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Some Remarks on Local Fractional Integral Inequalities Involving Mittag–Leffler Kernel Using Generalized (E, h)-Convexity

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Abstract: In the present work, we introduce two new local fractional integral operators involving Mittag–Leffler kernel on Yang's fractal sets. Then, we study the related generalized Hermite–Hadamard-type inequality using generalized (E, h)-convexity and obtain two identities pertaining to these operators, and the respective first- and second-order derivatives are given. In terms of applications, we provide some new generalized trapezoid-type inequalities for generalized (E, h)-convex functions. Finally, some special cases are deduced for different values of δ, E, and h.

Keywords: fractal sets; generalized (E, h)-convexity; local fractional integral and derivative; generalized Mittag–Leffler function

MSC: 52A41; 26A51; 26A33; 26D07

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1. Introduction

Convexity is a significant concept in various fields of mathematics and related areas, such as economics, finance, and biology. It provides a powerful framework for exploring diverse topics in both pure and applied sciences. The idea of convexity is closely connected with the development of the theory of inequalities, which is an essential tool for analyzing particular aspects of solutions to differential equations and estimating the errors of quadrature formulas.

The most famous inequality related to convex functions is the so-called Hermite–Hadamard integral inequality, which can be stated as follows [1]:

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b g(x)dx \leqslant \frac{g(a)+g(b)}{2} \tag{1}$$

where $g : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function on $I = [a, b]$, with $a < b$.

The class of convex functions has been extended to many more general classes, among which we can cite the class of h-convex functions introduced in [2] as follows:

Definition 1. *Let $h : [0, 1] \longrightarrow (0, \infty)$ be a positive function and $U \subseteq \mathbb{R}$ be an interval; then, $g : U \longrightarrow \mathbb{R}$ is called h-convex if g is non-negative and the following inequality holds:*

$$g(\vartheta u_1 + (1-\vartheta)u_2) \leqslant h(\vartheta)g(u_1) + h(1-\vartheta)g(u_2), \ \ \forall u_1, u_2 \in U, \ \ \vartheta \in [0, 1]. \tag{2}$$

The class of h-convex functions includes many types of convexity, such as classical convexity ($h(\vartheta) = \vartheta$), s-convexity ($h(\vartheta) = \vartheta^s$), Godunova–Levin functions ($h(\vartheta) = \frac{1}{\vartheta}$), and P-functions ($h(\vartheta) = 1$).

Modified h-convexity, generalized modified h-convexity, strong h-convexity, and generalized strongly modified h-convexity are studied in [3,4].

In [5], Sarikaya et al. provided the Hermite–Hadamard inequality related to h-convex functions.

Theorem 1 ([5])**.** *Let $g : U \longrightarrow \mathbb{R}$ be h-convex, and $a, b \in U$, with $a < b$ and $g \in L_1([a, b])$. Then,*

$$\frac{1}{h(\frac{1}{2})} g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b g(x)dx \leqslant [g(a) + g(b)] \int_0^1 h(x)dx. \tag{3}$$

In [6], the author introduces the class of E-convex functions as follows:

Definition 2 ([6])**.** *A function $g : [\rho_1, \rho_2] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be E-convex, where $E : [\rho_1, \rho_2] \longrightarrow [\rho_1, \rho_2]$, if*

$$g(\vartheta E(\mu_1) + (1-\vartheta)E(\mu_2)) \leqslant \vartheta g(E(\mu_1)) + (1-\vartheta) g(E(\mu_2)), \forall \mu_1, \mu_2 \in [\rho_1, \rho_2], \vartheta \in [0,1].$$

The Hermite–Hadamard inequality related to E-convex functions is presented in [7].

Theorem 2 ([7])**.** *Let us assume that $E : I_1 \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous increasing function and $a, b \in I_1$, with $a < b$. A function $g : I_2 \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be E-convex on $[a, b]$, if the following inequality holds:*

$$g\left(\frac{E(a) + E(b)}{2}\right) \leqslant \frac{1}{E(b) - E(a)} \int_{E(a)}^{E(b)} g(E(x))dE(x) \leqslant \frac{g(E(a)) + g(E(b))}{2} \tag{4}$$

Fractal sets and fractal theory have garnered considerable interest from scientists and engineers. According to Mandelbrot, a fractal set is one in which the Hausdorff dimension is greater than the topological dimension [8,9]. Several techniques of fractional calculus have been developed to study the properties of functions acting on fractal sets, such as those presented in [10–19]. The investigation and advancement of local fractional functions in fractal sets, including local fractional calculus and function monotonicity, are extensively studied by Yang in [20].

Conforming to Gao-Yang-Kang's notion [20], the real line number in fractal set $\mathbb{R}^\delta$ has the below properties.

If $\iota_1^\delta, \iota_2^\delta$, and $\iota_3^\delta \in \mathbb{R}^\delta$, where $0 < \delta \leqslant 1$, then:

1. $\iota_1^\delta + \iota_2^\delta \in \mathbb{R}^\delta, \iota_1^\delta \iota_2^\delta \in \mathbb{R}^\delta$.
2. $\iota_1^\delta + \iota_2^\delta = \iota_2^\delta + \iota_1^\delta = (\iota_1 + \iota_2)^\delta = (\iota_2 + \iota_1)^\delta$.
3. $\iota_1^\delta + (\iota_2^\delta + \iota_3^\delta) = (\iota_1^\delta + \iota_2^\delta) + \iota_3^\delta$.
4. $\iota_1^\delta \iota_2^\delta = \iota_2^\delta \iota_1^\delta = (\iota_1 \iota_2)^\delta = (\iota_2 \iota_1)^\delta$.
5. $\iota_1^\delta (\iota_2^\delta \iota_3^\delta) = (\iota_1^\delta \iota_2^\delta) \iota_3^\delta$.
6. $\iota_1^\delta (\iota_2^\delta + \iota_3^\delta) = (\iota_1^\delta \iota_2^\delta) + (\iota_1^\delta \iota_3^\delta)$.
7. $\iota_1^\delta + 0^\delta = 0^\delta + \iota_1^\delta = \iota_1^\delta$ and $\iota_1^\delta \cdot 1^\delta = 1^\delta \cdot \iota_1^\delta = \iota_1^\delta$.

The following definitions related to the local fractional calculus on $\mathbb{R}^\delta$ are given in [20]:

Definition 3. *The local fractional derivative of $g(x)$ of order δ at $x = x_0$ is given by*

$$g^\delta(x_0) = \frac{d^\delta g(x)}{dx^\delta} |_{x=x_0} = \lim_{x \longrightarrow x_0} \frac{\Gamma(\delta+1)(g(x) - g(x_0))}{(x - x_o)^\delta}.$$

Definition 4. *Let g be a local fractional continuous on $[a, b]$. The local fractional of function $g(x)$ of order δ is defined by*

$${}_aI_b^{(\delta)} g(x) = (\Gamma(1+\delta))^{-1} \int_a^b g(\tau)(d\tau)^\delta$$

$$= (\Gamma(1+\delta))^{-1} \lim_{\triangle\tau \longrightarrow 0} \sum_{n=0}^{m} g(\tau_n)(\triangle\tau_n)^{\delta} \tag{5}$$

where $\triangle\tau_n = \tau_{n+1} - \tau_n$ *and* $\triangle\tau = \max \triangle\tau_n : n = 1, 2, \ldots, m-1$, *where* $[\tau_n, \tau_{n+1}], n = 0, 1, \ldots, m-1$, $a = \tau_0 < \tau_1 < \cdots < \tau_{m-1} < \tau_m = b$ *is a partition on* $[a, b]$ *and* Γ *is the well-known Gamma function* $\Gamma(\chi) = \int_0^{\infty} \rho^{\chi-1} e^{-\rho} d\rho$.

The following defines the Mittag–Leffler function on fractal sets of order δ $(0 < \delta \leqslant 1)$ (see [20,21]):

$$\mathbb{E}_{\delta}(x^{\delta}) = \sum_{n=0}^{\infty} \frac{x^{n\delta}}{\Gamma(1+n\delta)}, x \in \mathbb{R}.$$

Lemma 1. *The local fractional derivative and integral of the Mittag–Leffler function are given by*

$$\frac{d^{\delta}}{dx^{\delta}} \mathbb{E}_{\delta}(\tau x^{\delta}) = \tau \mathbb{E}_{\delta}(\tau x^{\delta}),$$

$${}_{a}I_{b}^{(\delta)} \mathbb{E}_{\delta}(x^{\delta}) = \mathbb{E}_{\delta}(b^{\delta}) - \mathbb{E}_{\delta}(a^{\delta}),$$

where τ *is an arbitrary constant.*

Two local fractional integral operators involving Mittag–Leffler kernel are described in [20,21].

Definition 5. *Let us assume that* $g : [a, b] \longrightarrow \mathbb{R}^{\delta}$ *and* $g(x) \in I_x^{(\delta)}[a, b]$. *The local fractional left- and right-side integral operators of order* $\delta \in (0, 1)$ *are defined, respectively, by*

$$\Im_{a^+}^{\delta} g(x) = \frac{1}{\delta^{\delta}\Gamma(1+\delta)} \int_a^x \mathbb{E}_{\delta}(-\frac{1-\delta}{\delta}(x-s))^{\delta} g(s)(ds)^{\delta}, a < x. \tag{6}$$

$$\Im_{b^-}^{\delta} g(x) = \frac{1}{\delta^{\delta}\Gamma(1+\delta)} \int_x^b \mathbb{E}_{\delta}(-\frac{1-\delta}{\delta}(s-x))^{\delta} g(s)(ds)^{\delta}, x < b. \tag{7}$$

On Yang's fractal sets, the generalized h-convex function is defined in [22] as

Definition 6. *Let us consider that* $h : [0, 1] \longrightarrow \mathbb{R}_+$ *is a positive function; then,* $g : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{\delta}$ *is called generalized* h*-convex if* g *is non-negative and the following inequality holds:*

$$g(\vartheta\mu_1 + (1-\vartheta)\mu_2) \leqslant h^{\delta}(\vartheta)g(\mu_1) + h^{\delta}(1-\vartheta)g(\mu_2), \ \ \forall \mu_1, \mu_2 \in U, \ \ \vartheta \in [0, 1]. \tag{8}$$

Some Hermite–Hadamard-type integral inequalities for local fractional integral operators involving Mittag–Leffler kernel using generalized h-convexity are studied in [23].

On Yang's fractal sets, the generalized E-convex function is defined in [24] as

Definition 7. *A function* $g : \mathbb{R} \longrightarrow \mathbb{R}^{\delta}$ *is called a generalized* E*-convex function on a set* $U \subseteq \mathbb{R}$ *if there exists a map* $E : \mathbb{R} \longrightarrow \mathbb{R}$ *such that* U *is an* E*-convex set and*

$$g(\vartheta E(\mu_1) + (1-\vartheta)E(\mu_2)) \leqslant \vartheta^{\delta} g(E(\mu_1)) + (1-\vartheta)^{\delta} g(E(\mu_2)), \tag{9}$$

where $\forall \mu_1, \mu_2 \in U$, $\vartheta \in [0, 1]$, *and* $\delta \in (0, 1]$.

In this article, we focus on some local fractional integral inequalities involving Mittag–Leffler kernel using combined (E, h)-convexity. First, we introduce two local fractional integral operators; then, we establish generalized Hermite–Hadamard-type inequalities related to these operators for the aforementioned class of functions. Finally, using two novel identities involving first-order and second-order local fractional derivatives, we establish some trapezium-type inequalities for the same class of functions.

2. Main Results

We begin this section with the following definitions:

Definition 8. *Let us consider that $h : [0,1] \longrightarrow \mathbb{R}_+$ is a positive function and $E : \mathbb{R} \longrightarrow \mathbb{R}$; then, $g : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^\delta$ is called generalized (E,h)-convex if g is non-negative and the following inequality holds:*

$$g(\vartheta E(\mu_1) + (1-\vartheta)E(\mu_2)) \leqslant h^\delta(\vartheta)g(E(\mu_1)) + h^\delta(1-\vartheta)g(E(\mu_2)), \ \ \forall \mu_1,\mu_2 \in U, \ \ \vartheta \in [0,1]. \tag{10}$$

This type of convexity, which combines generalized h-convexity and E-convexity, defines a larger and more comprehensive class of functions. Essentially, for $h = I$, we obtain generalized E-convexity, and for $E = I$, we obtain generalized h-convexity. Therefore, both of these types of convexity imply generalized (E,h)-convexity, but the converse is false, as shown by the following example:

Example 1. *Let $g : [0,2] \to \mathbb{R}^\delta$ be a non negative function defined as follows:*

$$g(x) = \begin{cases} x^{\frac{\delta}{2}}; & x \in [0,1] \\ x^{2\delta}; & x \in [1,2] \end{cases}$$

Here, g is a generalized (E,h)-convex function with respect to $E(x) = \frac{x+2}{x+1}$ and $h(x) = x^{\frac{3}{2}}$, but not a generalized h-convex function.

Example 2. *Let us assume that*

$$g(x) = \begin{cases} 1^\delta; & x \neq \frac{a+b}{2}, \\ (\frac{1}{2})^\delta; & x = \frac{a+b}{2}, \end{cases}$$

where $g : [a,b] \longrightarrow \mathbb{R}^\delta$, $h(x) = (\frac{1}{x}), x > 0$, and $E : [a,b]^2 \longrightarrow \mathbb{R}$ such that $E(x,y) = \frac{a+b}{2}$, $\forall x,y \in [a,b]$. Consequently, g is a generalized (E,h)-convex function but not a generalized h-convex function.

Definition 9. *Let us assume that $g : [a,b] \longrightarrow \mathbb{R}^\delta$ and $g(x) \in I_x^{(\delta)}[a,b]$. The left- and right-side local fractional integral operators of order $\delta \in (0,1)$ are defined, respectively, by*

$$\begin{aligned} \Im^\delta_{E(a)^+} g(E(x)) = {} & \frac{1}{\delta^\delta \Gamma(1+\delta)} \int_{E(a)}^{E(x)} \mathbb{E}_\delta(-\frac{1-\delta}{\delta}(E(x)-E(s)))^\delta g(E(s))(dE(s))^\delta, \\ & E(a) < E(x). \end{aligned} \tag{11}$$

$$\begin{aligned} \Im^\delta_{E(b)^-} g(E(x)) = {} & \frac{1}{\delta^\delta \Gamma(1+\delta)} \int_{E(x)}^{E(b)} \mathbb{E}_\delta(-\frac{1-\delta}{\delta}(E(s)-E(x)))^\delta g(E(s))(dE(s))^\delta, \\ & E(x) < E(b). \end{aligned} \tag{12}$$

Remark 1.

1. *If $E = I$, then Definition 8 returns to Definition 2 in [23].*
2. *If $\delta = 1$, then*

$$\lim_{\delta\longrightarrow 1} \Im^{\delta}_{E(a)^+} g(E(x)) = \int_{E(a)}^{E(x)} g(E(s))(dE(s)), E(x) > E(a). \tag{13}$$

$$\lim_{\delta\longrightarrow 1} \Im^{\delta}_{E(b)^-} g(E(x)) = \int_{E(x)}^{E(b)} g(E(s))(dE(s)), E(b) > E(x). \tag{14}$$

In what follows, we set $\lambda = \frac{1-\delta}{\delta}(E(b) - E(a))$.

Theorem 3. *Let us assume that $g : [a,b] \longrightarrow \mathbb{R}^{\delta}$ is a positive function, with $0 < a < b$ and $g(x) \in I_x^{(\delta)}[a,b]$. If g is a generalized (E,h)-convex function on $[a,b]$, then the following inequalities hold:*

$$\begin{aligned} &\frac{1^{\delta} - \mathbb{E}_{\delta}(-\lambda)^{\delta}}{\lambda^{\delta} h^{\delta}(\frac{1}{2})} g\left(\frac{E(a)+E(b)}{2}\right) \\ \leqslant\ & \left(\frac{\delta}{E(b)-E(a)}\right)^{\delta}\left[\Im^{\delta}_{E(a)^+} g(E(b)) + \Im^{\delta}_{E(b)^-} g(E(a))\right] \\ \leqslant\ & [g(E(a)) + g(E(b))]\ {}_0 I_1^{(\delta)} \mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}[h^{\delta}(\vartheta) + h^{\delta}(1-\vartheta)]. \end{aligned} \tag{15}$$

Proof. Since g is a generalized (E,h)-convex function on $[a,b]$, then

$$g\left(\frac{E(x)+E(y)}{2}\right) \leqslant h^{\delta}\left(\frac{1}{2}\right) g(E(x)) + h^{\delta}\left(\frac{1}{2}\right) g(E(y)).$$

Let us put $E(x) = \vartheta E(a) + (1-\vartheta)E(b)$ and $E(y) = (1-\vartheta)E(a) + \vartheta E(b)$; then,

$$\frac{1^{\delta}}{h^{\delta}\left(\frac{1}{2}\right)} g\left(\frac{E(a)+E(b)}{2}\right) \leqslant g(\vartheta E(a) + (1-\vartheta)E(b)) + g((1-\vartheta)E(a) + \vartheta E(b)). \tag{16}$$

By multiplying $\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}$ on both sides of inequality (16) and integrating the result into $[0,1]$, we get

$$\begin{aligned} &\frac{1^{\delta}}{h^{\delta}\left(\frac{1}{2}\right)} g\left(\frac{E(a)+E(b)}{2}\right) \frac{1}{\Gamma(1+\delta)} \int_0^1 \mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta} (d\vartheta)^{\delta} \\ \leqslant & \frac{1}{\Gamma(1+\delta)} \int_0^1 \mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta} g(\vartheta E(a) + (1-\vartheta)E(b))(d\vartheta)^{\delta} \\ &+ \frac{1}{\Gamma(1+\delta)} \int_0^1 \mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta} g((1-\vartheta)E(a) + \vartheta E(b))(d\vartheta)^{\delta}. \end{aligned} \tag{17}$$

Let us put, $E(s) = \vartheta E(a) + (1-\vartheta)E(b)$ and $E(r) = (1-\vartheta)E(a) + \vartheta E(b)$; then,

$$\begin{aligned} &\frac{1^{\delta} - \mathbb{E}_{\delta}(-\lambda)^{\delta}}{\lambda^{\delta} h^{\delta}(\frac{1}{2})} g\left(\frac{E(a)+E(b)}{2}\right) \\ \leqslant & \left(\frac{1}{E(b)-E(a)}\right)^{\delta}\left[\frac{1}{\Gamma(1+\delta)} \int_{E(a)}^{E(b)} \mathbb{E}_{\delta}\left(-\lambda(\frac{E(b)-E(s)}{E(b)-E(a)})\right)^{\delta} g(E(s))(dE(s))^{\delta} \right. \\ &\left. + \frac{1}{\Gamma(1+\delta)} \int_{E(a)}^{E(b)} \mathbb{E}_{\delta}\left(-\lambda(\frac{E(r)-E(a)}{E(b)-E(a)})\right)^{\delta} g(E(r))(dE(r))^{\delta}\right] \end{aligned}$$

$$
\begin{aligned}
&=\left(\frac{1}{E(b)-E(a)}\right)^{\delta}\left[\frac{1}{\Gamma(1+\delta)}\int_{E(a)}^{E(b)}\mathbb{E}_{\delta}\left(-\frac{1-\delta}{\delta}\right)^{\delta}(E(b)-E(s))^{\delta}g(E(s))(dE(s))^{\delta}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\delta)}\int_{E(a)}^{E(b)}\mathbb{E}_{\delta}\left(-\frac{1-\delta}{\delta}\right)^{\delta}(E(r)-E(a))^{\delta}g(E(r))(dE(r))^{\delta}\right]\\
&=\left(\frac{\delta}{E(b)-E(a)}\right)^{\delta}\left[\Im^{\delta}_{E(a)^{+}}g(E(b))+\Im^{\delta}_{E(b)^{-}}g(E(a))\right].
\end{aligned}
\tag{18}
$$

Therefore, the first inequality of (15) holds.

Now, we prove the second inequality of (15). Since g is a generalized (E,h)-convex function for $t\in[0,1]$, then

$$g(\vartheta E(a)+(1-\vartheta)E(b))\leqslant h^{\delta}(\vartheta)g(E(a))+h^{\delta}(1-\vartheta)g(E(b)).$$

$$g((1-\vartheta)E(a)+\vartheta E(b))\leqslant h^{\delta}(1-\vartheta)g(E(a))+h^{\delta}(\vartheta)g(E(b)).$$

Adding the two sides of the above two inequalities gives

$$g(\vartheta E(a)+(1-\vartheta)E(b))+g((1-\vartheta)E(a)+\vartheta E(b))\leqslant[h^{\delta}(\vartheta)+h^{\delta}(1-\vartheta)][g(E(a))+g(E(b))]. \tag{19}$$

By multiplying $\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}$ on both sides of inequality (19) and integrating the result into $[0,1]$, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^{\delta}\\
&+\frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g((1-\vartheta)E(a)+\vartheta E(b))(d\vartheta)^{\delta}\\
&\leqslant[g(E(a))+g(E(b))]\frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda t)^{\delta}[h^{\delta}(\vartheta)+h^{\delta}(1-\vartheta)](d\vartheta)^{\delta}.
\end{aligned}
$$

From (17) and (18), the above inequality becomes

$$
\begin{aligned}
&\left(\frac{\delta}{E(b)-E(a)}\right)^{\delta}[\Im^{\delta}_{E(a)^{+}}g(E(b))+\Im^{\delta}_{E(b)^{-}}g(E(a))]\\
&\leqslant[g(E(a))+g(E(b))]\frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}[h^{\delta}(\vartheta)+h^{\delta}(1-\vartheta)](d\vartheta)^{\delta}\\
&=[g(E(a))+g(E(b))]\,{}_{0}I_{1}^{\delta}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}[h^{\delta}(\vartheta)+h^{\delta}(1-\vartheta)].
\end{aligned}
$$

Therefore, the second inequality of (15) holds. This completes the proof. □

Remark 2.

1. *For $\delta\longrightarrow 1$, in the above Theorem, we obtain the following inequalities:*

$$
\begin{aligned}
\frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{E(a)+E(b)}{2}\right)&\leqslant\frac{2}{E(b)-E(a)}\int_{E(a)}^{E(b)}g(E(s))ds\\
&\leqslant[g(E(a))+g(E(b))]\int_{0}^{1}[h(\vartheta)+h(1-\vartheta)]d\vartheta.
\end{aligned}
$$

2. *Let $E=I$; then, we obtain the local fractional integral inequalities (3.3) in [23].*

Corollary 1. *If $h^{\delta}(\vartheta)=\vartheta^{\delta}$ in Theorem 3, then we obtain*

$$g\left(\frac{E(a)+E(b)}{2}\right)\leqslant\frac{(1-\delta)^{\delta}}{2^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}[\Im^{\delta}_{E(a)^{+}}g(E(b))+\Im^{\delta}_{E(b)^{-}}g(E(a))]$$

$$\leqslant \frac{g(E(a)) + g(E(b))}{2^{\delta}}.$$

Proof. If $h^{\delta}(\vartheta) = \vartheta^{\delta}$, by using $E(s)^{\delta} + E(r)^{\delta} = (E(s) + E(r))^{\delta}$, we have

$$\begin{aligned} {}_{o}I_{1}^{(\delta)}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}[h^{\delta}(\vartheta) + h^{\delta}(1-\vartheta)] =& \frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}(\vartheta^{\delta} + (1-\vartheta)^{\delta})(d\vartheta)^{\delta} \\ =& \frac{1^{\delta} - \mathbb{E}_{\delta}(-\lambda)^{\delta}}{\lambda^{\delta}}. \end{aligned}$$

□

Lemma 2. *Let us assume that* $g : [a, b] \longrightarrow \mathbb{R}^{\delta}$ *is a local fractional differentiable function on* $[a, b]$*. If* $g^{\delta}(x) \in I_{x}^{(\delta)}[a, b]$*, then we obtain*

$$\begin{aligned} & \frac{g(E(a)) + g(E(b))}{2^{\delta}} - \frac{(1-\delta)^{\delta}}{2^{\delta}[1^{\delta} - \mathbb{E}_{\delta}(-\lambda)^{\delta}]}(\Im^{\delta}_{E(a)+}g(E(b)) + \Im^{\delta}_{E(b)-}g(E(a))) \\ =& \frac{(E(b) - E(a))^{\delta}}{2^{\delta}[1^{\delta} - \mathbb{E}_{\delta}(-\lambda)^{\delta}]}\left[\frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g^{(\delta)}(\vartheta E(a) + (1-\vartheta)E(b))(d\vartheta)^{\delta}\right. \\ & \left. - \frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}g^{(\delta)}(\vartheta E(a) + (1-\vartheta)E(b))(d\vartheta)^{\delta}\right]. \end{aligned}$$

Proof. Using local fractional integration by parts and putting $E(x) = \vartheta E(a) + (1-\vartheta)E(b)$, we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g^{(\delta)}(\vartheta E(a) + (1-\vartheta)E(b))(d\vartheta)^{\delta} \\ =& (\frac{1}{E(a) - E(b)})^{\delta}\left[\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g(\vartheta E(a) + (1-\vartheta)E(b))|_{0}^{1}\right. \\ & \left. - \frac{1}{\Gamma(1+\delta)}\int_{0}^{1}g(\vartheta E(a) + (1-\vartheta)E(b))(\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta})^{\delta}(d\vartheta)^{\delta}\right] \\ =& \left(\frac{1}{E(a) - E(b)}\right)^{\delta}\left[\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g(E(a)) - g(E(b))\right. \\ & \left. -(-\lambda)^{\delta}\frac{1}{\Gamma(1-\delta)}\int_{0}^{1}g(\vartheta E(a) + (1-\vartheta)E(b))\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}(d\vartheta)^{\delta}\right] \\ =& \frac{\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g(E(a)) - g(E(b))}{(E(a) - E(b))^{\delta}} \\ & -\lambda^{\delta}\delta^{\delta}\left(\frac{1}{E(a) - E(b)}\right)^{2\delta}\frac{1}{\delta^{\delta}\Gamma(1+\delta)}\int_{E(a)}^{E(b)}g(E(x))\mathbb{E}_{\delta}\left(-\frac{1-\delta}{\delta}(E(b) - E(x))\right)^{\delta}(dE(x))^{\delta} \\ =& \frac{\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g(E(a)) - g(E(b))}{(E(a) - E(b))^{\delta}} - \left(\frac{1-\delta}{E(a) - E(b)}\right)^{\delta}\Im^{\delta}_{E(a)+}g(E(b)). \end{aligned} \tag{20}$$

Similarly, we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}g^{(\delta)}(\vartheta E(a) + (1-\vartheta)E(b))(d\vartheta)^{\delta} \\ =& \frac{-g(E(a)) - \mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g(E(b))}{(E(a) - E(b))^{\delta}} + \left(\frac{1-\delta}{E(b) - E(a)}\right)^{\delta}\Im^{\delta}_{E(b)-}g(E(a)). \end{aligned} \tag{21}$$

By subtracting (21) from (20), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\delta)}\int_0^1 \mathbb{E}_\delta(-\lambda\vartheta)^\delta g^{(\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^\delta \\
&-\frac{1}{\Gamma(1+\delta)}\int_0^1 \mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta g^{(\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^\delta \\
=\;&\frac{[1^\delta-\mathbb{E}_\delta(-\lambda\vartheta)^\delta][g(E(a))+g(E(b))]}{(E(b)-E(a))^\delta}-\left(\frac{1-\delta}{E(b)-E(a)}\right)^\delta[\Im^\delta_{E(a)^+}g(E(b))+\Im^\delta_{E(b)^-}g(E(a))].
\end{aligned} \tag{22}$$

The proof is completed. □

Theorem 4. *Let $g:[a,b]\longrightarrow\mathbb{R}^\delta$ be a local fractional differentiable function on $[a,b]$. If $g^{(\delta)}(x)\in I_x^{(\delta)}[a,b]$ and $\mid g^{(\delta)}\mid$ is generalized (E,h)-convex on $[a,b]$, then we obtain*

$$\begin{aligned}
&\left|\frac{g(E(a))+g(E(b))}{2^\delta}-\frac{(1-\delta)^\delta}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]}(\Im^\delta_{E(a)^+}g(E(b))+\Im^\delta_{E(b)^-}g(E(a)))\right| \\
\leqslant\;&\frac{(E(b)-E(a))^\delta(|g^{(\delta)}(E(a))|+|g^{(\delta)}(E(b))|)}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]} \\
&\times\frac{1}{\Gamma(1+\delta)}\int_0^{\frac12}(\mathbb{E}_\delta(-\lambda\vartheta)^\delta-\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta)(h^\delta(\vartheta)+h^\delta(1-\vartheta))(d\vartheta)^\delta.
\end{aligned} \tag{23}$$

Proof. Since h is a positive function and $\left|g^{(\delta)}\right|$ is generalized (E,h)-convex on $[a,b]$, then by using Lemma 2, we have

$$\begin{aligned}
&\left|\frac{g(E(a))+g(E(b))}{2^\delta}-\frac{(1-\delta)^\delta}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]}(\Im^\delta_{E(a)^+}g(E(b))+\Im^\delta_{E(b)^-}g(E(a)))\right| \\
\leqslant\;&\frac{(E(b)-E(a))^\delta}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]}\frac{1}{\Gamma(1+\delta)} \\
&\times\int_0^1|\mathbb{E}_\delta(-\lambda\vartheta)^\delta-\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta||g^{(\delta)}(\vartheta E(a)+(1-\vartheta)E(b))|(d\vartheta)^\delta \\
\leqslant\;&\frac{(E(b)-E(a))^\delta}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]} \\
&\times\left\{\frac{1}{\Gamma(1+\delta)}\int_0^{\frac12}\left(\mathbb{E}_\delta(-\lambda\vartheta)^\delta-\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta\right)\left[h^\delta(\vartheta)|g^{(\delta)}(E(a))|+h^\delta(1-\vartheta)|g^{(\delta)}(E(b))|\right](d\vartheta)^\delta\right. \\
&\left.+\frac{1}{\Gamma(1+\delta)}\int_{\frac12}^1\left(\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta-\mathbb{E}_\delta(-\lambda\vartheta)^\delta\right)\left[h^\delta(\vartheta)|g^{(\delta)}(E(a))|+h^\delta(1-\vartheta)|g^{(\delta)}(E(b))|\right](d\vartheta)^\delta\right\} \\
=\;&\frac{(E(b)-E(a))^\delta}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]}\frac{1}{\Gamma(1+\delta)} \\
&\times\int_0^{\frac12}\left(\mathbb{E}_\delta(-\lambda\vartheta)^\delta-\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta\right)\left[h^\delta(\vartheta)+h^\delta(1-\vartheta)\right]\left[|g^{(\delta)}(E(a))|+|g^{(\delta)}(E(b))|\right](d\vartheta)^\delta \\
=\;&\frac{(E(b)-E(a))^\delta\left[|g^{(\delta)}(E(a))|+|g^{(\delta)}(E(b))|\right]}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]} \\
&\times\frac{1}{\Gamma(1+\delta)}\int_0^{\frac12}\left(\mathbb{E}_\delta(-\lambda\vartheta)^\delta-\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta\right)\left[h^\delta(\vartheta)+h^\delta(1-\vartheta)\right](d\vartheta)^\delta
\end{aligned}$$

This completes the proof. □

Remark 3.

1. *For* $\delta \longrightarrow 1$, *we have*

$$\begin{aligned}&\left|\frac{g(E(a))+g(E(b))}{2}-\frac{1}{E(b)-E(a)}\int_{E(a)}^{E(b)}g(x)dx\right|\\ \leqslant\;& \frac{(E(b)-E(a))(|g'(E(a))|+|g'(E(b))|)}{2}\int_0^{\frac{1}{2}}(1-2\vartheta)(h(\vartheta)+h(1-\vartheta))d\vartheta.\end{aligned}$$

2. *If* $E=I$, *then inequality (23) returns to inequality (3.17) in [23].*

Corollary 2. *If* $h^\delta(\vartheta)=\vartheta^\delta$ *in the last Theorem, we obtain*

$$\begin{aligned}&\left|\frac{g(E(a))+g(E(b))}{2^\delta}-\frac{(1-\delta)^\delta}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]}(\Im^\delta_{E(a)^+}g(E(b))+\Im^\delta_{E(b)^-}g(E(a)))\right|\\ \leqslant\;& \frac{(E(b)-E(a))^\delta|(|g^{(\delta)}(E(a))|+|g^{(\delta)}(E(b))|)}{2^\delta\lambda^\delta}\tanh_\delta\left(\frac{\lambda}{a}\right)^\delta.\end{aligned}$$

Proof. If $h^\delta(\vartheta)=\vartheta^\delta$, by using $E(r)^\delta+E(s)^\delta=(E(r)+E(s))^\delta$,

$$\begin{aligned}&\frac{1}{\Gamma(1+\delta)}\int_0^{\frac{1}{2}}(\mathbb{E}_\delta(-\lambda\vartheta)^\delta-\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta)(h^\delta(\vartheta)+h^\delta(1-\vartheta))(d\vartheta)^\delta\\ =\;& \frac{1}{\Gamma(1+\delta)}\int_0^{\frac{1}{2}}(\mathbb{E}_\delta(-\lambda\vartheta)^\delta-\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta)(d\vartheta)^\delta\\ =\;& \frac{1^\delta-2^\delta\mathbb{E}_\delta(\frac{-\lambda}{2})^\delta+\mathbb{E}_\delta(-\lambda)^\delta}{\lambda^\delta}\\ =\;& \frac{(1^\delta-\mathbb{E}_\delta(\frac{-\lambda}{2})^\delta)^2}{\lambda^\delta}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\frac{g(E(a))+g(E(b))}{2^\delta}-\frac{(1-\delta)^\delta}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]}(\Im^\delta_{E(a)^+}g(E(b))+\Im^\delta_{E(b)^-}g(E(a)))\right|\\ \leqslant\;& \frac{(E(b)-E(a))^\delta(|g^{(\delta)}(E(a))|+|g^{(\delta)}(E(b))|)}{2^\delta[1^\delta-\mathbb{E}_\delta(-\frac{\lambda}{2})^\delta][1^\delta-\mathbb{E}_\delta(-\frac{\lambda}{2})^\delta]}\frac{(1^\delta-\mathbb{E}_\delta(-\frac{\lambda}{2})^\delta)^2}{\lambda^\delta}\\ =\;& \frac{(E(b)-E(a))^\delta(|g^{(\delta)}(E(a))|+|g^{(\delta)}(E(b))|)}{2^\delta\lambda^\delta}\frac{(1^\delta-\mathbb{E}_\delta(-\frac{\lambda}{2})^\delta)^2}{(1^\delta+\mathbb{E}_\delta(-\frac{\lambda}{2})^\delta)}\\ =\;& \frac{(E(b)-E(a))^\delta(|g^{(\delta)}(E(a))|+|g^{(\delta)}(E(b))|)}{2^\delta\lambda^\delta}\tanh_\delta(\frac{\lambda}{4})^\delta.\end{aligned}$$

The proof is completed. □

Lemma 3. *Let us assume that* $g:[a,b]\longrightarrow\mathbb{R}^\delta$ *is a twice local fractional differentiable function on* $[a,b]$. *If* $g^{(2\delta)}(x)\in I_x^{(\delta)}[a,b]$, *we obtain*

$$\begin{aligned}&\tfrac{g(E(a))+g(E(b))}{2^\delta}-\tfrac{(1-\delta)^\delta}{2^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]}(\Im^\delta_{E(a)^+}g(E(b))+\Im^\delta_{E(b)^-}g(E(a)))\\ =\;& \tfrac{(E(b)-E(a))^{2\delta}}{2^\delta\lambda^\delta[1^\delta-\mathbb{E}_\delta(-\lambda)^\delta]}\left[\tfrac{1}{\Gamma(1+\delta)}\int_0^1(\mathbb{E}_\delta(-\lambda)^\delta+1^\delta-\mathbb{E}_\delta(-\lambda\vartheta)^\delta-\mathbb{E}_\delta(-\lambda(1-\vartheta))^\delta)g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^\delta\right].\end{aligned}\tag{24}$$

Proof. Using local fractional integration by parts, then

$$\frac{1}{\Gamma(1+\delta)}\int_0^1\mathbb{E}_\delta(-\lambda\vartheta)^\delta g^{(\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^\delta$$

$$
\begin{aligned}
&= (-\frac{1}{\lambda})^{\delta}\Big[g^{(\delta)}(\vartheta E(a)+(1-\vartheta)E(b))\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}|_{0}^{1} \\
&\quad -\frac{(E(a)-E(b))^{\delta}}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^{\delta}\Big] \\
&= (-\frac{1}{\lambda})^{\delta}\Big[g^{(\delta)}(E(a))\mathbb{E}_{\delta}(-\lambda)^{\delta}-g^{(\delta)}(E(b)) \\
&\quad -\frac{(E(a)-E(b))^{\delta}}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^{\delta}\Big]. \qquad (25)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}g^{(\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^{\delta} \\
&= (\frac{1}{\lambda})^{\delta}\Big[g^{(\delta)}(\vartheta E(a)+(1-\vartheta)E(b))\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}|_{0}^{1} \\
&\quad -\frac{(E(a)-E(b))^{\delta}}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^{\delta}\Big] \\
&= (\frac{1}{\lambda})^{\delta}\Big[g^{(\delta)}(E(a))-g^{(\delta)}(E(b))\mathbb{E}_{\delta}(-\lambda)^{\delta} \\
&\quad -\frac{(E(a)-E(b))^{\delta}}{\Gamma(1+\delta)}\int_{0}^{1}\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^{\delta}\Big]. \qquad (26)
\end{aligned}
$$

By substituting (25) and (26) in (22), we obtain

$$
\begin{aligned}
&\tfrac{g(E(a))+g(E(b))}{2^{\delta}}-\tfrac{(1-\delta)^{\delta}}{2^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}(\Im^{\delta}_{E(a)^{+}}g(E(b))+\Im^{\delta}_{E(b)^{1}}g(E(a))) \\
&= \tfrac{(E(b)-E(a))^{\delta}}{2^{\delta}\lambda^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}\Big[(\mathbb{E}_{\delta}(-\lambda)^{\delta}+1^{\delta})(g^{(\delta)}(E(a))-g^{(\delta)}(E(b))) \\
&\quad -frac(E(b)-E(a))^{\delta}\Gamma(1+\delta)\textstyle\int_{0}^{1}[\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}+\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}]\,g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^{\delta}\Big] \\
&= \tfrac{(E(b)-E(a))^{2\delta}}{2^{\delta}\lambda^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}\Big[\tfrac{1}{\Gamma(1+\delta)}\textstyle\int_{0}^{1}(\mathbb{E}_{\delta}(-\lambda)^{\delta}+1^{\delta}-\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}-\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta})g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))(d\vartheta)^{\delta}\Big].
\end{aligned}
$$

This completes the proof. □

Theorem 5. *Let $g:[a,b]\longrightarrow\mathbb{R}^{\delta}$ be a twice local fractional differentiable function on $[a,b]$. If $g^{(2\delta)}(x)\in I_{x}^{(\delta)}[a,b]$ and $|\,g^{(2\delta)}\,|$ is generalized (E,h)-convex on $[a,b]$, then we obtain*

$$
\begin{aligned}
&\left|\frac{g(E(a))+g(E(b))}{2^{\delta}}-\frac{(1-\delta)^{\delta}}{2^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}\left(\Im^{\delta}_{E(a)^{+}}g(E(b))+\Im^{\delta}_{E(b)^{-}}g(E(a))\right)\right| \\
&\leqslant \frac{(E(b)-E(a))^{2\delta}}{2^{\delta}\lambda^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}\frac{1}{\Gamma(1+\delta)}\left(\mathbb{E}_{\delta}(-\lambda)^{\delta}+1^{\delta}-2\mathbb{E}_{\delta}\left(-\frac{\lambda}{2}\right)^{\delta}\right) \\
&\quad\times\left[|g^{(2\delta)}(E(a))|\int_{0}^{1}h^{\delta}(\vartheta)(d\vartheta)^{\delta}+|g^{(2\delta)}(E(b))|\int_{0}^{1}h^{\delta}(1-\vartheta)(d\vartheta)^{\delta}\right].
\end{aligned}
$$

Proof. If we consider the function $\Lambda(\vartheta)=\mathbb{E}_{\delta}(-\lambda)^{\delta}+1^{\delta}-\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}-\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}$, then from [20], we have:

- Λ is non-negative on $[0,1]$.
- Λ is increasing on $[0,\frac{1}{2})$.
- Λ is decreasing on $(\frac{1}{2},1]$.

Then, function Λ reaches its maximum at $\vartheta=\frac{1}{2}$.

Consequently, we get

$$
\begin{aligned}
&\left|\frac{g(E(a))+g(E(b))}{2^{\delta}}-\frac{(1-\delta)^{\delta}}{2^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}(\Im^{\delta}_{E(a)^{+}}g(E(b))+\Im^{\delta}_{E(b)^{-}}g(E(a)))\right| \\
\leqslant\ & \frac{(E(b)-E(a))^{2\delta}}{2^{\delta}\lambda^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}\frac{1}{\Gamma(1+\delta)} \\
&\times\int_0^1\left|\mathbb{E}_{\delta}(-\lambda)^{\delta}+1^{\delta}-\mathbb{E}_{\delta}(-\lambda\vartheta)^{\delta}-\mathbb{E}_{\delta}(-\lambda(1-\vartheta))^{\delta}\right|\left|g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))\right|(d\vartheta)^{\delta}. \\
\leqslant\ & \frac{(E(b)-E(a))^{2\delta}}{2^{\delta}\lambda^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}\frac{1}{\Gamma(1+\delta)} \\
&\times\int_0^1\left(\mathbb{E}_{\delta}(-\lambda)^{\delta}+1^{\delta}-2\mathbb{E}_{\delta}\left(-\frac{\lambda}{2}\right)^{\delta}\right)\left|g^{(2\delta)}(\vartheta E(a)+(1-\vartheta)E(b))\right|(d\vartheta)^{\delta}.
\end{aligned}
$$

Now, since h is a positive function and $\left|g^{(2\delta)}\right|$ is generalized (E,h)-convex on $[a,b]$, then by using Lemma 3, we have

$$
\begin{aligned}
&\left|\frac{g(E(a))+g(E(b))}{2^{\delta}}-\frac{(1-\delta)^{\delta}}{2^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}(\Im^{\delta}_{E(a)^{+}}g(E(b))+\Im^{\delta}_{E(b)^{-}}g(E(a)))\right| \\
\leqslant\ & \frac{(E(b)-E(a))^{2\delta}}{2^{\delta}\lambda^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}\frac{1}{\Gamma(1+\delta)}\left(\mathbb{E}_{\delta}(-\lambda)^{\delta}+1^{\delta}-2\mathbb{E}_{\delta}\left(-\frac{\lambda}{2}\right)^{\delta}\right) \\
&\times\int_0^1\left[h^{\delta}(\vartheta)|g^{(2\delta)}(E(a))|+h^{\delta}(1-\vartheta)|g^{(2\delta)}(E(b))|\right](d\vartheta)^{\delta}. \\
=\ & \frac{(E(b)-E(a))^{2\delta}}{2^{\delta}\lambda^{\delta}[1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}\frac{1}{\Gamma(1+\delta)}\left(\mathbb{E}_{\delta}(-\lambda)^{\delta}+1^{\delta}-2\mathbb{E}_{\delta}\left(-\frac{\lambda}{2}\right)^{\delta}\right) \\
&\times\left[|g^{(2\delta)}(E(a))|\int_0^1 h^{\delta}(\vartheta)(d\vartheta)^{\delta}+|g^{(2\delta)}(E(b))|\int_0^1 h^{\delta}(1-\vartheta)(d\vartheta)^{\delta}\right].
\end{aligned}
$$

This completes the proof. □

3. Application Examples

In this section, we provide examples to illustrate the application of results in the previous section.

Example 3. *Let $0<a<b$; we can obtain the following inequalities:*

$$\frac{1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}}{2^{2\delta}\lambda^{\delta}}(a+b)^{3\delta}\leqslant\left(\frac{\delta}{b-a}\right)^{\delta}\left[\Im^{\delta}_{a^{+}}b^{3\delta}+\Im^{\delta}_{b^{-}}a^{3\delta}\right]\leqslant\frac{[a^{3\delta}+b^{3\delta}][1^{\delta}-\mathbb{E}_{\delta}(-\lambda)^{\delta}]}{\lambda^{\delta}}. \tag{27}$$

Proof. By taking $g(x)=x^{3\delta}, x\in(0,\infty)$, since g is a generalized (E,h)-convex function for $E(x)=x$ and $h(x)=x$, and by applying Theorem 3, inequalities (27) are obtained. □

Example 4. *Let $0<a<b$; we can obtain the following inequalities:*

$$
\begin{aligned}
&\left|\frac{(\lambda+1)(a^{\lambda}b-ab^{\lambda})+(\lambda-1)(b^{\lambda+1}-a^{\lambda+1})}{(\lambda+1)(b-a)}\right| \\
\leq\ & |\lambda|(b-a)\left(\left|a^{\lambda-1}\right|+\left|b^{\lambda-1}\right|\right)\left(\frac{1+2^{k}k}{2^{k}(k+1)(k+2)}\right).
\end{aligned} \tag{28}
$$

Proof. By taking $g(x)=x^{\lambda}, x\geq 0, \lambda\in\mathbb{R}$, $h(x)=x^{k}, k\in\mathbb{R}$ and $E=I$, g is a classical h-convex function. So, according to the result in Remark 3, we obtain inequalities (28). □

4. Conclusions

In this paper, new generalized Hermite–Hadamard-type inequalities related to local fractional integral operators involving Mittag–Leffler kernel are established for the class of (E, h)-convex functions on fractal sets. Then, some new trapezium-type inequalities are derived from two new identities using local fractional integrals involving first-order and second-order derivatives.

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