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Curvatures on Homogeneous Generalized Matsumoto Space

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Abstract: The curvature characteristics of particular classes of Finsler spaces, such as homogeneous Finsler spaces, are one of the major issues in Finsler geometry. In this paper, we have obtained the expression for S-curvature in homogeneous Finsler space with a generalized Matsumoto metric and demonstrated that the homogeneous generalized Matsumoto space with isotropic S-curvature has to vanish the S-curvature. We have also derived the expression for the mean Berwald curvature by using the formula of S-curvature.

Keywords: Minkowski space; Finsler space; homogeneous space; isometry group; Lie group; Matsumoto change

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1. Introduction

About eight decades ago, G. Randers [1] published a paper on an asymmetric metric at the four-dimensional space of general relativity. R. S. Ingarden identified this metric as a special kind of Finsler metric in 1957. By extending the Randers metric of Finsler space, M. Matsumoto [2] in 1972 developed the concept of an (α, β)-metric. The theory of Finsler spaces with an (α, β)-metric has been developed into a useful branch of Finsler geometry and studied by many geometers [3–8]. The (α, β)-metric is scalar function at the tangent bundle TM defined using $F = \alpha\phi(s)$ with $s = \beta/\alpha$, where $\beta = b_i(x)y^i$ is 1-form and $\alpha = \sqrt{a_{hk}(x)y^h y^k}$ denotes a Riemannian metric in the manifold M. The Randers metric, Matsumoto metric and Kropina metric are very important examples of (α, β)-metrics. Among them, the Matsumoto metric is one of the interesting examples with $\phi(s) = \frac{1}{1-s}$, given by M. Matsumoto [9] in 1989 by using the gradient of a slope, gravity and speed. The generalized Matsumoto metric is defined as $F = \alpha\phi(s)$ with

$$\phi(s) = \frac{1}{(1-s)^m}, \tag{1}$$

i.e., $F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$, where, $m \neq 0, -1$, and $s \neq 1$.

The germs of homogeneous Riemannian space were found in the Myers–Steenrod theorem [10] in 1939, which states that *"The group of isometries of a Riemannian manifold admits a differentiable structure such that it forms a Lie transformation group of the manifold"*. This theorem generalized the concept of the Lie group to homogeneous Riemannian manifolds. Further, in 2002, Deng and Hou [11] extended the above conclusion to the Finsler manifold. This result introduced the idea of using Lie theory in Finsler geometry. Some classical results in Riemannian homogeneous space to Finsler homogeneous space were extended by

Latifi and Razavi [12]. The properties of homogeneous Riemannian manifolds are studied in [13].

The exploration of the curvature characteristics of a special class of Finsler space is the central problem at Finsler geometry. In 1997, Z. Shen [14] discussed S-curvature, a non-Riemannian quantity, while studying the volume comparison in Riemann–Finsler geometry. It measures the change rate of volume form for a Finsler space along the geodesics. He has given an explicit formula for S-curvature in homogeneous Finsler manifold with an (α, β)-metric [15].

In 2006, the expression for S-curvature in the homogeneous Randers metric was derived by S. Deng [16]. Further, in 2010, S. Deng and X. Wang [17] obtained the expression for S-curvature of a G-invariant homogeneous Finsler manifold with an (α, β)-metric on the coset space of a Lie group G, and by using this formula, they have obtained the Berwald mean curvature E_{ij}. As an application, they have proved that a G-invariant homogeneous Finslerian space with an (α, β)-metric has isotropic S-curvature if and only if its S-curvature vanishes. The expression for S-curvature, which was derived by S. Deng and X. Wang [17], was modified by Shankar and Kaur [18] while studying homogeneous Finslerian space with an (α, β)-metric; they also discussed the curvature property of homogeneous Finsler space with an exponential metric [19]. Narasimhamurthy et al. [20] explored the curvature characteristics of homogeneous Kropina space in 2017. Further, in 2022, G. Shankar et al. [21] discussed curvatures on homogeneous generalized Kropina space. Narasimhamurthy et al. [22] also studied the curvature properties on homogeneous Matsumoto space. In the present paper, we obtained the expression for S-curvature in homogeneous Finsler space with the generalized Matsumoto metric, and we used this expression of S-curvature to obtain the expression for mean Berwald curvature.

This paper is structured as follows: we discuss the basic information on homogeneous Finsler space and Lie groups in Section 2. In Section 3, we obtain the expression for S-curvature in a homogeneous Finsler manifold with the generalized Matsumoto metric and demonstrated that the isotropic S-curvature in homogeneous generalized Matsumoto space must have vanishing S-curvature. An explicit formula for Berwald mean curvature E_{ij} is obtained in Section 4.

Definition 1. [23] *Suppose V is a real vector space of dimension n endowed with a smooth norm F defined on $V \backslash \{0\}$ that satisfies the following:*

(i) $F(e) \geq 0, \ \forall \ e \ \epsilon \ V$,

(ii) $F(\lambda e) = \lambda F(e)$, $\forall \lambda > 0$, *i.e., F is positively homogeneous,*

(iii) *Let $\{e_1, e_2, ...e_n\}$ be a basis of V such that $y = y^1 e_1 + y^2 e_2 + ... + y^n e_n$. Then the Hessian matrix*

$$g_{ij} = \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right),$$

is positive definite at each point of $V \backslash \{0\}$.

Then, F is called the Minkowski norm, and the pair (V, F) is termed as Minkowski space.

Definition 2. [23] *The Finsler metric at the smooth connected manifold M is a function $F : TM \to [0, \infty)$ satisfying the following conditions:*

(i) *Regular: F is smooth on the entire slit tangent bundle $TM \backslash \{0\}$,*

(ii) *The restriction of F to any tangent plane $T_x M$, $x \epsilon M$ is a Minkowskian norm.*

Then, the pair (M, F) is termed a Finsler manifold, or Finsler space.

The prerequisite for an (α, β)-metric to be a Finsler metric is stated by Z. Shen as follows:

Lemma 1. [23] *Let $F = \alpha\phi(s)$ with $s = \frac{\beta}{\alpha}$, where α stands for Riemannian metric and β represents a 1-form such that $b := \|\beta\|_\alpha < b_0$ for $b_0 \in (0, \infty)$. If the smooth positive function ϕ on $(-b_0, b_0)$ obeys the restriction*

$$(b^2 - s^2)\phi'' - s\phi' + \phi > 0, \quad for \ |s| \le b < b_0;$$

then, F becomes a Finsler metric and vice-versa.

Now, let us discuss the S-curvature on a Finsler manifold. Let F be the Minkowskian norm on an n-dimensional vector space V with the basis $\{e_i\}$. The distortion of Minkowski space (V, F) is defined as follows [14]:

$$\tau(y) \;=\; ln \frac{\sqrt{|g_{ij}(y)|}}{\sigma_F},$$

where

$$\sigma_F \;=\; \frac{Vol(B^n)}{Vol\{(y^i)\,\epsilon\,\mathbb{R}^n | F(y^i e^i) < 1\}}.$$

Furthermore, for any tangent vector $y\,\epsilon\,T_xM\backslash\{0\}$, let $\tau(x, y)$ be a distortion of a Minkowskian norm F on T_xM for any $x\,\epsilon\,M$. Let $\gamma(t)$ be a geodesic satisfying $\gamma(0) = x$, $\dot{\gamma}(0) = y$. The rate of change of the distortion along a geodesic γ is defined to be S-curvature and is given by

$$S(x, y) \;=\; \frac{d}{dt}[\tau(\gamma(t)), \dot{\gamma}(t)]|_{\,t=0}\,.$$

While studying the volume comparison in Riemann–Finsler geometry, Z. Shen and X. Cheng [15] defined the two different volume forms as follows:

Definition 3. [15] *The Holmes–Thompson volume form $dV_{HT} = \sigma_{HT}(x)d(x)$ is defined as*

$$\sigma_{TH} \;=\; \frac{1}{Vol B^n}\int_{\{y^i\,\epsilon\,\mathbb{R}^n | F(x, y^i e^i) < 1\}} |(g_{ij})| dy.$$

The Busemann–Hausdroff volume form $dV_{BH} = \sigma_{BH}(x)d(x)$ is defined as

$$\sigma_{BH} \;=\; \frac{Vol(B^n)}{Vol\{(y^i)\,\epsilon\,\mathbb{R}^n | F(x, y^i e^i) < 1\}}.$$

In the case of Riemannian metric, both volume forms coincide, i.e., $dV_{HT} \;=\; dV_{BH} \;=\; \sqrt{|g_{ij}(x)| dx}\,.$

An isometry in a Finslerian space (M, F) is defined as follows:

Definition 4. [11] *A diffeomorphism ϕ: $M \;\rightarrow\; M$ is said to be an isometry if*

$$F(\phi(x), d\phi_x y) = F(x, y), \quad \forall\; x\,\epsilon\,M,\; y\,\epsilon\,T_xM.$$

Deng and Hou [11] have shown that the above definition of isometry in the Finsler geometry (M, F) is equivalent to the following: "*An isometry of (M, F) is a mapping of M onto M which preserves the distance of each pair of points of M* ".

Let G be a smooth manifold satisfying the group properties. If the map $\lambda : G \times G \rightarrow G$, $\lambda(g_1, g_2) = g_1 g_2^{-1}$ $\forall\; g_1, g_2\,\epsilon\,G$ is smooth, then G is called a Lie group.

Let $\sigma : G \times M \rightarrow M$, $(g, x) \rightarrow g.x$ be an action of a Lie group G on a smooth manifold M. It is called a Lie group action (or smooth action) if the map σ is differentiable. Thus, Lie groups act on M as the left-translation given by $L_g(x) = gx$.

The orbit of a Lie group G on a manifold M is defined as $Gx \;=\; \{g.x \,|\, g\,\epsilon\,G\}$, at each point $x\,\epsilon\,M$. A Lie group action is called transitive if the Lie group possesses only one orbit.

Definition 5. [11] *A Finsler space (M, F) is said to be a homogeneous Finsler space if the group of isometries $I(M, F)$ acts transitively on the manifold M.*

Definition 6. *A Riemannian metric $\alpha : TM \bigoplus TM \to R$ on a manifold M is called G-invariant if*

$$\alpha_{gx}(gv, gu) = \alpha_x(v, u), \quad \forall\, v, u \,\epsilon\, T_xM,\ \forall\, x \,\epsilon\, M \text{ and } \ \forall\, g \,\epsilon\, G.$$

A Lie algebra $\mathfrak{g}$ is a vector space together with the Lie bracket (an alternating bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$) which satisfies the Jacobi identity. A Lie algebra is the tangent space of a Lie group at the identity.

Definition 7. *A Lie group G on a smooth manifold M is called a Lie transformation group if G has smooth action on M.*

One of Myers and Steenrod's most well-known results is that *"The group of isometries $I(M, F)$ of a Riemannian manifold is a Lie transformation group on M"* [10]. This result was extended to Finsler space by S. Deng and Z. Hau [11] as follows: *"Let (M, F) be a homogeneous Finsler space with the group of isometries $G = I(M, F)$ being a Lie transformation group of M. Then for $a \,\epsilon\, M$ the isotropic subgroup $I_a(M, F)$ of $I(M, F)$ at a is compact"*.

Let $N = I_a(M, F)$ be a closed isotropy subgroup of G; then, by using the above result, the subgroup N is compact and a Lie group itself. Therefore, a homogeneous Finsler manifold M can be written as a coset space G/N.

Definition 8. [11] *Let G/N be a homogeneous Finsler space, where N is a closed subgroup of a connected Lie group G with Lie algebras $\mathfrak{n}$ and $\mathfrak{g}$, respectively. The homogeneous Finsler space $(G/N, \alpha)$ with an invariant Riemannian metric α is called reductive in the sense of Nomizu [24,25] if the Lie algebra $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$ with $Ad(\mathfrak{q}) \subset \mathfrak{q}, \forall\ \mathfrak{n} \epsilon N$. Then, $\mathfrak{q}$ corresponds to the tangent space of G/N at the origin of N and β corresponds to a vector in $\mathfrak{q}$ which is invariant under the adjoint action of N on $\mathfrak{q}$.*

Definition 9. [23] *A Finsler Manifold M with Minkowaskian norm F is said to have almost isotropic S-curvature if there exists a closed 1-form η and a smooth function $k(x)$ on M that satisfies*

$$S(x, y) = (1 + n)[\eta(y) + k(x)F(y)], \quad \text{for } \ x \,\epsilon\, G/N,\ y \,\epsilon\, T_xM.$$

If $\eta(y) = 0$, then the Finsler space (M, F) is said to have isotropic S-curvature. In addition, if $\eta(y) = 0$ together with $k(x) = const$, then the Finsler space (M, F) is said to have constant S-curvature.

2. S-Curvature of Homogeneous Finsler Space with Generalized Matsumoto Metric

In 2009, Z. Shen and Cheng [15] obtained the formula for S-curvature for Finsler space with (α, β)-metric, which is given as follows:

$$S = (s_0 + r_0)\left(2\psi - \frac{f'(b)}{bf(b)}\right) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0), \tag{2}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'}, \qquad \psi = \frac{Q'}{2\Delta}, \qquad \Delta = Q'(b^2 - s^2) + Qs + 1, \tag{3}$$

$$\Phi = (Q - Q's)(\Delta n + Qs + 1) - (b^2 - s^2)(1 + Qs)Q'', \tag{4}$$

$$2r_{ij} = b_{i|j} + b_{j|i}, \quad b^i r_{ij} = r_j, \quad r_{ij}y^iy^j = r_{00},$$

$$2s_{ij} = b_{i|j} - b_{j|i}, \quad b^i s_{ij} = s_j, \quad s_iy^i = s_0,$$

and

$$f(b) = \begin{cases} \dfrac{\int_0^{\pi}(sin^{n-2})T(bcost)dt}{\int_0^{\pi}(sin^{n-2}t)dt}, & \text{if} \quad dV = dV_{HT}, \\ \dfrac{\int_0^{\pi} sin^{n-2}t}{\int_0^{\pi} \frac{sin^{n-2}t}{\phi(bcost)^n dt} dt}, & \text{if} \quad dV = dV_{BH}, \end{cases}$$

where dV_{HT} and dV_{BH} are the Holmes–Thompson volume and the Busemann–Haudroff volume form, respectively.

Z. Shen [15] proved that in the case of a constant Riemannian length b, the parameter $s_0 + r_0$ vanishes. Hence, Equation (2) reduces to

$$S = -\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0).$$

S. Deng and X. Wang deduced the above formula as follows:

Theorem 1. [17] *If a reductive homogeneous Finsler space G/N has a decomposition of its Lie algebra* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$ *with a G-invariant* (α, β)*-metric* $(F = \alpha\phi(s))$*, then the S-curvature is given as follows:*

$$S(N,y) = \frac{\Phi}{2\Delta^2\alpha}\{\alpha Q\langle[w,y]_{\mathfrak{q}}, w\rangle + c\langle[w,y]_{\mathfrak{q}}, y\rangle\},$$

where $w \epsilon \mathfrak{q}$ *corresponds to* β*. Here,* $\mathfrak{q}$ *corresponds to the tangent space* $T_N(G/N)$ *of* G/N *at origin N.*

Later on, G. Shankar and Kaur [18] modified the above expression of S-curvature as

$$S(N,y) = \frac{\Phi}{2\Delta^2\alpha}\{\alpha Q\langle[w,y]_{\mathfrak{q}}, w\rangle + \langle[w,y]_{\mathfrak{q}}, y\rangle\}. \tag{5}$$

By using Equation (5), we find the expression of S-curvature for the homogeneous generalized Matsumoto space.

In view of Equations (1), (3), and (4), we obtain the following parameters for the homogeneous Finsler space with generalized Matsumoto metric:

$$Q = \frac{m}{[1-s(m+1)]} \qquad Q' = \frac{m(m+1)}{[1-s(m+1)]^2} \qquad Q'' = -\frac{2m(m+1)^2}{[1-s(m+1)]^3} \tag{6}$$

$$\Delta = \frac{mb^2(m+1) + (1-s)^2 - sm(1+sm)}{[1-s(m+1)]^2}, \tag{7}$$

and

$$\Phi = \frac{2ms(1+m) - m}{[1-s(m+1)]^4}\Big\{[1-s(m+1)](1-s) + n[mb^2(m+1) + (1-s)^2 - sm(1+sm)]\Big\} + \frac{2m(m+1)^2(b^2-s^2)(1-s)}{[1-s(m+1)]^4},$$

which can be written as

$$\Phi = \frac{(s^3A + s^2B + sC - D)}{[1-s(m+1)]^4}, \tag{8}$$

where

$$A = 2m(m+1)[2-n(m-1)], \qquad B = -3m - m^2 - mn(m+1)(m^2+2m+5),$$
$$C = m(m+2)(n+1) + 2m(m+1)\Big\{1+n+b^2[mn^2+mn-m-1]\Big\},$$
$$D = m(n+1) + mb^2[mn^2+mn-2(m+1)^2].$$

Substituting the values from Equations (6)–(8) into Equation (5), we obtain the expression of S-curvature in the homogeneous generalized Matsumoto space as

$$S(N,y) = \frac{(s^3A + s^2B + sC - D)}{2\{mb^2(m+1) + (1-s)^2 - sm(1+sm)\}^2}\left[\frac{m\Big\langle [w,y]_{\mathfrak{q}}, w\Big\rangle}{[1-s(m+1)} + \frac{\Big\langle [w,y]_{\mathfrak{q}}, y\Big\rangle}{\alpha}\right]. \tag{9}$$

Thus we have

Theorem 2. *If a reductive homogeneous Finsler space G/N has a decomposition of Lie algebra* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$ *with a G-invariant generalized Matsumoto metric* ($F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$)*, then the S-curvature is given by* (9).

Next, we give an application of the above theorem.

Theorem 3. *If a reductive homogeneous Finsler space G/N has a decomposition of Lie algebra* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$ *with a G-invariant generalized Matsumoto metric* ($F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$)*, then the Finsler space* $(G/N, F)$ *has isotropic S-curvature if and only if the S-curvature vanishes.*

Proof. We have to prove only the necessary part. Let the Finsler space $(G/N, F)$ have isotropic S-curvature; then, in view of Definition 9, we get

$$S(x, y) = k(x)(n+1)F(y), \quad x \,\epsilon\, G/N \;\text{ and }\; y \,\epsilon\, T_xM.$$

□

In the case of a homogeneous space, it is enough to calculate S-curvature at origin, i.e., $x = N$, $y = w$. At origin, the right-hand side of the formula of S-curvature given in Equation (9) becomes zero as $\langle [w, w]_{\mathfrak{q}}\, w\rangle = 0$, giving $k(N) = 0$, which implies $S(N, y) = 0\ \forall\ \text{y}\ \epsilon\ T_xM$. Hence, the Finsler space G/N has zero S-curvature.

3. Mean Berwald Curvature

Another significant non-Riemannian Finslerian quantity is mean Berwald curvature. Z. Shen and S. S. Chern [23] have presented the mean Berwald curvature for the Finsler space as a family of symmetry space $E_y : T_xM \times T_xM \to \mathbb{R}$, defined by

$$E_y(u,v) = E_{ij}(x,y)u^iv^j,$$

where

$$2E_{ij} = \frac{\partial^2 S(x, y)}{\partial y^i \partial y^j}, \tag{10}$$

and $u = u^i\frac{\partial}{\partial x^i}$ and $v = v^i\frac{\partial}{\partial x^i} \,\epsilon\, T_xM$ for $x \,\epsilon\, M$. Then, $E = \{E_y : y \,\epsilon\, TM\backslash\{0\}\}$ is called the mean Berwald curvature or E-curvature. They [23] also defined the isotropic E-curvature as follows:

Definition 10. [23] *A homogeneous Finsler space* G/N *has isotropic E-curvature if*

$$2E = k(x)(n+1)hF^{-1},$$

where $h = h_{ij}(x,y)dx^i dx^j$ is the angular metric tensor.

At the origin , $a_{ij} = \delta^i_j$; therefore, we can easily get the following equalities:

$$y_i = y^i, \qquad \alpha_{y^i} = \frac{y_i}{\alpha}, \qquad \beta_{y^i} = b_i, \tag{11}$$

$$s_{y^i} = \frac{1}{\alpha}(b_i - s\frac{y_i}{\alpha}), \qquad s_{y^i y^j} = \frac{\partial}{\partial y^j}\left(\frac{\alpha b_i - s y_i}{\alpha^2}\right) = \frac{[3sy_i y_j - \alpha(b_i y_j + b_j y_i) - s\alpha^2\delta^i_j]}{\alpha^4}. \tag{12}$$

Let us consider

$$\xi = \frac{s^3 A + s^2 B + sC - D}{2\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^2}. \tag{13}$$

Differentiating (13) with respect to y^j and using Equations (11) and (12), we get

$$\begin{aligned}\frac{\partial \xi}{\partial y^j} &= \frac{\left(s^3 A + s^2 B + sC - D\right)[2(1-s) + m(1+2sm)]}{\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^3} s_{y^j} \\ &+ \frac{\left(3s^2 A + 2sB + C\right)}{2\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^2} s_{y^j},\end{aligned}$$

which can be simplified as

$$\frac{\partial \xi}{\partial y^j} = \frac{(s^4 R + s^3 P + s^2 T + sU + V)}{2\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^3} s_{y^j}, \tag{14}$$

where

$$R = 3A(m^2-1), \quad P = 2B(m^2-1) - A(m+2), \quad T = 3A[1 + mb^2(m+1)] + 3C(m^2-1),$$
$$U = 2B[1 + mb^2(m+1)] + C(m+2) - 4D(m^2-1), \quad V = C[1 + mb^2(m+1)] - 2D(m+2).$$

Differentiating (14) with respect to y^i, we get

$$\begin{aligned}\frac{\partial^2 \xi}{\partial y^i \partial y^j} &= \frac{(s^4 R + s^3 P + s^2 T + sU + V)}{2\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^3} s_{y^i y^j} \\ &+ \frac{(4s^3 R + 3s^2 P + 2sT + U)}{2\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^3} s_{y^i} s_{y^j} \\ &+ \frac{3[2(1-s) + m(1+2sm)]\left(s^4 R + s^3 P + s^2 T + sU + V\right)}{2\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^4} s_{y^i} s_{y^j},\end{aligned}$$

which can be simplified as

$$\begin{aligned}\frac{\partial^2 \xi}{\partial y^i \partial y^j} &= \frac{(s^4 R + s^3 P + s^2 T + sU + V)}{2\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^3} s_{y^i y^j} \\ &+ \frac{\{s^5\theta_1 + s^4\theta_2 + s^3\theta_3 + s^2\theta_4 + s\theta_5 + \theta_6\}}{2\{mb^2(m+1) + (1-s)^2 - ms(1+ms)\}^4} s_{y^i} s_{y^j},\end{aligned}$$

where

$$\begin{aligned}
\theta_1 &= 2R(m^2-1), \quad \theta_2 = 3P(m^2-1) - R(m+2), \\
\theta_3 &= 4R[1+mb^2(m+1)] + 4T(m^2-1), \\
\theta_4 &= 3P[1+mb^2(m+1)] + T(m+2) + 5U(m^2-1), \\
\theta_5 &= 2T[1+mb^2(m+1)] + 2U(m+2) + bV(m^2-1), \\
\theta_6 &= 3V(m+2) + [1+mb^2(m+1)]V.
\end{aligned}$$

In view of (13) Equation (9) can be rewritten as

$$S(N, y) = \frac{\xi m}{[1-s(m+1)]}\Big\langle [w, y]_{\mathfrak{q}}, w \Big\rangle + \frac{\xi}{\alpha}\Big\langle [w, y]_{\mathfrak{q}}, y \Big\rangle. \tag{15}$$

By using (15), Equation (10) can be rewritten as

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S(x, y)}{\partial y^i \partial y^j} = \frac{1}{2}\left[\frac{\partial^2 I}{\partial y^i \partial y^j} + \frac{\partial^2 II}{\partial y^i \partial y^j}\right], \tag{16}$$

where

$$I = \frac{\xi m}{[1-s(m+1)]}\Big\langle [w, y]_{\mathfrak{q}}, w \Big\rangle, \quad II = \frac{\xi}{\alpha}\Big\langle [w, y]_{\mathfrak{q}}, y \Big\rangle. \tag{17}$$

Now, we have to calculate each term of Equation (16) separately. Differentiating I with respect to y^j and using Equation (11), we obtain

$$\begin{aligned}
\frac{\partial I}{\partial y^j} &= \frac{\partial}{\partial y^j}\left\{\frac{\xi m}{[1-(m+1)s]}\Big\langle [w, y]_{\mathfrak{q}}, w \Big\rangle\right\} \\
&= \left[\frac{m}{[1-(m+1)s]}\frac{\partial \xi}{\partial y^j} + \frac{m(m+1)}{[1-(m+1)s]^2}\xi s_{y^j}\right]\Big\langle [w, y]_{\mathfrak{q}}, w \Big\rangle \\
&\quad + \frac{\xi m}{[1-(m+1)s]}\Big\langle [w, w_j]_{\mathfrak{q}}, w \Big\rangle.
\end{aligned} \tag{18}$$

Again, differentiating Equation (18) with respect to y^i, we obtain

$$\begin{aligned}
\frac{\partial^2 I}{\partial y^i \partial y^j} &= \Bigg[\frac{m}{[1-s(m+1)]}\frac{\partial^2 \xi}{\partial y^i \partial y^j} + \frac{m(m+1)}{[1-s(m+1)]^2}\frac{\partial \xi}{\partial y^j}s_{y^i} + \frac{m(m+1)}{[1-s(m+1)]^2}\frac{\partial \xi}{\partial y^i}s_{y^j} \\
&\quad + \frac{m(m+1)\xi}{[1-s(m+1)]^2}s_{y^i}s_{y^j} + \frac{m(m+1)^2\xi}{[1-s(m+1)]^3}s_{y^i}s_{y^j}\Bigg]\Big\langle [w, y]_{\mathfrak{q}}, w \Big\rangle \\
&\quad + \left[\frac{m}{[1-s(m+1)]}\frac{\partial \xi}{\partial y^j} + \frac{m(m+1)\xi}{[1-s(m+1)]^2}s_{y^j}\right]\Big\langle [w, w_i]_{\mathfrak{q}}, w \Big\rangle \\
&\quad + \left[\frac{m}{[1-s(m+1)]}\frac{\partial \xi}{\partial y^i} + \frac{m(m+1)\xi}{[1-s(m+1)]^2}s_{y^i}\right]\Big\langle [w, w_j]_{\mathfrak{q}}, w \Big\rangle.
\end{aligned} \tag{19}$$

Now differentiating II with respect to y^j and using Equation (11), we get

$$\begin{aligned}
\frac{\partial II}{\partial y^j} &= \frac{\partial}{\partial y^j}\left[\frac{\xi}{\alpha}\langle [w, y]_{\mathfrak{q}}, y \rangle\right] \\
&= \frac{\partial}{\partial y^j}\left(\frac{\xi}{\alpha}\right)[\langle [w, y]_{\mathfrak{q}}, y \rangle] + \frac{\xi}{\alpha}\frac{\partial}{\partial y^j}[\langle [w, y]_{\mathfrak{q}}, y \rangle] \\
&= \left[\frac{1}{\alpha}\frac{\partial \xi}{\partial y^j} - \frac{\xi y^j}{\alpha^3}\right][\langle [w, y]_{\mathfrak{q}}, y \rangle] + \frac{\xi}{\alpha}[\langle [\langle [w, y]_{\mathfrak{q}}, w_j \rangle + w, w_j]_{\mathfrak{q}}, y \rangle],
\end{aligned} \tag{20}$$

Again differentiating the Equation (20) with respect to y^i, we get

$$\begin{aligned}\frac{\partial^2 II}{\partial y^i \partial y^j} = &\left[\frac{1}{\alpha}\frac{\partial^2\xi}{\partial y^i\partial y^j} - \frac{\partial\xi}{\partial y^j}\frac{y_i}{\alpha^3} - \frac{\partial\xi}{\partial y^i}\frac{y_j}{\alpha^3} + \frac{3\xi y_i y_j}{\alpha^5} - \frac{\xi}{\alpha^3}\delta^i_j\right]\langle [w,y]_{\mathfrak{q}}\, y\rangle \\ &+ \left[\frac{1}{\alpha}\frac{\partial\xi}{\partial y^j} - \frac{\xi y^j}{\alpha^3}\right]\left[\langle [w,w_i]_{\mathfrak{q}}, y\rangle + \langle [w,y]_{\mathfrak{q}}, w_i\rangle\right] \\ &+ \left[\frac{1}{\alpha}\frac{\partial\xi}{\partial y^i} - \frac{\xi y^i}{\alpha^3}\right]\left[\langle [w,w_j]_{\mathfrak{q}}, y\rangle + \langle [w,y]_{\mathfrak{q}}, w_j\rangle\right] \\ &+ \frac{\xi}{\alpha}\left[\langle [w,w_i]_{\mathfrak{q}}, w_j\rangle + \langle [w,w_j]_{\mathfrak{q}}, w_i\rangle\right],\end{aligned} \tag{21}$$

where

$$\begin{aligned}\frac{\partial}{\partial y^j}\left[\langle [w,y]_{\mathfrak{q}}, y\rangle\right] &= \langle [w,y]_{\mathfrak{q}}, w_j\rangle + \langle [w,w_j]_{\mathfrak{q}}, y\rangle, \\ \frac{\partial^2}{\partial y_i\partial y^j}\left[\langle [w,y]_{\mathfrak{q}}, y\rangle\right] &= \langle [w,w_i]_{\mathfrak{q}}, w_j\rangle + \langle [w,w_j]_{\mathfrak{q}}, w_i\rangle,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y^j}\left[\langle [w,w_i]_{\mathfrak{q}}, w\rangle\right] &= 0, \\ \frac{\partial}{\partial y^i}\left[\langle [w,w_j]_{\mathfrak{q}}, w\rangle\right] &= 0.\end{aligned}$$

Plugging the value of (19) and (21) into Equation (16), we get an expression for the mean Berwald curvature E_{ij} as follows:

$$\begin{aligned}E_{ij} = \frac{1}{2}\Bigg\{ &\left[\frac{1}{\alpha}\frac{\partial^2\xi}{\partial y^i\partial y^j} - \frac{\partial\xi}{\partial y^i}\frac{y_j}{\alpha^3} - \frac{\partial\xi}{\partial y^j}\frac{y_i}{\alpha^3} + \frac{3\xi y_i y_j}{\alpha^5} - \frac{\xi}{\alpha^3}\delta^i_j\right]\langle [w,y]_{\mathfrak{q}}\, y\rangle \\ &+ \left[\frac{1}{\alpha}\frac{\partial\xi}{\partial y^j} - \frac{\xi y^j}{\alpha^3}\right]\left[\langle [w,y]_{\mathfrak{q}}, w_i\rangle + \langle [w,w_i]_{\mathfrak{q}}, y\rangle\right] + \frac{\xi}{\alpha}\left[\langle [w,w_i]_{\mathfrak{q}}, w_j\rangle + \langle [w,w_j]_{\mathfrak{q}}, w_i\rangle\right] \\ &+ \left[\frac{1}{\alpha}\frac{\partial\xi}{\partial y^i} - \frac{\xi y^i}{\alpha^3}\right]\left[\langle [w,y]_{\mathfrak{q}}, w_j\rangle + \langle [w,w_j]_{\mathfrak{q}}, y\rangle\right] \\ &- \left[\frac{m}{[1-s(m+1)]}\frac{\partial^2\xi}{\partial y^i\partial y^j} + \frac{m(m+1)}{[1-s(m+1)]^2}\frac{\partial\xi}{\partial y^j}s^i_y + \frac{m(m+1)\xi}{[1-s(m+1)]^2}s_{y^i}s_{y^j}\right. \\ &\left. + \frac{m(m+1)}{[1-s(m+1)]^2}\frac{\partial\xi}{\partial y^i}s_{y^j} + \frac{m(m+1)^2\xi}{[1-s(m+1)]^3}s_{y^i}s_{y^j}\right]\langle [w,y]_{\mathfrak{q}}, w\rangle \\ &- \left[\frac{m}{[1-s(m+1)]}\frac{\partial\xi}{\partial y^j} + \frac{m(m+1)\xi}{[1-s(m+1)]^2}s_{y^j}\right]\langle [w,w_i]_{\mathfrak{q}}, w\rangle \\ &- \left[\frac{m}{[1-s(m+1)]}\frac{\partial\xi}{\partial y^i} + \frac{m(m+1)\xi}{[1-s(m+1)]^2}s_{y^i}\right]\langle [w,w_j]_{\mathfrak{q}}, w\rangle\Bigg\}.\end{aligned} \tag{22}$$

Thus we have

Theorem 4. *If a reductive homogeneous Finsler space G/N has a decomposition of its Lie algebra* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$ *with a G-invariant generalized Matsumoto metric* ($F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$)*, then mean Berwald curvature is given by* (22)*.*

Z. Shen [26] has proved that if a Finsler space has almost isotropic S-curvature, then it has isotropic E-curvature. In Theorem 3, we have shown that *a reductive homogeneous generalized Matsumoto space has isotropic S-curvature if and only if the S-curvature vanishes*. In view of Equation (10), the E-curvature also vanishes; therefore, the space is weakly Berwald.

Thus, we have the following corollary.

Corollary 1. *Let a reductive homogeneous Finsler space G/N have a decomposition of its Lie algebra* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$ *with a G-invariant generalized Matsumoto metric* ($F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$). *If* $(G/N, F)$ *has isotropic S-curvature, then the space is weakly Berwald.*

4. Conclusions

We have obtained the result that if a reductive homogeneous generalized Matsumoto space has isotropic S-curvature, then both the S-curvature and the E-curvature vanish, and therefore the space is weakly Berwald. The question remains whether the above result is true for any reductive homogeneous Finsler space with an (α, β)-metric.

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