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Uniform Asymptotic Estimate for the Ruin Probability in a Renewal Risk Model with Cox–Ingersoll–Ross Returns

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Abstract: Consider an insurance risk model with arbitrary dependence structures between the claim sizes. Suppose that the risky investment in the insurer can be established by the Cox–Ingersoll–Ross model. When the claim-size distribution is heavy-tailed, a uniform asymptotic formula for ruin probability is obtained.

Keywords: uniform; asymptotics; the Cox–Ingersoll–Ross model; ruin probability; risk model

MSC: 62P05; 91G05

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1. Introduction

In this paper, we consider an insolvency problem for an insurance company that makes investments. The classic insurance risk model has the following relation:

$$R(t) = x + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{1}$$

where $R(t)$ denotes the surplus of the insurance company up to time t, $x > 0$ is the initial reserve, ct is the total premiums up to time t with premium rate $c > 0$, and $\sum_{i=1}^{N(t)} X_i$ is the total claim sizes up to time t. Here, $\{X_i, i \geq 1\}$ is the sequence of claim sizes whose common inter-arrival times $\{\theta_i, i \geq 1\}$ form a sequence of independent and identically distributed (i.i.d.) random variables. Then the arrival times of the successive claims $\tau_n = \sum_{i=1}^{n} \theta_i, n \geq 1$, constitute a renewal counting process $\left\{N(t) : N(t) = \sum_{i=1}^{\infty} \mathbb{I}_{[\tau_i \leq t]}; t \geq 0\right\}$ with renewal function $\lambda(t) = EN(t) = \sum_{i=1}^{\infty} P(\tau_i \leq t) < \infty$, where $\mathbb{I}_E$ is the indicator function of an event E.

Suppose that the insurance company is allowed to make risk-free or risky assets. From Guo and Wang [1], we can denote the investment return of the surplus of the insurance company by $\{e^{\xi(t)}; t \geq 0\}$, where $\xi(t)$ can be a stochastic process. Further, the insurance company invests one unit of capital into financial assets at time 0 and receives the benefits of $e^{\xi(t)}$ units at time t. By (1.1) of Guo and Wang [1], we can solve the stochastic differential equation satisfied by the surplus process of insurance risk process with investment, and then obtain the integrated risk process $U(t)$ of the insurance company,

$$U(t) = e^{\xi(t)} \left(x + \int_0^t e^{-\xi(s-)} dR(s) \right) = e^{\xi(t)} \left(x + c \int_0^t e^{-\xi(s)} ds - \sum_{i=1}^{N(t)} X_i e^{-\xi(\tau_i)} \right). \tag{2}$$

This paper considers $\xi(t) = \int_0^t r_s ds$ with $\xi(0) = 0$, where $\{r_t; t \geq 0\}$ is a stochastic short-rate process, and the evolution of the interest rate is given by the following Cox–Ingersoll–Ross (CIR) model:

$$dr_t = m(l - r_t)dt + \mu\sqrt{r_t}dW_t, \tag{3}$$

where m, l, and μ are positive constants, r_0 is a constant, and $\{W_t,\ t \geq 0\}$ is a standard Wiener process.

This paper adopts the following definition of ruin probability which is given by

$$\Psi(x,t) = P(T_{\max} \leq t \mid U(0) = x), \qquad 0 < t < \infty, \tag{4}$$

where $T_{\max} = \inf\{t > 0 : U(t) < 0\}$ denotes the ruin time with $\inf \emptyset = \infty$ by convention.

The ruin probability of insurance risk models has been a hot topic of risk theory and actuarial mathematics. The study on the uniform asymptotic estimates for ruin probabilities in insurance risk models with constant force of interest has achieved fruitful results. Some recent works include Chen et al. [2], Cheng and Yu [3], and Jiang et al. [4], etc. For the risk model with investment return process described by geometric Lévy process, there are many studies on the uniform asymptotic analysis for ruin probability. See, for example, Fu and Ng [5], Guo and Wang [1], Guo et al. [6], Li [7], and Tang et al. [8], among many others. However, all the aforementioned works did not pay special attention to the case of the uniform asymptotic formula for risk models with risky investments related to the CIR model. In the actuarial literature, many researchers assumed that the claim sizes in one line of business of the insurance company are independent. For example, Fu and Ng [5], Guo et al. [6], Jiang et al. [4], Li [7], and Tang et al. [8], etc. However, the assumption of independence between the claim sizes X_i, $i \geq 1$, is too strong. Therefore, some authors started to propose extensions with various dependence structures. Guo and Wang [1] considered that the claim size sequence $\{X_i, i \geq 1\}$ is bivariate upper tail independent. Chen et al. [2] investigated that the sequences of claim sizes are upper tail asymptotically independent (UTAI). Cheng and Yu [3] assumed that the sequences of claim sizes are tail asymptotically independent (TAI). Motivated by the references mentioned-above, in this paper we consider that the sequence of claim sizes $\{X_i, i \geq 1\}$ possesses an arbitrary dependent structure, and then we derive the uniform asymptotic formula of $\Psi(x,t)$ for model (2).

In the rest of this paper, Section 2 presents the main result after recalling some necessary preliminaries, Section 3 performs some simulations, Section 4 establishes some crucial lemmas, Section 5 proves the main result, and Section 6 restates the paper's context and discusses future work.

2. Preliminaries and Main Result

Hereafter, C always stands for a positive constant and may vary in different places. For two positive functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ satisfying $l_1 = \liminf_{x\to\infty} \inf_{t\in E\neq\emptyset} \frac{f(x,t)}{g(x,t)} \leq \limsup_{x\to\infty} \sup_{t\in E\neq\emptyset} \frac{f(x,t)}{g(x,t)} = l_2$, we say that $f(x,t) \lesssim g(x,t)$ holds uniformly for $t \in E$ if $l_2 \leq 1$; $f(x,t) \gtrsim g(x,t)$ holds uniformly for $t \in E$ if $l_1 \geq 1$; $f(x,t) \sim g(x,t)$ holds uniformly for $t \in E$ if $l_1 = l_2 = 1$; and $f(x,t) \asymp g(x,t)$ holds uniformly for $t \in E$ if $0 < l_1 \leq l_2 < \infty$. For two real numbers a and b, we write $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

Now recall the definition and some properties of distributions with regularly varying tails. A distribution F on $\mathbb{R}$ belongs to the class of distributions with regularly-varying tails if $\bar{F}(x) > 0$ for all $x \geq 0$ and

$$\lim_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = y^{-\alpha}, \qquad \text{for any } y > 0, \tag{5}$$

for some $\alpha > 0$, denoted by $F \in \mathcal{R}_{-\alpha}$. By Theorem 1.5.2 of Bingham et al. [9], the convergence in (5) is uniform over $[\varepsilon, \infty)$ for every fixed $\varepsilon > 0$; namely,

$$\bar{F}(xy) \sim y^{-\alpha}\bar{F}(x), \qquad \text{uniformly for } y \in [\varepsilon, \infty). \tag{6}$$

For a distribution $F \in \mathcal{R}_{-\alpha}$ with some $0 < \alpha < \infty$, according to Bingham et al. [9] (Proposition 2.2.1) we know that, for any $p > \alpha$, there exist two positive constants $C_p > 1$ and D_p such that

$$\frac{\bar{F}(y)}{\bar{F}(x)} \leq C_p \left(\frac{x}{y}\right)^p \tag{7}$$

holds uniformly for all $x \geq y \geq D_p$. From the relation (7), it follows that

$$x^{-p} = o(\bar{F}(x)), \qquad x \to \infty. \tag{8}$$

In addition, suppose that a nonnegative random variable X with distribution function $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ and a nonnegative random variable Y are independent. Then, by Lemma 3.2 of Heyde and Wang [10], there exists some constant $C > 0$ without relation to Y and δ such that, for arbitrarily fixed $\delta > 0$ and $\alpha < p < \infty$,

$$P(XY > \delta x | Y) \leq C\bar{F}(x)\left(\delta^{-p} Y^p \mathbb{I}_{[Y \geq \delta]} + \mathbb{I}_{[Y < \delta]}\right), \qquad \text{for all large } x. \tag{9}$$

For more details on the distribution class of regular variation, we refer to Bingham et al. [9] and Embrechts et al. [11].

For $\xi(t) = \int_0^t r_s ds$ and the relation (3), according to the relation (3.30) of Guo and Wang [1], we obtain

$$Ee^{b\xi(t)} = \exp\left\{\hat{c}(b,t)r_0 + \frac{(m - \hat{\Omega}(b))mlt}{\mu^2} - \frac{2ml}{\mu^2}\ln\left(\frac{\hat{\zeta}(b) - e^{-\hat{\Omega}(b)t}}{\hat{\zeta}(b) - 1}\right)\right\}, \quad m^2 > 2\mu^2 b, \tag{10}$$

where

$$\hat{c}(b,t) = \tfrac{m - \hat{\Omega}(b)}{\mu^2} - \tfrac{2\hat{\Omega}(b)}{\mu^2} \cdot \tfrac{1}{\hat{\zeta}(b)e^{\hat{\Omega}(b)t} - 1}, \quad \hat{\Omega}(b) = \sqrt{m^2 - 2\mu^2 b}, \quad \hat{\zeta}(b) = 1 - \tfrac{2\hat{\Omega}(b)}{\hat{\Omega}(b) - m}.$$

Denote by Λ the set of all t for which $0 < \lambda(t) < \infty$. With $\underline{t} = \inf\{t : P(\tau_1 \leq t) > 0\}$, it is clear that

$$\Lambda = \begin{cases} [\underline{t}, \infty] & \text{if } P(\tau_1 = \underline{t}) > 0, \\ (\underline{t}, \infty] & \text{if } P(\tau_1 = \underline{t}) = 0. \end{cases}$$

For notational convenience, we write $\Lambda_T = (0, T] \cap \Lambda$ for every fixed $T \in \Lambda$. As usual, assume that $\{X_i, i \geq 1\}$, $\{\theta_i, i \geq 1\}$ and $\{\xi(t), t \geq 0\}$ are independent. We are now ready to state the main result of this paper:

Theorem 1. *Consider the insurance risk model (2) in which the claim sizes form a sequence of identically distributed but not necessarily independent random variables with common distribution $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$. Furthermore, we allow arbitrary dependence structures between the claim sizes. If there exists constant $\beta > 0$ such that $m^2 > 2\mu^2\beta$. Then it holds uniformly for all $t \in \Lambda_T$ that*

$$\Psi(x,t) \sim \bar{F}(x)\int_0^t \exp\left\{\hat{c}(-\alpha,s)r_0 + \frac{(m - \hat{\Omega}(-\alpha))mls}{\mu^2} - \frac{2ml}{\mu^2}\ln\left(\frac{\hat{\zeta}(-\alpha) - e^{-\hat{\Omega}(-\alpha)s}}{\hat{\zeta}(-\alpha) - 1}\right)\right\} d\lambda(s). \tag{11}$$

An insurance company has to hold enough risk capital so that the ruin probability is sufficiently low. Furthermore, the pricing of related insurance products may use the ruin probability as a trigger. Both cases require an assessment of the ruin probability, making the relation (11) plays an important role in guidance to insurers and regulators for risk capital calculation.

From Theorem 1, there exists $\beta > 0$ such that $m^2 > 2\mu^2\beta$, hence we have by (10) that, for $T \in \Lambda$ and large $A > 0$, $e^{\beta A} := \sup_{t\in(0,T]} Ee^{\beta\xi(t)} + 1$. Consequently,

$$\sup_{t\in(0,T)} P(\xi(t) > A) = \sup_{t\in(0,T)} P\left(e^{\xi(t)} > e^A\right) \leq \frac{\sup_{t\in(0,T)} Ee^{\beta\xi(t)}}{e^{\beta A}} < 1. \tag{12}$$

Fix $b = -\kappa$ for $\kappa > \alpha$ in (10), then the following relation holds for $H > 0$:

$$\sup_{t\in(0,T)} Ee^{-\kappa\xi(t)} < H. \tag{13}$$

Fix $b = -1$ in (10), then we obtain

$$E\left(\int_0^T e^{-\xi(s)}ds\right) = \int_0^T Ee^{-\xi(s)}ds < \infty. \tag{14}$$

3. Numerical Simulations

In this section, we illustrate the accuracy of the relation (11) in Theorem 1. All the numerical simulations are carried out on R software.

For simplicity, we assume that the claim inter-arrival times are independent and identically distributed exponential random variables with parameter $\lambda > 0$. Moreover, the successive claims form a sequence of i.i.d. random variables with the generic random variable X. Let X follow a Pareto distribution F with shape parameter α and scale parameter κ, which means $F(x) = 1 - (\kappa/(x+\kappa))^\alpha \in \mathcal{R}_{-\alpha}$, $x > 0$. Then, the asymptotic estimation of ruin probability $\Psi(x,t)$ can be rewritten as

$$\begin{aligned}\Psi(x,t) \sim \lambda\bar{F}(x)\int_0^t \exp\left\{\hat{c}(-\alpha,s)r_0 + \frac{(m-\hat{\Omega}(-\alpha))mls}{\mu^2} - \frac{2ml}{\mu^2}\ln\left(\frac{\hat{\zeta}(-\alpha) - e^{-\hat{\Omega}(-\alpha)s}}{\hat{\zeta}(-\alpha)-1}\right)\right\}ds \\ := \Psi_1(x,t).\end{aligned}$$

The parameters in this subsection are given as: $\kappa = 1$, $\alpha = 1.5$, $t = 1$, $\lambda = 0.5$, $m = 1$, $\mu = 0.5$, $l = 1$, $r_0 = 0.5$, $c = 10$, and the initial capital $x = 200, 400, 600, 800, 1000$.

We employ $\Psi_2(x,t)$ to represent the Monte Carlo (MC) simulation result of the ruin probability $\Psi(x,t)$. The procedure of simulation of $\Psi_2(x,t)$ is the following:

1. Assign a value for variable x and set $\tau_0 = 0$, $N = n = 0$, and $S = 0$;
2. Generate random variable X with X following Pareto distribution F and θ following exponential distribution with parameter λ, and then set $\tau_1 = \theta$;
3. Set $\tau_1 = \tau_0 + \tau_1$. If $\tau_1 > t$, set $N = N + 1$. If $\tau_1 \leq t$, divide the interval $[\tau_0, \tau_1]$ into 30 pieces, and denote these points as $t_0 = \tau_0, t_1, \cdots, t_{30} = \tau_1$. According to Glasserman [12] (p. 124), we can simulate $r_1, \cdots, r_{30}$, and then we can simulate $\xi_1, \cdots, \xi_{30}$ from $\xi_i = \xi_{i-1} + r_i(t_i - t_{i-1})$, $i = 1, \cdots, 30$. Calculate
$$S = S + Xe^{-\xi_{\tau_1}} - c\sum_{i=1}^{30} e^{-\xi_{t_i}}(t_i - t_{i-1}).$$
If $S > x$, then $n = n + 1$ and $N = N + 1$. If not, set $\tau_0 = \tau_1$, and repeat Steps 2 and 3;
4. Set $\tau_0 = 0$ and $S = 0$. Repeat Steps 2 and 3 until $N = 10^6$;
5. Calculate $\Psi_2(x,t) = n/N$.

When the initial capital $x = 200, 400, 600, 800, 1000$, the comparison of asymptotic estimate and MC simulation of $\Psi(x,t)$ can be seen in Table 1. From Table 1, we observe that the ruin probability decreases as x becomes large, and the ratio $\Psi_2(x,t)/\Psi_1(x,t)$ becomes closer to 1 as x becomes large.

Table 1. Asymptotic versus simulated values.

x	200	400	600	800	1000
$\Psi_1(x,t)$	1.15×10^{-4}	4.07×10^{-5}	2.22×10^{-5}	1.44×10^{-5}	1.03×10^{-5}
$\Psi_2(x,t)$	0.84×10^{-4}	3.3×10^{-5}	2×10^{-5}	1.4×10^{-5}	1.0×10^{-5}
$\Psi_2(x,t)/\Psi_1(x,t)$	0.73	0.81	0.90	0.97	0.97

4. Lemmas

Lemma 1. *Under the conditions of Theorem 1, it holds uniformly for $t\in\Lambda_T$ that, for every $j>i\geq 1$,*

$$P\Big(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>x,X_je^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\leq t]}>x\Big)=o(1)\bar{F}(x)P(\tau_i\leq t).$$

Proof. According to (13), we have by Hölder's inequality that

$$\sup_{t\in(0,T)}Ee^{-l\xi(t)}\leq\sup_{t\in(0,T)}\Big(Ee^{-\kappa\xi(t)}\Big)^{l/\kappa}\leq H^{l/\kappa},\quad 0<l\leq\kappa. \tag{15}$$

Then for every $t\in\Lambda_T$,

$$E\Big(e^{-l\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}\Big)=\int_0^t Ee^{-l\xi(s)}P(\tau_i\in ds)\leq P(\tau_i\leq t)\sup_{s\in(0,t]}Ee^{-l\xi(s)}\leq H^{\frac{l}{\kappa}}P(\tau_i\leq t). \tag{16}$$

For any $\varepsilon>0$ and $\alpha<p<\kappa$, there exists large $L>1$ satisfying $L^{-\alpha}+L^{-\kappa(1-p/\kappa)}<\varepsilon$. Then for $i<j$,

$$\begin{aligned}&P\Big(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>x,X_je^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\leq t]}>x\Big)\\&\leq P\Big(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>x,X_je^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\leq t]}>x,e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}\leq L,e^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\leq t]}\leq L\Big)\\&\quad+P\Big(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>x,e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>L\Big)+P\Big(X_je^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\leq t]}>x,e^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\leq t]}>L\Big)\\&:=M_1+M_2+M_3.\end{aligned}$$

For M_1, we obtain that

$$\begin{aligned}M_1&\leq P\Big(X_i>\frac{x}{L},X_j>\frac{x}{L},e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}\leq L,e^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\leq t]}\leq L\Big)\\&\leq P\Big(X_i>\frac{x}{L}\Big)P(\tau_i\leq t)\sim L^{-\alpha}\bar{F}(x)P(\tau_i\leq t)\leq\varepsilon\bar{F}(x)P(\tau_i\leq t)\end{aligned}$$

holds uniformly for $t\in\Lambda_T$.

For M_2, by (9), Hölder's inequality, Markov's inequality and (16), the following relation holds uniformly for $t\in\Lambda_T$ and large x:

$$\begin{aligned}M_2&=E\Big\{\mathbb{I}_{[e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>L]}P\Big(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>x|e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}\Big)\Big\}\\&\leq C\bar{F}(x)E\Big(e^{-p\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}\mathbb{I}_{[e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>L]}\Big)\\&\leq C\bar{F}(x)\Big[E\Big(e^{-\kappa\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}\Big)\Big]^{p/\kappa}\Big[P(e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}>L)\Big]^{1-p/\kappa}\\&\leq C\bar{F}(x)L^{-\kappa(1-p/\kappa)}E\Big(e^{-\kappa\xi(\tau_i)}\mathbb{I}_{[\tau_i\leq t]}\Big)\\&\leq CHL^{-\kappa(1-p/\kappa)}\bar{F}(x)P(\tau_i\leq t)\leq C\varepsilon\bar{F}(x)P(\tau_i\leq t).\end{aligned}$$

Similarly, M_3 can be proved. That is to say, $M_3\leq C\varepsilon\bar{F}(x)P(\tau_j\leq t)\leq C\varepsilon\bar{F}(x)P(\tau_i\leq t)$ holds uniformly for $t\in\Lambda_T$. This completes the proof. □

Lemma 2. *Under the conditions of Theorem 1, for integer n, it holds uniformly for $t \in \Lambda_T$ that*

$$P\left(\sum_{i=1}^{n} X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x\right) \sim \sum_{i=1}^{n} P\left(X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x\right) \sim \bar{F}(x) \int_0^t E e^{-\alpha \xi(s)} \sum_{i=1}^{n} P(\tau_i \in ds).$$

Proof. From (12), there exists a constant $0 < u_0 < 1$ such that

$$\sup_{t \in (0,T]} P(\xi(t) > A) \le 1 - u_0$$

holds for $T \in \Lambda$ and large A. Then for $0 < l \le \kappa$ and $t \in \Lambda_T$,

$$\begin{aligned} \inf_{s \in (0,t]} E e^{-l\xi(s)} & \ge \inf_{s \in (0,t]} E\left(e^{-l\xi(s)} \mathbb{I}_{[e^{-l\xi(s)} \ge e^{-lA}]}\right) \\ & \ge e^{-lA} \cdot \inf_{s \in (0,t]} P\left(e^{-l\xi(s)} \ge e^{-lA}\right) \\ & = e^{-lA} \cdot \inf_{s \in (0,t]} P(\xi(s) \le A) \\ & = e^{-lA} \cdot \left[1 - \sup_{s \in (0,t]} P(\xi(s) > A)\right] \\ & \ge u_0 e^{-lA}. \end{aligned} \tag{17}$$

Therefore,

$$E\left(e^{-l\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]}\right) = \int_0^t E e^{-l\xi(s)} P(\tau_i \in ds) \ge \inf_{s \in (0,t]} E e^{-l\xi(s)} P(\tau_i \le t) \ge u_0 e^{-lA} P(\tau_i \le t). \tag{18}$$

By Tang and Yang [13] (Lemma 5.3), (16) and (18), the following relation holds uniformly for $t \in \Lambda_T$ and fixed $i \ge 1$:

$$P\left(X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x\right) \sim \bar{F}(x) \int_0^t E e^{-\alpha \xi(s)} P(\tau_i \in ds) \asymp \bar{F}(x) P(\tau_i \le t). \tag{19}$$

To prove this lemma, we know from (19) that it suffices to prove

$$P\left(\sum_{i=1}^{n} X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x\right) \sim \sum_{i=1}^{n} P\left(X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x\right).$$

Firstly, we shall show the upper bound of $P\left(\sum_{i=1}^{n} X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x\right)$. Due to $F \in \mathcal{R}_{-\alpha}$, there exists a constant $0 < v_0 < 1$ satisfying $(1 - v_0)^{-\alpha} < 1 + \varepsilon$ for any $\varepsilon > 0$. Denote

$$A = \bigcup_{j=1}^{n} \left(X_j e^{-\xi(\tau_j)} \mathbb{I}_{[\tau_j \le t]} > (1 - v_0) x\right).$$

Then by (19) and Lemma 1, the following relation holds uniformly for $t \in \Lambda_T$:

$$\begin{aligned} & P\left(\sum_{i=1}^{n} X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x\right) = P\left(\sum_{i=1}^{n} X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x, A \bigcup A^c\right) \\ & \le \sum_{j=1}^{n} P\left(X_j e^{-\xi(\tau_j)} \mathbb{I}_{[\tau_j \le t]} > (1 - v_0) x\right) \\ & \quad + P\left(\sum_{i=1}^{n} X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > x, \bigcap_{j=1}^{n} \left(X_j e^{-\xi(\tau_j)} \mathbb{I}_{[\tau_j \le t]} \le (1 - v_0) x\right)\right) \end{aligned}$$

$$\begin{aligned}
&\sim \sum_{j=1}^{n}(1-v_0)^{-\alpha}\bar{F}(x)\int_0^t Ee^{-\alpha\xi(s)}P(\tau_j\in ds)\\
&\quad + P\left(\sum_{i=1}^{n}X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x,\bigcap_{j=1}^{n}\left(X_je^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\le t]}\le(1-v_0)x\right)\right)\\
&\le(1+\varepsilon)\sum_{i=1}^{n}\bar{F}(x)\int_0^t Ee^{-\alpha\xi(s)}P(\tau_i\in ds)\\
&\quad+\sum_{1\le i\ne j\le n}P\left(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>\frac{v_0x}{n},X_je^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\le t]}>\frac{v_0x}{n}\right)\\
&\le(1+C\varepsilon)\sum_{i=1}^{n}P\left(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x\right).
\end{aligned}$$

We next show the lower bound of $P\left(\sum_{i=1}^n X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x\right)$. By Lemma 1 and (19), it holds uniformly for $t\in\Lambda_T$ that

$$\begin{aligned}
&P\left(\sum_{i=1}^{n}X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x\right)\ge P\left(\bigcup_{i=1}^{n}\left(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x\right)\right)\\
&\ge\sum_{i=1}^{n}P\left(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x\right)-\sum_{1\le j<i\le n}P\left(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x,X_je^{-\xi(\tau_j)}\mathbb{I}_{[\tau_j\le t]}>x\right)\\
&\sim\sum_{i=1}^{n}P\left(X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x\right).
\end{aligned}$$

This completes the proof. □

5. Proof of Theorem 1

Proof. Notice that $\sum_{i=1}^{\infty}i^q[P(\theta\le t)]^{i-1}<\infty$ for $q\ge0$ holds uniformly for $t\in\Lambda_T$. Then there exists $n_1>0$ such that $\sum_{i=n_1+1}^{\infty}i^q[P(\theta\le t)]^{i-1}<\varepsilon$ for any $\varepsilon>0$. Then by (18), we obtain

$$\begin{aligned}
\sum_{i=n_1+1}^{\infty}i^qP(\tau_i\le t)&\le P(\tau_1\le t)\textstyle\sum_{i=n_1+1}^{\infty}i^q[P(\theta\le t)]^{i-1}\\
&\le C\varepsilon P(\tau_1\le t)\\
&\le C\varepsilon\textstyle\int_0^tEe^{-\alpha\xi(s)}P(\tau_1\in ds).
\end{aligned}\tag{20}$$

We formulate the uniform asymptotic formula of $\Psi(x,t)$ into two steps. Firstly, we prove the upper bound version of the relation (11). Since $F\in\mathcal{R}_{-\alpha}$ for some $0<\alpha<\infty$, there exists a constant v_1 satisfying $0<(1-v_1)^{-\alpha}<1+\varepsilon$. Then we obtain

$$\begin{aligned}
\Psi(x,t)&\le P\left(\sum_{i=1}^{\infty}X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>x\right)\\
&\le P\left(\sum_{i=1}^{n_1}X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>(1-v_1)x\right)+P\left(\sum_{i=n_1+1}^{\infty}X_ie^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}>v_1x\right)\\
&:=I_1+I_2.
\end{aligned}\tag{21}$$

For I_1, by Lemma 2, the following relation holds uniformly for $t\in\Lambda_T$:

$$I_1\sim(1-v_1)^{-\alpha}\bar{F}(x)\int_0^tEe^{-\alpha\xi(s)}\sum_{i=1}^{n_1}P(\tau_i\in ds)\le(1+\varepsilon)\bar{F}(x)\int_0^tEe^{-\alpha\xi(s)}d\lambda(s).$$

For I_2, we have by (9), (16), and (21) that, for $\alpha < p < \kappa$,

$$\begin{aligned}
I_2 &\le P\left(\sum_{i=n_1+1}^{\infty} X_i e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]} > \sum_{i=n_1+1}^{\infty}\frac{v_1 x}{2i^2}\right) \\
&\le \sum_{i=n_1+1}^{\infty} P\left(X_i e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]} > \frac{v_1 x}{2i^2}\right) = \sum_{i=n_1+1}^{\infty} E\left\{P\left(X_i e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]} > \frac{v_1 x}{2i^2}\Big| e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}\right)\right\} \\
&\le C\bar{F}(x)\sum_{i=n_1+1}^{\infty}\left\{v_1^{-p}2^p i^{2p} E\left(e^{-p\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}\right) + E\left(\mathbb{I}_{[e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]}<\frac{v_1}{2i^2}]}\right)\right\} \\
&\le C\bar{F}(x)\left(\sum_{i=n_1+1}^{\infty} i^{2p}P(\tau_i\le t) + \sum_{i=n_1+1}^{\infty}P(\tau_i\le t)\right) \\
&\le C\varepsilon\bar{F}(x)\int_0^t Ee^{-\alpha\xi(s)}P(\tau_1\in ds) \\
&\le C\varepsilon\bar{F}(x)\int_0^t Ee^{-\alpha\xi(s)}d\lambda(s)
\end{aligned}$$

holds uniformly for $t \in \Lambda_T$. Therefore, we obtain

$$\Psi(x,t) \le (1+C\varepsilon)\bar{F}(x)\int_0^t Ee^{-\alpha\xi(s)}d\lambda(s)$$

uniformly for $t \in \Lambda_T$.

Next we prove the lower bound version of the relation (11). Choose sufficiently large $A' > 0$ satisfying $(A')^{-(1-p/\kappa)} < \varepsilon$ for $\alpha < p < \kappa$. Then

$$\begin{aligned}
\Psi(x,t) &\ge P\left(\sum_{i=1}^{\infty} X_i e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]} - c\int_0^t e^{-\xi(s)}ds > x\right) \\
&\ge P\left(\sum_{i=1}^{n_1} X_i e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]} > x + cA', \int_0^t e^{-\xi(s)}ds \le A'\right) \\
&= P\left(\sum_{i=1}^{n_1} X_i e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]} > x + cA'\right) - P\left(\sum_{i=1}^{n_1} X_i e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]} > x + cA', \int_0^t e^{-\xi(s)}ds > A'\right) \\
&:= I_1' - I_2'.
\end{aligned} \tag{22}$$

For I_1', choose $0 < v_2 < 1$ satisfying $(1+v_2)^{-\alpha} \ge 1-\varepsilon$ and $v_2 x > cA'$ for large x. Then by Lemma 2, the following relation holds uniformly for $t \in \Lambda_T$:

$$\begin{aligned}
I_1' &\ge P\left(\sum_{i=1}^{n_1} X_i e^{-\xi(\tau_i)}\mathbb{I}_{[\tau_i\le t]} > (1+v_2)x\right) \\
&\sim (1+v_2)^{-\alpha}\bar{F}(x)\int_0^t Ee^{-\alpha\xi(s)}\sum_{i=1}^{n_1}P(\tau_i\in ds) \\
&\ge (1-\varepsilon)\bar{F}(x)\int_0^t Ee^{-\alpha\xi(s)}\left(\sum_{i=1}^{\infty} - \sum_{i=n_1+1}^{\infty}\right)P(\tau_i\in ds) \\
&\ge (1-C\varepsilon)\bar{F}(x)\int_0^t Ee^{-\alpha\xi(s)}d\lambda(s),
\end{aligned}$$

where the last step is due to

$$\sum_{i=n_1+1}^{\infty} P(\tau_i\le t) \le C\varepsilon\int_0^t Ee^{-\alpha\xi(s)}P(\tau_1\in ds) \le C\varepsilon\int_0^t Ee^{-\alpha\xi(s)}d\lambda(s)$$

by (21).

For I_2', by (9), Hölder's inequality, Markov's inequality, (15), (14), and (18), it holds uniformly for $t \in \Lambda_T$ that

$$\begin{aligned}
I_2' &\le \sum_{i=1}^{n_1} P\left(X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t]} > \frac{x}{n_1}, \int_0^t e^{-\xi(s)} ds > A'\right) \\
&= \sum_{i=1}^{n_1} P\left(X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t, \int_0^t e^{-\xi(s)} ds > A']} > \frac{x}{n_1}\right) \\
&= \sum_{i=1}^{n_1} E\left\{P\left(X_i e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t, \int_0^t e^{-\xi(s)} ds > A']} > \frac{x}{n_1} \Big| e^{-\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t, \int_0^t e^{-\xi(s)} ds > A']}\right)\right\} \\
&\le C\bar{F}(x) \sum_{i=1}^{n_1} E\left(n_1^p e^{-p\xi(\tau_i)} \mathbb{I}_{[\tau_i \le t, \int_0^t e^{-\xi(s)} ds > A']} + \mathbb{I}_{[\tau_i \le t, \int_0^t e^{-\xi(s)} ds > A']}\right) \\
&= C\bar{F}(x) \sum_{i=1}^{n_1} \int_0^t E\left(e^{-p\xi(s)} \mathbb{I}_{[\int_0^t e^{-\xi(u)} du > A']}\right) P(\tau_i \in ds) \\
&\quad + C\bar{F}(x) \sum_{i=1}^{n_1} P(\tau_i \le t) P\left(\int_0^t e^{-\xi(s)} ds > A'\right) \\
&\le C\bar{F}(x) \sum_{i=1}^{n_1} \int_0^t \left(E e^{-\kappa\xi(s)}\right)^{p/\kappa} (A')^{-(1-p/\kappa)} \left[E\left(\int_0^t e^{-\xi(s)} ds\right)\right]^{1-p/\kappa} P(\tau_i \in ds) \\
&\quad + C\bar{F}(x)(A')^{-1} E\left(\int_0^t e^{-\xi(s)} ds\right) \sum_{i=1}^{n_1} P(\tau_i \le t) \\
&\le C\varepsilon\bar{F}(x) \sum_{i=1}^{n_1} P(\tau_i \le t) = C\varepsilon\bar{F}(x) \sum_{i=1}^{n_1} \int_0^t P(\tau_i \in ds) \\
&\le C\varepsilon\bar{F}(x) \int_0^t E e^{-\alpha\xi(s)} d\lambda(s).
\end{aligned}$$

Therefore, we have that $\Psi(x,t) \ge (1 - C\varepsilon)\bar{F}(x) \int_0^t E e^{-\alpha\xi(s)} d\lambda(s)$ holds uniformly for $t \in \Lambda_T$. Consequently,

$$\Psi(x,t) \sim \bar{F}(x) \int_0^t E e^{-\alpha\xi(s)} d\lambda(s),$$

which together with (10) ends the proof. □

6. Future Work

An insurance risk model with investment returns described by the CIR model—namely, insurance risk model (2)—was investigated in this paper. With arbitrary dependent structure between the claim sizes, we established the uniform asymptotic relation of finite-time ruin probability for the insurance risk model (2), while in a more practical and real situation, the investment return process is rather wider than these investment return processes mentioned above (constant force of interest, geometric Lévy process, and CIR model). Therefore, future research will focus on whether the investment return process can be extended to the càdlàg (right-continuous with left limits) process. Furthermore, we attempt to find conditions for the càdlàg process, which can guarantee uniform asymptotic estimates for ruin probabilities. Furthermore, these conditions are weak enough to be satisfied by many important stochastic processes, including the Lèvy process and the CIR model.

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