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On the Positive Decreasing Solutions of Half-Linear Delay Differential Equations of Even Order

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Abstract: In this paper, we derive new properties for the decreasing positive solutions of half-linear delay differential equations of even order. The positive-decreasing solutions have a great influence on the study of qualitative properties, which include oscillation, convergence, etc.; therefore, we take care of finding sufficient conditions to exclude these solutions. In addition, we present new criteria for testing the oscillation of the studied equation.

Keywords: differential equations of even order; decreasing positive solutions; oscillatory behavior; non-canonical

MSC: 34C10; 34K11

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1. Introduction

Differential equations were and still are the link between mathematics and real-life phenomena. In different branches of science, differential equations are used to model and describe life phenomena and problems, see, for example, [1–4]. One of the main branches of differential equations is the functional differential equations, from which the delay differential equations (DDE) are derived. DDEs are a better way to describe phenomena as they take into account the phenomenon's memory. It is worth noting that the even-order differential equations appear in mathematical models of various physical, biological, and chemical phenomena, including, for instance, problems of elasticity, deformation of structures, or soil settlement.

The study of the asymptotic and oscillatory behavior of solutions of DDEs has been and still attracts the attention of many researchers. Finding adequate conditions to guarantee that all differential equation solutions oscillate is one of the main objectives of oscillation theory. Ladas [5] is one of the earliest monographs on oscillation theory, covering the findings up until 1984. This book's main focus is on how deviating arguments affect the oscillation of solutions; neutral delay equations are not discussed in this book. It should be noted that monographs [6,7] are important and vital contributions to the oscillation theory. For further results, techniques, and references, monographs [8,9] covered and summarized many of the findings reported in the literature up until the past ten years.

Recently, it is easy to notice the many developments in the methods and techniques for studying the oscillatory behavior of second-order DDEs, see [10–16]. While the differential equations of higher-order have been studied with development as well, it is much less compared with what the equations of the second order have enjoyed.

This study aims to derive new criteria for testing the oscillation of the non-canonical DDE

$$\left(a(t)\cdot\left(v^{(n-1)}(t)\right)^{k}\right)'+\rho(t)\cdot\left(v^{k}(h(t))\right)=0, \tag{1}$$

where $t\geq t_0$, $n\geq 4$ is an even natural number, $k\geq 1$ is a result of dividing two odd natural numbers, and the following assumptions hold:

P1: $a,h\in C^1([t_0,\infty),(0,\infty))$, $\rho\in C([t_0,\infty),[0,\infty))$, $h'(t)\geq 0$, $h(t)\leq t$, and

$$\int_{t_0}^{\infty}a^{-1/k}(v)\mathrm{d}v<\infty.$$

A continuous function u that is differentiable n times is said to be a solution to Equation (1) if it satisfies this equation. In this study, we consider only non-zero solutions as well as those that do not vanish eventually. A solution of a differential equation is called an oscillatory solution if it has arbitrary large zeros. Other than that, it is called non-oscillatory.

Notation 1. *For the sake of brevity, we define the following operator and functions:*

$$\Lambda[F;\ u,v]=\int_u^v F(s)\mathrm{d}s,$$

$$\eta_0(t):=\Lambda\left[a^{-1/k};\ t,\infty\right]\ \textit{and}\ \eta_{s+1}=\Lambda[\eta_s;\ t,\infty]\ \textit{for}\ s=0,1,\ldots,n-3.$$

In both the delay and advanced cases, Koplatadze et al. [17] found asymptotic classification for non-oscillatory solutions of the equation

$$v^{(n)}+\rho\cdot(v\circ h)=0,\ t\geq 0. \tag{2}$$

One of their results for the delay case is that they prove that Equation (2) oscillates if

$$\limsup_{t\to\infty}\left[h\cdot\Lambda\left[h^{n-2}\cdot\rho;\ t,\infty\right]+\Lambda\left[h^{n-2}\cdot\rho;\ h,t\right]\right.$$
$$\left.+\frac{1}{h}\Lambda\left[v\cdot h^{n-1}\cdot\rho;\ 0,h\right]\right]>(n-1)!.$$

In 2011, Agarwal et al. [18] checked the oscillatory behavior of the solutions of the equation.

$$\left(\left|v^{(n-1)}\right|^{k-1}\cdot v^{(n-1)}\right)'+F(t,v\circ h)=0, \tag{3}$$

in the canonical case, where $k>0$, $F\in C^1([t_0,\infty)\times\mathbb{R})$, $vF(t,v)>0$ and $F(t,v)$ sgn $v\geq\rho(t)|v|^{\beta}$ for $v\neq 0$. Using comparison and Riccati substitution techniques, they obtained various forms of criteria that ensure the oscillation of all solutions. Moreover, they presented some oscillation condition for the more general equations

$$\left(\left|v^{(n-1)}\right|^{k-1}\cdot v^{(n-1)}\right)'+F\left(t,v\circ h,v'\circ g\right)=0,$$

and

$$\left(\left|z^{(n-1)}\right|^{k-1}\cdot z^{(n-1)}\right)'+F(t,v\circ h)=0,$$

where $z:=v+p\cdot(v\circ\tau)$. Using the averaging technique, Xu and Xia [19] established some oscillation conditions for Equation (3).

In the even-order equations, the study of the non-canonical case differs from the canonical case in the probability of the existence of decreasing positive solutions, and this requires the presence of an additional condition to exclude it. In the canonical and non-canonical cases, Baculíková et al. [20] used the comparison technique to test the oscillation of the solutions of the equation

$$(a\cdot(v^{(n-1)})^k)' + \rho\cdot(G\circ v\circ h) = 0, \tag{4}$$

where $G \in C([t_0,\infty),\mathbb{R})$, $G'(t) \geq 0$, and $G(uv) \geq G(u)G(v)$ for $uv > 0$. They excluded decreasing positive solutions by assuming that there is a $\varrho \in C^1([t_0,\infty))$ with $\varrho(t) > t$, $\varrho'(t) \geq 0$ and $(\varrho_{n-2}\circ h) < t$ such that

$$u' + a^{-1/k}\cdot(\Lambda[\rho;\ t_0,t])^{1/k}\cdot(L_{n-2}\circ h)\cdot(u\circ\varrho_{n-2}\circ h) = 0 \tag{5}$$

is oscillatory, where

$$\varrho_1 = \varrho,\ \varrho_{i+1} = \varrho_i\circ\varrho,\ L_1 = \varrho - t \text{ and } L_{i+1}(t) = \Lambda[L_i;\ t,\varrho],$$

for $i = 1,2,\ldots,n-3$. Later, in 2013, Zhang et al. [21] set criteria for the oscillation of Equation (4) when $G(u) = u^k$, where they excluded decreasing positive solutions using the condition

$$\limsup_{t\to\infty}\Lambda\left[\left(\rho\cdot\rho^k - \frac{(k/(k+1))^{k+1}(\rho'_+)^{k+1}}{\rho\cdot\rho_1^k}\right);\ t_0,t\right] = \infty, \tag{6}$$

where

$$\rho(t) = \frac{1}{(n-3)!}\Lambda\left[(v-t)^{n-3}\cdot\eta_0;\ t,\infty\right]$$

and

$$\rho_1(t) = \frac{1}{(n-4)!}\Lambda\left[(v-t)^{n-4}\cdot\eta_0;\ t,\infty\right].$$

Very recently, using the comparison technique with a first-order equation, Muhib et al. [22] introduced a standard that ensures that there are no positive decreasing solutions to (1). They proved that if there is a $\epsilon_0 > 0$ such that $\eta_{n-2}^{k+1}\cdot\rho \geq \epsilon_0 k\eta_{n-3}(t)$, and

$$\liminf_{t\to\infty}\Lambda\left[\rho\cdot\eta_{n-2}^k;\ h,t\right] > \frac{k(1-\epsilon_0)}{\mathrm{e}},$$

then there are no positive decreasing solutions to Equation (1). By using a generalized Riccati substitution, Moaaz et al. [23] presented a new criterion for oscillation of solutions of fourth-order quasi-linear differential equations

$$\left(a(t)(v'''(t))^k\right)' + f(t,v(h(t))) = 0$$

in the non-canonical case. They utilized an approach that gives rise to two or three independent conditions, eliminating non-oscillatory solutions. Moreover, they establish conditions in a non-traditional form ($\limsup(\cdot) > 1$), while condition ($\limsup(\cdot) = \infty$) cannot be applied.

Nabih et al. [24] iteratively deduced new monotonic properties of (). They discuss the non-canonical case in which there are possible decreasing positive solutions. Then, they found iterative criteria that exclude the existence of these positive decreasing solutions. Using these new criteria and based on the comparison and Riccati substitution methods, they created sufficient conditions to ensure that all solutions of the studied equation oscillate.

In this paper, we extend Koplatadze's results to DDEs in the non-canonical case. We begin by setting conditions that exclude positive decreasing solutions to the studied equation. Then, we derive some new monotonic properties through which we can find improved relationships between the solution with delay and without delay. In addition, we present additional standards that ensure that all solutions to the investigated equation oscillate. Finally, we apply the new results to some special cases to clarify their importance and applicability.

2. Positive Decreasing Solutions

In this section, we obtain a new condition that ensures that all positive decreasing solutions of (1) are excluded. For this, we need the following lemma, which presents nontraditional monotonic properties of the derivatives of decreasing positive solutions.

Lemma 1. *Assume that v is an eventually positive decreasing solution of (1). Then, eventually,*

$$(-1)^s \left(\frac{v^{(n-2-s)}(t)}{\eta_s(t)} \right)' \geq 0, \text{ for } s = 0, 1, \ldots, n-2. \tag{7}$$

Proof. Suppose that v is an eventually positive decreasing solution of (1). Since v is eventually positive and $h \to \infty$ as $t \to \infty$, we have that $v \circ h$ is also positive. Thus, from (1),

$$\left(a \cdot \left(v^{(n-1)} \right)^k \right)' \leq 0. \tag{8}$$

According to Lemma 1.1 in [25], which classifies the signs of derivatives of positive solutions, the class of decreasing positive solutions achieves

$$(-1)^s v^{(s)}(t) > 0, \text{ for } s = 0, 1, \ldots, n-1. \tag{9}$$

Using (8) and (9), we get

$$\begin{aligned} -v^{(n-2)}(t) &\leq \Lambda\left[v^{(n-1)};\, t, \infty \right] \\ &= \Lambda\left[a^{-1/k} \cdot \left(a \cdot \left(v^{(n-1)} \right)^k \right)^{1/k};\, t, \infty \right] \\ &\leq a^{1/k}(t) v^{(n-1)}(t) \eta_0(t). \end{aligned} \tag{10}$$

Therefore,

$$\left(\frac{v^{(n-2)}}{\eta_0} \right)' = \frac{\eta_0 \cdot v^{(n-1)} + a^{-1/k} \cdot v^{(n-2)}}{\eta_0^2} \geq 0,$$

which leads to

$$\begin{aligned} -v^{(n-3)}(t) &\geq \Lambda\left[v^{(n-2)};\, t, \infty \right] = \Lambda\left[\eta_0 \frac{v^{(n-2)}}{\eta_0};\, t, \infty \right] \\ &\geq \frac{v^{(n-2)}(t)}{\eta_0(t)} \eta_1(t). \end{aligned}$$

Then,

$$\left(\frac{v^{(n-3)}}{\eta_1} \right)' = \frac{\eta_1 \cdot v^{(n-2)} + \eta_0 \cdot v^{(n-3)}}{\eta_1^2} \leq 0.$$

Repeating the same approach, we get the desired result. □

Theorem 1. *If*

$$\limsup_{t\to\infty}\left[(\eta_{n-2}\circ h)\Lambda\left[\frac{\rho}{\eta_{n-2}^{1-k}};\ t_0,h\right]+\Lambda\left[\eta_{n-2}^{k}\cdot\rho;\ h,t\right]\right.$$
$$\left.+\frac{1}{(\eta_{n-2}\circ h)}\Lambda\left[\eta_{n-2}^{k}\cdot\rho\cdot(\eta_{n-2}\circ h);\ t,\infty\right]\right]>k, \tag{11}$$

then there are no eventually positive decreasing solutions of Equation (1).

Proof. Suppose the contrary that v is an eventually positive decreasing solution of (1). As in the proof of Lemma 1, we get that (9) and (10) hold. Applying the operator $\Lambda[\ \cdot\ ;\ t,\infty]$ $(n-3)$ times to (10), we arrive at

$$v'\leq a^{1/k}\cdot v^{(n-1)}\cdot\eta_{n-3}. \tag{12}$$

By applying $\Lambda[\ \cdot\ ;\ t,\infty]$ one more time, we obtain

$$v\geq -a^{1/k}\cdot v^{(n-1)}\cdot\eta_{n-2}. \tag{13}$$

Now, let $w:=a^{1/k}\cdot v^{(n-1)}\cdot\eta_{n-2}+v$, which from (13) gives $w\geq 0$. Then,

$$w'=\left(a^{1/k}\cdot v^{(n-1)}\right)'\cdot\eta_{n-2}-a^{1/k}\cdot v^{(n-1)}\cdot\eta_{n-3}+v'.$$

Hence, from (12), we arrive at

$$w'\leq\left(a^{1/k}\cdot v^{(n-1)}\right)'\cdot\eta_{n-2}\leq 0. \tag{14}$$

From (1) and (13), we find

$$\begin{aligned}-\rho\cdot\left(v^{k}\circ h\right) &= \left(\left(a^{1/k}\cdot v^{(n-1)}\right)^{k}\right)' \\ &= k\left(a^{1/k}\cdot v^{(n-1)}\right)^{k-1}\left(a^{1/k}\cdot v^{(n-1)}\right)' \\ &\geq k\frac{v^{k-1}\circ h}{\eta_{n-2}^{k-1}}\left(a^{1/k}\cdot v^{(n-1)}\right)',\end{aligned}$$

and so,

$$\left(a^{1/k}\cdot v^{(n-1)}\right)'\leq -\frac{1}{k}\eta_{n-2}^{k-1}\cdot\rho\cdot(v\circ h). \tag{15}$$

Applying $\Lambda[\ \cdot\ ;\ t_1,t]$ to (15), we have

$$a^{1/k}(t)v^{(n-1)}(t)\leq -\frac{1}{k}\Lambda\left[\eta_{n-2}^{k-1}\cdot\rho\cdot(v\circ h);\ t_1,t\right]. \tag{16}$$

Combining (14) and (15), we get

$$w'\leq -\frac{1}{k}\eta_{n-2}^{k}\cdot\rho\cdot(v\circ h).$$

By applying $\Lambda[\ \cdot\ ;\ t,\infty]$ to the last inequality, we obtain

$$w(t)\geq\frac{1}{k}\Lambda\left[\eta_{n-2}^{k}\cdot\rho\cdot(v\circ h);\ t,\infty\right],$$

which, from the definition of w, yields

$$v(t) \geq -a^{1/k}(t)v^{(n-1)}(t)\eta_{n-2}(t) + \frac{1}{k}\Lambda\Big[\eta_{n-2}^{k} \cdot \rho \cdot (v \circ h);\ t, \infty\Big].$$

Hence, from (16), we arrive at

$$v(t) \geq \frac{1}{k}\eta_{n-2}(t)\Lambda\Big[\eta_{n-2}^{k-1} \cdot \rho \cdot (v \circ h);\ t_1, t\Big] + \frac{1}{k}\Lambda\Big[\eta_{n-2}^{k} \cdot \rho \cdot (v \circ h);\ t, \infty\Big],$$

and, thus,

$$\begin{aligned} k\, v \circ h \ &\geq\ (\eta_{n-2} \circ h)\Lambda\Big[\eta_{n-2}^{k-1} \cdot \rho \cdot (v \circ h);\ t_1, h\Big] + \Lambda\Big[\eta_{n-2}^{k} \cdot \rho \cdot (v \circ h);\ h, \infty\Big] \\ &\geq\ (\eta_{n-2} \circ h)\Lambda\Big[\eta_{n-2}^{k-1} \cdot \rho \cdot (v \circ h);\ t_1, h\Big] + \Lambda\Big[\eta_{n-2}^{k} \cdot \rho \cdot (v \circ h);\ h, t\Big] \\ &\quad + \Lambda\Big[\eta_{n-2}^{k} \cdot \rho \cdot (v \circ h);\ t, \infty\Big]. \end{aligned} \tag{17}$$

From Lemma 1, we have that (7) holds. It follows form (7) that

$$(v \circ h)(u) \geq \frac{(\eta_{n-2} \circ h)(u)}{(\eta_{n-2} \circ h)(t)}(v \circ h)(t).$$

Thus, (17) becomes

$$\begin{aligned} k \ &\geq\ (\eta_{n-2} \circ h)\Lambda\Big[\eta_{n-2}^{k-1} \cdot \rho;\ t_1, h\Big] + \Lambda\Big[\eta_{n-2}^{k} \cdot \rho;\ h, t\Big] \\ &\quad + \frac{1}{(\eta_{n-2} \circ h)}\Lambda\Big[\eta_{n-2}^{k} \cdot \rho \cdot (\eta_{n-2} \circ h);\ t, \infty\Big], \end{aligned}$$

which contradicts assumption (11). □

It is easy to notice the importance of the monotonic properties of the positive solution in obtaining the exclusion criterion. By extending the approach followed in [10], these properties can be iteratively improved, as in the following lemmas.

Lemma 2. *Assume that v is an eventually positive decreasing solution of (1). If there exists a $\delta_0 > 0$ such that*

$$\rho(t) \geq k\delta \frac{\eta_{n-3}(t)}{\eta_{n-2}^{k+1}(t)}, \tag{18}$$

then, eventually,

$$\left(\frac{v}{\eta_{n-2}^{\delta}}\right)' \leq 0.$$

Proof. Suppose that v is an eventually positive decreasing solution of (1). From Lemma 1 and its proof, we get that (7), (9), and (10) hold.
First, we need to prove that $\lim_{t\to\infty} v(t) = 0$. From the fact that $v(t) > 0$ and $v'(t) < 0$, we have that $\lim_{t\to\infty} v(t) = c \geq 0$. Now, we suppose that $c > 0$. Then, $(v \circ h)(t) \geq c$, for t large enough. We can see that (15) and (16) hold if we proceed as we did in the proof of Theorem 1. From (16) and (18), we arrive at

$$\begin{aligned} a^{1/k}(t)v^{(n-1)}(t) \ &\leq\ -\frac{c}{k}\Lambda\Big[\eta_{n-2}^{k-1} \cdot \rho;\ t_1, t\Big] \\ &\leq\ -c\delta\Lambda\left[\frac{\eta_{n-3}(t)}{\eta_{n-2}^{2}(t)};\ t_1, t\right] \\ &=\ -c\delta\left(\frac{1}{\eta_{n-2}(t)} - \frac{1}{\eta_{n-2}(t_1)}\right). \end{aligned} \tag{19}$$

Since $\lim_{t\to\infty}\eta_{n-2}^{-1}(t)=\infty$, we obtain that $\eta_{n-2}^{-1}(t)-\eta_{n-2}^{-1}(t_1)\geq l\eta_{n-2}^{-1}(t)$ for $l\in(0,1)$ and t large enough. Thus, from (12) and (19), we find

$$v'(t)\leq a^{1/k}(t)v^{(n-1)}(t)\eta_{n-3}(t)\leq -c\delta l\frac{\eta_{n-3}(t)}{\eta_{n-2}(t)},$$

and after applying $\Lambda[\,\cdot\,;\,t_1,t]$ to this inequality, we conclude that

$$v(t)\leq v(t_1)-c\delta l\ln\frac{\eta_{n-2}(t_1)}{\eta_{n-2}(t)}\to\infty \text{ as } t\to\infty,$$

which is a contradiction. Therefore,

$$\lim_{t\to\infty}v(t)=0. \tag{20}$$

Next, applying $\Lambda[\,\cdot\,;\,t_1,t]$ to (15), we have

$$\begin{aligned}a^{1/k}(t)v^{(n-1)}(t) &\leq a^{1/k}(t_1)v^{(n-1)}(t_1)-\frac{1}{k}\Lambda\Big[\eta_{n-2}^{k-1}\cdot\rho\cdot(v\circ h);\,t_1,t\Big]\\ &\leq a^{1/k}(t_1)v^{(n-1)}(t_1)-\frac{1}{k}v(t)\Lambda\Big[\eta_{n-2}^{k-1}\cdot\rho;\,t_1,t\Big],\end{aligned}$$

which from (18) yields

$$a^{1/k}(t)v^{(n-1)}(t)\leq a^{1/k}(t_1)v^{(n-1)}(t_1)-\delta v(t)\left(\frac{1}{\eta_{n-2}(t)}-\frac{1}{\eta_{n-2}(t_1)}\right). \tag{21}$$

It follows from (20) that $a^{1/k}(t_1)v^{(n-1)}(t_1)+\delta v(t)/\eta_{n-2}(t_1)\leq 0$. Then,

$$a^{1/k}\cdot v^{(n-1)}\leq -\frac{\delta v}{\eta_{n-2}}. \tag{22}$$

Combining (12) and (22), we see that

$$\eta_{n-2}\cdot v'(t)\leq \eta_{n-2}\cdot\left(a^{1/k}\cdot v^{(n-1)}\right)\cdot\eta_{n-3}\leq -\delta v\cdot\eta_{n-3}. \tag{23}$$

This implies

$$\left(\frac{v}{\eta_{n-2}^{\delta}}\right)'=\frac{\eta_{n-2}v'+\delta\eta_{n-3}v}{\eta_{n-2}^{\delta+1}}\leq 0.$$

This is the desired result. □

Lemma 3. *Let v is a solution of (1), v is eventually positive and decreasing, and there is a $\delta\in(0,1)$ such that (18) holds. If there are $m\in\mathbb{N}$ and $\lambda>1$ such that $\delta_i<\delta_{i+1}<1$ for $i=0,1,\ldots,m-1$, and $\eta_{n-2}(h(t))\geq\lambda\eta_{n-2}(t)$, then*

$$\left(\frac{v}{\eta_{n-2}^{\delta_m}}\right)'\leq 0, \tag{24}$$

eventually, where $\delta_0=\delta$ and

$$\delta_{i+1}=\frac{\delta\lambda^{\delta_i}}{1-\delta_i},\ i=0,1,\ldots,m-1.$$

Proof. Suppose that v is an eventually positive decreasing solution of (1). From the proof of Theorem 1, we get that (12), (13), and (15) hold.
First, we need to prove that $\lim_{t\to\infty}\left(v(t)\eta_{n-2}^{-\delta_0}(t)\right)=0$. From the fact that $v(t)\eta_{n-2}^{-\delta_0}(t)>0$

and $\left(v(t)\eta_{n-2}^{-\delta_0}(t)\right)' \leq 0$, we have that $\lim_{t\to\infty}\left(v(t)\eta_{n-2}^{-\delta_0}(t)\right) = c_1 \geq 0$. Now, we suppose that $c_1 > 0$. Then,

$$v(t)\eta_{n-2}^{-\delta_0}(t) \geq c_1, \tag{25}$$

for $t \geq t_1$, where t_1 large enough. Now, we define the function

$$\psi := \frac{a^{1/k} \cdot v^{(n-1)} \cdot \eta_{n-2} + v}{\eta_{n-2}^{\delta_0}},$$

which from (13) gives $\psi \geq 0$, eventually. Then,

$$\psi' = \frac{\left(a^{1/k} \cdot v^{(n-1)}\right)' \cdot \eta_{n-2} - a^{1/k} \cdot v^{(n-1)} \cdot \eta_{n-3} + v'}{\eta_{n-2}^{\delta_0}} + \frac{a^{1/k} \cdot v^{(n-1)} \cdot \eta_{n-2} + v}{\eta_{n-2}^{\delta_0+1}}\delta_0\eta_{n-3}.$$

Hence, from (12) and (15), we arrive at

$$\psi' \leq -\frac{1}{k}\frac{\eta_{n-2}^{k} \cdot \rho \cdot (v \circ h)}{\eta_{n-2}^{\delta_0}} + \delta_0\frac{\eta_{n-3} \cdot \left(a^{1/k} \cdot v^{(n-1)}\right)}{\eta_{n-2}^{\delta_0}} + \delta_0\frac{\eta_{n-3} \cdot v}{\eta_{n-2}^{\delta_0+1}}.$$

It follows from (18) that

$$\psi' \leq -\delta_0\frac{\eta_{n-3} \cdot (v \circ h)}{\eta_{n-2}^{\delta_0+1}} + \delta_0\frac{\eta_{n-3} \cdot \left(a^{1/k} \cdot v^{(n-1)}\right)}{\eta_{n-2}^{\delta_0}} + \delta_0\frac{\eta_{n-3} \cdot v}{\eta_{n-2}^{\delta_0+1}},$$

with the fact that $v'(t) < 0$ yields

$$\psi' \leq \delta_0\frac{\eta_{n-3} \cdot \left(a^{1/k} \cdot v^{(n-1)}\right)}{\eta_{n-2}^{\delta_0}}. \tag{26}$$

From the proof of Lemma 2, we obtain that (23) holds. Using (13) and (25), inequality (26) becomes

$$\psi' \leq -\delta_0^2\frac{\eta_{n-3}}{\eta_{n-2}} \cdot \frac{v}{\eta_{n-2}^{\delta_0}} \leq -\delta_0^2 c_1\frac{\eta_{n-3}}{\eta_{n-2}}. \tag{27}$$

Integrating (27) from t_1 to t, we find

$$\psi(t_1) \geq \delta_0^2 c_1 \ln\frac{\eta_{n-2}(t_1)}{\eta_{n-2}(t)}.$$

Then, $\psi(t_1) \to \infty$ as $t \to \infty$, is a contradiction. Therefore, $c_1 = 0$.

Next, applying $\Lambda[\ \cdot\ ;\ t_1, t]$ to (15), we get

$$a^{1/k}(t)v^{(n-1)}(t) \leq a^{1/k}(t_1)v^{(n-1)}(t_1) - \frac{1}{k}\Lambda\left[\eta_{n-2}^{k-1} \cdot \rho \cdot (v \circ h);\ t_1, t\right]. \tag{28}$$

From Lemma 2, we get that $\left(v(t)\eta_{n-2}^{-\delta_0}(t)\right)' \leq 0$. Thus, (28) turns into

$$\begin{aligned} a^{1/k}(t)v^{(n-1)}(t) &\leq a^{1/k}(t_1)v^{(n-1)}(t_1) - \frac{1}{k}\Lambda\left[\eta_{n-2}^{k-1} \cdot \rho \cdot \frac{\eta_{n-2}^{\delta_0} \circ h}{\eta_{n-2}^{\delta_0}} \cdot v;\ t_1, t\right] \\ &\leq a^{1/k}(t_1)v^{(n-1)}(t_1) - \frac{1}{k}\frac{v(t)}{\eta_{n-2}^{\delta_0}(t)} \cdot \Lambda\left[\eta_{n-2}^{k-1} \cdot \rho \cdot \left(\eta_{n-2}^{\delta_0} \circ h\right);\ t_1, t\right] \end{aligned}$$

with (18) gives

$$\begin{aligned} a^{1/k}(t)v^{(n-1)}(t) &\leq a^{1/k}(t_1)v^{(n-1)}(t_1) - \delta \frac{v(t)}{\eta_{n-2}^{\delta_0}(t)} \cdot \Lambda\left[\frac{\eta_{n-3}(t)}{\eta_{n-2}^{2-\delta_0}(t)} \cdot \frac{\eta_{n-2}^{\delta_0} \circ h}{\eta_{n-2}^{\delta_0}};\ t_1, t\right] \\ &\leq a^{1/k}(t_1)v^{(n-1)}(t_1) - \delta\lambda^{\delta_0} \frac{v(t)}{\eta_{n-2}^{\delta_0}(t)} \cdot \Lambda\left[\frac{\eta_{n-3}(t)}{\eta_{n-2}^{2-\delta_0}(t)};\ t_1, t\right] \\ &\leq a^{1/k}(t_1)v^{(n-1)}(t_1) - \frac{\delta\lambda^{\delta_0}}{1-\delta_0} \frac{v(t)}{\eta_{n-2}^{\delta_0}(t)} \cdot \left(\frac{1}{\eta_{n-2}^{1-\delta_0}(t)} - \frac{1}{\eta_{n-2}^{1-\delta_0}(t_1)}\right). \end{aligned}$$

Using the fact that $\lim_{t\to\infty}\left(v(t)\eta_{n-2}^{-\delta_0}(t)\right) = 0$, we find

$$a^{1/k} \cdot v^{(n-1)} \leq -\delta_1 \frac{v}{\eta_{n-2}}.$$

From (23), we arrive at $\eta_{n-2} \cdot v' \leq -\delta_1 \eta_{n-3} \cdot v$, and, hence,

$$\left(\frac{v}{\eta_{n-2}^{\delta_1}}\right)' \leq 0.$$

If $\delta_1 < 1$, then we can repeat the same previous procedures and, thus, we get (24). □

Theorem 2. *Assume that there is a $\delta \in (0,1)$ such that (18) holds. If there are $m \in \mathbb{N}$ and $\lambda > 1$ such that $\delta_i < \delta_{i+1} < 1$ for $i = 0, 1, \ldots, m-1$, $\eta_{n-2}(h(t)) \geq \lambda\eta_{n-2}(t)$, and*

$$\begin{aligned} \limsup_{t\to\infty} \Big[&\left(\eta_{n-2}^{1-\delta_m} \circ h\right) \cdot \Lambda\left[\eta_{n-2}^{k-1} \cdot \rho \cdot \left(\eta_{n-2}^{\delta_m} \circ h\right);\ t_1, h\right] \\ &+ \left(\eta_{n-2}^{-\delta_m} \circ h\right) \cdot \Lambda\left[\eta_{n-2}^{k} \cdot \rho \cdot \left(\eta_{n-2}^{\delta_m} \circ h\right);\ h, t\right] \\ &+ \left(\eta_{n-2}^{-1} \circ h\right) \cdot \Lambda\left[\eta_{n-2}^{k} \cdot \rho \cdot (\eta_{n-2} \circ h);\ t, \infty\right]\Big] > k, \end{aligned} \tag{29}$$

then there are no eventually positive decreasing solutions of Equation (1).

Proof. Suppose the contrary that v is an eventually positive decreasing solution of (1). As in the proof of Theorem 1, we arrive at (17). From the facts that $\left(v(t)\eta_{n-2}^{-1}(t)\right)' \geq 0$ and $\left(v(t)\eta_{n-2}^{-\delta_m}(t)\right)' \leq 0$, we get

$$v(h(s)) \geq \frac{\eta_{n-2}^{\delta_m}(h(s))}{\eta_{n-2}^{\delta_m}(h(t))} v(h(t)) \text{ for } s \leq t,$$

and

$$v(h(s)) \geq \frac{\eta_{n-2}(h(s))}{\eta_{n-2}(h(t))} v(h(t)) \text{ for } s \geq t.$$

Therefore, (17) becomes

$$\begin{aligned} k \geq\ &\left(\eta_{n-2}^{1-\delta_m} \circ h\right) \cdot \Lambda\left[\eta_{n-2}^{k-1} \cdot \rho \cdot \left(\eta_{n-2}^{\delta_m} \circ h\right);\ t_1, h\right] \\ &+ \left(\eta_{n-2}^{-\delta_m} \circ h\right) \cdot \Lambda\left[\eta_{n-2}^{k} \cdot \rho \cdot \left(\eta_{n-2}^{\delta_m} \circ h\right);\ h, t\right] \\ &+ \left(\eta_{n-2}^{-1} \circ h\right) \Lambda\left[\eta_{n-2}^{k} \cdot \rho \cdot (\eta_{n-2} \circ h);\ t, \infty\right], \end{aligned}$$

which contradicts assumption (29). □

3. Asymptotic Behavior and Oscillation

Based on the proofs of Lemmas 2 and 3, we can directly obtain the following result:

Lemma 4. *Assume that there exists a $\delta \in (0,1)$ such that (18) holds. If there are $m \in \mathbb{N}$ and $\lambda > 1$ such that $\delta_i < \delta_{i+1} < 1$ for $i = 0,1,\ldots,m-1$, and $\eta_{n-2}(h(t)) \geq \lambda\eta_{n-2}(t)$, then all eventually positive decreasing solution of (1) converge to zero, moreover,*

$$\lim_{t\to\infty} \frac{v(t)}{\eta_{n-2}^{\delta_m}(t)} = 0.$$

Based on the assumption that the solution v of Equation (1) is ultimately positive, according to Lemma 1.1 in [25], positive solutions can be classified into the following three classes:

(a) $v^{(i)} > 0$ for $i = 0, 1, n-1$ and $v^{(n)} < 0$;
(b) $v^{(i)} > 0$ for $i = 0, 1, n-2$ and $v^{(n-1)} < 0$;
(c) $(-1)^i v^{(i)} > 0$ for $i = 0, 1, \ldots, n-1$.

In the following, we review some of the previous results in the literature that provided criteria that ensure the oscillation of the solutions of Equation (1).

Theorem 3. *([20], Theorem 4 with $f(x) = x^k$). Assume that the DDEs*

$$\phi' + \rho \cdot \left(\frac{\ell_0 h^{n-1}}{(n-1)!(a^{1/k} \circ h)} \right)^k \cdot (\phi \circ h) = 0 \tag{30}$$

and

$$\varphi' + \frac{\ell_1}{(n-2)! a^{1/\kappa}} \cdot \Lambda^{1/k} \left[\rho \cdot \left(h^{n-2} \right)^k ;\ t_0; t \right] \cdot (\varphi \cdot h) = 0, \tag{31}$$

are oscillatory, for some $\ell_0, \ell_1 \in (0,1)$. If there is a $\varrho \in C^1([t_0,\infty))$ with $\varrho(t) > t$, $\varrho'(t) \geq 0$ and $(\varrho_{n-2} \circ h) < t$ such that the DDE (5) is oscillatory, then all solutions of (1) are oscillatory.

Theorem 4. *([21], Theorem 2.1 with $\alpha = \beta$). Assume that the DDE (30) is oscillatory, and*

$$\limsup_{t\to\infty} \Lambda \left[\left(\rho \cdot \eta_0^k \cdot \left(\frac{\ell_2 h^{n-2}}{(n-2)!} \right)^k - \frac{(k/(k+1))^{k+1}}{\eta_0 \cdot a^{1/k}} \right) ; t_0, t \right] = \infty, \tag{32}$$

for some $\ell_0, \ell_2 \in (0,1)$. If (6) holds, then every solution of (1) is oscillatory.

To obtain our oscillation criteria, we combine the criteria that exclude cases (a) and (b) of the derivatives of the positive solution with the conditions obtained in the previous section.

Theorem 5. *Assume that (11) holds, and the DDEs (30) and (31) are oscillatory, for some $\ell_0, \ell_1 \in (0,1)$. Then, all solutions of (1) are oscillatory.*

Theorem 6. *Assume that (11) and (32) hold for some $\ell_2 \in (0,1)$, and the DDE (30) is oscillatory, for some $\ell_0 \in (0,1)$. Then, all solutions of (1) are oscillatory.*

Theorem 7. *Assume that (18) holds for some $\delta \in (0,1)$, and there are $m \in \mathbb{N}$ and $\lambda > 1$ such that $\delta_i < \delta_{i+1} < 1$ for $i = 0,1,\ldots,m-1$, $\eta_{n-2}(h(t)) \geq \lambda\eta_{n-2}(t)$, and (29) holds. If the DDEs (30) and (31) are oscillatory, for some $\ell_0, \ell_1 \in (0,1)$, then all solutions of (1) are oscillatory.*

Theorem 8. *Assume that (18) holds for some $\delta \in (0,1)$, and there are $m \in \mathbb{N}$ and $\lambda > 1$ such that $\delta_i < \delta_{i+1} < 1$ for $i = 0,1,\ldots,m-1$, $\eta_{n-2}(h(t)) \geq \lambda\eta_{n-2}(t)$, and (29) holds. If (32) holds for* *some $\ell_2 \in (0,1)$, and the DDE (30) is oscillatory, for some $\ell_0 \in (0,1)$, then all solutions of (1) are oscillatory.*

Example 1. *Consider a special case of Equation (1), namely the fourth-order DDE*

$$\left(t^5\left(v'''(t)\right)\right)' + \rho_0 t v(h_0 t) = 0, \tag{33}$$

where $t \geq 1$, $\rho_0 > 0$ and $h_0 \in (0,1)$. It is easy to verify that $a(t) = t^5$, $\rho(t) = \rho_0 t$ and $h(t) = h_0 t$. Then, we have $\eta_2 = \frac{1}{36t^2}$.
If we let $\delta_m = 0.5$, then Equations (30) and (31) are oscillatory if

$$\rho_0 > \frac{16h_0^2}{\mathrm{e}\ln\frac{1}{h_0}},$$

and

$$\rho_0 > \frac{8}{h_0^2\mathrm{e}\ln\frac{1}{h_0}},$$

respectively. Condition (11) reduces to

$$\rho_0 > \frac{36}{\left(1+\ln\frac{1}{1/h_0}\right)}.$$

By Theorem 6, Equation (33) is oscillatory if

$$\rho_0 > \max\left\{\frac{16h_0^2}{\mathrm{e}\ln\frac{1}{h_0}}, \frac{8}{h_0^2\mathrm{e}\ln\frac{1}{h_0}}, \frac{36}{\left(1+\ln\frac{1}{h_0}\right)}\right\}. \tag{34}$$

Remark 1. *From Theorem 2.4 in [23] it follows that (33) is oscillatory if*

$$\rho_0 > \max\left\{12, \frac{8}{h_0^2}, \frac{6h_0^2}{\mathrm{e}\ln(1/h_0)}\right\}. \tag{35}$$

For $h_0 = 1/2$, criteria (34) and (35) reduce to $\rho_0 > 21.262$ and $\rho_0 > 32.0$, respectively. Thus, our results improve the results in [23].

4. Conclusions

Numerous phenomena can be modeled but the resulting differential equations are frequently nonlinear and cannot be solved in closed form. Therefore, it is very useful to study the qualitative properties of these equations. The study of qualitative characteristics helps understand, interpret, and analyze these phenomena.

In this work, we focus on studying some properties of positive decreasing solutions of an even-order delay differential equation. The non-canonical case was taken into account, as the canonical case guarantees that there are no positive decreasing solutions of the studied equation. A new condition that ensures that there are no decreasing positive solutions has been added. Moreover, we improved the properties of the studied separation of solutions so that we can obtain criteria of iterative nature. It would be interesting, as a future research issue, to extend the results of this paper to the neutral and advanced cases.

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