mathematics

MDPI

Article

A Novel Approach to Solving Fractional-Order Kolmogorov and Rosenau–Hyman Models through the q-Homotopy Analysis Transform Method

Laila F. Seddek [1,2], Essam R. El-Zahar [1,3], Jae Dong Chung [4] and Nehad Ali Shah [4,*]

1 Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, P.O. Box 83, Al-Kharj 11942, Saudi Arabia
2 Department of Engineering Mathematics and Physics, Faculty of Engineering, Zagazig University, Zagazig 44519, Egypt
3 Department of Basic Engineering Science, Faculty of Engineering, Menoufia University, Shebin El-Kom 32511, Egypt
4 Department of Mechanical Engineering, Sejong University, Seoul 05006, Republic of Korea
* Correspondence: nehadali199@sejong.ac.kr

Abstract: In this study, a novel method called the q-homotopy analysis transform method (q-HATM) is proposed for solving fractional-order Kolmogorov and Rosenau–Hyman models numerically. The proposed method is shown to have fast convergence and is demonstrated using test examples. The validity of the proposed method is confirmed through graphical representation of the obtained results, which also highlights the ability of the method to modify the solution's convergence zone. The q-HATM is an efficient scheme for solving nonlinear physical models with a series solution in a considerable admissible domain. The results indicate that the proposed approach is simple, effective, and applicable to a wide range of physical models.

Keywords: Laplace transform; fractional-order Kolmogorov; Rosenau–Hyman equations; Atangana-Baleanu-Caputo derivative; q-homotopy analysis transform method

MSC: 83C15; 35A20; 35C05; 35C07; 35C08

check for updates

Citation: Seddek, L.F.; El-Zahar, E.R.; Chung, J.D.; Shah, N.A. A Novel Approach to Solving Fractional-Order Kolmogorov and Rosenau–Hyman Models through the q-Homotopy Analysis Transform Method. *Mathematics* **2023**, *11*, 1321. https://doi.org/10.3390/math11061321

Academic Editors: Svetozar Margenov and Stanislav Harizanov

Received: 6 February 2023
Revised: 3 March 2023
Accepted: 6 March 2023
Published: 9 March 2023

1. Introduction

Fractional calculus (FC) is a branch of mathematical analysis that deals with derivatives and integrals of a non-integer order. The fractional derivative of a function represents its rate of change with respect to a non-integer order derivative operator. Fractional calculus has found applications in various fields such as physics, engineering, finance, and biology. The core principle of FC is that natural phenomenon modeling is done via fractional operators rather than integer operators. As a result, fractional calculus focuses on phenomena that standard theory cannot model [1–5]. Fractional partial differential equations (FPDEs) have received a number of notable contributions in the past. In many different domains, such as unification of diffusion, dynamical systems, wave propagation phenomenon, heat transfer, control theory, image processing, mixed convection flows, and mechanical systems, these equations are more useful for analyzing and describing a variety of phenomena [6–13]. In recent times, a variety of nonlinear fractional differential equations that do not possess exact analytical solutions have been approximately solved by means of numerical methods such as the variational iteration method (VIM), residual power series method (RPSM), reproducing kernel method (RKM), Laplace Adomian decomposition method (LADM), Laplace variational iteration method (LVIM) and Adomian decomposition method (ADM) for further details on the approaches and numerical strategies used to solve fractional differential equations [14–20].

Fractional calculus is a rapidly growing field of mathematics that deals with fractional-order differentiation and integration. The application of fractional calculus in modeling real-world systems has been a topic of research for several years [21–23]. The Fractional-Order Kolmogorov and Rosenau–Hyman models are two such models that have gained significant attention in recent years. The Fractional-Order Kolmogorov model is a fractional generalization of the classical Kolmogorov model, which is a stochastic process that describes the evolution of a probability density function. The Fractional-Order Kolmogorov model is used to model complex systems such as biological, ecological, and financial systems [24–26].

The Rosenau–Hyman model is another fractional-order model that is used to describe the dynamics of complex systems. This model is particularly useful for systems that exhibit long-range interactions, such as plasma physics, fluid dynamics, and geophysics. Both models have demonstrated their effectiveness in capturing the behavior of complex systems that cannot be fully explained by classical models. In this context, this article will explore the Fractional-Order Kolmogorov and Rosenau–Hyman models, their properties, and applications [27–33].

Fractional calculus (FC) is a classical extension of calculus that deals with arbitrary order differentiation and integration. It focuses on phenomena that standard theory cannot model using fractional operators rather than integer operators. Fractional partial differential equations (FPDEs) have received significant attention in various domains, such as diffusion, wave propagation, and heat transfer, due to their usefulness in analyzing and describing a variety of phenomena. However, no technique gives an explicit solution for FPDEs due to the complexity of fractional calculus [1–5]. Numerical techniques such as the variational iteration method (VIM), residual power series method (RPSM), Adomian decomposition method (ADM), Laplace Adomian decomposition method (LADM), reproducing kernel method (RKM) and Laplace variational iteration method (LVIM) have been used to solve these equations approximately [6–13].

The homotopy analysis method (HAM) is a powerful technique for solving differential and integral equations of fractional and classical order. Unlike other methods, HAM does not require perturbation or linearization and has been successfully applied to various nonlinear models in science and technology. However, the drawback of HAM is that it requires a significant amount of computer memory and processing time. To address this issue, some researchers have suggested combining HAM with previously used transform methods. By using transform methods, it is possible to simplify the complexity of the problem, which can reduce the computational burden of HAM. Furthermore, the combination of HAM with transform methods can provide more accurate solutions and improve the convergence rate. Overall, the combination of HAM with transform methods is a promising direction for future research in this area. It has the potential to make HAM more efficient and effective for solving various nonlinear models, which can have a significant impact on scientific and technological advancements [14–20].

In the current work, q-HATM was taken into consideration when trying to solve challenges that were predicted inside the FC framework. Singh et al. [34], using the Laplace transform and the q-homotopy analysis approach, suggest this method. This solution technique deals with numerous operators that may help regulate the convergence province and we modify the accuracy of the produced findings. It is not necessary to meet any of the aforementioned objectives. The suggested method is innovative in that it provides a straightforward method for locating the solution, a sizable convergence zone, and a non-local influence in the discovered solution. The suggested method manipulates and regulates the acquired solution, which, contrary to other conventional methods, swiftly converges to the analytical solution in a constrained acceptable zone.

2. Preliminaries

Here some basic definitions about fractional derivative and Laplace transform [35–39] and integrals [40] are discussed here.

Definition 1. *Let* $\mathcal{U} \in \mathbb{H}^1(\mu, \epsilon)(\epsilon > \mu)$, $\varkappa \in [0,1]$ *be differentiable, then the Atangana-Baleanu derivative of order* $\varkappa$ *in Caputo sense is given by*

$$ {}_{\mu}^{ABC}D_{\varsigma}^{\varkappa}(\mathcal{U}(\varsigma)) = \frac{\mathfrak{N}[\varkappa]}{1-\varkappa}\frac{d}{d\varsigma}\int_{\mu}^{\varsigma}\mathcal{U}'(\varphi)E_{\varkappa}\left[\varkappa\frac{(\varsigma-\varphi)^{\varkappa}}{\varkappa-1}\right]d\varphi, \tag{1}$$

where, the function $\mathfrak{N}$ *is a normalization of the function satisfies* $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$. *Noting that*

$$E_{\varkappa}(\varsigma^{\varkappa}) = \sum_{\vartheta=0}^{\infty}\frac{\varsigma^{\varkappa\vartheta}}{\Gamma(\varkappa\vartheta+1)}.$$

Definition 2. *The AB fractional integral is defined by*

$$ {}_{\mu}^{AB}I_{\varsigma}^{\varkappa}(\mathcal{U}(\varsigma)) = \frac{1-\varkappa}{\mathfrak{N}[\varkappa]}\mathcal{U}(\varsigma) + \frac{\varkappa}{\mathfrak{N}[\varkappa]\Gamma(\varkappa)}\int_{\mu}^{\varsigma}\mathcal{U}(\varphi)(\varsigma-\varphi)^{\varkappa-1}d\varphi. \tag{2}$$

Definition 3. *Fractional derivative Laplace transform (LT) is given by*

$$\mathcal{L}[{}_{\mu}^{ABC}D_{\varsigma}^{\varkappa}(\mathcal{U}(\varsigma)) = \frac{\mathfrak{N}[\varkappa]}{1-\varkappa}\frac{s^{\varkappa}\mathcal{L}[\mathcal{U}(\varsigma)] - s^{\varkappa-1}\mathcal{U}(0)}{s^{\varkappa}+\frac{\varkappa}{(1-\varkappa)}},\ \ 0<\varkappa\leq 1. \tag{3}$$

3. Methodology

The general methodology of q-HATM [41–44] for fractional Kolmogorov IVP

$$ {}^{ABC}D_{\varsigma}^{\varkappa}\mathcal{U}(\varrho,\varsigma) = \mathrm{R}[\mathcal{U}(\varrho,\varsigma)] + \mathrm{N}[\mathcal{U}(\varrho,\varsigma)],\quad 0<\varkappa\leq 1, \tag{4}$$

with initial condition

$$\mathcal{U}(\varrho,0) = f(\varrho), \tag{5}$$

where ${}^{ABC}D_{\varsigma}^{\varkappa}\mathcal{U}(\varrho,\varsigma)$ symbolise the AB derivative of $\mathcal{U}(\varrho,\varsigma)$, R and N is linear and nonlinear functions.

On using the LT on Equation (4), we have after simplification

$$\mathcal{L}[\mathcal{U}(\varrho,\varsigma)] = \frac{f(\varrho)}{s} + \frac{1}{\mathfrak{N}[\varkappa]}\left(1-\varkappa+\frac{\varkappa}{s^{\varkappa}}\right)\mathcal{L}[\mathrm{R}[\mathcal{U}(\varrho,\varsigma)] + \mathrm{N}[\mathcal{U}(\varrho,\varsigma)]]. \tag{6}$$

The non-linear operator is defined as follows

$$\begin{aligned} N[\phi(\varrho,\varsigma;q)] =& \mathcal{L}[\mathcal{U}(\varrho,\varsigma)] - \frac{f(\varrho)}{s} \\ &+ \frac{1}{\mathfrak{N}[\varkappa]}\left(1-\varkappa+\frac{\varkappa}{s^{\varkappa}}\right)\mathcal{L}[\mathrm{R}\phi(\varrho,\varsigma;q) + \mathrm{N}\phi(\varrho,\varsigma;q)]. \end{aligned} \tag{7}$$

Here, $\phi(\varrho,\varsigma;q)$ is the real-valued function with respect to ϱ, ς and $q \in [0, \frac{1}{n}]$. Now, we define a homotopy as follows

$$(1-nq)\mathcal{L}[\phi(\varrho,\varsigma;q) - \mathcal{U}_0(\varrho,\varsigma)] = \hbar q\mathfrak{N}[\phi(\varrho,\varsigma;q)], \tag{8}$$

where $\hbar$ is an auxiliary parameter, $\mathcal{L}$ is LT, $q \in [0, \frac{1}{n}]$ $(n \geq 1)$ is the embedding parameter. For $q = 0$ and $q = \frac{1}{B}$, the below hold true

$$\phi(\varrho,\varsigma;0) = \mathcal{U}_0(\varrho,\varsigma),\ \ \phi(\varrho,\varsigma;\frac{1}{n}) = \mathcal{U}(\varrho,\varsigma). \tag{9}$$

Thus, by intensifying q from 0 to $\frac{1}{n}$, the solution $\phi(\varrho,\varsigma;q)$ varies from initial guess $\mathcal{U}_0(\varrho,\varsigma)$ to $\mathcal{U}(\varrho,\varsigma)$. We defining $\phi(\varrho,\varsigma;q)$ with respect to q by using the Taylor theorem, we get

$$\phi(\varrho,\varsigma;q) = \mathcal{U}_0(\varrho,\varsigma) + \sum_{m=1}^{\infty} \mathcal{U}_m(\varrho,\varsigma)q^m, \tag{10}$$

where

$$\mathcal{U}_m = \frac{1}{m!}\frac{\partial^m \phi(\varrho,\varsigma;q)}{\partial q}|_{q=0}. \tag{11}$$

The series (8) converges at $q = \frac{1}{n}$ for the proper choice of $\mathcal{U}_0(\varrho,\xi,\varsigma)$, n and $\hbar$. Then

$$\mathcal{U}(\varrho,\varsigma) = \mathcal{U}_0(\varrho,\varsigma) + \sum_{m=1}^{\infty} \mathcal{U}_m(\varrho,\varsigma)\left(\frac{1}{n}\right)^m. \tag{12}$$

Taking the derivative of Equation (8) with respect to the embedding parameter q and then putting $q = 0$, later dividing by $m!$, we obtain

$$\mathcal{L}[\mathcal{U}(\varrho,\varsigma) - k_m\mathcal{U}_{m-1}(\varrho,\varsigma)] = \hbar\Re_m(\overrightarrow{\mathcal{U}}_{m-1}), \tag{13}$$

where the vectors are defined as

$$\overrightarrow{\mathcal{U}}_m = [\mathcal{U}_0(\varrho,\varsigma),\mathcal{U}_1(\varrho,\varsigma),\cdots,\mathcal{U}_m(\varrho,\varsigma)]. \tag{14}$$

On applying inverse LT on Equation (13), one can get

$$\mathcal{U}_m(\varrho,\xi,\varsigma) = k_m\mathcal{U}_{m-1}(\varrho,\xi,\varsigma) + \hbar\mathcal{L}^{-1}[\Re_m(\overrightarrow{\mathcal{U}}_{m-1})], \tag{15}$$

where

$$\Re_m(\overrightarrow{\mathcal{U}}_{m-1}) = \mathcal{L}[\mathcal{U}_{m-1}(\varrho,\varsigma)] - \left(1-\frac{k_m}{n}\right)\left(\frac{f(\varrho)}{s}\right) + \frac{1}{\mathfrak{N}[\varkappa]}\left(1-\varkappa+\frac{\varkappa}{s^{\varkappa}}\right)\mathcal{L}\left[\mathrm{N}\mathcal{U}(\varrho,\varsigma)\right], \tag{16}$$

and

$$k_m = \begin{cases} 0, & m \le 1, \\ n, & m > 1. \end{cases} \tag{17}$$

Using the Equations (15) and (16), one can get the series of $\mathcal{U}_m(\varkappa,\varsigma)$. Lastly, the series q-HATM solution is defined as

$$\mathcal{U}(\varrho,\varsigma) = \sum_{m=0}^{\infty} \mathcal{U}_m(\varrho,\varsigma). \tag{18}$$

4. Numerical Problems

Problem 1. *Consider the following non-linear time-fractional Kolmogorov IVP:*

$$\begin{cases} D_{\varsigma}^{\varkappa}\mathcal{U}(\varrho,\varsigma) = (\varrho+1)D_{\varrho}\mathcal{U}(\varrho,\varsigma) + \varrho^2 e^{\varsigma} D_{\varrho}^2\mathcal{U}(\varrho,\varsigma),\ 0<\varkappa\le 1,\ (\varrho,\varsigma)\in[0,1]\times\mathbb{R}, \\ \mathcal{U}(\varrho,0) = \varrho+1. \end{cases} \tag{19}$$

The exact solution at $\varkappa = 1$ is given by

$$\mathcal{U}(\varrho,\varsigma) = (\varrho+1)e^{\varsigma}. \tag{20}$$

Applying the Laplace transform on Equation (19) and using initial condition, we get

$$\mathcal{L}(\mathcal{U}) = \frac{(\varrho+1)}{s} + \frac{1}{\mathfrak{N}[\varkappa]}\left((1-\varkappa)+\frac{\varkappa}{s^{\varkappa}}\right)\mathcal{L}\left[(\varrho+1)\frac{\partial\mathcal{U}}{\partial\varrho} + \varrho^2 e^{\varsigma}\frac{\partial^2\mathcal{U}}{\partial\varrho^2}\right]. \tag{21}$$

The non-linear operator is defined as

$$\begin{aligned}\mathfrak{N}[\Phi(\varrho,\varsigma;q)] =& \mathcal{L}(\Phi(\varrho,\varsigma;q)) - \frac{(\varrho+1)}{s} \\ &- \frac{1}{\mathfrak{N}[\varkappa]}\Big((1-\varkappa)+\frac{\varkappa}{s^{\varkappa}}\Big)\mathcal{L}\Big[(\varrho+1)\frac{\partial\Phi(\varrho,\varsigma;q)}{\partial\varrho}+\varrho^2 e^{\varsigma}\frac{\partial^2\Phi(\varrho,\varsigma;q)}{\partial\varrho^2}\Big].\end{aligned} \tag{22}$$

The m*th order deformation equation define by the assist of suggested method as follows*

$$\mathcal{L}[\mathcal{U}_m(\varrho,\varsigma)-k_m\mathcal{U}_{m-1}(\varrho,\varsigma)] = \hbar\mathcal{R}_m(\overrightarrow{\mathcal{U}}_{m-1}), \tag{23}$$

where $\mathcal{R}_m(\overrightarrow{\mathcal{U}})$ *is*

$$\begin{aligned}\mathcal{R}_m(\overrightarrow{\mathcal{U}}) =& \mathcal{L}(\mathcal{U}_{m-1}) - \Big(1-\frac{k_m}{n}\Big)\mathcal{L}(\varrho+1) \\ &- \frac{1}{\mathfrak{N}[\varkappa]}\Big((1-\varkappa)+\frac{\varkappa}{s^{\varkappa}}\Big)\mathcal{L}\Big[(\varrho+1)\frac{\partial\mathcal{U}_{m-1}}{\partial\varrho}+\varrho^2 e^{\varsigma}\frac{\partial^2\mathcal{U}_{m-1}}{\partial\varrho^2}\Big].\end{aligned} \tag{24}$$

Applying inverse LT on Equation (31), we get

$$\mathcal{U}_m(\varrho,\varsigma) = k_m\mathcal{U}_{m-1}(\varrho,\varsigma)+\hbar\mathcal{L}[\mathcal{R}_m(\overrightarrow{\mathcal{U}}_{m-1})]. \tag{25}$$

Solving the above equations, we get

$$\begin{aligned}
\mathcal{U}_0(\varrho,\varsigma) =& \varrho+1,\\
\mathcal{U}_1(\varrho,\varsigma) =& \frac{\hbar\Big(-1-\varrho+\Big(-\frac{(\varrho+1)\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}+\varrho+1\Big)\varkappa\Big)}{\mathfrak{N}[\varkappa]},\\
\mathcal{U}_2(\varrho,\varsigma) =& \frac{n\hbar\Big(-1-\varrho+\Big(-\frac{(\varrho+1)\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}+\varrho+1\Big)\varkappa\Big)}{\mathfrak{N}[\varkappa]}+\frac{\hbar^2}{\mathfrak{N}[\varkappa]^2}\Bigg(1+\varkappa^2\Big(1+\frac{\varsigma^{2\varkappa}}{\Gamma(2\varkappa+1)}\Big)+\mathfrak{N}[\varkappa](-1+\varkappa)\\
&+\Big(-2+\frac{(-\mathfrak{N}[\varkappa]-2\varkappa+2)\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}\Big)\varkappa\Bigg)(\varrho+1),\\
\mathcal{U}_3(\varrho,\varsigma) =& n\Bigg(\frac{n\hbar\Big(-1-\varrho+\Big(-\frac{(\varrho+1)\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}+\varrho+1\Big)\varkappa\Big)}{\mathfrak{N}[\varkappa]}+\frac{\hbar^2}{\mathfrak{N}[\varkappa]^2}\Bigg(1+\varkappa^2\Big(1+\frac{\varsigma^{2\varkappa}}{\Gamma(2\varkappa+1)}\Big)+\mathfrak{N}[\varkappa](-1+\varkappa)\\
&+\Big(-2+\frac{(-\mathfrak{N}[\varkappa]-2\varkappa+2)\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}\Big)\varkappa\Bigg)(\varrho+1)\Bigg)+\hbar\Bigg(\frac{1}{\mathfrak{N}[\varkappa]^2}\Big(\hbar\Big(n+2\hbar+\varkappa^2\Big(2\hbar+n+\frac{(2\hbar+n)\varsigma^{2\varkappa}}{\Gamma(2\varkappa+1)}\\
&+\varrho(2\hbar+n)\Big)+\Big(-4\hbar-2n+\frac{(4\hbar+2n-\mathfrak{N}[\varkappa](2\hbar+n)-2\varkappa(2\hbar+n))\varsigma^{\varkappa}(\varrho+1)}{\Gamma(\varkappa+1)}-2\varrho(2\hbar+n)\Big)\varkappa\\
&+\varrho(2\hbar+n)\Big)\Big)+\frac{1}{\mathfrak{N}[\varkappa]^3}\Big(\hbar(\varrho+1)\Big((-1+\varkappa)\Big(\mathfrak{N}[\varkappa]^2(\hbar+n)+\frac{3\hbar\varkappa^2\varsigma^{2\varkappa}}{\Gamma(2\varkappa+1)}\Big)+\Big(\Big(1-\frac{\varsigma^{3\varkappa}}{\Gamma(3\varkappa+1)}\Big)\varkappa^3\\
&-3\varkappa^3+3\Big(-\frac{(-1+\varkappa)^2\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}+1\Big)\varkappa-1\Big)\hbar\Big)\Big)\Bigg),\\
&\vdots
\end{aligned} \tag{26}$$

In Figure 1, the exact and analytical solutions graph at $\varkappa=1$ *for Problem 1. Figure 2, three dimensional various fractional order graph of* $\varkappa$*. Table 1, absolute error of Exact and fourth terms approximate solutions for Problem 1 with various* ς *and* ϱ *for* $\varkappa=1$.

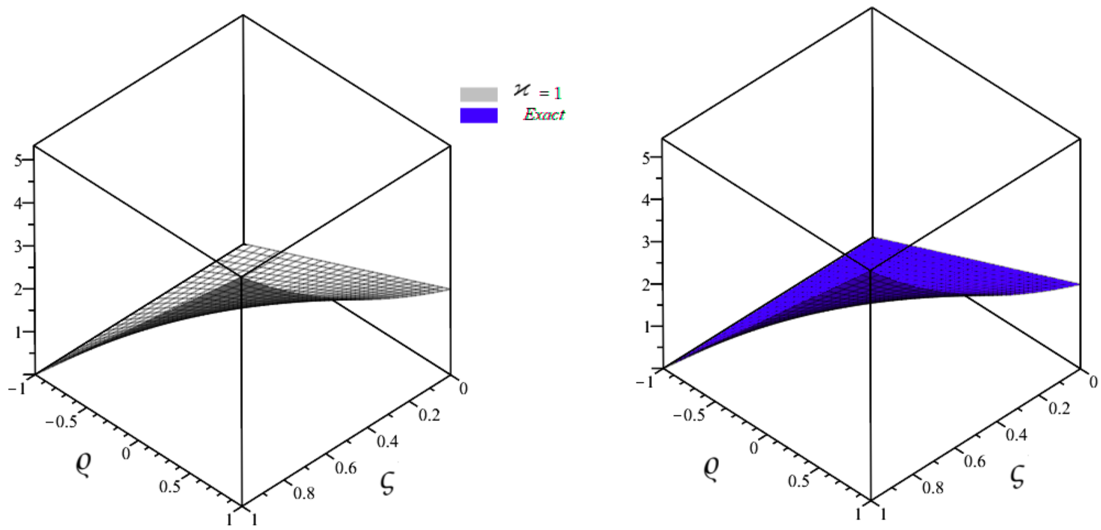

Figure 1. The exact and analytical solutions graph at $\varkappa = 1$ for Problem 1.

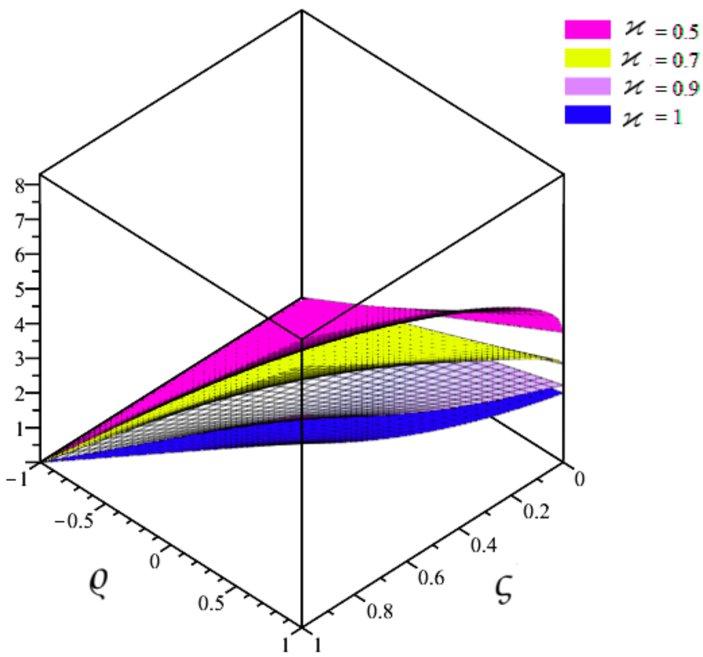

Figure 2. The three dimensional various fractional order graph of $\varkappa$.

Table 1. Absolute error of Exact and fourth terms approximate solutions for Problem 1 with various ς andϱ for $\varkappa = 1$.

ς	ϱ	$Approximate_{\varkappa=1}$	Exact	AE
0.1	0	1.105166667	1.105170918	4.251×10^{-6}
	0.3	1.326200000	1.326205102	5.102×10^{-6}
	0.5	1.547233333	1.547239285	5.952×10^{-6}
	0.7	1.547239285	1.768273469	6.802×10^{-6}
	0.9	1.989300000	1.989307652	7.652×10^{-6}
	1	2.210333333	2.210341836	8.503×10^{-6}

Table 1. *Cont.*

ς	ϱ	$Approximate_{\varkappa=1}$	Exact	AE
0.3	0	1.221333333	1.221402758	6.9425×10^{-5}
	0.3	1.465600000	1.465683310	8.3310×10^{-5}
	0.5	1.709866667	1.709963861	9.7194×10^{-5}
	0.7	1.954133333	1.954244413	1.11080×10^{-4}
	0.9	2.198400000	2.198524964	1.24964×10^{-4}
	1	2.442666667	2.442805516	1.38849×10^{-4}

Problem 2. *Consider the following non-linear time-fractional Rosenau–Hyman IVP:*

$$\begin{cases} D_{\varsigma}^{\varkappa}\mathcal{U}(\varrho,\varsigma) = \mathcal{U}(\varrho,\varsigma)D_{\varrho}^{3}\mathcal{U}(\varrho,\varsigma) + \mathcal{U}(\varrho,\varsigma)D_{\varrho}\mathcal{U}(\varrho,\varsigma) + 3D_{\varrho}\mathcal{U}(\varrho,\varsigma)D_{\varrho}^{2}\mathcal{U}(\varrho,\varsigma), \ 0 < \varkappa \leq 1, \ (\varrho,\varsigma) \in [0,1] \times \mathbb{R}, \\ \mathcal{U}(\varrho,0) = -\frac{8\mathcal{C}}{3}\cos^2\left(\frac{\varrho}{4}\right). \end{cases} \tag{27}$$

The exact solution at $\varkappa = 1$ is given by

$$\mathcal{U}(\varrho,\varsigma) = -\frac{8}{3}\cos^2\left(\frac{\varrho - \mathcal{C}\varsigma}{4}\right). \tag{28}$$

Applying the Laplace transform on Equation (27) and using initial condition, we get

$$\mathcal{L}(\mathcal{U}) = \frac{\left(-\frac{8\mathcal{C}}{3}\cos^2\left(\frac{\varrho}{4}\right)\right)}{s} + \frac{1}{\mathfrak{N}[\varkappa]}\left((1-\varkappa) + \frac{\varkappa}{s^{\varkappa}}\right)\mathcal{L}\left[\mathcal{U}D_{\varrho}^{3}\mathcal{U} + \mathcal{U}D_{\varrho}\mathcal{U} + 3D_{\varrho}\mathcal{U}D_{\varrho}^{2}\mathcal{U}\right]. \tag{29}$$

The non-linear operator is defined as

$$\begin{aligned} \mathfrak{N}[\Phi(\varrho,\varsigma;q)] =& \mathcal{L}(\Phi(\varrho,\varsigma;q)) - \frac{\left(-\frac{8\mathcal{C}}{3}\cos^2\left(\frac{\varrho}{4}\right)\right)}{s} - \frac{1}{\mathfrak{N}[\varkappa]}\left((1-\varkappa) + \frac{\varkappa}{s^{\varkappa}}\right)\mathcal{L}\Bigg[\Phi(\varrho,\varsigma;q)D_{\varrho}^{3}\Phi(\varrho,\varsigma;q) \\ &+ \Phi(\varrho,\varsigma;q)D_{\varrho}\Phi(\varrho,\varsigma;q) + 3D_{\varrho}\Phi(\varrho,\varsigma;q)D_{\varrho}^{2}\Phi(\varrho,\varsigma;q)\Bigg]. \end{aligned} \tag{30}$$

The m*th order deformation equation define by the assist of suggested method as follows*

$$\mathcal{L}[\mathcal{U}_m(\varrho,\varsigma) - k_m\mathcal{U}_{m-1}(\varrho,\varsigma)] = \hbar\mathcal{R}_m(\overrightarrow{\mathcal{U}}_{m-1}), \tag{31}$$

where $\mathcal{R}_m(\overrightarrow{\mathcal{U}})$ is

$$\begin{aligned} \mathcal{R}_m(\overrightarrow{\mathcal{U}}) =& \mathcal{L}(\mathcal{U}_{m-1}) - \left(1 - \frac{k_m}{n}\right)\mathcal{L}\left(-\frac{8\mathcal{C}}{3}\cos^2\left(\frac{\varrho}{4}\right)\right) - \frac{1}{\mathfrak{N}[\varkappa]}\left((1-\varkappa) + \frac{\varkappa}{s^{\varkappa}}\right)\mathcal{L}\Bigg[\sum_{i=0}^{m-1}\mathcal{U}_iD_{\varrho}^{3}\mathcal{U}_{m-i-1} \\ &+ \sum_{i=0}^{m-1}\mathcal{U}_iD_{\varrho}\mathcal{U}_{m-i-1} + 3\sum_{i=0}^{m-1}D_{\varrho}\mathcal{U}_iD_{\varrho}^{2}\mathcal{U}_{m-i-1}\Bigg] \end{aligned} \tag{32}$$

Applying inverse LT on Equation (31), we get

$$\mathcal{U}_m(\varrho,\varsigma) = k_m\mathcal{U}_{m-1}(\varrho,\varsigma) + \hbar\mathcal{L}[\mathcal{R}_m(\overrightarrow{\mathcal{U}}_{m-1})]. \tag{33}$$

Solving the above equations, we get

$$
\begin{aligned}
\mathcal{U}_0(\varrho,\varsigma) =& -\frac{8\mathcal{C}}{3}\cos^2\left(\frac{\varrho}{4}\right),\\
\mathcal{U}_1(\varrho,\varsigma) =& \frac{4\hbar\mathcal{C}^2\cos\left(\frac{\varrho}{4}\right)\sin\left(\frac{\varrho}{4}\right)\left(1+\varkappa\left(-1+\frac{\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}\right)\right)}{3\mathfrak{N}[\varkappa]},\\
\mathcal{U}_2(\varrho,\varsigma) =& \frac{4n\hbar\mathcal{C}^2\cos\left(\frac{\varrho}{4}\right)\sin\left(\frac{\varrho}{4}\right)\left(1+\varkappa\left(-1+\frac{\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}\right)\right)}{3\mathfrak{N}[\varkappa]}\\
&+\frac{1}{3}\left(\hbar^2\left(\mathcal{C}^3\left(\frac{\varsigma^{2\varkappa}\varkappa^2\left(\cos\left(\frac{\varrho}{4}\right)^2-\sin\left(\frac{\varrho}{4}\right)^2\right)}{\Gamma(2\varkappa+1)}+\frac{2(-1+\varkappa)\left(-2\cos\left(\frac{\varrho}{4}\right)^2+1\right)\varkappa\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}\right.\right.\right.\\
&\left.\left.\left.+\left(2\cos\left(\frac{\varrho}{4}\right)^2-1\right)(-1+\varkappa)^2\right)\right)+\frac{4\mathcal{C}^2\cos\left(\frac{\varrho}{4}\right)\sin\left(\frac{\varrho}{4}\right)\left(1+\varkappa\left(-1+\frac{\varsigma^{\varkappa}}{\Gamma(\varkappa+1)}\right)\right)}{\mathfrak{N}[\varkappa]}\right),\\
&\vdots
\end{aligned}
\tag{34}
$$

Figure 3, the analytical solutions of different fractional order $\varkappa$ of Problem 2. Figure 4, the analytical solutions of different fractional order $\varkappa$ of Problem 2. Table 2, absolute error of exact and q-HATM of Problem 2 of different fractional order of ς and ϱ for $\varkappa = 1$.

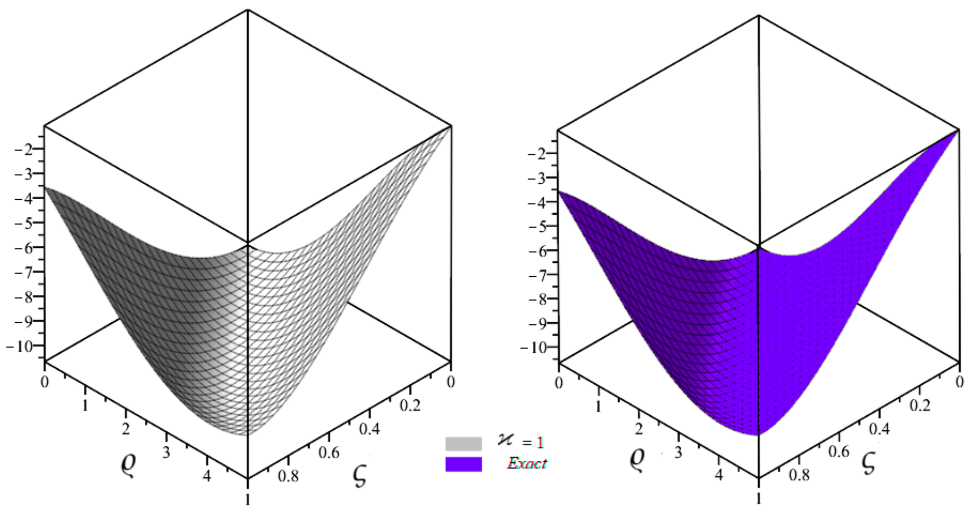

Figure 3. The exact and analytical solution graph of Problem 2.

Table 2. The absolute error of exact and q-HATM of Problem 2 of different fractional order of ς and ϱ for $\varkappa = 1$.

ς	ϱ	$Approximate_{\varkappa=1}$	Exact	AE
0.1	0	10.56035556	10.56035508	4.8×10^{-8}
	0.3	10.64002126	10.64002221	9.5×10^{-7}
	0.5	10.66666431	10.66666667	2.36×10^{-7}
	0.7	10.64001846	10.64002221	3.75×10^{-7}
	0.9	10.56034998	10.56035508	5.10×10^{-7}
	1	10.42845487	10.40846247	6.400×10^{-7}
0.2	0	10.24568889	10.24565863	3.026×10^{-6}
	0.3	10.42833314	10.48246127	1.616×10^{-6}
	0.5	10.45689126	10.56035508	7.043×10^{-6}
	0.7	10.63991712	10.64002221	1.0408×10^{-5}
	0.9	10.66651797	10.66666667	1.3770×10^{-5}
	1	10.63983139	10.46003331	1.8972×10^{-5}

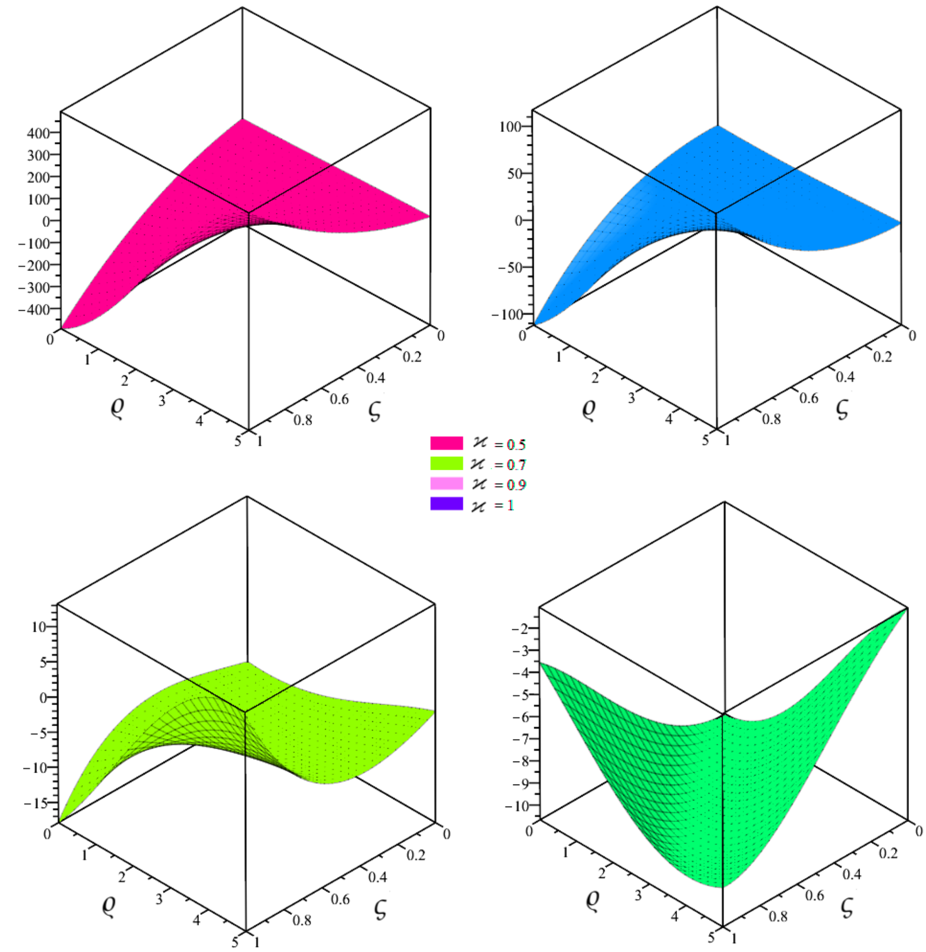

Figure 4. The analytical solutions of different fractional order $\varkappa$ of Problem 2.

5. Conclusions

The present research has effectively solved the fractional-order Kolmogorov and Rosenau–Hyman models using q-HATM without any difficulty. The q-HATM has numerous advantages and efficiency due to its ability to provide a broad convergence area, uncomplicated solution process, and high precision in obtaining accurate results. The q-HATM approach is more robust and systematic than other analytical methods, and it can be used to investigate nonlinear mathematical models that describe real-world issues. The auxiliary parameter h used in the proposed method explains the non-local convergence and provides an illuminated solution in series form without any ambiguity.

Author Contributions: Methodology, L.F.S., E.R.E.-Z. and N.A.S.; Software, L.F.S.; Investigation, E.R.E.-Z. and J.D.C.; Data curation, J.D.C.; Writing—original draft, N.A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2023/R/1444). This work was supported by the Technology Innovation Program (20018869, Development of Waste Heat and Waste Cold Recovery Bus Air-conditioning System to Reduce Heating and Cooling Load by 10%) funded By the Ministry of Trade, Industry & Energy (MOTIE, Republic of Korea).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zhang, X.; Yu, L.; Jiang, J.; Wu, Y.; Cui, Y. Solutions for a singular Hadamard-type fractional differential equation by the spectral construct analysis. *J. Funct. Spaces* **2020**, *2020*, 8392397. [CrossRef]
2. He, J.; Zhang, X.; Liu, L.; Wu, Y.; Cui, Y. A singular fractional Kelvin–Voigt model involving a nonlinear operator and their convergence properties. *Bound. Value Probl.* **2019**, *2019*, 112. [CrossRef]
3. Caputo, M. Linear models of dissipation whose Q is almost frequency independent—II. *Geophys. J. Int.* **1967**, *13*, 529–539. [CrossRef]
4. Oldham, K.B.; Spanier, J. The Fractional Calculus. In *Integrations and Differentiations of Arbitrary Order*; Descartes Press: Cambridge, MA, USA, 1974.
5. Xie, Z.; Feng, X.; Chen, X. Partial Least Trimmed Squares Regression. *Chemom. Intell. Lab. Syst.* **2022**, *221*, 104486. [CrossRef]
6. Al-Smadi, M.; Abu Arqub, O.; Hadid, S. An attractive analytical technique for coupled system of fractional partial differential equations in shallow water waves with conformable derivative. *Commun. Theor. Phys.* **2020**, *72*, 085001. [CrossRef]
7. Jleli, M.; Kumar, S.; Kumar, R.; Samet, B. Analytical approach for time fractional wave equations in the sense of Yang-Abdel-Aty-Cattani via the homotopy perturbation transform method. *Alex. Eng. J.* **2020**, *59*, 2859–2863. [CrossRef]
8. Chen, X.; Xu, Y.; Meng, L.; Chen, X.; Yuan, L.; Cai, Q.; Huang, G. Non-parametric Partial Least Squares-Discriminant Analysis Model Based on Sum of Ranking Difference Algorithm for Tea Grade Identification Using Electronic Tongue Data. *Sensors Actuators Chem.* **2020**, *311*, 127924. [CrossRef]
9. Hasan, S.; El-Ajou, A.; Hadid, S.; Al-Smadi, M.; Momani, S. Atangana-Baleanu fractional framework of reproducing kernel technique in solving fractional population dynamics system. *Chaos Solitons Fractals* **2020**, *133*, 109624. [CrossRef]
10. Al-Smadi, M.; Abu Arqub, O.; Momani, S. A computational method for two-point boundary value problems of fourth-order mixed integrodifferential equations. *Math. Probl. Eng.* **2013**, *2013*, 832074. [CrossRef]
11. Al-Smadi, M.; Dutta, H.; Hasan, S.; Momani, S. On numerical approximation of Atangana-Baleanu-Caputo fractional integrodifferential equations under uncertainty in Hilbert Space. *Math. Model. Nat. Phenom.* **2021**, *16*, 41. [CrossRef]
12. Qin, X.; Zhang, L.; Yang, L.; Cao, S. Heuristics to Sift Extraneous Factors in Dixon Resultants. *J. Symb. Comput.* **2022**, *112*, 105–121. [CrossRef]
13. Al-Smadi, M.; Abu Arqub, O.; Gaith, M. Numerical simulation of telegraph and Cattaneo fractional-type models using adaptive reproducing kernel framework. *Math. Methods Appl. Sci.* **2021**, *44*, 8472–8489. [CrossRef]
14. Li, X.; Dong, Z.; Wang, L.; Niu, X.; Yamaguchi, H.; Li, D.; Yu, P. A Magnetic Field Coupling Fractional Step Lattice Boltzmann Model for the Complex Interfacial Behavior in Magnetic Multiphase Flows. *Appl. Math. Model.* **2023**, *117*, 219–250. [CrossRef]
15. Odibat, Z.; Momani, S. Application of variational iteration method to nonlinear differential equations of fractional order. *Int. J. Nonlinear Sci. Numer. Simul.* **2006**, *7*, 27–34. [CrossRef]
16. Sun, L.; Hou, J.; Xing, C.; Fang, Z. A Robust Hammerstein-Wiener Model Identification Method for Highly Nonlinear Systems. *Processes* **2022**, *10*, 2664. [CrossRef]
17. Hasan, S.; Al-Smadi, M.; El-Ajou, A.; Momani, S.; Hadid, S.; Al-Zhour, Z. Numerical approach in the Hilbert space to solve a fuzzy Atangana-Baleanu fractional hybrid system. *Chaos Solitons Fractals* **2021**, *143*, 110506. [CrossRef]
18. Liu, K.; Yang, Z.; Wei, W.; Gao, B.; Xin, D.; Sun, C.; Wu, G. Novel Detection Approach for Thermal Defects: Study on Its Feasibility and Application to Vehicle Cables. *High Volt.* **2022**, 1–10. [CrossRef]
19. Kumar, S. A new fractional modeling arising in engineering sciences and its analytical approximate solution. *Alex. Eng. J.* **2013**, *52*, 813–819. [CrossRef]
20. Xu, K.; Guo, Y.; Liu, Y.; Deng, X.; Chen, Q.; Ma, Z. 60-GHz Compact Dual-Mode On-Chip Bandpass Filter Using GaAs Technology. *IEEE Electron Device Lett.* **2021**, *42*, 1120–1123. [CrossRef]
21. Mofarreh, F.; Khan, A.; Abdeljabbar, A. A Comparative Analysis of Fractional-Order Fokker-Planck Equation. *Symmetry* **2023**, *15*, 430. [CrossRef]
22. Naeem, M.; Yasmin, H.; Shah, N.A.; Nonlaopon, K. Investigation of Fractional Nonlinear Regularized Long-Wave Models via Novel Techniques. *Symmetry* **2023**, *15*, 220. [CrossRef]
23. Alshehry, A.S.; Ullah, R.; Shah, N.A.; Nonlaopon, K. Implementation of Yang residual power series method to solve fractional non-linear systems. *AIMS Math.* **2023**, *8*, 8294–8309. [CrossRef]
24. Alderremy, A.A.; Shah, R.; Aly, S.; Nonlaopon, K. Comparison of two modified analytical approaches for the systems of time fractional partial differential equations. *AIMS Math.* **2023**, *8*, 7142–7162. [CrossRef]
25. Alyobi, S.; Khan, A.; Shah, N.A.; Nonlaopon, K. Fractional Analysis of Nonlinear Boussinesq Equation under Atangana-Baleanu-Caputo Operator. *Symmetry* **2022**, *14*, 2417. [CrossRef]
26. Alshehry, A.S.; Shah, R.; Dassios, I. A reliable technique for solving fractional partial differential equation. *Axioms* **2022**, *11*, 574. [CrossRef]
27. Sukhinov, A.; Chistyakov, A.; Nikitina, E.T.E.A.; Belova, Y. The Construction and Research of the Modified Upwind Leapfrog Difference Scheme with Improved Dispersion Properties for the Korteweg-de Vries Equation. *Mathematics* **2022**, *10*, 2922. [CrossRef]
28. Sukhinov, A.; Chistyakov, A.; Kuznetsova, I.; Belova, Y.; Rahimbaeva, E. Development and Research of a Modified Upwind Leapfrog Scheme for Solving Transport Problems. *Mathematics* **2022**, *10*, 3564. [CrossRef]

29. Salnikov, N.N. Construction of Weight Functions of the Petrov-Galerkin Method for Convection-Diffusion-Reaction Equations in the Three-Dimensional Case. *Cybern. Syst. Anal.* **2014**, *50*, 805–814. [CrossRef]
30. Siryk, S.V.; Salnikov, N.N. Numerical solution of Burgers' equation by Petrov-Galerkin method with adaptive weighting functions. *J. Autom. Inf. Sci.* **2012**, *44*, 50–67. [CrossRef]
31. Salnikov, N.N.; Siryk, S.V.; Tereshchenko, I.A. On construction of finite-dimensional mathematical model of convection-diffusion process with usage of the Petrov-Galerkin method. *J. Autom. Inf. Sci.* **2010**, *42*, 67–83. [CrossRef]
32. AbdulRidha, M.W.; Kashkool, H.A. Space-Time Petrov-Discontinuous Galerkin Finite Element Method for Solving Linear ConvectionDiffusion Problems. *J. Phys. Conf. Ser.* **2022**, *2322*, 012007. [CrossRef]
33. Saadoon, J.J.; Kashkool, H.A. hp-discontinuous Galerkin Finite Element Method for Incompressible Miscible Displacement in Porous Media. *J. Phys. Conf. Ser.* **2020**, *1530*, 012001.
34. Singh, J.; Kumar, D.; Swroop, R. Numerical solution of time- and space-fractional coupled Burgers' equations via homotopy algorithm. *Alex. Eng. J.* **2016**, *55*, 1753–1763. [CrossRef]
35. Liu, Y.; Xu, K.; Li, J.; Guo, Y.; Zhang, A.; Chen, Q. Millimeter-Wave E-Plane Waveguide Bandpass Filters Based on Spoof Surface Plasmon Polaritons. *IEEE Trans. Microw. Theory Tech.* **2022**, *70*, 4399–4409. [CrossRef]
36. Jin, H.; Wang, Z. Boundedness, Blowup and Critical Mass Phenomenon in Competing Chemotaxis. *J. Differ. Equations* **2016**, *260*, 162–196. [CrossRef]
37. Jin, H.Y.; Wang, Z. Asymptotic Dynamics of the One-Dimensional Attraction-Repulsion Keller-Segel Model. *Math. Methods Appl. Sci.* **2015**, *38*, 444–457. [CrossRef]
38. Ciancio, A. Analysis of time series with wavelets. *Int. J. Wavelets Multiresolut. Inf. Process.* **2007**, *5*, 241–256. [CrossRef]
39. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *arXiv* **2016**, arXiv:1602.03408.
40. Srivastava, H.M.; Kumar, D.; Singh, J. An efficient analytical technique for fractional model of vibration equation. *Appl. Math. Model.* **2017**, *45*, 192–204. [CrossRef]
41. Liao, S.J. The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems. Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, China, 1992.
42. Liao, S.J. Homotopy analysis method and its applications in mathematics. *J. Basic Sci. Eng.* **1997**, *5*, 111–125.
43. Liao, S. *Beyond Perturbation: Introduction to Homotopy Analysis Method*; CRC Press: Boca Raton, FL, USA, 2000.
44. Liao, S.J. Notes on the homotopy analysis method: Some definitions and theorems. *Commun. Nonlinear Sci. Numer. Simul.* **2009**, *14*, 83–97. [CrossRef]