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Integrity on *m*-Polar Fuzzy Graphs and Its Application

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Abstract: Integrity for crisp graph theory is a well-defined topic. However, the integrity concept for fuzzy graphs has only recently been defined and extensively researched. However, in *m*-polar fuzzy graphs (*m*PFG), each node as well as edges has *m* components. So, defining integrity in the *m*PF environment needs a new concept. As in the *m*-polar fuzzy environment, each node and edge has *m* components, so we have more flexibility to address the uncertainty rather than fuzzy as well as other uncertain environments. In this article, we developed a brand-new idea known as node integrity on *m*PFG and went in-depth on a few of their related properties. We have thoroughly covered some of their related properties as well as a brand-new idea called dominating integrity on *m*PFG. Different types of integrity on *m*PFG such as node integrity, dominating integrity, and edge integrity are discussed thoroughly along with some of its interesting facts have been introduced. Under isomorphism, their properties have also been studied. We also discussed the interrelation between them. A new type of *m*PFG called efficient *m*PFG which is directly related to dominating integrity concept has also been introduced. Several facts about efficient *m*PFG have also been studied here along with details descriptions. Finally, a real-world mobile network application that is directly related to the integrity of the *m*PFG concept has been discussed.

Keywords: *m*-polar fuzzy graph; node integrity; edge integrity; dominating set; dominating integrity

MSC: 05C72; 03E72

Citation: Muhiuddin, G.; Mahapatra, T.; Pal, M.; Alshahrani, O.; Mahboob, A. Integrity on *m*-Polar Fuzzy Graphs and Its Application. *Mathematics* **2023**, *11*, 1398. https://doi.org/10.3390/math11061398

Academic Editor: Aleksandar Aleksic

Received: 3 February 2023
Revised: 5 March 2023
Accepted: 7 March 2023
Published: 14 March 2023

1. Introduction

1.1. Research Background and Related Works

In technical development, fuzzy graph (FG) theory has an important role. The way of many rule-based expert systems for engineers have been made from FG theory. It is seen in the maximum time that graph theory is found as an essential part of connectivity in some fields of geometry, algebra, topology, number theory, computer science, operations research and optimization. In real life, many problems have been solved using data that come from different origins or sources. This type of data collection represents multi-polarity. In this type of polarity, we can not be structured well by the conception of fuzzy models or bipolar fuzzy models. For example, if we consider a mobile networking model which assures minimum installations of towers to cover the whole area so that no loss of signal throughout the area. For this, we assign the node membership value (MV) based on the situation of the mobile towers as (maximum capacity of a tower unit, distance covered for signaling systems, and material used for installation of a mobile tower). In nature, these terms are uncertain. To represent this situation, we need to use the 3-polar fuzzy model. Using a fuzzy model or an intuitionistic fuzzy model, or a bipolar fuzzy model, this situation can not be handled.

To address uncertainty and fuzziness, Zadeh [1] introduced fuzzy sets. From that point forward, the hypothesis of fuzzy sets has become a space of exploration in different orders. In 1994, Zhang [2,3] expanded the thought of fuzzy sets to bipolar fuzzy sets, which have numerous applications in mathematical speculations just as real-life problems. The set concept in a m-dimension fuzzy environment was then introduced by Chen et al. [4] as an extension of bipolar fuzzy sets. Utilizing Zadeh's procedure, Kauffman [5] fostered an incredible strategy of fuzzy graph theory in 1973. After that, Rosenfeld [6] gave another explained idea of a fuzzy graph. Several concepts on fuzzy graphs have been investigated on [7–11]. In 2014, the influential thought of the m-PFS was proposed by Chen et al. [4], which is an augmentation of bipolar fuzzy sets to manage these sorts of properties of algebraic and graphical models. Some basic properties on mPFG has been inspected by Ghorai and Pal [12,13]. Different properties of graphs under m-polar fuzzy environment have been studied by Akram et al. [14–16]. The conception of domination integrity and efficient FG was initiated by Mariappan et al. [17]. The conception of Concept of integrity as well as its value of FGs was introduced by Saravanana et al. [18]. They also developed a study on regular FGs and the integrity of FG [19]. Next, Mahapatra et al. [20,21] presented coloring on different uncertainty environments.

In real life, many problems have been solved using data that come from different origins or sources. This type of data collection represents multi-polarity. In this type of polarity, we can not be structured well by the conception of fuzzy models or bipolar fuzzy models. For example, we consider a mobile networking model which assures minimum installations of the tower to cover the whole area so that no loss of signal throughout the area. For this, we assign the node membership value (MV) based on the situation of the mobile towers as (maximum capacity of a tower unit, distance covered for signaling systems, and material used for installation of the mobile tower). In nature, these terms are uncertain. To represent this situation, we need to use the 3-polar fuzzy model. Using a fuzzy model or an intuitionistic fuzzy model, or a bipolar fuzzy model, this situation can not be handled. Thus, the integrity concept on FGs is not appropriate for operation in these kinds of situations. To overcome this situation, we are interested in working using the concepts of integrity concept in a m-polar fuzzy environment. Anyone can analyze the MVs in a multi-polar fuzzy environment in a certain way. Since in our consideration, we consider three components for each node as well as edges, therefore we can not handle this type of situation using a fuzzy model as there is a single component for this concept. Again, we can not apply a bipolar or intuitionistic fuzzy graph model as each edge or node have just two components. Thus, these mPFG models give more efficient fuzziness results than another fuzzy model. Furthermore, it is very interesting to develop and analyze such types of mPFGs with examples and related theorems. These definitions and theorems are definitely improving the existing concepts of mPFGs and are more reliable for solving any complicated real-life problem.

1.2. Contribution of This Study

The formation of this article is as follows: Section 2 mentions some useful concepts which are essential for this article. In Section 3, we have defined the different types of integrity on mPFG and provided some theorems on their aspect. In Section 3.1, Node integrity on mPFG is a brand-new idea that we have introduced along with some of its related properties. In Section 3.2, We have thoroughly covered some of their related properties as well as a brand-new idea called dominating integrity on mPFG. Here, we also investigated some relations between dominating integrity and node integrity. In Section 3.3, we have introduced a new concept called node integrity on mPFG and also discussed some of their related properties thoroughly. We investigated some features based on the above conception. In Section 4, we have discussed new types of mPFG called efficient mPFG along with its details description. In Section 5, we have discussed an application based on a mobile networking problem. In Section 7, we have discussed the advantages and limitations of the proposed model. Finally, the conclusion of the study has been presented in Section 8.

2. Preliminaries

Here, we briefly call again some definitions connected to mPFG, such as complete mPFG, strong mPFG, and path in mPFG.

Throughout this article $p_i : [0,1]^m \to [0,1]$ indicated ith material of projection mapping. Furthermore, $i = 1(1)m$ indicates that $i = 1, 2, \ldots, m$.

Definition 1 ([13]). *An mPFG* $\Gamma = (\tilde{A}, \sigma, \mu)$ *having underlying crisp graph (UCG)* $\Gamma^* = (\tilde{A}, \tilde{B})$, *where* $\sigma : \tilde{A} \to [0,1]^m$ *and* $\mu : \tilde{A} \times \tilde{A} \to [0,1]^m$ *indicate an mPFS of* $\tilde{A}$ *and* $\tilde{A} \times \tilde{A}$, *respectively, and which follows the relation such that for all* $i = 1(1)m$, $p_i \circ \mu(b,c) \leq \{p_i \circ \sigma(b) \wedge p_i \circ \sigma(c)\}$ *for all* $(b,c) \in \tilde{A} \times \tilde{A}$ *as well as* $\mu(b,c) = 0$ *for all* $(b,c) \in (\tilde{A} \times \tilde{A} - \tilde{B})$.

Definition 2 ([22]). $\Gamma = (V, \sigma, \mu)$ *is conferred as complete mPFG provided* $p_i \circ \mu(b,d) = \{p_i \circ \sigma(b) \wedge p_i \circ \sigma(d)\}, i = 1(1)m, \forall b, d \in V$.

Definition 3 ([13]). $\Gamma = (V, \sigma, \mu)$ *is conferred as a mPF strong graph if*

$$p_i \circ \mu(b,d) = \{p_i \circ \sigma(b) \wedge p_i \circ \sigma(d)\},$$

$i = 1(1)m, \forall (b,d) \in E$.

Definition 4 ([15]). *Let* $\Gamma = (V, \sigma, \mu)$ *is an mPFG as well as* $P : b_1, b_2, \ldots, b_k$ *be a path in* G. $S(P)$ *indicates the strength of* P, *defined as* $S(P) = (\min_{1 \leq i < j \leq k} p_1 \circ \mu(b_i, b_j), \min_{1 \leq i < j \leq k} p_2 \circ \mu(b_i, b_j), \ldots, \min_{1 \leq i < j \leq k} p_m \circ \mu(b_i, b_j)) = (\mu_1^n(b_i, b_j), \mu_2^n(b_i, b_j), \ldots, \mu_m^n(b_i, b_j))$.

The strength of connectedness (SC) of the path in between b_1 *and* b_k *is given in the following way:* $CONN_G(b_1, b_k) = (p_1 \circ \mu(b_i, b_j)^\infty, p_2 \circ \mu(b_i, b_j)^\infty, \ldots, p_m \circ \mu(b_i, b_j)^\infty)$, *where* $(p_i \circ \mu(b_i, b_j)^\infty) = \max_{n \in N}(\mu_i^n(b_i, b_j))$.

Definition 5 ([20]). *An edge* $(b,d), b, d \in V$ *is regarded as independently strong for an mPFG* $\Gamma = (V, \sigma, \mu)$ *if* $\frac{1}{2}\{p_i \circ \sigma(b) \wedge p_i \circ \sigma(d)\} \leq p_i \circ \mu(b,d), i = 1(1)m$. *It is considered weak independently if not. In order to determine how strong the edge* (b,d) *is,*

$$p_i \circ I(b,d) = \frac{p_i \circ \mu(b,d)}{p_i \circ \sigma(b) \wedge p_i \circ \sigma(d)}, i = 1(1)m.$$

Definition 6 ([22]). *Consider* Γ *and* Γ' *be two mPFGs of the UCG* $\Gamma^* = (\tilde{A}, \tilde{B})$ *and* $\Gamma'^* = (\tilde{A}', \tilde{B}')$, *respectively. A bijective mapping is an isomorphism between* Γ *and* Γ *is* $f : \tilde{A} \to \tilde{A}'$ *which satisfies*

$$p_i \circ \sigma(t) = p_i \circ \sigma'(f(t)) \text{ and } p_i \circ \mu(t,u) = p_i \circ \mu'(f(t), f(u))$$

$\forall t, u \in \tilde{A}$ *and every* $i = 1(1)m$. *Then* Γ *is called to be isomorphic with* Γ'.

3. Differents Types of Integrity on *m*-Polar Fuzzy Graph

Here, we will discuss a new concept of mPFGs such as node integrity, edge integrity, and dominating integrity such that they fulfill a specific criterion on the node set or edge set or dominating set (DS).

3.1. Node Integrity on m-Polar Fuzzy Graph

In this subsection, we will discuss a new idea of mPFGs node integrity such that they fulfill a specific criterion on the node set.

Definition 7. *Let Γ be an mPFG having UGC $\Gamma^* = (V, E)$. Suppose $A \subset V$ is the set of node bumps whose omission disconnects Γ^*. Now, $p_i \circ m(\Gamma - A)$ indicates the set of the order of the topmost element in the graph $V - A$. Then the node integrity of Γ is indicated by $\tilde{I}(\Gamma)$ and is conferred by $p_i \circ \tilde{I}(\Gamma) = \min_{A \subset V}\{p_i \circ |A| + p_i \circ m(\Gamma - A)\}, \forall i = 1(1)m$.*

Example 1. *Let Γ be a 3PFG displayed in Figure 1. The set of nodes $V = \{a, b, c, d\}$ such that $\sigma(a) = (0.5, 0.4, 0.3), \sigma(b) = (0.3, 0.3, 0.4), \sigma(c) = (0.3, 0.5, 0.2), \sigma(d) = (0.6, 0.5, 0.2)$ with $\mu(a, b) = (0.1, 0.1, 0.05), \mu(b, c) = (0.3, 0.3, 0.2), \mu(c, d) = (0.3, 0.5, 0.2), \mu(d, a) = (0.5, 0.4, 0.2), \mu(b, d) = (0.3, 0.3, 0.2)$.*

The node integrity value in 3PFG Γ, shown in Figure 1 is $\tilde{I}_V(\Gamma) = (1.4, 1.3, 0.9)$ and integrity set $\tilde{I}_V$ of Γ is $\{b, d\}$ which disconnects the graph Γ.

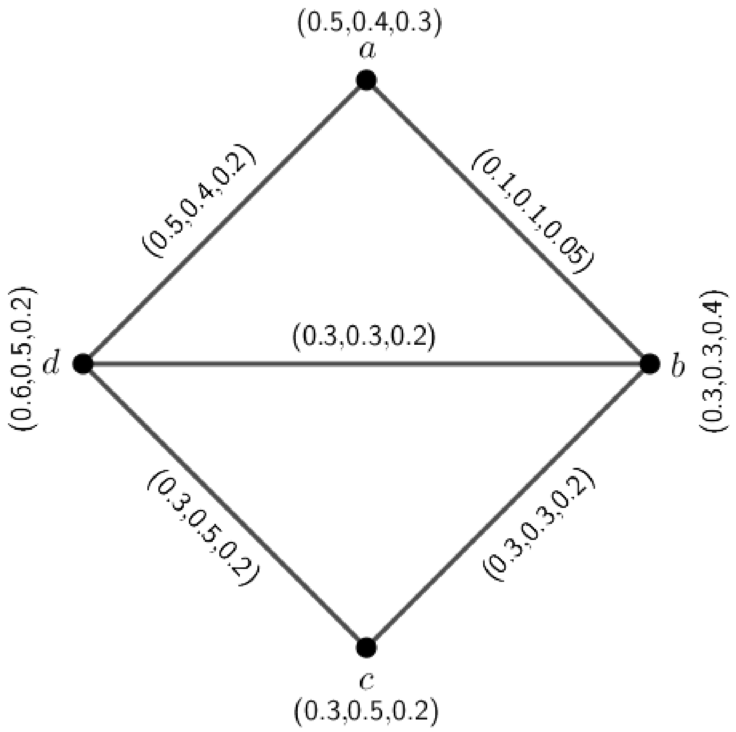

Figure 1. Node integrity of 3PFG Γ.

Theorem 1. *Let P be an mPF subgraph of Γ. Then $\tilde{I}(P) \leq \tilde{I}(\Gamma)$.*

Proof. Assume P is a full mPF subgraph of Γ. Then there is a node, let us say a, of P that has a lower MV than the MV of Γ; otherwise, P has fewer nodes than Γ. Consequently $|P| < |\Gamma|$. However, for any mPFG P, $\tilde{I}(P) \leq |P| < |\Gamma|$. Let an integrity set A of P be such that $\tilde{I}(\Gamma) \leq \tilde{I}(P)$. As a result, $p_i \circ \tilde{I}(P) = \{p_i \circ |A| + p_i \circ m(P - A)\}$, for all $i = 1(1)m$. As a result, $p_i \circ \tilde{I}(P) - \{p_i \circ |A| \geq p_i \circ m(P - A)\}$, for all $i = 1(1)m$. If A is also an integrity set of Γ, then $p_i \circ m(P - A) \leq p_i \circ m(\Gamma - A)$, for all $i = 1(1)m$, which is not feasible, as P is subset of Γ. If A is without integrity set of Γ then $p_i \circ \tilde{I}(\Gamma) - p_i \circ |A| \leq p_i \circ m(\Gamma - A)$, for all $i = 1(1)m$, which is a contradictory. Hence, the theorem. □

Theorem 2. *Node integrity is invariant under isomorphism between two mPFG.*

Proof. Let Γ_1 and Γ_2 be two mPFG and $\phi : \Gamma_1 \to \Gamma_2$ be the isomorphism, that is, $\forall i = 1(1)m$.

1. $p_i \circ \sigma_1(b) = p_i \circ \sigma_2(\phi(b)), \forall a \in V_1$.
2. $p_i \circ \mu_1(b, d) = p_i \circ \mu_2(\phi(b), \phi(d)), \forall (b, d) \in \widetilde{V_1 \times V_1}$.

Suppose, A_1 be set in Γ_1 such that $p_i \circ \tilde{I}(\Gamma_1) = \min_{A_1 \subset V}\{p_i \circ |A_1| + p_i \circ m(G - A_1)\}$. Since Γ_1 and Γ_2 are isomorphic therefore from Equations (i) and (ii) its preserves the node and edge MVs. Hence, clearly $p_i \circ \tilde{I}(\Gamma_2) = \min_{\phi(A_1) \subset V}\{p_i \circ |\phi(A_1)| + p_i \circ m(\Gamma_2 - \phi(A_1))\}$ holds for $\phi(A_1)$. Therefore, Γ_2 is also a node integrity. □

Theorem 3. *Node integrity of a complete mPFG is its order.*

Proof. Let Γ be the full mPFG. Every node is then adjacent to the rest of the remaining nodes. Removing any number of p nodes from Γ yields a single component graph with many $n - p$ nodes. Therefore, for each subset A, the value of $\{p_i \circ |A| + p_i \circ m(\Gamma - A)\}$, $\forall i = 1(1)m$ remains fixed as well as equals to the total MV of the node. Thus, the node integrity of Γ is the same as the order of Γ. □

Theorem 4. *An integrity of a star mPFG $K_{1,n}$ is equal to the max $p_i \circ \sigma(a) + p_i \circ \sigma(c)$, for each $i = 1(1)m$ and $(a,c) \in E$.*

Proof. Each edge of Γ is connected to an intermediate node on the star diagram. So removing this intermediate node leaves all other nodes isolated. So, according to Theorem 3, the maximum value of the remaining nodes MV defines the maximum degree of the remaining graph. Summing this with the MVs of the intermediate nodes gives the completeness of the star graph $K_{1,n}$. Since this central node is linked to this node, it is defined as the integrity of the star graph $K_{1,n}$ is equal to the $max\, p_i \circ \sigma(a) + p_i \circ \sigma(d)$, $\forall i = 1(1)m$ and $(a,d) \in E$. □

3.2. Dominating Integrity on m-Polar Fuzzy Graph

In this section, we'll talk about a novel notion of mPFGs dominating integrity by meeting a particular DS criterion.

Definition 8. *Let us assume that Γ be an mPFG having UGC $\Gamma^* = (V,E)$. Assume that $A \subset V$ is a DS. The set of the largest component's order in the graph $\Gamma - A$ is now indicated by the expression $m(\Gamma - A)$. Then the dominating integrity (DI) of Γ is indicated by $\tilde{DI}(\Gamma)$ and is conferred by $p_i \circ \tilde{DI}(\Gamma) = \min_{A \subset V}\{p_i \circ |A| + p_i \circ m(\Gamma - A)\}$, $\forall i = 1(1)m$.*

Definition 9. *If every node in $V - A$ has at least one strong nbd in A, then the set A is said to be the DS of an mPFG Γ. The minimal cardinality of such a DS is the dominance number of Γ denoted by $\gamma(\Gamma)$, and such a set is called minimal dominance.*

Definition 10. *Any subset of A of $V(\Gamma)$ that is an integrity set $\tilde{I}$-set of Γ if $p_i \circ \tilde{DI}(\Gamma) = \{p_i \circ |A| + p_i \circ m(\Gamma - A)\}$, $\forall i = 1(1)m$.*

Example 2. *Let Γ be a 3PFG depcited in Figure 2. The set of nodes $V = \{a,b,c,d\}$ such that $\sigma(a) = (0.5, 0.4, 0.3), \sigma(b) = (0.3, 0.3, 0.4), \sigma(c) = (0.3, 0.5, 0.2), \sigma(d) = (0.6, 0.5, 0.2)$ with $\mu(a,b) = (0.1, 0.1, 0.05), \mu(b,c) = (0.3, 0.3, 0.2), \mu(c,d) = (0.3, 0.5, 0.2)$, $\mu(d,a) = (0.5, 0.4, 0.2), \mu(b,d) = (0.3, 0.3, 0.2)$.*

Clearly, the arcs $(a,d), (b,d), (b,c)$ and (c,d) are strong. The DS A, $\Gamma - A$, $m(\Gamma - A)$, $|A|$ and $|A| + m(\Gamma - A)$ all are given in Table 1.

From Table 1, we see that $min(|A| + m(\Gamma - A)) = (1.4, 1.3, 0.9)$, which corresponds to the domination set $\{b,d\}$. As a result, $\{b,d\}$ is the minimal DI set of 3PFG Γ.

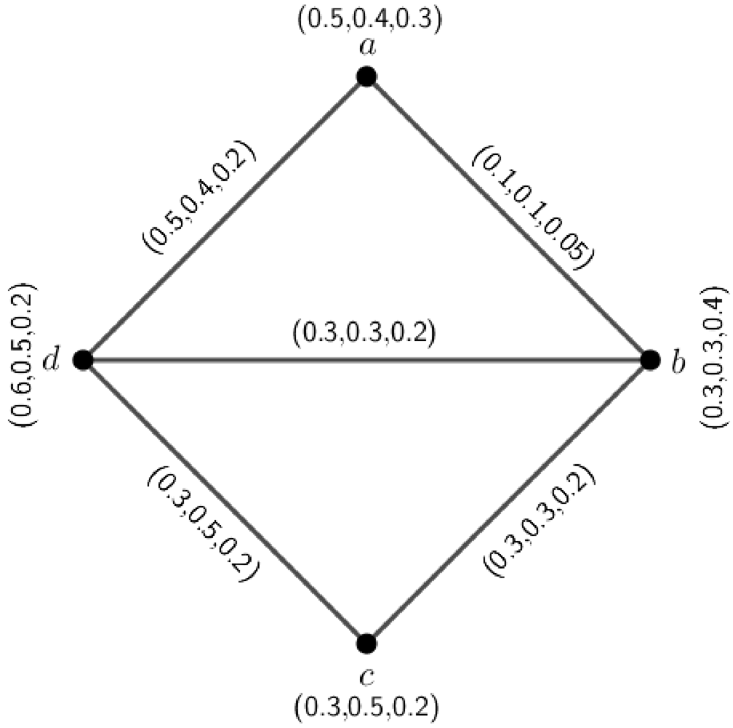

Figure 2. Domination integrity of 3PFG Γ.

Table 1. Domination integrity of 3PFG Γ.

A	$\Gamma - A$	$m(\Gamma - A)$	$\lvert A\rvert$	$\lvert A\rvert + m(\Gamma - A)$
$\{d\}$	$a - b - c$	$(1.1, 1.2, 0.9)$	$(0.6, 0.5, 0.2)$	$(1.7, 1.7, 1.1)$
$\{a, b\}$	$d - c$	$(0.9, 1, 0.4)$	$(0.8, 0.7, 0.7)$	$(1.7, 1.7, 1.1)$
$\{a, c\}$	$d - b$	$(0.9, 0.8, 0.6)$	$(0.8, 0.9, 0.5)$	$(1.7, 1.7, 1.1)$
$\{a, d\}$	$b - c$	$(0.6, 0.8, 0.6)$	$(1.1, 0.9, 0.5)$	$(1.7, 1.7, 1.1)$
$\{c, d\}$	$b - a$	$(0.8, 0.7, 0.7)$	$(0.9, 1, 0.4)$	$(1.7, 1.7, 1.1)$
$\{b, d\}$	$\{a\}, \{c\}$	$(0.5, 0.5, 0.3)$	$(0.9, 0.8, 0.6)$	$(1.4, 1.3, 0.9)$
$\{a, b, c\}$	$\{d\}$	$(0.6, 0.5, 0.2)$	$(1.1, 1.2, 0.9)$	$(1.7, 1.7, 1.1)$
$\{a, c, d\}$	$\{b\}$	$(0.3, 0.3, 0.4)$	$(1.4, 1.4, 0.7)$	$(1.7, 1.7, 1.1)$
$\{b, c, d\}$	$\{a\}$	$(0.5, 0.4, 0.3)$	$(1.2, 1.3, 0.8)$	$(1.7, 1.7, 1.1)$
$\{a, b, d\}$	$\{c\}$	$(0.3, 0.5, 0.2)$	$(1.4, 1.2, 0.9)$	$(1.7, 1.7, 1.1)$
$\{a, b, c, d\}$	ϕ	$(0, 0, 0)$	$(1.7, 1.7, 1.1)$	$(1.7, 1.7, 1.1)$

Theorem 5. *For any mPFG G, $p_i \circ \gamma(G) \leq p_i \circ \tilde{D}I(G)$, for $i = 1(1)m$.*

Proof. Only the cardinality of DS affects the dominance of mPFG. The dominance number, on the other hand, is influenced by DS A and the greatest order of the corresponding components, $G - S$. From this $p_i \circ \gamma(G) \leq p_i \circ \tilde{D}I(G)$, for $i = 1(1)m$. Equality holds for the mPFG with isolated nodes only. Equality applies only to mPFG with isolated nodes. The entire set of nodes is the only DS in graphs with isolated nodes. For this set $p_i \circ m(G^* - A) = 0$, for $i = 1(1)m$. □

Remark 1. *The integrity of G is equal to the maximum MV of G nodes if an mPFG G is null.*

Theorem 6. *DI is invariant under isomorphism between two mPFG.*

Proof. Let Γ_1 and Γ_2 be two mPFG and $\phi : \Gamma_1 \to \Gamma_2$ be the isomorphism, that is, $\forall i = 1(1)m$

1. $p_i \circ \sigma_1(b) = p_i \circ \sigma_2(\phi(b)), \forall a \in V_1$.
2. $p_i \circ \mu_1(b,d) = p_i \circ \mu_2(\phi(b), \phi(d)), \forall (b,d) \in \widetilde{V_1 \times V_1}$.

Suppose, A_1 be a DS in Γ_1 such that $p_i \circ \tilde{D}I(\Gamma_1) = \min_{A_1 \subset V} \{p_i \circ |A_1| + p_i \circ m(\Gamma_1 - A_1)\}$. Since Γ_1 and Γ_2 are isomorphic therefore from Equations (i) and (ii) its preserves the node and edge MVs. Hence, clearly $p_i \circ \tilde{D}I(\Gamma_2) = \min_{\phi(A_1) \subset V} \{p_i \circ |\phi(A_1)| + p_i \circ m(\Gamma_1 - \phi(A_1))\}$ holds for $\phi(A_1)$. Therefore, Γ_2 is also a DI. □

Theorem 7. *Let Γ be an mPFG. $p_i \circ \tilde{D}I(\Gamma) = t$, for $i = 1(1)m$ iff Γ is either complete mPFG or $\bar{\Gamma}$, where $(t, t, \ldots, t)$ is the order of the mPFG.*

Proof. Let Γ be the complete mPFG. Any dominating subset V of Γ, and all that is left of the graph is a single component made up of all the remaining nodes. As a result, the DI of Γ is only the order of Γ. Assume that the entire mPFG is complemented by Γ. Then, the graph with a set of isolated nodes is indicated as G. Thus, $N(a) = \phi$; for all $a \in V$. The total node set is the only DS of Γ. For this DS A, is $p_i \circ m(\Gamma - A) = 0$, for $i = 1(1)m$. DI of Γ is therefore nothing more than Γ order. Thus, the only DS is V. As a result, $p_i \circ \tilde{D}I(\Gamma) = t$, for $i = 1(1)m$.

Consider, however, that A is the DS of Γ and that the order of Γ is the DI number. Since the sum of the MV of A as well as $m(\Gamma - A)$ is not greater than the order of Γ, if $\Gamma - A$ has more than one connected component, then $\Gamma - A$ must only have one connected component. This holds for all DS, but especially for any set of singletons. Given that this singleton node set controls all of Γ's other nodes, it is possible that Γ is a full mPFG. □

Theorem 8. *Let Γ be a complete bi-partiated mPFG $K_{\{\sigma_1, \sigma_2\}}$. Then*

$$p_i \circ \tilde{D}I(\Gamma) = min\{p_i \circ |V_1| + p_i \circ max|\sigma_2|, p_i \circ |V_2| + p_i \circ max|\sigma_1|\},$$

where $V = V_1 \cup V_2$.

Proof. Since Γ is a bi-partiated mPFG, therefore its node set can be partitioned into V_1, V_2. Suppose Γ_1 and Γ_2. Let A be a DS in Γ. Then three cases may arise which are discussed as follows:

Case 1: Suppose $A \in V_1$. The collection of all isolated nodes in V_2 is then called $(\Gamma - A)$. Hence, $p_i \circ m(\Gamma - A) = p_i \circ max|\sigma_2(V_2)|$.

Case 2: Suppose $A \in V_2$. The collection of all isolated nodes in V_1 is then called $(\Gamma - A)$. Hence, $p_i \circ m(\Gamma - A) = p_i \circ max|\sigma_1(V_1)|$.

Case 3: If A is a DS with a node from V_1 and the remainder from V_2, $\Gamma - A$ will still be the only connected component. With one node from V_1 and the remaining nodes from V_2, we can disregard the DS A by taking into account the minimum value for the three DSs mentioned above.

Combining all these cases we get

$$p_i \circ \tilde{D}I(\Gamma) = min\{p_i \circ |V_1| + p_i \circ max|\sigma_2|, p_i \circ |V_2| + p_i \circ max|\sigma_1|\},$$

where $V = V_1 \cup V_2$. □

Theorem 9. *Assume that Γ is a strong mPFG and that $\bar{\Gamma}$ is Γs complement. When Γ and $\bar{\Gamma}$ are united, the node integrity of the union equals the order of the complete mPFG on Γ.*

Proof. As Γ is a strong mPFG, according to the Definition of complement of mPFG, $\bar{\Gamma}$ also qualifies as a strong mPFG. Each node in the resulting graph is next to every other node and has an edge MV that is lower than the MVs of its neighbors. As a result, we know that the order of the mPFG determines the integrity of the mPFG produced by the union of Γ and its complement. As a result, the order of the entire mPFG form in Γ is the integrity of the union of Γ and its complement. □

3.3. Edge Integrity on m-Polar Fuzzy Graph

In this subsection, we will discuss a new idea of mPFGs edge integrity such that they fulfill a specific criterion on the edge set.

Definition 11. *Let us assume that Γ is a mPFG with UGC $\Gamma^* = (V, E)$. Assume that $A \subset E$ is an edge set that, if removed, would render Γ^* as disconnected. The set of the order of the largest component in the graph $\Gamma - A$ is now indicated by $p_i \circ m(\Gamma - A)$. Then the edge integrity of Γ is indicated by $\tilde{I_E}(\Gamma)$ and granted by $p_i \circ \tilde{I_E}(\Gamma) = \min_{A \subset E}\{p_i \circ |A| + p_i \circ m(\Gamma - A)\}$, $\forall i = 1(1)m$.*

Example 3. *Let Γ be a 3PFG depicted in Figure 3. The nodes sets $V = \{a, b, c, d\}$ such that $\sigma(a) = (0.2, 0.5, 0.3), \sigma(b) = (0.6, 0.4, 0.2), \sigma(c) = (0.5, 0.3, 0.2), \sigma(d) = (0.2, 0.3, 0.2)$ with $\mu(a, b) = (0.2, 0.4, 0.1), \mu(b, c) = (0.5, 0.3, 0.2), \mu(c, d) = (0.2, 0.3, 0.2), \mu(d, a) = (0.1, 0.2, 0.2)$, $\mu(a, c) = (0.2, 0.3, 0.2)$.*

Let $B_1 = \{(a, d), (a, c), (b, c)\}$ and $B_2 = \{(d, c), (a, c), (a, b)\}$ be two subsets of edge set of G which disconnect the graph G. Now, $|B_1| = (0.8, 0.8, 0.6)$, $|B_2| = (0.6, 1, 0.5)$, $m(G - B_1) = (0.2, 0.4, 0.2)$ and $m(G - B_2) = (0.5, 0.3, 0.2)$. For the set B_1, the edge integrity of G is $\tilde{I}_{B_1}(G) = (1, 1.2, 0.8)$ and for B_2 is $\tilde{I}_{B_2}(G) = (1.1, 1.3, 0.7)$. Thus, the edge integrity value of the 3PFG G is $\tilde{I}_E(G) = (1, 1.2, 0.7)$ and the integrity sets are $B_1 = \{(a, d), (a, c), (b, c)\}$ and $B_2 = \{(d, c), (a, c), (a, b)\}$.

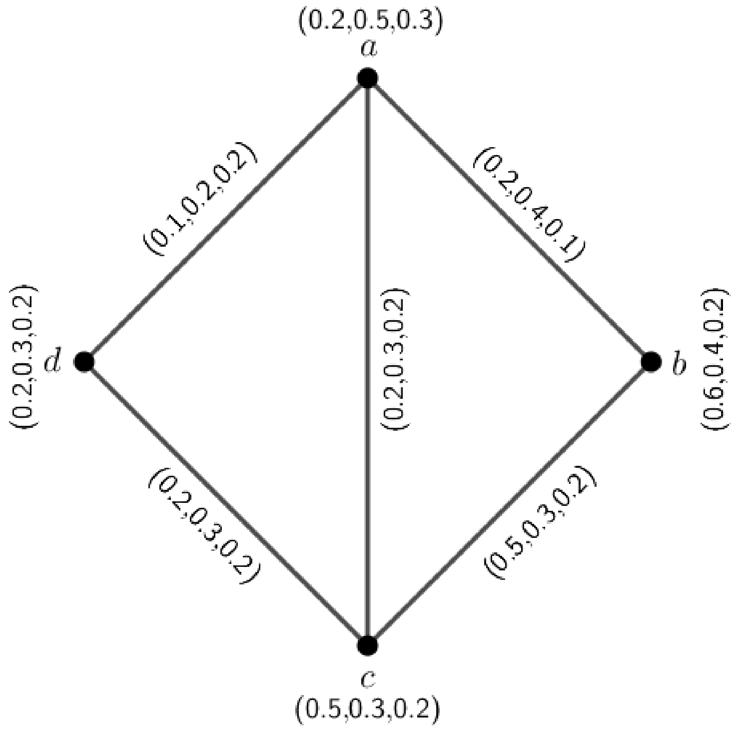

Figure 3. Edge integrity of 3PFG Γ.

Theorem 10. *Let Γ be a complete mPFG. max$\{p_i \circ \sigma(v)\}$ $\forall vs. \in V$ is the edge integrity of complement of the complete mPFG, $\forall vs. \in V$.*

Proof. Complement of complete mPFGs is, obviously, a mPFG without edges. As a result, $\bar{\Gamma} = (V, \bar{\sigma}, \bar{\mu})$ is a set of nodes without edges. Maximum node MV in $\bar{\Gamma} = (V, \bar{\sigma}, \bar{\mu})$

determines the edge integrity of this graph. Thus, for all $I_E\tilde{(\Gamma)} = max\{p_i \circ \sigma(v)\}$, $v \in V$. □

Theorem 11. *The edge integrity of a complete mPFG is its order.*

Proof. Let Γ be the full mPFG. Each node is then connected to the rest of the other nodes. Removing any number of p nodes from Γ yields a single component graph with many $n - p$ nodes. Therefore, for each subset A, the value of $\{p_i \circ |A| + p_i \circ m(\Gamma - A)\}$, $\forall i = 1(1)m$ remains fixed as well as equals to the total value of node MV. As a result, the order of Γ is equal to the edge integrity of Γ. □

Theorem 12. *Let us have a strong mPFG Γ. Order of the entire mPFG formed in Γ is the edge integrity of the union of Γ as well as $\bar{\Gamma}$.*

Proof. The union of Γ and $\bar{\Gamma}$ is understood to be a complete mPFG. Theorem 11 establishes the order of the edge integrity of the entire mPFG. As a result, the edge integrity of the union of Γ as well as $\bar{\Gamma}$ depends on the order of the entire mPFG created in Γ. □

4. Efficient *m*-Polar Fuzzy Graph

Each UCG is undeniably a member of a unique class of mPFG. A mPFG becomes a crisp graph if the σ and μ values are the same for each node and edge. The possibility of the DSs of the mPFG and the UCG being identical is an intriguing discovery. We create the effective mPFG class of mPFGs as a result.

Definition 12. *An efficient mPFG is an mPFG that shares equal DS as its crisp graph, other than V.*

Example 4. *Assume that Γ is the 3PFG depicted in Figure 4. The collection of nodes is $V = \{a, b, c\}$ where $\sigma(a) = (0.3, 0.5, 0.4), \sigma(b) = (0.2, 0.3, 0.5), \sigma(c) = (0.6, 0.5, 0.3)$ with $\mu(a, b) = (0.2, 0.2, 0.3), \mu(b, c) = (0.2, 0.3, 0.3), \mu(c, a) = (0.3, 0.5, 0.2)$.*

Each arc of Γ is strong. There is no doubt that the sets $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{c, a\}$ are the domination sets of Γ^*. Now, $N_A(a) = \{b, c\}$, $N_A(b) = \{c, a\}$, $N_A(c) = \{a, b\}$, $N_A(\{a, b\}) = \{c\}$, $N_A(\{b, c\}) = \{a\}$, $N_A(\{c, a\}) = \{b\}$, which are also the DSs of Γ. Therefore the DSs of Γ^* and Γ are the same. Hence Γ is an efficient 3PFG.

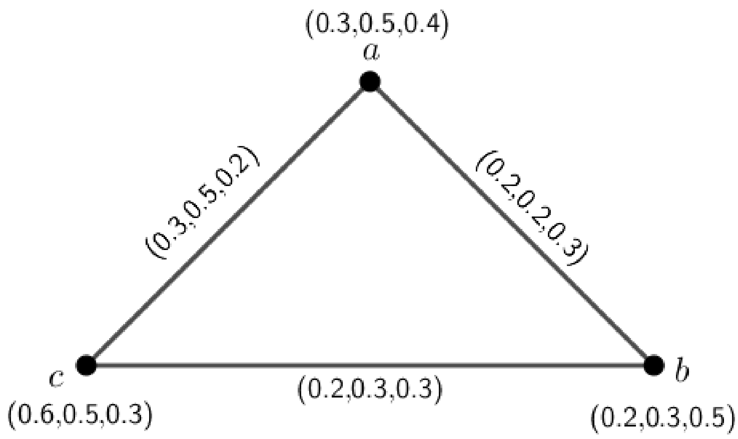

Figure 4. Efficient 3PFG Γ.

Remark 2. *If we take the arc (a, b) in Figure 4 which is not strong, then the set $\{a\}$ and $\{b\}$ can not dominate the whole graph Γ. So $\{a\}$ and $\{b\}$ are not DS s of Γ but they are also DS s of Γ^*. In this case, Γ is not an efficient 3PFG.*

Theorem 13. *Every complete mPFG is an efficient mPFG.*

Proof. Every node set in a complete graph is a DS of Γ. Every edge in the given mPFG Γ is a strong arc since it is a complete mPFG. The edge and node sets of Γ and Γ^* are identical. As a result, they share a DS. Therefore, each complete mPFG is an efficient mPFG. □

Theorem 14. *Every mPFG with a constant μ value is a mPFG that works effectively.*

Proof. According to the idea of a strong arc, arcs are β strong if all of their MV same. All of the arcs are strengthened as a result. Therefore, both the crisp graph and the mPFG graph have the same closed nbd set for each node. As a result of Γ and Γ^* sharing the same DS, Γ is an efficient mPFG. □

Theorem 15. *Let Γ be a mPFG having UCG as a path P_n. Then it is efficient.*

Proof. Since Γ is an mPFG having UCG a path. Therefore we get two cases:-
Case 1: Every arc is an effective arc Hence, Γ and UCG Γ^* share equal node sets and edge sets. Therefore, they contain equal DS. Hence, Γ is an efficient mPFG.
Case 2: Let us say all the arcs are ineffective. Here, we need to demonstrate that all non-efficient arcs must also be strong arcs. In P_n, there is unquestionably one and only one path connecting any two nodes. The graph becomes disconnected when an edge is removed. Therefore, between adjacent nodes, the SC will decrease. Therefore, the edge needs to be strong. Therefore, the strong arcs in P_n are the ineffective arcs. As a result, Γ is an effective graph. □

5. Application

The mPFG is a key mathematical framework for visualizing interconnected real-world phenomena, where nodes and edges are represented by mPF information. In this section, we have illustrated a mobile network installation issue using the idea of integrity on mPFG.

5.1. Model Construction

Suppose you need to cover a group of villages on your mobile phone Telephone network tower. Let us say every village has at least one tower. The graph theory problem looks like this: Every village is a landmark. Villages are connected by edges if the tower installed in the village is close by. Now the issue is identifying the dominant group. However, we cannot guarantee that the best models are produced by a dominant set. Everyone is at risk of network failure during a natural disaster. You will need to cover the biggest area, even if some of the towers fail. As a result, network providers need to think about their networks. A team that covers the most ground. As a graph consistency, this idea came into existence. Both honesty and dominance are taken into consideration by the new parameters. The advantage of the integrity set (set of towers) is that it ensures complete coverage and a more stable network with the widest possible coverage area. The situation with multiple towers in real time

Here, we use the seven villages a, b, c, d, e, f, g as the installation nodes for mobile towers. If the mobile network covers the signal from the closest mobile towers, there will be an edge between two nodes. To solve the allocation issue in this case, dominating integrity in 3PFG $G = (V, \sigma, \mu)$ is used. Three factors are taken into account when calculating the node MVs. Those criteria are as follows: {maximum capacity of a tower unit, distance covered for signaling systems, material used for installation of mobile tower}. Three factors are taken into account when calculating the edge MV. These are listed below: {signal strength between two towers, internet speed, hazards occurrence due to unavoidable circumstances}. The 3PFG model is displayed in Figure 5.

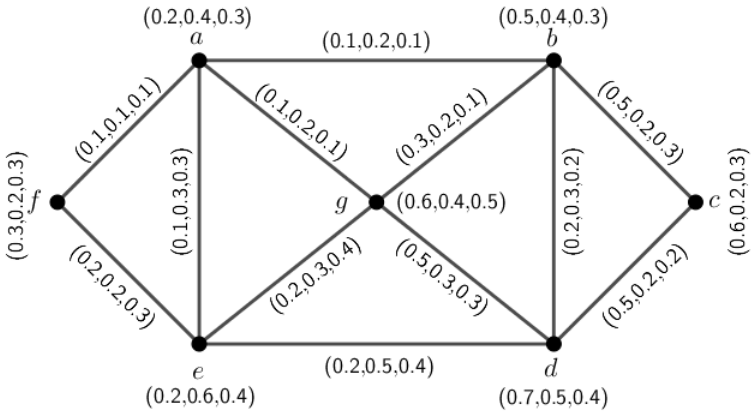

Figure 5. Domination integrity of 3PFG G.

5.2. Illustration of Membership Value

Clearly, the arcs $(e,g),(g,d),(e,d),(b,c),(d,c),(e,f)$ and (a,e) are strong arcs. The DS A, $G-A$, $m(G-A)$, $|A|$ and $|A|+m(G-A)$ all are given in Table 2.

Table 2. Model 3PF domination integrity graph G.

A	$G-A$	$m(G-A)$	$\|A\|$	$\|A\|+m(G-A)$
$\{e,b,d\}$	$a-f-g,\{c\}$	$(1.1,1,1.1)$	$(1.4,1.5,1.1)$	$(2.5,2.5,2.2)$
$\{e,d,c\}$	$f-a-g-b$	$(1.6,1.4,1.4)$	$(1.5,1.3,1.1)$	$(3.1,2.7,2.5)$
$\{e,g,b\}$	$a-f,d-c$	$(1.3,0.7,0.7)$	$(0.7,1,0.7)$	$(2,1.7,1.4)$
$\{e,g,c\}$	$f-a-b-d$	$(1.7,1.5,1.3)$	$(1.4,1.2,1.2)$	$(3.1,2.7,2.3)$
$\{a,b,g,f\}$	$e-d-c$	$(1.5,1.3,1.1)$	$(1.6,1.4,1.4)$	$(3.1,2.7,2.5)$
$\{f,e,g,b\}$	$\{a\},d-c$	$(1.3,0.7,0.7)$	$(1.6,1.6,1.5)$	$(2.9,2.3,2.2)$
$\{a,g,e,c\}$	$b-d,\{f\}$	$(1.2,0.9,0.7)$	$(1.6,1.6,1.5)$	$(2.8,2.5,2.2)$
$\{f,e,d,c\}$	$a-g-b$	$(1.3,1,0.7)$	$(1.8,1.5,1.4)$	$(3.1,2.5,2.1)$
$\{f,a,b,c\}$	$e-g-d$	$(1.5,1.5,1.3)$	$(1.6,1.2,1.2)$	$(3.1,2.7,2.5)$
$\{a,c,d,e\}$	$b-g,\{f\}$	$(1.1,0.8,0.8)$	$(1.7,1.7,1.4)$	$(2.8,2.5,2.2)$
$\{b,c,d,e\}$	$f-a-g$	$(1.1,1,1.1)$	$(2,1.7,1.4)$	$(3.1,2.7,2.5)$
$\{b,d,e,f\}$	$a-g,\{c\}$	$(0.8,0.8,0.8)$	$(1.7,1.7,1.4)$	$(2.5,2.5,2.2)$
$\{a,b,c,d,e\}$	$\{f\},\{g\}$	$(0.6,0.4,0.5)$	$(2.2,2.1,1.7)$	$(2.8,2.5,2.3)$
$\{a,c,d,e,f\}$	$b-g$	$(1.1,0.8,0.8)$	$(2,1.9,1.7)$	$(3.1,2.7,2.5)$
$\{a,d,e,f,g\}$	$b-c$	$(1.1,0.6,0.6)$	$(2,2.1,1.9)$	$(3.1,2.7,2.5)$
$\{b,c,d,e,f\}$	$a-g$	$(0.8,0.8,0.8)$	$(2.3,1.9,1.7)$	$(3.1,2.7,2.5)$
$\{b,d,e,f,g\}$	$\{a\},\{c\}$	$(0.6,0.4,0.3)$	$(2.3,2.1,1.9)$	$(2.9,2.5,2.2)$
$\{c,d,e,f,g\}$	$a-b$	$(0.7,0.8,0.6)$	$(2.4,1.9,1.9)$	$(3.1,2.7,2.5)$
$\{a,b,c,d,e,f\}$	$\{g\}$	$(0.6,0.4,0.5)$	$(2.5,2.3,2)$	$(3.1,2.7,2.5)$
$\{a,c,d,e,f,g\}$	$\{b\}$	$(0.5,0.4,0.3)$	$(2.4,2.3,2.2)$	$(2.9,2.7,2.5)$
$\{b,c,d,e,f,g\}$	$\{a\}$	$(0.2,0.4,0.3)$	$(2.7,2.3,2.2)$	$(2.9,2.7,2.5)$
V	ϕ	$(0,0,0)$	$(2.9,2.7,2.5)$	$(2.9,2.7,2.5)$

5.3. Decision Making

From Table 2, we see that $min(|A|+m(G-A))=(2,1.7,1.4)$, which corresponds to the domination set $\{e,g,b\}$. Hence $\{e,g,b\}$ is the minimal dominating integrity set of 3PFG G. As a result, these sets are trustworthy and dominant in the network. The way the tower was constructed. Compared to other nodes, these nodes offer better protection even during natural disasters.

Through the discussion above, we can conclude that dominating integrity on mPFG actually plays a significant role in this kind of allocation problem. Furthermore, we acknowledge that in the allocation problem, dominating integrity on mPFG is more applicable than dominating integrity on FG.

6. Comparative Study

At first Saravanan et al. [19] introduced an integrity idea for fuzzy graph theory. Next, Mariappan et al. [17] studied the integrity concept and its value in fuzzy graphs. Mariappan et al. [18] then also discussed dominating integrity and efficient fuzzy graph concept. So, all the results discussed earlier are not applicable when the model is considered in another environment like in m-polar fuzzy sets. If we consider a mobile networking model which assures minimum installations of towers to cover the whole area so that no loss of signal throughout the area. For this, we assign the node membership value (MV) based on the situation of the mobile towers as (maximum capacity of a tower unit, distance covered for signaling systems, and material used for installation of the mobile tower). In nature, these terms are uncertain. To represent this situation, we need to use the 3-polar fuzzy model. Using a fuzzy model or an intuitionistic fuzzy model, or a bipolar fuzzy model, this situation can not be handled. Thus, the integrity concept of FGs is not appropriate for operating in these kinds of situations. To overcome this situation, we are interested in working using the concepts of integrity concept in a m-polar fuzzy environment. This is why the proposed model in this paper plays a significant role in such situations to give better results.

7. Advantages and Limitations of the Proposed Work

Some of the advantages of the proposed are as follows:

1. This work mainly depends on m-polar fuzzy logic network system.
2. Many important definitions and theorems are presented in this study which is very useful.
3. A real application of m-polar fuzzy integrity is presented on a resource power controlling system.

Some of the limitations of this study are given as follows:

1. This work mainly focuses on m-polar fuzzy graph.
2. If the membership value of the characters is given in different interval-valued m-polar fuzzy environment, then m-polar fuzzy integrity graph cannot be used.
3. This type of proposed work mains used in control systems.

8. Conclusions

In this article, we developed a brand-new idea known as node integrity on mPFG and went in-depth on a few of their related properties. We have thoroughly covered some of their related properties as well as a brand-new idea called dominating integrity on mPFG. Here, we also investigated some relations between dominating integrity and node integrity. We investigated some features based on the above conception. We studied a new type of mPFG called efficient mPFG along with its details description. A new type of mPFG called efficient mPFG which is directly related to dominating integrity concept has also been introduced. Several facts about efficient mPFG have also been studied here. Finally, a real-world mobile network application that is directly related to the integrity of the mPFG concept has been discussed. We will extend our research work on a more generalized mPFG concept.

Author Contributions: Conceptualization, G.M. and M.P.; Methodology, T.M. and A.M.; Software, O.A.; Validation, T.M.; Investigation, A.M.; Data curation, T.M., A.M. and O.A.; Writing—original

draft, G.M.; Visualization, M.P. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Scientific Research at University of Tabuk for funding this work through Research no. S-0136-1443.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their sincere thanks to the anonymous reviewers for their valuable comments and helpful suggestions which greatly improved the quality of this paper. The authors extend their appreciation to the Deanship of Scientific Research at University of Tabuk for funding this work through Research no. S-0136-1443.

Conflicts of Interest: The authors declare no conflict of interest.

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