_j}(q) \supset Q$, we have $\bigcup\limits_{j=1}^{M} \mathbf{S}_{h_j}(S) \supset Q$. Thus, we obtain the contradiction with the condition (51). $\square$

Therefore, the statement of the Theorem 10 is the consequence of the Lemma 24. $\square$

Corollary 18. *Let the condition (3) be satisfied. Then the equality $\lambda_{\mathcal{E}}(A) = \lambda_{\mathcal{E}}(A) = 0$ holds for any set $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$.*

The Theorem 9 follows from corollaries 13, 16, 18.

6. Isometry-Invariant Measure

Let $\mathbb{I}$ be the group of isometries of the space E. The group $\mathbb{I}$ is generated by the group $\mathbb{S}$ of shifts on a vector of the space E and the orthogonal group $\mathbb{O}$ of orthogonal mappings in the space E.

Let $\mathcal{S}$ be a set of ONB in Hilbert space E. Theorems 8 and 9 imply the following statement.

Theorem 11. *Let $\mathcal{E}, \mathcal{F} \in \mathcal{S}$. Then $\lambda_{\mathcal{E}}|_{\mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}} = \lambda_{\mathcal{F}}|_{\mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}}$. In particular, if the condition (14) is satisfied, then $\mathcal{R}_{\mathcal{E}} = \mathcal{R}_{\mathcal{F}}$ and $\lambda_{\mathcal{E}} = \lambda_{\mathcal{F}}$. If the condition (14) is violated, then $\lambda_{\mathcal{F}}(A) = \lambda_{\mathcal{E}}(A) = 0$ for every set $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$.*

Consider the family $\{R_{\mathcal{E}}, \mathcal{E} \in \mathcal{S}\}$ of rings of subsets of the space E. Let $\mathcal{M} = \bigcup\limits_{\mathcal{E} \in \mathcal{S}} \mathcal{R}_{\mathcal{E}}$. Let us define the function of a set $\lambda: \ \mathcal{M} \to [0, +\infty]$ by the equality $\lambda(A) = \lambda_{\mathcal{E}}(A) \ \forall\, A \in R_{\mathcal{E}}$. The function λ is correctly defined since $\lambda_{\mathcal{E}}(A) = \lambda_{\mathcal{F}}(A)$ for every $A \in R_{\mathcal{E}} \bigcap R_{\mathcal{F}}$ due to the Theorem 11. Let r be the ring generated by the family of sets $\mathcal{M}$. Let us study the problem of extension of the function λ from the collection of sets $\mathcal{M}$ to the ring r [51].

Let us introduce the relation $\sim$ on the set $\mathcal{S}$ of ONB by the following way. ONB $\mathcal{E}$ and $\mathcal{F}$ are in the relation $\sim$ if the condition (14) is satisfied for ONB $\mathcal{E}$ and $\mathcal{F}$.

Definition 2. *ONB $\mathcal{E}$ and $\mathcal{F}$ are called equivalent if they satisfy the condition (14).*

The relation $\sim$ in Definition 2 is obviously reflexive. According to the Theorem 6 the relation $\sim$ is symmetric. Now we prove that the relation $\sim$ is transitive. Let us assume that the pairs of ONB $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{F}$, $\mathcal{G}$ satisfy the relation $\sim$ (14). Then $\mathcal{R}_{\mathcal{E}} = \mathcal{R}_{\mathcal{F}}$ and $\mathcal{R}_{\mathcal{F}} = \mathcal{R}_{\mathcal{G}}$ according to the Theorem 11. Therefore, $\mathcal{R}_{\mathcal{E}} = \mathcal{R}_{\mathcal{G}}$. Hence, ONB $\mathcal{E}, \mathcal{G}$ satisfy the condition (14) according to the Theorem 11. Hence, the relation $\sim$ is transitive.

Let Σ be a set of equivalence classes with respect to relation $\sim$: $\Sigma = \mathcal{S}/\sim$. For every ONB $\mathcal{E} \in \mathcal{S}$ the space $\mathcal{H}_{\mathcal{E}} = L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ of quadratically integrable with respect to the measure $\lambda_{\mathcal{E}}$ complex valued functions is introduced by the standard way (see [17]). If $\{\mathcal{E}\} \in \Sigma$ and $\mathcal{E}', \mathcal{E}'' \in \{\mathcal{E}\}$, then $\mathcal{H}_{\mathcal{E}'} = \mathcal{H}_{\mathcal{E}''}$ according to the Theorem 11 and definition of the spaces $\mathcal{H}_{\mathcal{E}}$, $\mathcal{E} \in \mathcal{S}$. The symbol $\mathcal{H}_{\{\mathcal{E}\}}$ denotes the space $\mathcal{H}_{\mathcal{E}}$ for arbitrary choice of an ONB $\mathcal{E} \in \{\mathcal{E}\}$.

Now we describe the ring generated by the family of subsets $\mathcal{R}_{\mathcal{E}} \bigcup \mathcal{R}_{\mathcal{F}}$ for a pair of ONB $\mathcal{E}$, $\mathcal{F}$ belonging to different classes $\{\mathcal{E}\}, \{\mathcal{F}\} \in \Sigma$. Moreover, the sum of spaces $\mathcal{H}_{\mathcal{E}}$ and $\mathcal{H}_{\mathcal{F}}$ will be defined.

The intersection of rings $\mathcal{R}_{\mathcal{E}}$ and $\mathcal{R}_{\mathcal{F}}$ is the ring which is denoted by the symbol $\mathcal{R}_{\mathcal{E}} \bigcap \mathcal{F}$. Then, $\lambda_{\mathcal{E}}(A) = \lambda_{\mathcal{F}}(A) \equiv \lambda_{\mathcal{E}\mathcal{F}}(A) \ \ \forall\, A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ according to the Theorem 11. Then, the space $\mathcal{H}_{\mathcal{E} \bigcap \mathcal{F}} = L_2(E, \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}, \lambda_{\mathcal{E}\mathcal{F}}, \mathbb{C})$ is the subspace of Hilbert spaces $\mathcal{H}_{\mathcal{E}}$ and $\mathcal{H}_{\mathcal{F}}$. Let $\mathcal{H}_{\mathcal{E}}^{\perp \mathcal{F}}$ and $\mathcal{H}_{\mathcal{F}}^{\perp \mathcal{E}}$ be orthogonal complements of the space $\mathcal{H}_{\mathcal{E} \bigcap \mathcal{F}}$ up to spaces $\mathcal{H}_{\mathcal{E}}$ and $\mathcal{H}_{\mathcal{F}}$ respectively. Then, Hilbert space $\mathcal{H}_{\mathcal{E}\mathcal{F}}$ is defined as the direct sum of three orthogonal subspaces

$$\mathcal{H}_{\mathcal{E}\mathcal{F}} = \mathcal{H}_{\mathcal{E}}^{\perp \mathcal{F}} \oplus \mathcal{H}_{\mathcal{E} \bigcap \mathcal{F}} \oplus \mathcal{H}_{\mathcal{F}}^{\perp \mathcal{E}}. \tag{52}$$

Lemma 25. *Let $\{\mathcal{E}\}$, $\{\mathcal{F}\}$ be different equivalence classes of ONB in the space E. Let $\mathcal{E} \in \{\mathcal{E}\}$, $\mathcal{F} \in \{\mathcal{F}\}$. Then $\mathcal{H}_{\mathcal{E}\mathcal{F}} = \mathcal{H}_{\mathcal{E}} \oplus \mathcal{H}_{\mathcal{F}}$. Moreover, there is the shift-invariant measure $\lambda_{\mathcal{E}\mathcal{F}}: \ \mathcal{R}_{\mathcal{E}\mathcal{F}} \to [0, +\infty)$ such that $\mathcal{H}_{\mathcal{E}} \oplus \mathcal{H}_{\mathcal{F}} = L_2(E, \mathcal{R}_{\mathcal{E}\mathcal{F}}, \lambda_{\mathcal{E}\mathcal{F}}, \mathbb{C})$. Here the ring $\mathcal{R}_{\mathcal{E}\mathcal{F}}$ is generated by the family of sets $\mathcal{R}_{\mathcal{E}} \bigcup \mathcal{R}_{\mathcal{F}}$.*

Proof. If equivalence classes $\{\mathcal{E}\}, \{\mathcal{F}\} \in \Sigma$ are different, then the equality $\lambda_{\mathcal{E}\mathcal{F}}(A) = 0$ holds for any $A \in \mathcal{R}_{\mathcal{E}} \bigcap \mathcal{R}_{\mathcal{F}}$ according to the Theorem 11. Hence, the space $\mathcal{H}_{\mathcal{E} \bigcap \mathcal{F}}$ is trivial and $\mathcal{H}_{\mathcal{E}\mathcal{F}} = \mathcal{H}_{\mathcal{E}} \oplus \mathcal{H}_{\mathcal{F}}$. $\square$

Thus, every pair of Hilbert spaces $\mathcal{H}_{\{\mathcal{E}\}}$ and $\mathcal{H}_{\{\mathcal{F}\}}$ defines the Hilbert space

$$\mathcal{H}_{\{\mathcal{E}\}\{\mathcal{F}\}} = \mathcal{H}_{\{\mathcal{E}\}} \oplus \mathcal{H}_{\{\mathcal{F}\}}. \tag{53}$$

Let $\mathcal{R}_{\{\mathcal{E}\}\{\mathcal{F}\}}$ be the ring which is generated by the collection of sets $\mathcal{R}_{\{\mathcal{E}\}} \bigcup \mathcal{R}_{\{\mathcal{F}\}}$. The equality (53) defines the scalar product in the space $\mathcal{H}_{\{\mathcal{E}\}\{\mathcal{F}\}}$. This scalar product defines (see [17]) the extension of the measures $\lambda_{\{\mathcal{E}\}}$, $\lambda_{\{\mathcal{F}\}}$ to the shift-invariant measure $\lambda_{\{\mathcal{E}\}\{\mathcal{F}\}}: \ \mathcal{R}_{\{\mathcal{E}\}\{\mathcal{F}\}} \to [0, +\infty)$ according to the following condition. The value of the measure $\lambda_{\{\mathcal{E}\}\{\mathcal{F}\}}$ on a set $A \bigcap B$ is given by the equality

$$\lambda_{\{\mathcal{E}\}\{\mathcal{F}\}}(A \bigcap B) = (\chi_A, \chi_B)_{\mathcal{H}_{\mathcal{E}\mathcal{F}}} \tag{54}$$

for every sets $A \in \mathcal{R}_{\{\mathcal{E}\}}$, $B \in \mathcal{R}_{\{\mathcal{F}\}}$. The value of the measure $\lambda_{\{\mathcal{E}\}\{\mathcal{F}\}}$ on other sets of the ring $\mathcal{R}_{\{\mathcal{E}\}\{\mathcal{F}\}}$ is defined by the additivity condition. Therefore, the function $\lambda_{\{\mathcal{E}\}\{\mathcal{F}\}}$:

$\mathcal{R}_{\{\mathcal{E}\}\{\mathcal{F}\}} \to R$ is the finitely additive measure. This measure is shift-invariant by the construction. Moreover, if $A \in \mathcal{R}_{\{\mathcal{E}\}}$, $B \in \mathcal{R}_{\{\mathcal{F}\}}$, then $\lambda_{\{\mathcal{E}\}\{\mathcal{F}\}}(A \bigcap B) = 0$ and $\lambda_{\{\mathcal{E}\}\{\mathcal{F}\}}(A \bigcup B) = \lambda_{\{\mathcal{E}\}}(A) + \lambda_{\{\mathcal{F}\}}(B)$ according to (53) and (54) . $\square$

Let us endow the linear hull $\mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma)$ with the Euclidean norm of direct sum of Hilbert spaces. Let $u = \sum\limits_{k=1}^{m} v_k$, where $v_k \in \mathcal{H}_{\{\mathcal{E}\}_k}$, $k = 1, \dots, m$. Then, the intersection $\mathcal{H}_{\{\mathcal{E}\}_k} \bigcap \mathcal{H}_{\{\mathcal{E}\}_j}$, $k \neq j$, is trivial subspace according to Lemma 25. Hence, the representation of an element $u \in \mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma)$ in the form $u = \sum\limits_{k=1}^{m} v_k$, where $v_k \in \mathcal{H}_{\{\mathcal{E}\}_k}$, $k = 1, \dots, m$, is unique.

For any vector $u \in \mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma)$ of the form $u = \sum\limits_{k=1}^{m} v_k, v_k \in \mathcal{H}_{\{\mathcal{E}\}_k}$, $k = 1, \dots, m$, let us define

$$\|u\| = (\sum_{k=1}^{m} \|v_k\|_{\{\mathcal{E}\}_k})^{\frac{1}{2}}. \tag{55}$$

Then, the function (55) is the Euclidean norm on the space $\mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma)$. By the construction the linear hull $\mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma)$ with the norm (55) is invariant both with respect to a shift of the argument of functions $u \in \mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma)$ on any vector of the space E and with respect to a transformation of argument of functions $u \in \mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma)$ by any orthogonal operator in the space E. The completion of Euclidean space $\mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma)$ by the norm (55) is the Hilbert space $\mathbb{H}$.

Theorem 12. *There is the unique finitely additive measure $\lambda:\ r \to [0, +\infty)$ which is an additive continuation of measures $\lambda_{\{\mathcal{E}\}}$, $\{\mathcal{E}\} \in \Sigma$, to the ring r generated by the family of sets $\bigcup\limits_{\{\mathcal{E}\}\in\Sigma} \mathcal{R}_{\{\mathcal{E}\}}$. The completion $\lambda:\ \mathcal{R} \to [0, +\infty)$ of the measure $\lambda:\ r \to [0, +\infty)$ is σ-finite, locally finite, invariant both with respect to shift on any vector of the space E and with respect to any orthogonal transformation of the space E. But the measure λ is not countably additive. The measure λ is connected with the space $\mathbb{H}$ by the equality $\mathbb{H} = L_2(E, \mathcal{R}, \lambda, \mathbb{C})$.*

Proof. Let r be the ring of subsets of the space E which is generated by the system of sets $\{\mathcal{R}_{\{\mathcal{E}\}},\ \{\mathcal{E}\} \in \Sigma\}$. Hence, the ring r is generated by the semi-ring

$$s = \{A_0 \backslash \bigcup_{j=1}^{N} A_j,\ N \in \mathbb{N},\ A_0 \in \mathcal{R}_{\{\mathcal{E}\}_0},\ A_j \in \mathcal{R}_{\{\mathcal{E}\}_j},\ \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_N \in \Sigma\}. \tag{56}$$

Since the system of sets $\mathcal{R}_{\{\mathcal{E}\}_0}$ is the ring, we can assume that $\{\mathcal{E}\}_j \neq \{\mathcal{E}\}_0$ for every $j = 1, \dots, N$. Hence, $\lambda_{\mathcal{E}}(A_0 \bigcap (\bigcup\limits_{j=1}^{N} A_j)) = 0$ according to the Theorem 11. Thus, the there is the unique additive continuation λ of measures $\lambda_{\{\mathcal{E}\}}$, $\{\mathcal{E}\} \in \Sigma$, to the semi-ring s which is defined by the equality $\lambda(A_0 \backslash \bigcup\limits_{j=1}^{N} A_j) = \lambda_{\mathcal{E}}(A_0)$ for any set $A_0 \backslash \bigcup\limits_{j=1}^{N} A_j$ from the semi-ring (56). Then, $\lambda(A) = \|\chi_A\|^2_{\mathbb{H}}\ \forall\ A \in s$. Additive function $\lambda:\ s \to [0, +\infty)$ on the semi-ring s admits the unique additive extension to the measure $\lambda:\ r \to [0, +\infty)$ on the ring r. Moreover, the measure λ satisfy the condition $\lambda(A) = \|\chi_A\|^2_{\mathbb{H}}\ \forall\ A \in r$.

The semi-ring s and the generated by this semi-ring ring r are invariant with respect to both a shift on a vector of the space E and an orthogonal mapping of the space E. The measure $\lambda:\ r \to [0, +\infty)$ is both rotation- and shift-invariant measure on the space E by its construction.

According to [17] (see also [40]) the measure λ takes zero values on a ball of the space E with sufficiently small radius $\rho \in (0, \frac{1}{\sqrt{2}})$ ([44]). Therefore, the measure λ is locally finite. Its σ-finiteness is the consequence of its locally finiteness and the separability of the space E. Moreover, since $\lambda(B_\rho(a)) = 0$ for any ball $B_\rho(a) = \{x \in E:\ \|x - a\|_E < \rho\}$ where

$a \in E$, $\rho = \frac{1}{4}$, the measure λ is not countably additive according to the separability of the space E.

The measure λ is not complete. Its completion $\lambda : \mathcal{R} \to [0, +\infty)$ is defined by the standard scheme by means of external and internal measures (see [17]). Here $\mathcal{R}$ is the completion of the ring r by the measure $\lambda : r \to [0, +\infty)$.

According to the construction of measure λ the following equality $\lambda|_{\mathcal{R}_{\{\mathcal{E}\}}} = \lambda_{\{\mathcal{E}\}}$ holds for every $\{\mathcal{E}\} \in \Sigma$. Therefore, $L_2(E, \mathcal{R}, \bar{\lambda}, \mathcal{C}) \supset \mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma)$. Linear manifold $\mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma)$ is dense in the space $\mathbb{H}$ since the space $\mathbb{H}$ is defined as the completion of the Euclidean space $(\mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma), \| \cdot \|)$.

The linear hull $\text{span}(\chi_A,\ A \in \mathcal{R})$ of the family of indicator functions of sets from the ring $\mathcal{R}$ is the dense linear manifold in the space $L_2(E, \mathcal{R}, \lambda, \mathbb{C})$ according to definition of this space. Since the ring $\mathcal{R}$ is the completion of the ring r with respect to measure $\lambda : r \to [0, +\infty)$, the linear hull of the family of indicator functions $\text{span}(\chi_A,\ A \in r)$ is dense in the space $L_2(E, \mathcal{R}, \lambda, \mathbb{C})$.

The ring r is generated by families of sets $\{\mathcal{R}_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma\}$. Therefore, the linear manifold $\text{span}(\chi_A,\ A \in r_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma)$ is dense in linear space $\text{span}(\chi_A,\ A \in r)$ equipped with the Euclidean norm of the space $L_2(E, \mathcal{R}, \lambda, \mathbb{C})$.

Since the linear space $\text{span}(\chi_A,\ A \in r_{\mathcal{E}})$ is the dense linear manifold in the space $\mathcal{H}_{\mathcal{E}}$ for every $\{\mathcal{E}\} \in \Sigma$, the linear manifold

$$\text{span}(\text{span}(\chi_A,\ A \in r_{\mathcal{E}}), \{\mathcal{E}\} \in \Sigma) = \text{span}(\chi_A,\ A \in r_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma)$$

dense in the linear space

$$\mathcal{L}(\mathcal{H}_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma) = \text{span}(\mathcal{H}_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma)$$

equipped with the norm $\| \cdot \|_{\mathcal{H}}$.

Therefore, $L_2(E, \mathcal{R}, \lambda, \mathbb{C}) = \mathbb{H}$ since the norm $\| \cdot \|_{\mathcal{H}}$ and the norm of the space $L_2(E, \mathcal{R}, \lambda, \mathbb{C})$ are coincide on the vectors of linear manifold $\text{span}(\chi_A,\ A \in r_{\{\mathcal{E}\}}, \{\mathcal{E}\} \in \Sigma)$ and this linear manifold is dense both in the space $L_2(E, \mathcal{R}, \lambda, \mathbb{C})$ and in the space $\mathbb{H}$. □

The Theorem 12 gives the orthogonal decomposition of the space $\mathbb{H}$. According to the Lemma 25 the condition $\{\mathcal{E}\} \neq \{\mathcal{F}\}$ implies $\mathcal{H}_{\{\mathcal{E}\}} \perp \mathcal{H}_{\{\mathcal{F}\}}$. Thus, we obtain the following

Corollary 19.

$$\mathbb{H} = \oplus_{\{\mathcal{E}\} \in \Sigma} \mathcal{H}_{\{\mathcal{E}\}}.$$

In fact, the Hilbert space $\mathbb{H}$ is defined as the completion of linear hull of the family of spaces $\mathcal{H}_{\mathcal{F}}$, $\mathcal{F} \in \mathcal{S}$ equipped with the scalar product (55).

The decomposition of the space $\mathbb{H}$ to the orthogonal sum of the invariant with respect to the group of shift subspaces as well as the decomposition of the space $\mathcal{H}_{\mathcal{E}}$ to the orthogonal sum of $\mathcal{S}_1$-invariant subspaces in the Theorem 2 present analogs of the lamination of the phase space of a dynamical system to invariant manifolds in the work [52].

Corollary 20. *Let $\mathcal{E}, \mathcal{F} \in \mathcal{S}$. If bases $\mathcal{E}, \mathcal{F}$ satisfy the condition (14), then $\mathcal{H}_{\mathcal{E}} = \mathcal{H}_{\mathcal{F}}$. If the condition (14) is violated for bases $\mathcal{E}, \mathcal{F}$, then $\mathcal{H}_{\mathcal{E}} \perp \mathcal{H}_{\mathcal{F}}$.*

Theorem 13. *Shift- and rotation-invariant measure $\lambda : \mathcal{R} \to [0, +\infty)$ on the space E from the Theorem 12 admits the decomposition*

$$\lambda = \sum_{\{\mathcal{F}\} \in \Sigma} \nu_{\{\mathcal{F}\}} \tag{57}$$

into the sum of mutually singular shift-invariant measures $\nu_{\{\mathcal{F}\}}$, $\{\mathcal{F}\} \in \Sigma$. Here for every $\{\mathcal{F}\} \in \Sigma$ the measure $\nu_{\{\mathcal{F}\}}$ is given by the equality

$$\nu_{\{\mathcal{F}\}}(A) = \sup_{B \in \mathcal{R}_{\mathcal{F}},\, B \subset A} \lambda_{\{\mathcal{F}\}}(B),\ A \in \mathcal{R}. \tag{58}$$

Proof. In the proof of Theorem 12 we show that if $A \in \mathcal{R}$, then for any $\epsilon > 0$ there are $A'_1 \in \mathcal{R}_{\{\mathcal{E}_1\}}, \dots, A'_m \in \mathcal{R}_{\{\mathcal{E}_m\}}$ and $A''_1 \in \mathcal{R}_{\{\mathcal{E}_1\}}, \dots, A''_m \in \mathcal{R}_{\{\mathcal{E}_m\}}$ such that $\bigcup\limits_{j=1}^{m} A'_j \subset A \subset \bigcup\limits_{j=1}^{m} A''_j$ and $\lambda((\bigcup\limits_{j=1}^{m} A''_j) \setminus (\bigcup\limits_{j=1}^{m} A'_j)) < \epsilon$. This fact and the Theorem 11 together imply the equality (57) where measures $\nu_{\mathcal{F}}$ are defined by (58). The mutually singularity of measures $\nu_{\{\mathcal{E}\}}, \nu_{\{\mathcal{F}\}}$ under the assumption $\{\mathcal{E}\} \neq \{\mathcal{F}\}$ is the consequence of the decomposition (57) and the Theorem 11. □

Remark 4. *If* $\mathbf{U}$ *is the unitary operator in the space* E*, then* $\nu_{\{\mathcal{F}\}}(\mathbf{U}A) = \nu_{\{\mathbf{U}^{-1}\mathcal{F}\}}(A)$ *for any set* $A \in \mathcal{R}$ *and for any class of ONB* $\{\mathcal{F}\} \in \Sigma$.

Let us obtain the ring-ergodic decomposition of isometry-invariant measure for a subgroup of the group of isometries $\mathbb{I}$.

Lemma 26. *The family of orthogonal mappings* $\mathbb{O}^1_{\mathcal{E}}(E) = \{\mathbf{U} \in \mathbb{O}(E) :\ \mathbf{U}(\mathcal{R}_{\mathcal{E}}) = \mathcal{R}_{\mathcal{E}}\}$ *forms a subgroup of the orthogonal group* $\mathbb{O}(E)$.

Proof. The unit operator $\mathbf{I}$ belongs to $\mathbb{O}^1_{\mathcal{E}}(E)$. According to the Theorem 11, if $\mathbf{U} \sim \mathbf{I}$ and $\mathbf{V} \sim \mathbf{I}$, then $\mathbf{UV} \sim \mathbf{I}$ and $\mathbf{U}^{-1} \sim \mathbf{I}$. □

Let $\mathcal{E}$ be an ONB in the space E. Let $\mathbb{I}_1(\mathcal{E})$ be the group of isometries which is generated by the groups $\mathbb{S}$ and the group $\mathbb{O}^1_{\mathcal{E}}(E)$. Let $\mathcal{K}_{0,\mathcal{E}}$ be the class of absolutely measurable rectangles which are $L_1(\mathcal{E})$-equivalent to centered rectangles. Then, according to Theorem 3 the measure $\lambda_{0,\mathcal{E}} :\ \mathcal{R}_{0,\mathcal{E}} \to [0, +\infty)$ is $\mathbb{O}^1_{\mathcal{E}}(E)$-invariant and ring-ergodic with respect to the group of shifts $\mathbb{S}$.

Corollary 21. *Let* $\mathcal{E}$ *be an ONB in the space* E*. Then the measure* $\lambda_{0,\mathcal{E}}$ *is ring-ergodic invariant measure on the space* E *with respect to the group of isometries* $\mathbb{I}_1(E)$.

The proof of Corollary 21 repeats the proof of Theorem 3.

7. Linear Operators in the Space $\mathbb{H}$ Generated by Orthogonal Transformations of Argument

Let $\mathbf{U}(t)$, $t \in R$, be the one-parametric family of operators in the space $\mathbb{H}$ which is given by the following way. Consider an ONB $\mathcal{E} = \{e_j,\ j \in \mathbb{N}\}$ in the space E. Let $E_k = \text{span}(e_{2k-1}, e_{2k})$, $k \in \mathbb{N}$, be a sequence of two-dimensional orthogonal subspaces of the space E. Let $\{a_k\}$ be a sequence of real numbers. Let us consider the group $\mathbf{\Lambda}(t)$, $t \in \mathbb{R}$ of orthogonal transformations of the space E such that, at first, subspaces E_k, $k \in \mathbb{N}$ are invariant with respect to operators of this group and, at second, for every $k \in \mathbb{N}$ the restriction $\mathbf{\Lambda}(t)|_{E_k}$, $t \in \mathbb{R}$ has the matrix

$$\begin{pmatrix} \cos(a_k t) & \sin(a_k t) \\ -\sin(a_k t) & \cos(a_k t) \end{pmatrix}.$$

Let $\mathbf{U}(t)$, $t \in \mathbb{R}$, be a one-parametric family of operators in the space E such that for every $t \in \mathbb{R}$ the operator $\mathbf{U}(t)$ is given by the equality

$$\mathbf{U}(t)u(x) = u(\mathbf{\Lambda}(t)x),\ x \in E,\ u \in \mathbb{H}. \tag{59}$$

Since the measure $\lambda :\ \mathcal{R} \to [0, +\infty)$ is invariant with respect to any orthogonal transformation of the space E, the equality (59) defines the unitary operator $\mathbf{U}(t)$ in the space $\mathbb{H}$ for every $t \in \mathbb{R}$. The one-parametric family of operators $\mathbf{U}(t)$, $t \in \mathbb{R}$ forms the one-parametric unitary group in the space $\mathbb{H}$.

It is easy to check that the matrix of the orthogonal operator $\mathbf{\Lambda}(t)$, $t \neq 0$, in an ONB $\mathcal{E}$ satisfies the condition (14) if and only if $\{a_k\} \in \ell_1$.

Lemma 27. *If $\{a_k\} \notin \ell_1$, then one-parametric group $\mathbf{U}(t)$, $t \in \mathbb{R}$, of unitary operators in the space $\mathbb{H}$ is not strongly continuous.*

Proof. Let $A \in \mathcal{R}_{\mathcal{E}}$ be a set such that $\lambda(A) = \lambda_{\mathcal{E}}(A) > 0$. Let us assume the opposite, that the one-parametric group $\mathbf{U}(t)$, $t \in \mathbb{R}$ is strongly continuous. Then, the function $(\mathbf{U}(t)\chi_A, \chi_A)_{\mathbb{H}}$, $t \in \mathbb{R}$, is continuous.

But this function has the discontinuity point $t_0 = 0$ since

$$(\mathbf{U}(t)\chi_A, \chi_A)_{\mathbb{H}}|_{t=0} = \lambda(A) > 0,$$

and $(\mathbf{U}(t)\chi_A, \chi_A)_{\mathbb{H}} = 0 \, \forall \, t \neq 0$. In fact, $\chi_A \in \mathcal{H}_{\{\mathcal{E}\}}$, $\mathbf{U}(t)\chi_A \in \mathcal{H}_{\{\mathbf{\Lambda(t)}(\mathcal{E})\}}$. Since the orthogonal mapping $\mathbf{\Lambda}(t)$ does not satisfy the condition (14), subspaces $\mathcal{H}_{\mathcal{E}}$ and $\mathcal{H}_{\{\mathbf{\Lambda}(t)(\mathcal{E})\}}$ are orthogonal and $(\mathbf{U}(t)\chi_A, \chi_A)_{\mathbb{H}} = 0$. The obtained contradiction proves the statement. □

Lemma 28. *If $\{a_k\} \in \ell_1$, then a subspace $\mathcal{H}_{\mathcal{E}}$ is invariant with respect to operators of one-parametric group $\mathbf{U}(t)$, $t \in \mathbb{R}$. Moreover, the group $\mathbf{U}(t)|_{\mathcal{H}_{\mathcal{E}}}$, $t \in \mathbb{R}$ is strongly continuous unitary group in the space $\mathcal{H}_{\mathcal{E}}$.*

Proof. The condition $\{a_k\} \in \ell_1$ implies that for every $t \in \mathbb{R}$ the matrix of orthogonal mapping $\mathbf{\Lambda}(t)$ in the basis $\mathcal{E}$ satisfies the condition (14). Therefore, ONB $\mathcal{E}$ and $\mathbf{\Lambda}(t)(\mathcal{E})$ are equivalent and hence $\mathbf{U}(t)\mathcal{H}_{\mathcal{E}} = \mathcal{H}_{\mathcal{E}} \, \forall \, t \in \mathbb{R}$.

Let us fix a number $\epsilon > 0$. Let $\phi \in \mathcal{H}_{\mathcal{E}}$. For every $m \in \mathbb{N}$ the equality $\{a_k\} = \{a'_k(m)\} + \{a''_k(m)\}$ holds. Here $\{a'_k(m)\} = \{a_1, \dots, a_m, 0, \dots\}$, $\{a''_k(m)\} = \{0, \dots, 0, a_{m+1}, \dots\}$. Hence, $\mathbf{\Lambda}(t) = \mathbf{\Lambda}''_m(t) \circ \mathbf{\Lambda}'_m(t)$, $t \in \mathbb{R}$ and $\mathbf{U}(t)|_{\mathcal{H}_{\mathcal{E}}} = \mathbf{U}''_m(t)|_{\mathcal{H}_{\mathcal{E}}} \circ \mathbf{U}'_m(t)|_{\mathcal{H}_{\mathcal{E}}}$, $t \in \mathbb{R}$.

Therefore, for any $t > 0$ there is a number $m \in \mathbb{N}$ such that $\sup\limits_{\tau \in [0,t]} \|\mathbf{U}''(\tau)_m\phi - \phi\|_{\mathcal{H}_{\mathcal{E}}} < \epsilon$. In fact, for every $\Pi \in \mathcal{K}_{\mathcal{E}}$ we have $\|\mathbf{U}''(t)_m\chi_\Pi - \chi_\Pi\|^2_{\mathbb{H}} = 2\lambda((\mathbf{\Lambda}''_m(t)\Pi)\backslash\Pi)$ and $\lim\limits_{m\to\infty} \sup\limits_{\tau \in [0,t]} \lambda((\mathbf{\Lambda}''_m(\tau)\Pi)\backslash\Pi) = 0$ according to Lemma 14. The strong continuity of the group $\mathbf{U}'_m(t)|_{\mathcal{H}_{\mathcal{E}}}$, $t \in \mathbb{R}$ is the consequence of the decomposition $\mathcal{H}_{\mathcal{E}} = L_2(\mathbb{R}^{2m}) \oplus \mathcal{H}_{\mathcal{E}''}$ (see [40]) and the strong continuity of the group of orthogonal transformations of the argument of a quadratically integrable function on a finite-dimensional Euclidean space $\mathbb{R}^{2m}$. □

Remark 5. *Let $\{a_k\} \in \ell_1$. Then a subspace $\mathcal{H}_{\mathcal{F}}$ can be not invariant with respect to operators of the one-parametric group (59) for every choice of ONB $\mathcal{F} \in \mathcal{S}$. Moreover, the group (59) of unitary operators in the space $\mathcal{H}$ can be discontinuous. This fact is shown by the following example.*

Let the operator $\mathbf{\Lambda}(t)$ in the ONB $\mathcal{E}$ has the matrix

$$\|\mathbf{\Lambda}(t)\|_{\mathcal{E}} = \begin{pmatrix} \cos(at) & \sin(at) & 0 & 0 & 0 & \dots \\ -\sin(at) & \cos(at) & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The operator $\mathbf{I}-\mathbf{\Lambda}(t)$ is trace class operator and the statement of the Lemma 28 takes place. Let us consider the ONB $\mathcal{F}$ which is the image of the ONB $\mathcal{E}$ under the action of the orthogonal mapping $\mathbf{V}$ of the space E with the following matrix in ONB $\mathcal{E}$

$$\|\mathbf{V}\|_{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & c_{2,2} & c_{2,3} & c_{2,4} & \dots \\ 0 & c_{3,2} & c_{3,3} & c_{3,4} & \dots \\ 0 & c_{4,2} & c_{4,3} & c_{4,4} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Rows of the matrix $\|\mathbf{V}\|_{\mathcal{E}}$ forms the orthonormal system in the space ℓ_2. There is the choice of this orthonormal system such that the following condition

$$\sum_{k=1}^{\infty} \mid c_{2,k} \mid = +\infty \tag{60}$$

holds. Let us consider the orthogonal mapping $\mathbf{V}$ such that the condition (60) is satisfied. Then, the matrix of the orthogonal operator $\mathbf{\Lambda}(t)$ in the ONB $\mathcal{F}$ is

$$\|\mathbf{\Lambda}(t)\|_{\mathcal{F}} = \|\mathbf{V}\|_{\mathcal{E}}^{-1} \|\mathbf{\Lambda}(t)\|_{\mathcal{E}} \|\mathbf{V}\|_{\mathcal{E}} =$$

$$= \begin{pmatrix} \cos(at) & c_{2,2}\sin(at) & c_{2,3}\sin(at) & c_{2,4}\sin(at) & \dots \\ -c_{2,2}\sin(at) & . & . & . & \dots \\ -c_{2,3}\sin(at) & . & . & . & \dots \\ -c_{2,4}\sin(at) & . & . & . & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Hence, the condition (14) is not satisfied for bases $\mathcal{F}$ and $\mathcal{F}_t = \mathbf{\Lambda}(t)\mathcal{F}$ such that $\sin(at) \neq 0$. If $t \in \mathbb{R}$ and $\sin(at) \neq 0$, then bases $\mathcal{E}$ and $\mathcal{E}_t = \mathbf{\Lambda}(t)\mathcal{E}$ are equivalent in the sense of the definition 2, but bases $\mathcal{F}$ and $\mathcal{F}_t = \mathbf{\Lambda}(t)\mathcal{F}$ are not equivalent. Hence, subspaces $\mathcal{H}_{\mathcal{F}}$ and $\mathbf{U}(t)\mathcal{H}_{\mathcal{F}} = \mathcal{H}_{\mathcal{F}_{-t}}$ are orthogonal in the space $\mathcal{H}$ for any $t :\ \sin(at) \neq 0$ according to the corollary 20. Then, the group of unitary operators (59) in the space $\mathbb{H}$ is discontinuous since the function $t \to (u, \mathbf{U}(t)u)_{\mathbb{H}}$, $t \in \mathbb{R}$, is discontinuous for any nontrivial vector $u \in \mathcal{H}_{\mathcal{F}}$.

Remark 6. *The condition (14) on the orthogonal mapping $\mathbf{\Lambda}$ and ONB $\mathcal{E}$ in the space E should not be considered as the condition on the operator $\mathbf{\Lambda}$ only. In particular, the condition (14) is not the consequence of the belonging of the operator $\mathbf{I}-\mathbf{\Lambda}$ to the space of trace class operators. This fact is shown by the example in the remark 5. In fact, the operator $\mathbf{\Lambda}(t)-\mathbf{I}$ is trace class operator (as well as the operator $\mathbf{V}^{-1}\mathbf{\Lambda}(t)\mathbf{V}-\mathbf{I}$). Nevertheless, the operator $\mathbf{\Lambda}(t)$ and the basis $\mathcal{E}$ satisfy the condition (14), but the operator $\mathbf{\Lambda}(t)$ and the basis $\mathcal{F} = \mathbf{V}\mathcal{E}$ are not satisfy the condition (14).*

Remark 7. *The unitary group of operators in the space $\mathbb{H}$ generated by the group of orthogonal mappings (59) in the space E has the strong continuity property describing by Lemmas 27, 28 and Remark 5. This property is similar to the strong continuity property of the unitary group of operators in the space $\mathbb{H}$ generated by the group of shifts of argument according to the formula*

$$\mathbf{S}_{th}u(x) = u(x-th),\ t \in \mathbb{R}. \tag{61}$$

If $h \neq 0$, then the group of unitary operators (61) is not continuous in the space $\mathbb{H}$. But it has the family of invariant subspaces $\mathcal{H}_{\mathcal{F}}$, $\mathcal{F} \in \mathcal{S}$ and the restriction $\mathbf{S}_{th}|_{\mathcal{H}_{\mathcal{F}}}$, $t \in \mathbb{R}$, is strongly continuous group in the space $\mathcal{H}_{\mathcal{F}}$ if and only if $\{(h, f_k)\} \in l_1$ (see [53]).

8. Measures Invariant with Respect to Some Group of Simplectomorphisms

8.1. Symplectic Structure

Now we introduce standard notations for the symplectic space and Hamiltonian systems that are used in [27]. Symplectic form on a real separable Hilbert space E is a non-degenerate closed differential 2-form on the space E. If a symplectic form on a Hilbert space E is invariant with respect to a shift, then it is given by a non-degenerate skew-symmetric bilinear form ω on the space E. A Hilbert space E is identified with its conjugate space. Let $B(E)$ be the Banach space of bounded linear operators $E \to E$ endowed with the operator norm. If linear operator $\mathbf{J} \in B(E)$ is associated with the bilinear form ω, then $\mathbf{J}$ is the non-degenerate skew-symmetric operator ([54]). Shift-invariant symplectic form ω on a real separable Hilbert space E is called standard, if there is an orthonormal basis (ONB) $\{e_k\} \equiv \mathcal{E}$ such that $\omega(e_{2k-1}, e_j) = \delta_{j,2k}$, $k, j \in \mathbb{N}$ where $\delta_{j,i}$ is the Kronecker symbol.

The standard symplectic form ω defines decomposition $E = Q \oplus P$ of the space E to the direct sum of two subspaces Q, P such that the following properties take place. There is a pair of ONB $\mathcal{F} = \{f_k\}$, $\mathcal{G} = \{g_k\}$ in the subspaces Q and P respectively such that $e_{2k-1} = f_k, k \in \mathbb{N}$ and $e_{2k} = g_k, k \in \mathbb{N}$. Then,

$$\omega(f_i, f_j) = 0, \quad \omega(g_i, g_j) = 0, \quad \omega(f_i, g_j) = \delta_{i,j}, \quad \forall\, i, j \in \mathbb{N} \tag{62}$$

(see [30]). In the above case the basis $\mathcal{E} = \{e_i,\ i \in \mathbb{N}\} = \{f_j, g_k;\ j, k \in \mathbb{N}\}$ is called symplectic basis of the symplectic form ω in the space E. Symplectic form ω on the space E with the symplectic basis $\{f_j, g_k;\ j \in \mathbf{N},\ k \in \mathbf{N}\}$ is given by bilinear form of skew-symmetric symplectic operator $\mathbf{J}$ which is associated with symplectic form ω by the condition

$$\omega(z_1, z_2) = (\mathbf{J}z_1, z_2)\ \forall\, z_1, z_2 \in E. \tag{63}$$

Then, the symplectic operator is defined by equalities $\mathbf{J}(e_j) = f_j$, $\mathbf{J}(f_k) = -e_k$, $j \in \mathbf{N}$, $k \in \mathbf{N}$. Spaces Q and P are called configuration and momentum space respectively. Any of this two spaces is conjugate to the other. (see [30,54,55]).

Hamiltonian system is defined as the following triplet $(E, \mathbf{J}, h)$. Here $(E, \mathbf{J})$ is a Hilbert space with the symplectic structure, $h : E_1 \to \mathbb{R}$ is the real-valued function which is continuously differentiable in the sense of Gateau on a dense subspace E_2 of the space E. The function h in this triplet is called Hamilton function [30,55].

For example, if Hamilton function h is defined by the equality

$$h(x) = \sum_{k=1}^{\infty} \lambda_k x_k^2, \tag{64}$$

then $E_1 = \{x \in E : \sum\limits_{k=1}^{\infty} |\lambda_k| x_k^2 < \infty\}$, $E_2 = \{x \in E : \sum\limits_{k=1}^{\infty} |\lambda_k|^2 x_k^2 < \infty\}$. Here $\{\lambda_k\} \in \mathbb{R}^{\mathbb{N}}$, $x_k = (x, e_k)$, $k \in \mathbb{N}$ and $\{e_k\}$ is an ONB in the space E.

A densely defined vector field $\mathbf{v} : E_2 \to E$ is called Hamiltonian vector field if there is a function $h : E_1 \to \mathbb{R}$, such that $\mathbf{v}(z) = \mathbf{J}Dh(z)$, $z \in E_2$. Here the Hamilton function h is Gateau differentiable on the dense subspace E_2 of the space E. In this case $Dh : E_2 \times E_2 \to E$ is the differential of the function h. In this case the differential equation $z'(t) = \mathbf{J}(h'(z(t)))$, $t \in \Delta$, on the unknown function on a segment Δ $z : \Delta \to E_2$ is called Hamilton equation for the Hamiltonian system $(E, \mathbf{J}, h)$ ([54,55]).

A linear Schrödinger equation is the Hamilton equation of a Hamiltonian system with quadratic Hamilton function such that the operator of quadratic form commute with the symplectic operator. In this case the phase space is the reification of complex Hilbert space of a quantum system ([54]).

A Hamiltonian vector field $\mathbf{v} : E_2 \to E$ generates the one-parametric group Φ_t, $t \in \mathbb{R}$, of continuously differential transformation of the space E_2 such that

$$\frac{d}{dt}\mathbf{\Phi}_t(q, p) = \mathbf{v}(\mathbf{\Phi}_t(q, p)),\ (q, p) \in E_2,\ t \in \mathbb{R}.$$

One-parametric group Φ_t, $t \in \mathbb{R}$, of transformation of the space E_2 is called smooth Hamiltonian flow in the space E_2

If a Hamiltonian flow in the space E_2 admits the unique continuous continuation to the space E, then this continuation $\mathbf{\Phi}$ is called generalized Hamiltonian flow in the space E generated by the Hamiltonian vector field $\mathbf{v}$ (by the Hamilton function h). this continuous continuation of a smooth Hamiltonian flow $\mathbf{\Phi}$ of linear operators to a generalized Hamiltonian flow exists if values of the smooth flow $\mathbf{\Phi}$ are contraction operators in the space E. The described situation is realized in the case of Hamiltonian system connected with a linear Schrodinger equation.

8.2. Symplectomorphism-Invariant Measures

Now we consider measures on a real separable Hilbert space E with a shift-invariant symplectic form ω such that these measures are invariant with respect to some group of symplectomorphisms (see [27,56]). Let $E = Q \oplus P$ and let $\mathcal{E} = \mathcal{F} \bigcup \mathcal{G}$ be the symplectic basis of the form ω (see (62)).

Definition 3. *A set $\Pi \subset E$ is called absolutely measurable symplectic rectangle in the Hilbert space E if there is a symplectic form ω on the space E with a symplectic basis $\{f_j, g_k,\ j \in \mathbb{N},\ k \in \mathbb{N}\}$ such that the set Π is given by the equality*

$$\Pi = \{z \in E : \ ((z, f_i), (z, g_i)) \in B_i,\ i \in \mathbb{N}\}, \tag{65}$$

where B_i are Lebesgue-measurable sets in a plane $\mathbb{R}^2$ such that the condition

$$\sum_{j=1}^{\infty} \max\{\ln(\lambda_2(B_j)), 0\} < +\infty$$

holds (here λ_2 is the Lebesgue measure on $\mathbb{R}^2$).

Let $\mathcal{K}(E)$ be the set of absolutely measurable symplectic rectangles in Hilbert space E.

Let us note that symplectic basis in the definition 3 depends on the choice of symplectic rectangle. Let us fix a symplectic basis $\mathcal{E} = \mathcal{F} \bigcup \mathcal{G}$. Let $\mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \equiv \mathcal{K}_{\mathcal{E}}(E)$ be the set of absolutely measurable symplectic rectangles such that any of these rectangles has the form (65) in the basis $\mathcal{F} \bigcup \mathcal{G}$.

Let $\lambda_{\mathcal{F},\mathcal{G}} : \ \mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \to [0, +\infty)$ be a function of a set which is given by the equality

$$\lambda_{\mathcal{F},\mathcal{G}}(\Pi) = \prod_{j=1}^{\infty} \lambda_2(B_j) = \exp(\sum_{j=1}^{\infty} \ln(\lambda_2(B_j)))$$

under the condition $\Pi \neq \oslash$; $\lambda_{\mathcal{F},\mathcal{G}}(\Pi) = 0$ in the case $\Pi = \oslash$.

It is easy to check that if $A, B \in \mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ for some ONB $\mathcal{F} \bigcup \mathcal{G}$, then $A \bigcap B \in \mathcal{K}_{\mathcal{F},\mathcal{F}}(G)$. Moreover, the class of sets $\mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ is invariant with respect to a shift on any vector of the space E. The function of a set $\lambda_{\mathcal{F},\mathcal{G}} : \ \mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \to [0, +\infty)$ is shift-invariant too. A set $\Pi \in \mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ in (65) is denoted by the symbol $\times_{j=1}^{\infty} B_j$.

Lemma 29 ([27]). *The function of a set $\lambda : \ \mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \to [0, +\infty)$ is finite additive.*

Let $r_{\mathcal{F},\mathcal{G}}$ be a ring generated by the system of sets $\mathcal{K}_{\mathcal{F},\mathcal{G}}$. It is easy to check the statement.

Lemma 30 ([20]). *The class Λ of sets $A = \Pi \backslash (\bigcup\limits_{i=1}^{n} \Pi_i)$, where $n \in \mathbb{N}_0$, $\Pi, \Pi_1, \dots, \Pi_n \in \mathcal{K}_{\mathcal{F},\mathcal{G}}$, is the semi-ring.*

Corollary 22 ([20]). *Let $r_{\mathcal{F},\mathcal{G}}$ be the ring generated by the class of sets $\mathcal{K}_{\mathcal{F},\mathcal{G}}$. Then the ring $r_{\mathcal{F},\mathcal{G}}$ consists of finite union of sets from the semi-ring Λ.*

Let us define the collection Λ_n of the sets of the type $A = \Pi \backslash (\bigcup_{i=1}^{n} \Pi_i)$ for any $n \in \mathbb{N}_0$, $\Pi, \Pi_1, \ldots, \Pi_n \in \mathcal{K}_{\mathcal{F},\mathcal{G}}$. In addition, for any $n \in \mathbb{N}$ we introduce the collection V_n of sets of the type $A = \bigcup_{i=1}^{n} \Pi_i$, where $\Pi_1, \ldots, \Pi_n \in \mathcal{K}_{\mathcal{F},\mathcal{G}}$. Then, $\Lambda_n \supset \Lambda_{n-1}$ for every $n \in \mathbb{N}$ and the equality $\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n$ holds.

Lemma 31 ([27]). *Let $\Pi, Q \in \mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ and $Q \subset \Pi$. Then, for any $\epsilon > 0$ there is a number $N \in \mathbb{N}$ such that $\Pi \supset Q_N \supset Q$, $\lambda(Q_N) - \lambda(Q) < \epsilon$ and there are pairwise disjoint symplectic rectangles $\Pi_1, \ldots, \Pi_m \in \mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ such that $\Pi \backslash Q_N = \bigcup_{j=1}^{m} \Pi_j$.*

Lemma 32 ([27]). *Let $\Pi, Q \in \mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ and $Q \subset \Pi$. Then there is a sequence $\{\Pi_k\}$ of pairwise disjoint symplectic rectangles from the class $\mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ such that $\Pi \backslash Q = \bigcup_{k=1}^{\infty} \Pi_k$ and the equality $\lambda(\Pi) = \lambda(Q) + \sum_{k=1}^{\infty} \lambda(\Pi_k)$ holds.*

Theorem 14 ([27]). *The additive function of a set $\lambda : \mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \to [0, +\infty)$ has the unique additive extension on the ring $r_{\mathcal{F},\mathcal{G}}$. The completion of the measure $\lambda : r_{\mathcal{F},\mathcal{G}} \to [0, +\infty)$ is the complete measure $\lambda_{\mathcal{F},\mathcal{G}} : \mathcal{R}_{\mathcal{F},\mathcal{G}} \to [0, +\infty)$, which is invariant with respect to a smooth symplectorphism $\Phi : E \to E$ which preserves two-dimensional symplectic subspaces $E_k = \mathrm{span}(f_k, g_k)$, $k \in \mathbb{N}$ of the decomposition $E = \oplus_{k=1}^{\infty} E_k$.*

The completion of the measure $\lambda : \Lambda \to [0, +\infty)$ is the complete measure $\lambda_{\mathcal{F},\mathcal{G}} : \mathcal{R}_{\mathcal{F},\mathcal{G}} \to [0, +\infty)$. The ring Λ defines the ring $\mathcal{R}_{\mathcal{F},\mathcal{G}}$ in the following way. Internal $\underline{\lambda}$ and external $\overline{\lambda}$ measures are defined by the measure $\lambda : \Lambda \to [0, +\infty)$ on the collection of arbitrary subsets of the space E. Then, $\mathcal{R}_{\mathcal{F},\mathcal{G}} = \{A \subset E : \underline{\lambda}(A) = \overline{\lambda}(A) \in \mathbb{R}\}$.

Remark 8. *The measure $\lambda_{\mathcal{F},\mathcal{G}} : \mathcal{R}_{\mathcal{F},\mathcal{G}} \to [0, +\infty)$ defines (see [27]) the space $\mathcal{H} = L_2(E, \mathcal{R}_{\mathcal{F},\mathcal{G}}, \lambda_{\mathcal{F},\mathcal{G}}, \mathbb{C})$ by the standard way as the completion in euclidean norm of the space $S_2(E, \mathcal{R}_{\mathcal{F},\mathcal{G}}, \lambda_{\mathcal{F},\mathcal{G}}, \mathbb{C})$ of equivalence classes of simple functions.*

8.3. Invariance of the Symplectic Measure with Respect to Hamiltonian Flows

Let $h : E \to R$ be a non-degenerate quadratic form on the space E. Let us consider function h as the Hamilton function on the symplectic space (E, ω). Quadratic form h on the space E has the canonical basis $\mathcal{E}$ such that its quadratic form is diagonal on the basis of $\mathcal{E}$. Let us assume that the linear operator associated with the form H commutes with the symplectic operator. Then the basis $\mathcal{E}$ can be chosen as the symplectic basis of the symplectic form ω ([54]). Hence, the bilinear form ω satisfies equalities $\omega(e_{2k-1}, e_{2k}) = -\omega(e_{2k}, e_{2k-1}) = 1$ and $\omega(e_l, e_m) = 0$ in other cases. Let us introduce the orthonormal systems $\mathcal{F}$, $\mathcal{G}$ in the subspaces P, Q such that $e_{2k-1} = f_k$, $e_{2k} = g_k$, $k \in \mathbb{N}$.

Let us consider a countable system of non-interacting oscillators.

Lemma 33 ([27]). *Let $\mathcal{E}$ be a symplectic basis of the form ω such that conditions (62) hold. Let a quadratic form h be diagonal on the basis of $\mathcal{E}$:*

$$h = \sum_{k=1}^{\infty} \lambda_k (p_k^2 + q_k^2),\ D(h) = \{(q, p) \in E : \sum_{k=1}^{\infty} |\lambda_k| (p_k^2 + q_k^2) < +\infty\}, \tag{66}$$

where $\{\lambda_k\} : \mathbb{N} \to \mathbb{R}$. Then the Hamiltonian vector field $\mathbf{v} = \mathbf{J} \nabla h$ is defined on the space $D^2(h) = \{(q, p) \in E : \sum_{k=1}^{\infty} \lambda_k^2 (q_k^2 + p_k^2) < +\infty\}$. This vector field generates the smooth Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$ in the space E_2. The flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$ has the unique continuation to *the generalized Hamiltonian flow in space E. The symplectic measure λ_ω is invariant with respect to the generalized Hamiltonian flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$.*

Proof. The dynamics of the Hamiltonian system (66) is defined by the countable system of ordinary differential equations

$$q_k' = h'_{p_k} = \omega_k p_k;\ p_k' = -h'_{q_k} = -\omega_k q_k, \quad k \in \mathbb{N}. \tag{67}$$

The Hamiltonian system (67) has the first integral $h(u) = (u, \mathbf{H}u)$, $u \in D(h)$. Here, $\mathbf{H}$ is the self-adjoint operator in the real Hilbert space E such that the spectrum of operator H is the sequence of eigenvalues $\{\lambda_k\}$ and $\mathrm{Ker}(\mathbf{H} - \lambda_k \mathbf{I}) = \mathrm{span}(e_k, f_k)$ for any eigenvalue λ_k. Then, there is the group $\mathbf{\Phi}_t = \exp(\mathbf{JH}t)$, $t \in \mathbb{R}$ of orthogonal operators defined in space E. For any $k \in \mathbb{N}$, the subspaces E_k are invariant subspaces of this group. The restriction $\mathbf{\Phi}_t|_{E_k}$, $t \in \mathbb{R}$ is the two-dimensional Hamiltonian flow $\mathbf{\Phi}_{t,k}$, $t \in \mathbb{R}$ of orthogonal operators in space E_k. For any $k \in \mathbb{N}$, the two-dimensional Hamiltonian flow $\mathbf{\Phi}_{t,k}$, $t \in \mathbb{R}$ is defined by the Hamiltonian function $h_k = \lambda_k(q_k^2 + p_k^2)$, $(q_k, p_k) \in E_k$.

The subspaces $D(h)$, $D^2(h)$ are invariant with respect to the group of operators $\mathbf{\Phi}_t$, $t \in \mathbb{R}$ of the Hamiltonian flow. Therefore, the restriction $(\mathbf{\Phi}_t)|_{E_2}$, $t \in \mathbb{R}$ is the smooth Hamiltonian flow in the space $D^2(h)$, which is the domain of the vector field $\mathbf{v}$. The group of operators $\mathbf{\Phi}_t$, $t \in \mathbb{R}$ are the unique continuations of the smooth Hamiltonian flow in the space $D^2(h)$.

If $A \in \mathcal{K}_{\mathcal{F},\mathcal{G}}$, then $\mathbf{\Phi}_t(A) \in \mathcal{K}_{\mathcal{F},\mathcal{G}}$ and $\lambda_{\mathcal{F},G}(\mathbf{\Phi}_t(A)) = \lambda_{\mathcal{F},G}(A)$ for all $t \in \mathbb{R}$. (Here, $\mathcal{E} = \mathcal{F} \bigcup \mathcal{G}$). Therefore, the ring $\mathcal{R}_{\mathcal{F},\mathcal{G}}$ is invariant with respect to the generalized flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$ and equalities $\lambda_{\mathcal{F},G} \circ \mathbf{\Phi}_t = \lambda_{\mathcal{F},G}$, $t \in \mathbb{R}$ hold. □

The flow $\mathbf{\Phi}_t$, $t \in \mathbb{R}$ in the space E from Lemma 33 defines the one-parametric group

$$\mathbf{U}_{\mathbf{\Phi}_t} u(x) = u(\mathbf{\Phi}_{-t}(x)),\ x \in E,\ u \in S(E, \mathcal{R}_{\mathcal{F},G}, \mathbb{C}),\ t \in \mathbb{R},$$

of linear isometric operators in the space of simple functions $S_2(E, \mathcal{R}_{\mathcal{F},G}, \lambda_{\mathcal{F},G}, \mathbb{C})$. The group of isometries $\mathbf{U}_{\mathbf{\Phi}_t}$, $t \in \mathbb{R}$ in the space $S_2(E, \mathcal{R}_{\mathcal{F},G}, \lambda_{\mathcal{F},G}, \mathbb{C})$ is the unique continuous extension of the unitary group in the space $\mathcal{H}_{\mathcal{F},G}$ such that

$$\mathbf{U}_{\mathbf{\Phi}_t} u(x) = u(\mathbf{\Phi}_{-t}(x)),\ t \in \mathbb{R},\ u \in \mathcal{H}_{\mathcal{F},\mathcal{G}},\ x \in E. \tag{68}$$

The unitary group (68) is called the Koopman representation of the Hamiltonian flow $\mathbf{\Phi}$.

8.4. Koopman Group in the Space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ and Its Generator

Let the Hamilton function $\mathbb{H}$ of the flow Φ be the reification of the quadratic form of a positive operator $\mathbf{\Lambda}$ in the space $H = \mathbf{R}(E)$ with the discrete spectrum $\{\lambda_k\}$. Then, $\mathbb{H}$ is the Hamiltonian of the countable system of oscillators in the symplectic space $(E, \mathbf{J})$:

$$\mathbb{H}(q, p) = \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k (p_k^2 + q_k^2),\ (q, p) \in E_1 = D(\mathbb{H}).$$

The Hamiltonian flow Φ preserves the two-dimensional symplectic subspace E_k, $k \in \mathbb{N}$ of the space E. Moreover, it preserves the measure $\lambda_{\mathcal{F},\mathcal{G}}$.

Example 1. *Let $u = \chi_{\Pi_{[-\frac{1}{2},\frac{1}{2}]}}$. Then, the function $(\mathbf{U}_{\mathbf{\Phi}}(t)u, u)_{\mathcal{H}_{\mathcal{F},\mathcal{G}}}$, $t \in \mathbb{R}$, is continuous if $\{\lambda_k\} \in l_1$.*

Lemma 34. *The Koopman group $\mathbf{U}_{\mathbf{\Phi}}$ is the unitary group in the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ which is strongly continuous if and only if the sequence $\{\lambda_k\}$ is finite.*

Proof. According to [56], the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$ is the tensor product $\mathcal{H}_{\mathcal{F},\mathcal{G}} = L_2(R^{2n}) \otimes \mathcal{H}_{\mathcal{F}^n,\mathcal{G}^n}$. Here, R^{2n} is the linear hull of the first $2n$ vectors of ONB $\{f_1, g_1, f_2, g_2, \dots$ and E^{2n} is the orthogonal complement of R^{2n}. Let $\mathcal{F}^n, \mathcal{G}^n$ be a part of ONB $\mathcal{F}, \mathcal{G}$ which belongs to the space E^{2n}. Let $\mathcal{H}_{\mathcal{F}^n,\mathcal{G}^n} = L_2(E^{2n}, \mathcal{R}_{\mathcal{F}^n,\mathcal{G}^n}, \lambda_{\mathcal{F}^n,\mathcal{G}^n}, \mathbb{C})$.

Let $\{\lambda_k\} \in c_0$. Let the flow Φ be generated by the Hamiltonian $\mathbb{H}$. Then, for every $t \in \mathbb{R}$, the mapping Φ_t is the tensor product of an orthogonal mapping in the space R^{2n} and the identical operator in the space E^{2n}. Therefore, if $\lambda_k = 0 \ \forall\, k > n$, then $\mathbf{U_\Phi} = \mathbf{U}_{\mathbf{\Phi}_{2n}} \otimes \mathbf{I}^{2n}$ where $\mathbf{U_\Phi}$ is the Koopman group of Hamiltonian flow Φ in the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$, $\mathbf{U}_{\mathbf{\Phi}_{2n}}$ is the Koopman group of the system of n harmonical oscillators in the space $L_2(\mathbb{R}^{2n})$, and $\mathbf{I}^{2n}$ is the identical operator in the space $\mathcal{H}_{\mathcal{E}^n,\mathcal{F}^n}$. It is well-known that the Koopman group of a system of n oscillators is the strong (and uniformly) continuous one-parametric unitary group in the space $L_2(\mathbb{R}^{2n})$. Hence, group $\mathbf{U_\Phi}$ of countable system of oscillators with a finite sequence of frequences $\{\lambda_k\}$ is the strong continuous unitary group in the space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$.

Conversely, let $\{\lambda_k\} \notin c_0$. Without loss of generality, we can assume that $\{\lambda_k\} \neq 0 \ \forall\, k$. The flow Φ is the tensor product of two-dimensional flows in the space Φ_k. For every $k \in \mathbb{N}$ the group, Φ_k is the rotation in the plane $E_{(k)}$ with the angular velocity λ_k. Let us consider the round K_k of radius $\frac{\sqrt{2}}{\sqrt{\pi}}$ in every plane $E_{(k)}$, $k \in \mathbb{N}$. Let $\{m_k\}$ be a sequence with values in the set $\mathbb{N}$. For every $k \in \mathbb{N}$, the round K_j, $j = 1, 2, \dots, 2^{m_k}$ is subdivided into 2^{m_k} sequentially numbered congruent sectors of square $\frac{1}{2^{m_k}}$. Let A_k be the union of sectors with even numbers. Then, $\lambda_{\mathbb{R}^2}(A_k) = 1$, $k \in \mathbb{N}$. Let $\Pi = A_1 \times A_2 \times \dots$. Then, $\Pi \in K_{\mathcal{F},\mathcal{G}}$ is the symplectic rectangle and $\lambda_{\mathcal{F},\mathcal{G}}(\Pi) = 1$.

The absence of the strong (weak) continuity of an operator-valued function $\mathbf{U_\Phi}(t)$, $t \in \mathbb{R}$ is the consequence of the discontinuity of scalar function $(\mathbf{U_\Phi}(t)\chi_\Pi, \chi_\Pi)_{\mathcal{H}_{\mathcal{F},\mathcal{G}}}$, $t \in \mathbb{R}$.

Let us fix a number $\delta \in (0, \frac{1}{3})$. For every $k \in \mathbb{N}$, the condition $(\mathbf{U}_{\mathbf{\Phi}_k}(t)\chi_{A_k}, \chi_{A_k}) \in [0, 1-\delta)$ holds for every $t \in \bigcup\limits_{n\in\mathbb{Z}} \Delta_k^n$, where

$$\Delta_k^n = (\frac{1}{\lambda_k}(\frac{1}{3}2^{-m_k}\pi + 2^{-m_k}\pi n), \frac{1}{\lambda_k}(\frac{2}{3}2^{-m_k}\pi + 2^{-m_k}\pi n)).$$

Hence, there are $\mathbb{N}$-valued sequences $\{m_k\}$ and $\{n_k\}$ such that, for every $k \in \mathbb{N}$, there is a number $n_k \in \mathbb{N}$ such that $\Delta_k^{n_k} \subset (0, \frac{1}{k})$ and $\Delta_k^{n_k} \supset \Delta_{k+1}^{n_{k+1}}$.

Therefore, there is a sequence $\{m_k\} : \mathbb{N} \to \mathbb{N}$ such that every right half-neighborhood of a zero point contains a point τ such that $(\mathbf{U}_{\mathbf{\Phi}_k}(\tau)\chi_{A_k}, \chi_{A_k}) \in [0, 1-\delta)$ for infinitely many numbers $k \in \mathbb{N}$.

Hence, there is a sequence $\{\tau_n\}$ such that $\tau_n \to +0$ and $(\mathbf{U_\Phi}(\tau_n)\chi_\Pi, \chi_\Pi)_{\mathcal{H}_{\mathcal{F},\mathcal{G}}} = 0$ for any $n \in \mathbb{N}$. This fact implies the discontinuity of the function $(\mathbf{U_\Phi}(t)\chi_\Pi, \chi_\Pi)_{\mathcal{H}_{\mathcal{F},\mathcal{G}}}$, $t \in \mathbb{R}$ in point $t = 0$, since $(\mathbf{U_\Phi}(0)\chi_\Pi, \chi_\Pi)_{\mathcal{H}_{\mathcal{F},\mathcal{G}}} = 1$. □

To define the strong continuity subspaces, we use the spectral properties of Koopman generator.

Let $L_{2,r}(0, +\infty)$ be the Hilbert space of Lebesgue measurable functions $(0, +\infty) \to \mathbb{C}$, which are quadratically integrable with the weight $\omega = \frac{1}{r}$. Let $(\mathbb{N} \to \mathbb{Z})_0$ be the space of finite sequences with values in the set of integer numbers $\mathbb{Z}$.

Theorem 15. *The Koopman group $\mathbf{U_\Phi}$ has the invariant subspace $\mathcal{H}_\Phi \subset \mathcal{H}_{\mathcal{F},\mathcal{G}}$ such that the group $\mathbf{U_\Phi}|_{\mathcal{H}_\Phi}$ is strongly continuous in the space $\mathcal{H}_\Phi$. The generator $\mathbf{H}_\Phi$ of the strongly continuous group $\mathbf{U_\Phi}|_{\mathcal{H}_\Phi}$ has the countable set of eigenvalues*

$$\lambda_{m_1,\dots,m_N} = m_1\lambda_1 + \dots + m_N\lambda_N, \quad N \in \mathbb{N},\ m_1, \dots, m_N \in \mathbb{Z}.$$

Every eigenvalue $\lambda_{m_1,\dots,m_N}$ has the proper subspace

$$\mathrm{Ker}(\mathbf{H}_\Phi - \lambda_{m_1,\dots,m_N}\mathbf{I}) \equiv \mathcal{H}_{\vec{m}} = \mathrm{span}(\prod_{k=1}^{\infty} v_{j_k}(r_k)e^{im_k\phi_k}), \tag{69}$$

where $\vec{m} \in (\mathbb{N} \to \mathbb{Z})_0$, $\{v_j\}$ is an ONB in the space $L_{2,r}([0, +\infty))$, and $\{j_k\} : \mathbb{N} \to \mathbb{N}$.

The Hilbert space $\oplus_{\vec{m}}\mathcal{H}_{\vec{m}}$ is the invariant subspace of strong continuity for the group $\mathbf{U}_{\mathbf{\Phi}}$.

Proof. It is directly calculated that

$$\mathbf{U}_{\mathbf{\Phi}}(t)(v_{j_k}(r_k)e^{im_k\phi_k}) = e^{it\lambda_k m_k}v_{j_k}(r_k)e^{im_k\phi_k},\ t \in \mathbb{R}.$$

Hence, the generator $\mathbf{H}_{\Phi}$ of the strongly continuous group $\mathbf{U}_{\mathbf{\Phi}}$ has the countable set of eigenvalues $\lambda_{m_1,\dots,m_N} = m_1\lambda_1 + \dots + m_N\lambda_N,\quad N \in \mathbb{N},\ m_1,\dots,m_N \in \mathbb{Z}$. Moreover, every eigenvalue $\lambda_{\vec{m}}$ has the infinite dimensional proper space (69). If $\vec{m} \neq \vec{n}$, then it is easy to check that subspaces $\mathcal{H}_{\vec{m}}$ and $\mathcal{H}_{\vec{n}}$ are orthogonal. If $\lambda_{\vec{m}} = \lambda_{\vec{n}}$, then the eigenvalue $\lambda_{\vec{m}}$ has the proper space $\mathcal{H}_{\vec{m}} \oplus \mathcal{H}_{\vec{n}}$. Every proper space $\mathcal{H}_{\vec{m}}$ is invariant with respect to the group $\mathbf{U}_{\mathbf{\Phi}}$ and the restriction $\mathbf{U}_{\mathbf{\Phi}}|_{\mathcal{H}_{\vec{m}}}$ is strongly continuous group in the space $\mathcal{H}_{\vec{m}}$. Therefore, if $\mathcal{H}_{\Phi} = \oplus_{\vec{m}}\mathcal{H}_{\vec{m}}$, then the space $\mathcal{H}_{\Phi}$ is invariant with respect to the group $\mathbf{U}_{\mathbf{\Phi}}$ and the restriction $\mathbf{U}_{\mathbf{\Phi}}|_{\mathcal{H}_{\Phi}}$ is strongly continuous group in the space $\mathcal{H}_{\Phi}$. □

Remark 9. *If the sequence $\{\lambda_k\}$ is not finite, then the Koopman unitary group $\mathbf{U}_{\mathbf{\Phi}}$ is not continuous on the whole space $\mathcal{H}_{\mathcal{F},\mathcal{G}}$. However, it has the invariant subspace of strong continuity $\mathcal{H}_{\Phi}$. Some parts of the space $\mathcal{H}_{\Phi}$ can be defined using the spectral properties of the unitary group $\mathbf{U}_{\mathbf{\Phi}}$.*

8.5. Measure with the Property of Orthosymplectic Invariance

We see that the shift-invariant measure $\lambda_{\mathcal{E}}$ on the Hilbert space has continuations to measures on more wide rings such that a continued measure is invariant with respect to a mere wide group. One of these continuations is the invariant with respect to the group of isometry measures λ. Another continuation is the measure $\lambda_{\mathcal{F},\mathcal{G}}$ which is invariant with respect to the group of symplectomorphisms.

Lemma 35. *There is no continuation of the measure $\lambda_{\mathcal{F},\mathcal{G}}$ which is invariant both to the group of symplectomorphisms and to the orthogonal group.*

Proof. Let us assume the opposite, that there is a measure ν which is defined on ring $\hat{\mathcal{R}}$ of the subsets of the space E such that

(1) $\hat{\mathcal{R}} \supset \mathcal{R}$, $\hat{\mathcal{R}} \supset \mathcal{R}_{\mathcal{F},\mathcal{G}}$ and $\nu|_{\mathcal{R}} = \lambda$, $\nu|_{\mathcal{R}_{\mathcal{F},\mathcal{G}}} = \lambda_{\mathcal{F},\mathcal{G}}$.
(2) $\mathbf{U}(A) \in \hat{\mathcal{R}}$ and $\nu(\mathbf{U}(A)) = \nu(A)\ \forall\ A \in \hat{\mathcal{R}}$ and for any orthogonal mapping $\mathbf{U}$.
(3) $\mathbf{\Phi}(A) \in \hat{\mathcal{R}}$ and $\nu(\mathbf{\Phi}(A)) = \nu(A)\ \forall\ A \in \hat{\mathcal{R}}$ and for any symplectomorphism $\mathbf{\Phi}$ preserving two-dimensional symplectic subspaces.

Let $\Pi = B_1 \times B_2 \times \dots$ be a symplectic rectangle such that $B_k = [0,2) \times [0,\frac{1}{2})$. Since $\lambda_2(B_k) = 1\ \forall\ k \in \mathbb{N}$, Π is an absolutely measurable symplectic rectangle, $\Pi \in \mathcal{K}_{\mathcal{F},\mathcal{G}}$ and $\nu(\Pi) = \lambda_{\mathcal{F},\mathcal{G}}(\Pi) = 1$.

Let $\mathbf{U}$ be an orthogonal mapping of the space E which changes the order of the vectors of the symplectic orthonormal basis $\mathcal{E} = \{e_1, e_2, e_3, e_4, \dots\} = \{f_1, g_1, f_2, g_2, \dots\}$ only. Then, according to our assumption, $\mathbf{U}(\Pi) \in \hat{\mathcal{R}}$ and $\nu(\mathbf{U}(\Pi)) = 1$.

Let us pose that $\mathbf{U}(e_1) = e_1$, $\mathbf{U}(e_3) = e_2$, $\mathbf{U}(e_{2_k+3}) = e_{2k+1}\,\forall\, k \in \mathbb{N}$, $\mathbf{U}(e_{2m}) = e_{2m+2}\,\forall\, m \in \mathbb{N}$. Then, $\mathbf{U}(\Pi) = \Pi' = B_1' \times B_2' \times \dots$ where $B_1' = [0,2) \times [0,2)$ and $B_k' = B_k$, $\forall\, k = 2,3,\dots$. Therefore, $\mathbf{U}$ is orthogonal mapping, $\mathbf{U}(\Pi) \in \mathcal{K}_{\mathcal{F},\mathcal{G}}$ but $\nu(\mathbf{U}(\Pi)) \neq \nu(\Pi)$. Thus, there is no continuation of the symplectomorphism-invariant measure $\lambda_{\mathcal{F},\mathcal{G}}$ such that this continuation is invariant with respect to the orthogonal group. □

9. Conclusions

In this paper, we constructed the finitely additive measure λ on an infinite-dimensional real Hilbert space such that this measure is shift- and rotation-invariant. Moreover, the introduced measure is locally finite and σ-finite. However, it is not countably additive and Borel measureable. The decomposition of the measure λ into the sum of mutually singular shift-invariant measures was obtained.

By means of the constructed measure, the unitary representations of the group of shifts and the orthogonal group in the space which is quadratically integrable with respect to invariant measure functions were obtained.

The notion of the ring ergodicity of a measure with respect to a group was introduced. The ring-ergodic decomposition of a shift-invariant measure was obtained. The parametrization of the family of shift-invariant measures was given by the obtained ring-ergodic decomposition. Thus, the infinite dimensional analog of the Ruziewicz problem from Section 1.2 was solved. Every ring-ergodic component in the above decomposition defines the separable Hilbert space of functions that are quadratically integrable with respect to ring-ergodic invariant measures.

It was shown that the representation of the group of shifts is not continuous in the strong operator topology. The subgroup of the group of shifts with the strongly continuous representation was described.

The invariantness of a measure with respect to Hamiltonian flows was studied in the Hilbert space endowed with the shift-invariant symplectic form. The shift-invariant measure was extended to the measure which is invariant with respect to the group of symplectomorphisms preserving every two-dimensional symplectic subspace. By means of the symplectic-invariant measure, the Koopman unitary representation of the above group of symplectomorphisms was obtained.

The Koopman representation of the Hamiltonian flow of the countable system of harmonic oscillators was studied. The subspaces of strong continuity in the Koopman unitary group were described in terms of the spectrum of its generator.

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