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Event-Triggered Control for Intra/Inter-Layer Synchronization and Quasi-Synchronization in Two-Layer Coupled Networks

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Abstract: This paper studies the intra/inter-layer synchronization and quasi-synchronization in two-layer coupled networks via event-triggered control, in which different layers have mutually independent topologies. First, based on Lyapunov stability theory and event-triggered thoughts, hybrid controllers are designed, respectively, for intra-layer synchronization (ALS) and inter-layer synchronization (RLS). Second, a novel event-triggered rule is proposed, under which intra-layer quasi-synchronization (ALQS) and inter-layer quasi-synchronization (RLQS) can be respectively realized, and the event-triggered frequency can be greatly reduced. Moreover, the upper bound of the synchronization error can be flexibly adjusted by changing the parameters in event-triggered conditions, and the Zeno phenomenon about event-triggered control is also discussed in this paper. Finally, numerical examples are provided to confirm the correctness and validity of the proposed scheme.

Keywords: multi-layer networks; event-triggered control; intra-layer synchronization; inter-layer synchronization; quasi-synchronization

MSC: 37N35; 93C10; 93D20; 34H05

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1. Introduction

Synchronization is a ubiquitous collective behavior in nature, which has been widely studied. Recently, the synchronization of complex networks has become one of the most popular topics, and plenty of related research results were introduced in [1–16]. It is noted that most of these studies focus on single-layer networks. In the real world, however, different networks also interact with each other, such as the Internet and railway network, trunk and branch networks of urban public transport, etc. Multi-layer networks (MLNs) can reflect the characteristics of these real networks more roundly in comparison with traditional single-layer networks. Not only the internal topology and coupling strength of each layer, but also the interaction between different layers of network are considered in the multi-layer network model. At present, the research on MLNs has received extensive attention. For example, in [17], Wu et al. studied the synchronization of MLNs with different inner-coupling matrices. Furthermore, in [18,19], the authors considered the synchronization problem of MLNs with continuous and impulsive couplings between different nodes. Zhao et al. explored the synchronization of MLNs with delay, disturbance, or noise in [20–22]. In addition, finite/fixed-time synchronization issues of MLNs were discussed in [23,24]. Jiang et al. [25] studied the controllability of MLNs. MLNs have progressively become a significant research direction in the field of complex networks.

The synchronization of MLNs can be classified into three categories: inter-layer synchronization (RLS) [26–29], intra-layer synchronization (ALS) [30–33], and complete synchronization [34]. Among them, RLS requires the corresponding nodes of different layers to achieve synchronization, whereas ALS requires that nodes in the same layer can realize synchronization, and complete synchronization means that both RLS and ALS are satisfied at the same time. Actually, RLS and ALS are independent and not necessarily related. Thus, it is of great significance to study RLS and ALS problems. Rakshit et al. investigated RLS and ALS about time-varying MLNs in [35]. Zhang et al. discussed ALS and RLS of MLNs with fractional order in [36]. In the actual situation, not all MLNs can realize synchronization naturally, so it is necessary to impose certain control means on the network to achieve synchronization. In [17,30], the relationship between network coupling parameters and synchronization was studied. This can be regarded as a control problem through the adjustment of network parameters to reach network synchronization. However, to control the synchronization of the network by coupling strength, the boundary of coupling strength needs to be determined, which needs to be obtained by complicated calculation for specific system equations. And in practical application, the adjustment of coupling strength may not be easy to achieve. On the other hand, network synchronization through the addition of external control input has also received a lot of attention. Impulsive control and other methods were used to realize network synchronization in [21,32,37]. The external control input can feedback the state information of the network. The form of control is given, and the control parameters can be obtained through only simple calculation. It is simple and effective to realize synchronization in this way. However, most of the current synchronization control strategies for MLNs are actually time-based mechanisms. This will likely cause the controller to update at some point when the system does not need it. For example, when control signals are updated periodically, if system states are the same at the adjacent signal transmission time, the generated feedback control signals will be also the same, which means that the update of the controller is unnecessary, and some signal transmissions are redundant in this process.

In order to utilize network bandwidth more effectively and reduce communication pressure, researchers have proposed an event-triggered mechanism. According to event-triggered conditions, this mechanism determines whether to send state information to the controller for updating the control signal, so as to avoid unnecessary information transmission and reduce the waste of resources. However, in the process of event-triggered control, there may be an infinitesimal interval between two triggers, so that the event can be triggered for an infinite number of times in a finite time, which is often called the Zeno phenomenon. Therefore, it is necessary to exclude the Zeno phenomenon when discussing event-triggered control. Event-triggered control strategy has been widely applied because of its merits. For example, Tabuada studied a simple event-triggered strategy and showed how to ensure system performance [38]. Event-triggered control strategy of multi-agent systems and complex networks was studied in [39–49]. To the best of the authors' knowledge, nevertheless, the event-triggered mechanism has not received adequate attention in the synchronization control problem for MLNs. However, quasi-synchronization means that the synchronization error of each node has a non-zero bound. This concept provides a new thought for the research of multi-layer network synchronization and has great application value in actual engineering. For instance, quasi-synchronization is applied in military communication networks to enhance the anti-dilapidated ability. Few research studies have paid attention to intra-layer quasi-synchronization (ALQS) and inter-layer quasi-synchronization (RLQS) of MLNs [37]. In particular, ALQS and RLQS of MLNs is one of the problems discussed in this paper.

Motivated by the above, we attempt to study the event-triggered control for intra/inter-layer synchronization and quasi-synchronization in a class of two-layer coupled networks. The main contributions and obtained results of this paper are stated as below:

(i) In this paper, two kinds of mesoscale synchronization behaviors, namely ALS and RLS, are studied in two-layer coupled networks. For the non-synchronous two-layer network, different from the previous time-based control strategy [28,30], appropriate event-triggered controllers are designed to realize ALS and RLS, respectively.

(ii) In order to further reduce the event-triggered frequency and the burden of network communication, a novel event-triggered condition is proposed to realize ALQS and RLQS in two-layer coupled networks. Control parameters b and C_0 can be flexibly adjusted to change the final synchronization error threshold and event-triggered frequency.

(iii) Based on the Lyapunov stability theory, in a class of two-layer coupled networks, the sufficient conditions for realizing ALS, RLS, ALQS, and RLQS are strictly given. It is proven from both theory and simulation that the above network control process can exclude the Zeno phenomenon.

This paper is structured as follows. In Section 2, preliminaries such as system model and some definitions are introduced. In Section 3, the main content on intra/inter-layer synchronization and quasi-synchronization is presented. In Section 4, numerical simulations are given to further verify the correctness and validity of the main results. The conclusions are given in Section 5.

Notations: In this paper, $(\cdot)^{\mathrm{T}}$ denotes the transpose of a vector or a matrix; $\|\cdot\|$ stands for the Euclidean norm; $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent the n-dimensional real vector and the $m\times n$-dimensional real matrix, respectively; t_k is the event-triggered time of the corresponding node, and $K_i = \mathrm{diag}(k_i, k_i, \cdots, k_i) \in \mathbb{R}^{n\times n}(i = 1, 2, 3)$ is the control gain matrix; the Kronecker product is represented by $\otimes$; E_n is the $n\times n$ identity matrix; $M \leq 0$ means that M is negative semi-definite real matrix; $(\varepsilon h)_{min} = \min\{\varepsilon h_i | i = 1, 2, \cdots, Q\}$, $(\varepsilon h)_{max} = \max\{\varepsilon h_i |\ i = 1, 2, \cdots, Q\}$; $\lambda_{min}(\cdot)$ are the minimum eigenvalues of the corresponding real matrix.

2. Model Formulation and Preliminaries

Consider a two-layer coupled network with one-to-one inter-layer connections. The first layer and the second layer are represented as the x-layer and y-layer, respectively. Each layer consists of Q nodes. The network can be described by

$$\begin{cases} \dot{x}_i = f(x_i) - c_1 \sum_{j=1}^{Q} a_{ij} H x_j + \varepsilon H(y_i - x_i) + u_{xi}, \\ \dot{y}_i = g(y_i) - c_2 \sum_{j=1}^{Q} b_{ij} H y_j + \varepsilon H(x_i - y_i) + u_{yi}, \end{cases} \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^{\mathrm{T}} \in \mathbb{R}^n$ indicates the state of the ith node in x-layer, and $y_i = (y_{i1}, y_{i2}, \cdots, y_{in})^{\mathrm{T}} \in \mathbb{R}^n$ indicates the state of the i-th node in y-layer, $i = 1, 2, \cdots, Q$. Function f and g are the self-dynamics of each node in the x-layer and y-layer, respectively. Constant c_1 and c_2 are the intra-layer coupling strength of the x-layer and y-layer, respectively. The inter-layer coupling strength is represented by constant ε. The Laplace matrix of x-layer is represented by matrix $A = (a_{ij}) \in \mathbb{R}^{Q\times Q}$. If there is a connection between the ith node and the jth node ($i \neq j$), then $a_{ij} = a_{ji} = -1$, or $a_{ij} = a_{ji} = 0$. Furthermore, let $a_{ii} = -\sum_{j=1}^{Q} a_{ij}$. Similarly, the Laplace matrix of the y-layer is represented by matrix $B = (b_{ij}) \in \mathbb{R}^{Q\times Q}$; $H = \mathrm{diag}(h_1, h_2, \cdots, h_n) \in \mathbb{R}^{n\times n}$ is the inner coupling matrix, which describes the coupling between state components of the nodes; u_{xi} and u_{yi} are the controllers of the corresponding nodes.

Remark 1. *In the system model (1), the presence of the third item, the inter-layer connection, destroys the synchronization within each layer. Similarly, the second item, the intra-layer connection, is not conducive to synchronization of the corresponding nodes between different layers. Therefore, it is necessary to introduce external control input to reach ALS or RLS in MLNs better.*

In this paper, ALS, RLS, ALQS, and RLQS of network system (1) are represented as follows.

Definition 1 ([32]). *For network (1), assume that the synchronization target of the x-layer and y-layer are τ_x and τ_y, respectively. If $\|x_i - \tau_x\| \to 0$ and $\|y_i - \tau_y\| \to 0$ as $t \to +\infty$, ($i = 1, 2, \cdots, Q$), then network (1) realizes ALS.*

Definition 2. *For network (1), assume that the synchronization target of the x-layer and y-layer are τ_x and τ_y, respectively. If there exists $\eta > 0, \delta > 0$ such that $\|x_i - \tau_x\| \le \eta$ and $\|y_i - \tau_y\| \le \eta$, when $t \ge \delta$, $(i = 1, 2, \cdots, Q)$, then network (1) achieves ALQS.*

Definition 3. *Network (1) is said to achieve RLS if $\|x_i - y_i\| \to 0$ as $t \to +\infty$, for $i = 1, 2, \cdots, Q$.*

Definition 4. *Network (1) is said to achieve RLQS if there exists $\eta > 0$, $\delta > 0$ such that $\|x_i - y_i\| \le \eta$, when $t \ge \delta$, for $i = 1, 2, \cdots, Q$.*

Definition 5 ([50]). *If $\lim_{k\to\infty} t_k = \sum_{i=0}^{\infty}(t_{k+1} - t_k)$ converges, then network (1) is said to have the Zeno phenomenon.*

Assumption 1 ([51]). *For the functions $f(x)$, $g(x)$, and any two vectors $\nu_1, \nu_2 \in \mathbb{R}^n$, there exist positive constants ρ_1, ρ_2 such that*

$$(\nu_1 - \nu_2)^{\mathrm{T}}[f(\nu_1) - f(\nu_2)] \le \rho_1(\nu_1 - \nu_2)^{\mathrm{T}}(\nu_1 - \nu_2),$$
$$(\nu_1 - \nu_2)^{\mathrm{T}}[g(\nu_1) - g(\nu_2)] \le \rho_2(\nu_1 - \nu_2)^{\mathrm{T}}(\nu_1 - \nu_2).$$

3. Main Results

In this section, intra/inter-layer synchronization and quasi-synchronization of the two-layer coupled network (1) will be investigated based on Lyapunov stability theory and event-triggered control.

3.1. Intra-Layer Synchronization and Quasi-Synchronization

Denote $z = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \cdots, x_Q^{\mathrm{T}}, y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \cdots, y_Q^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathcal{L} = \begin{bmatrix} c_1 A + \varepsilon E_Q & -\varepsilon E_Q \\ -\varepsilon E_Q & c_2 B + \varepsilon E_Q \end{bmatrix}$. Then network system (1) can be rewritten as

$$\begin{cases} \dot{x}_i = f(x_i) - \sum_{j=1}^{2Q} \mathcal{L}_{ij} H z_j + u_{xi}, \\ \dot{y}_i = g(y_i) - \sum_{j=1}^{2Q} \mathcal{L}_{(i+Q,j)} H z_j + u_{yi}. \end{cases} \tag{2}$$

Define the synchronization target τ_x and τ_y to satisfy

$$\begin{cases} \dot{\tau}_x = f(\tau_x) + \varepsilon H(\tau_y - \tau_x), \tau_x(0) = \tau_{x0}, \\ \dot{\tau}_y = g(\tau_y) + \varepsilon H(\tau_x - \tau_y), \tau_y(0) = \tau_{y0}. \end{cases}$$

The following event-triggered mechanism is proposed:

$$\begin{cases} t_{k+1}^{xi} = sup\{t > t_k^{xi} : \|E_{xi}(t)\| \le \zeta_{x1}\}, \\ t_{k+1}^{yi} = sup\{t > t_k^{yi} : \|E_{yi}(t)\| \le \zeta_{y1}\}, \end{cases} \tag{3}$$

where $\zeta_{x1} = [k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}(t)\| + \exp(-t)$, $\zeta_{y1} = [k_2 + \lambda_{min}(\mathcal{D}) - \rho_2 - 1]\|e_{yi}(t)\| + \exp(-t)$, $\mathcal{D} = [(\mathcal{L} \otimes H)^{\mathrm{T}} + (\mathcal{L} \otimes H)]/2$, and

$$\begin{cases} E_{xi}(t) = K_1[e_{xi}(t) - e_{xi}(t_k^{xi})], t \in [t_k^{xi}, t_{k+1}^{xi}], \\ E_{yi}(t) = K_2[e_{yi}(t) - e_{yi}(t_k^{yi})], t \in [t_k^{yi}, t_{k+1}^{yi}], \end{cases}$$

with $k_1 \ge \rho_1 - \lambda_{min}(\mathcal{D}) + 1$, $k_2 \ge \rho_2 - \lambda_{min}(\mathcal{D}) + 1$, $e_{xi}(t) = x_i(t) - \tau_x(t)$, $e_{yi}(t) = y_i(t) - \tau_y(t)$.

Theorem 1. *Under Assumption 1, for the two-layer coupled network (2), if u_{xi} and u_{yi} are designed as*

$$\begin{cases} u_{xi} = -K_1 e_{xi}(t_k^{xi}), t \in [t_k^{xi}, t_{k+1}^{xi}], \\ u_{yi} = -K_2 e_{yi}(t_k^{yi}), t \in [t_k^{yi}, t_{k+1}^{yi}], \end{cases} \tag{4}$$

together with the event-triggered mechanism (3), then the network (2) achieves ALS and there is no Zeno phenomenon.

Proof. (i). Synchronization analysis of closed-loop system

Let $e(t) = [e_{x1}^{\mathrm{T}}, e_{x2}^{\mathrm{T}}, \cdots, e_{xQ}^{\mathrm{T}}, e_{y1}^{\mathrm{T}}, e_{y2}^{\mathrm{T}}, \cdots, e_{yQ}^{\mathrm{T}}]^{\mathrm{T}}$, the ALS error system is given

$$\begin{cases} \dot{e}_{xi} = f(x_i) - f(\tau_x) - \sum_{j=1}^{2Q} \mathcal{L}_{ij} H e_j + u_{xi}, \\ \dot{e}_{yi} = g(y_i) - g(\tau_y) - \sum_{j=1}^{2Q} \mathcal{L}_{(i+Q,j)} H e_j + u_{yi}. \end{cases}$$

How to obtain the error system is discussed in the appendix in [32]. Construct the Lyapunov function as

$$V(e(t)) = \frac{1}{2} e(t)^{\mathrm{T}} e(t) = \frac{1}{2} \sum_{i=1}^{Q} e_{xi}(t)^{\mathrm{T}} e_{xi}(t) + \frac{1}{2} \sum_{i=1}^{Q} e_{yi}(t)^{\mathrm{T}} e_{yi}(t).$$

Differentiating $V(e(t))$ with respect to t can be calculated as follows:

$$\begin{aligned} \dot{V}(e(t)) &= \sum_{i=1}^{Q} e_{xi}(t)^{\mathrm{T}} \dot{e}_{xi}(t) + \sum_{i=1}^{Q} e_{yi}(t)^{\mathrm{T}} \dot{e}_{yi}(t) \\ &= \sum_{i=1}^{Q} e_{xi}(t)^{\mathrm{T}} [f(x_i) - f(\tau_x) - \sum_{j=1}^{2Q} \mathcal{L}_{ij} H e_j + u_{xi}] + \sum_{i=1}^{Q} e_{yi}(t)^{\mathrm{T}} [g(y_i) - g(\tau_y) \\ &\quad - \sum_{j=1}^{2Q} \mathcal{L}_{(i+Q,j)} H e_j + u_{yi}]. \end{aligned}$$

With Assumption 1, one obtains

$$\begin{aligned} \dot{V}(e(t)) \leq& (\rho_1 - k_1) \sum_{i=1}^{Q} e_{xi}(t)^{\mathrm{T}} e_{xi}(t) + \sum_{i=1}^{Q} e_{xi}(t)^{\mathrm{T}} E_{xi}(t) + (\rho_2 - k_2) \sum_{i=1}^{Q} e_{yi}(t)^{\mathrm{T}} e_{yi}(t) \\ &+ \sum_{i=1}^{Q} e_{yi}(t)^{\mathrm{T}} E_{yi}(t) - e(t)^{\mathrm{T}} (\mathcal{L} \otimes H) e(t). \end{aligned}$$

Therefore, one has

$$\begin{aligned} \dot{V}(e(t)) \leq& (\rho_1 - k_1 - \lambda_{min}(\mathcal{D})) \sum_{i=1}^{Q} e_{xi}(t)^{\mathrm{T}} e_{xi}(t) + \sum_{i=1}^{Q} e_{xi}(t)^{\mathrm{T}} E_{xi}(t) + (\rho_2 - k_2 - \lambda_{min}(\mathcal{D})) \\ & \sum_{i=1}^{Q} e_{yi}(t)^{\mathrm{T}} e_{yi}(t) + \sum_{i=1}^{Q} e_{yi}(t)^{\mathrm{T}} E_{yi}(t) \\ \leq& \sum_{i=1}^{Q} \|e_{xi}(t)\| \{[\rho_1 - k_1 - \lambda_{min}(\mathcal{D})] \|e_{xi}(t)\| + \|E_{xi}(t)\|\} + \sum_{i=1}^{Q} \|e_{yi}(t)\| \{[\rho_2 - k_2 - \\ & \lambda_{min}(\mathcal{D})] \|e_{yi}(t)\| + \|E_{yi}(t)\|\}. \end{aligned}$$

Under the even-triggered condition (3), it holds

$$\dot{V}(e(t)) \leq \sum_{i=1}^{Q} \|e_{xi}(t)\|(\exp(-t) - \|e_{xi}(t)\|) + \sum_{i=1}^{Q} \|e_{yi}(t)\|(\exp(-t) - \|e_{yi}(t)\|).$$

If $\exp(-t) \geq \|e_{xi}(t)\|$, it means that $\|e_{xi}(t)\|$ will asymptotically converge to 0. The same is true for $\|e_{yi}(t)\|$. Therefore, network (2) can realize ALS under the event-triggered controller (4). Otherwise, one has $\dot{V}(e(t)) \leq 0$. It is easy to know that the largest invariant set of $\{\dot{V}(e(t)) = 0\}$ is $\{\|e(t)\| = 0\}$. Therefore, according to LaSalle's principle [52], the synchronization errors asymptotically converge to 0, that is, ALS can be realized.

(ii). Analysis of the inter-execution times

Assume that there is a Zeno phenomenon in system (2). Therefore, there must be x_i satisfying $\lim_{k\to\infty} t_k^{xi} = T$ in this network, where T is a constant and $t_k^{xi} < T$. The derivative of $\|E_{xi}(t)\|$ on the interval (t_k^{xi}, t_{k+1}^{xi}) is obtained:

$$\frac{\mathrm{d}\|E_{xi}(t)\|}{\mathrm{d}t} = \frac{E_{xi}(t)^{\mathrm{T}} E_{xi}(t)}{\|E_{xi}(t)\|} \leq \frac{\|\dot{E}_{xi}(t)\|\|E_{xi}(t)\|}{\|E_{xi}(t)\|} = \|\dot{e}_{xi}(t)\| \leq \omega_{xi}, \tag{5}$$

where $\omega_{xi} = \sup_{0\leq t<T} \{\|\dot{x}_i - \dot{\tau}_x\|\}$. Suppose $\xi(t)$ is a nonnegative function and satisfies

$$\dot{\xi}(t) = \omega_{xi}, \xi(0) = \|E_{xi}(t_k^{xi})\| = 0.$$

By (5) and using the comparison principle of differential equations, one has $\|E_{xi}(t)\| \leq \omega_{xi}(t - t_k^{xi})$. One can get that if $\|E_{xi}(t)\| \leq [k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}(t)\| + \frac{1}{2}\exp(-t)$, then the event-triggered condition (3) will not be reached at time t. As a consequence, the evolution time of $\omega_{xi}(t - t_k^{xi})$ from 0 to $[k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}(t)\| + \frac{1}{2}\exp(-t)$ can be denoted as τ_{xi}. Therefore, the triggering interval $t_{k+1}^{xi} - t_k^{xi}$ is greater than or equal to τ_{xi}. That is, τ_{xi} can be found from the following equation:

$$\omega_{xi}\tau_{xi} = [k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}(t)\| + \frac{1}{2}\exp(-t). \tag{6}$$

Since $\tau_{xi} \leq t_{k+1}^{xi} - t_k^{xi}$ and $\lim_{k\to\infty} t_k^{xi} = T$, a contradiction that $0 = [k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}(T)\| + \frac{1}{2}\exp(-T)$ can be obtained from (6) as $m \to \infty$. Therefore, all nodes in system (2) will not have a Zeno phenomenon. □

Remark 2. *The controller in this paper is the feedback of state information to the network. By introducing the event-triggered mechanism, the control signal u remains unchanged in* $[t_k, t_{k+1}]$*, and the corresponding update will be carried out only when the event-triggered condition is reached (see, for instance, Figure 1). For example, in (3), if* $\|E_{xi}(t_r)\| > \zeta_{x1}(t_r)$*, and* $\|E_{xi}(t)\| \leq \zeta_{x1}(t)$ *for* $t < t_r$*, the control signal* u_{xi} *will be updated with the state information at* t_r*. The controller will make the solutions of the partial node systems approximately become the same function after a certain time T to achieve the purpose of synchronization (if T is large enough, it will become exactly the same function).*

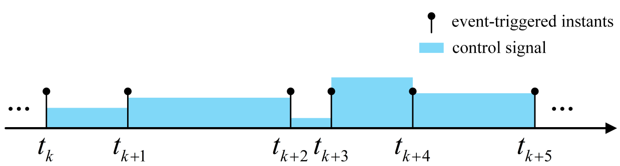


Figure 1. Schematic diagram of event-triggered control.

Remark 3. *In practical applications, the system may not have such high requirements for synchronization performance. In order to further reduce the burden of signal transmission, a new event-triggered rule is proposed.*

Theorem 2. *Under Assumption 1, for the two-layer coupled network (2), if u_{xi} and u_{yi} are designed as (4), together with the event-triggered mechanism*

$$\begin{aligned} t^{xi}_{k+1} &= \begin{cases} sup\{t > t^{xi}_k : \|E_{xi}(t)\| \le \zeta_{x2}\}, \|e_{xi}\| \ge b, \\ sup\{t > t^{xi}_k : \|E_{xi}(t)\| \le \eta_{x2}\}, \|e_{xi}\| < b, \end{cases} \\ t^{yi}_{k+1} &= \begin{cases} sup\{t > t^{yi}_k : \|E_{yi}(t)\| \le \zeta_{y2}\}, \|e_{yi}\| \ge b, \\ sup\{t > t^{yi}_k : \|E_{yi}(t)\| \le \eta_{y2}\}, \|e_{yi}\| < b, \end{cases} \\ &\begin{cases} E_{xi}(t) = K_1[e_{xi}(t) - e_{xi}(t^{xi}_k)], t \in [t^{xi}_k, t^{xi}_{k+1}] \\ E_{yi}(t) = K_2[e_{yi}(t) - e_{yi}(t^{yi}_k)], t \in [t^{yi}_k, t^{yi}_{k+1}] \end{cases} \end{aligned} \tag{7}$$

where $\zeta_{x2} = [k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}(t)\| + \exp(-t)$, $\eta_{x2} = [k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}(t)\| + \exp(-t) + C_0$, $\zeta_{y2} = [k_2 + \lambda_{min}(\mathcal{D}) - \rho_2 - 1]\|e_{yi}(t)\| + \exp(-t)$, $\eta_{y2} = [k_2 + \lambda_{min}(\mathcal{D}) - \rho_2 - 1]\|e_{yi}(t)\| + \exp(-t) + C_0$, $k_1 \ge \rho_1 - \lambda_{min}(\mathcal{D}) + 1$, $k_2 \ge \rho_2 - \lambda_{min}(\mathcal{D}) + 1$, $e_{xi}(t) = x_i(t) - \tau_x(t)$, $e_{yi}(t) = y_i(t) - \tau_y(t)$, $(i = 1, 2, \cdots, Q)$ and C_0 is a positive constant, then the system (2) realizes ALQS and the synchronization error is bounded by a positive constant b. There is no Zeno phenomenon.

Proof. When $\|e_{xi}\| \ge b$ or $\|e_{yi}\| \ge b$, the proof of Theorem 1 shows that $\|e_{xi}\|$ and $\|e_{yi}\|$ can decay below b. When $\|e_{xi}\| < b$ or $\|e_{yi}\| < b$, the analysis process is as follows.

(i). Synchronization analysis of closed-loop system

There are probably some intervals where $\dot{V} > 0$ in the case of Theorem 2. In detail, it needs to be discussed in two cases.

When $\|e_{xi}\| < b$, there is $t_c \in [t^{xi}_k, t^{xi}_{k+1}]$. When $t > t_c$, $\dot{V}$ may be greater than 0. Therefore, the value of $\|e_{xi}\|$ may increase.

(a). $\|E_{xi}(t)\| = [k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}\| + \exp(-t) + C_0$ occurs before $\|e_{xi}\| = b$.

Event ($sup\{t > t^{xi}_k : \|E_{xi}(t)\| \le \zeta_{x2}\}, \|e_{xi}\| \ge b$) will be triggered. Then we have $\dot{V} \le 0$, so the value of $\|e_{xi}\|$ will not continue to increase. Therefore, $\|e_{xi}\|$ will not be greater than b.

(b). $\|e_{xi}\| = b$ occurs before $\|E_{xi}(t)\| = [k_1 + \lambda_{min}(\mathcal{D}) - \rho_1 - 1]\|e_{xi}\| + \exp(-t) + C_0$.

Event ($sup\{t > t^{xi}_k : \|E_{xi}(t)\| \le \eta_{x2}\}, \|e_{xi}\| < b$) will be triggered. Then we have $\dot{V} \le 0$, so the value of $\|e_{xi}\|$ will not continue to increase. Therefore, $\|e_{xi}\|$ will not be greater than b.

When $\|e_{yi}\| < b$, analysis process is the same as the above.

(ii). Analysis of the inter-execution times

It is easy to know that event-triggered mechanism (7) fires fewer times than mechanism (3). Combined with the proof of Theorem 1, no matter what the value of $\|e_{xi}\|$ or $\|e_{yi}\|$ is, the Zeno phenomenon will not appear under triggering mechanism (7). □

3.2. Inter-Layer Synchronization and Quasi-Synchronization

In this section, RLS and RLQS of the two-layer coupled network (1) will be discussed when $g(x) = f(x)$. The nodes in the y-layer can unidirectionally receive information from the corresponding nodes in the x-layer through the control input u. Then network system (1) can be rewritten as

$$\begin{cases} \dot{x}_i = f(x_i) - c_1 \sum_{j=1}^{Q} a_{ij} H x_j + \varepsilon H(y_i - x_i), \\ \dot{y}_i = f(y_i) - c_2 \sum_{j=1}^{Q} b_{ij} H y_j + \varepsilon H(x_i - y_i) + u. \end{cases} \tag{8}$$

The following event-triggered mechanism is proposed:

$$t^i_{k+1} = sup\{t > t_k : \|E_i(t)\| \le \zeta_{y3}\}, \tag{9}$$

where $\zeta_{y3} = [k_3 + 2(\varepsilon h)_{min} - \rho_1 - 1]\|e_i(t)\| + \exp(-t)$, and

$$E_i(t) = -K_3[e_i(t^i_k) - e_i(t)] + \sum_{j=1}^{Q} c_1 a_{ij} H[x_j(t^i_k) - x_j(t)] - \sum_{j=1}^{Q} c_2 b_{ij} H[y_j(t^i_k) - y_j(t)],$$

with $k_3 \ge \rho_1 - 2(\varepsilon h)_{min} + 1$, $e_i(t) = x_i(t) - y_i(t)$ for $t \in [t^i_k, t^i_{k+1}]$.

Theorem 3. *Under Assumption 1, for the two-layer coupled network system (8), if u is designed as*

$$u = K_3[x_i(t^i_k) - y_i(t^i_k)] - \sum_{j=1}^{Q} c_1 a_{ij} H x_j(t^i_k) + \sum_{j=1}^{Q} c_2 b_{ij} H y_j(t^i_k),\ t \in [t^i_k, t^i_{k+1}], \tag{10}$$

together with the event-triggered mechanism (9), then the network (8) achieves RLS and there is no Zeno phenomenon.

Proof. (i). Synchronization analysis of closed-loop system
From $e_i(t) = x_i(t) - y_i(t)$, the RLS error systems are given

$$\begin{aligned}
\dot{e}_i(t) &= \dot{x}_i(t) - \dot{y}_i(t) \\
&= f(x_i) - f(y_i) + \varepsilon H(y_i - x_i) - \varepsilon H(x_i - y_i) - K_3[x_i(t^i_k) - y_i(t^i_k)] + \sum_{j=1}^{Q} c_1 a_{ij} H[x_j(t^i_k) \\
&\quad - x_j(t)] - \sum_{j=1}^{Q} c_2 b_{ij} H[y_j(t^i_k) - y_j(t)] \\
&= f(x_i) - f(y_i) - 2\varepsilon H e_i(t) - K_3[e_i(t^i_k) - e_i(t) + e_i(t)] + \sum_{j=1}^{Q} c_1 a_{ij} H[x_j(t^i_k) - x_j(t)] \\
&\quad - \sum_{j=1}^{Q} c_2 b_{ij} H[y_j(t^i_k) - y_j(t)] \\
&= f(x_i) - f(y_i) - (2\varepsilon H + K_3)e_i(t) + E_i(t).
\end{aligned}$$

Denote $e(t) = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \cdots, e_Q^{\mathrm{T}}]^{\mathrm{T}}$, and construct the Lyapunov function as

$$V(e(t)) = \frac{1}{2} e(t)^{\mathrm{T}} e(t) = \frac{1}{2} \sum_{i=1}^{Q} e_i(t)^{\mathrm{T}} e_i(t).$$

Differentiation of $V(e(t))$ with respect to t can be calculated as follows:

$$\begin{aligned}
\dot{V}(e(t)) &= \sum_{i=1}^{Q} e_i(t)^{\mathrm{T}} \dot{e}_i(t) = \sum_{i=1}^{Q} e_i(t)^{\mathrm{T}} [f(x_i) - f(y_i) - (2\varepsilon H + K_3)e_i(t) + E_i(t)] \\
&\le \rho_1 \sum_{i=1}^{Q} e_i(t)^{\mathrm{T}} e_i(t) - \sum_{i=1}^{Q} e_i(t)^{\mathrm{T}} \begin{bmatrix} 2\varepsilon h_1 + k_3 & 0 & \cdots & 0 \\ 0 & 2\varepsilon h_2 + k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2\varepsilon h_n + k_3 \end{bmatrix} e_i(t) \\
&\quad + \sum_{i=1}^{Q} e_i(t)^{\mathrm{T}} E_i(t)
\end{aligned}$$

$$\leq \sum_{i=1}^{Q} \|e_i(t)\|[\rho_1 - 2(\varepsilon h)_{min} - k_3]\|e_i(t)\| + \|E_i(t)\|.$$

Under the even-triggered condition (9), it holds

$$\dot{V}(e(t)) \leq \sum_{i=1}^{Q} \|e_i(t)\|(\exp(-t) - \|e_i(t)\|).$$

If $\exp(-t) \geq \|e_i(t)\|$, it means that $\|e_i(t)\|$ will asymptotically converge to 0. Therefore, network (8) can realize RLS under the event-triggered controller (10). Otherwise, one has $\dot{V}(e(t)) \leq 0$. It is easy to know that the largest invariant set of $\{\dot{V}(e(t)) = 0\}$ is $\{\|e(t)\| = 0\}$. Therefore, according to LaSalle's principle [52], the synchronization errors asymptotically converges to 0, that is, RLS can be realized.

(ii). Analysis of the inter-execution times

The following proof is similar to Theorem 1. □

Remark 4. *In order to further reduce the event-triggered frequency and communication burden, similarly, the following content is proposed.*

Theorem 4. *Under Assumption 1, for the two-layer coupled network system (8), if u is designed as (9), together with the event-triggered mechanism*

$$t_{k+1}^{i} = \begin{cases} sup\{t > t_k^i : \|E_i(t)\| \leq \zeta_{y4}\}, \|e_i\| \geq b, \\ sup\{t > t_k^i : \|E_i(t)\| \leq \eta_{y4}\}, \|e_i\| < b, \end{cases} \tag{11}$$

$$E_i(t) = -K_3[e_i(t_k^i) - e_i(t)] + \sum_{j=1}^{Q} c_1 a_{ij} H[x_j(t_k^i) - x_j(t)] - \sum_{j=1}^{Q} c_2 b_{ij} H[y_j(t_k^i) - y_j(t)],$$

where $\zeta_{y4} = [k_3 + 2(\varepsilon h)_{min} - \rho_1 - 1]\|e_i(t)\| + \exp(-t)$, $\eta_{y4} = [k_3 + 2(\varepsilon h)_{min} - \rho_1 - 1]\|e_i(t)\| + \exp(-t) + C_0$, $k_3 \geq \rho_1 - 2(\varepsilon h)_{min} + 1$, $e_i(t) = x_i(t) - y_i(t)$, *and* C_0 *is a positive constant, then network (8) achieves RLQS and the synchronization error is bounded by a positive constant b. There is no Zeno phenomenon.*

Proof. Because it is similar to the proof process of Theorem 2, it is omitted here. □

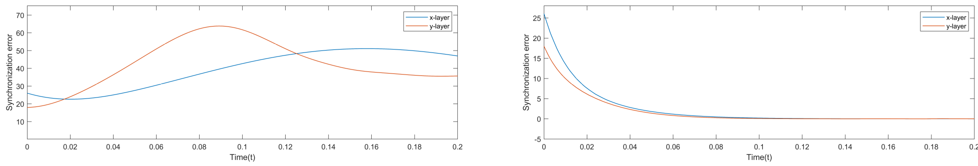
Remark 5. *The control mechanism proposed in this paper realizes mesoscale synchronization behavior in MLNs by feedback of state information. Using event-triggered strategy, the control signal does not need to be updated at all times, but only when the event-triggered condition is satisfied, which can greatly reduce the communication pressure in network systems. In addition, according to the specific properties of MLNs, such as coupling strength, topology structure, etc., the appropriate parameters can be obtained through simple calculation, so that the controller can be adjusted flexibly.*

4. Numerical Simulations

In this section, the validity of the proposed control scheme is further confirmed by simulation experiment. We consider a multi-layer network consisting of two layers. The network has one-to-one connections between layers, and the number of nodes is $Q = 5$ in each layer. The Laplacian matrices of the x-layer and y-layer, respectively, are

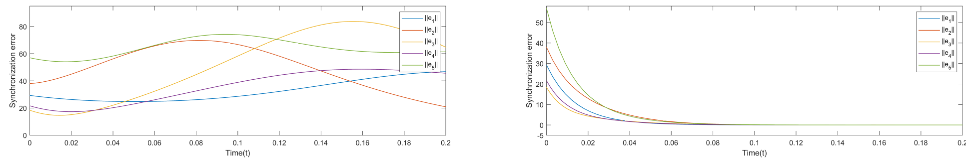
$$A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}.$$

The definition of the Laplacian matrices here are the same as in Section 2. The inner coupling matrix is $H = E_3$. In addition, the coupling strength inside each layer is assumed to be $c_1 = c_2 = 0.1$, and the coupling strength between the different layers is assumed to be $\varepsilon = 0.1$. Under such coupling strength, the network cannot realize ALS or RLS without controllers, which can be clearly seen from Figures 2a and 3a.



(**a**) (**b**)

Figure 2. Intra-layer synchronization (ALS) errors. (**a**) The two−layered network without control. (**b**) The two−layered network under event−triggered control (4).



(**a**) (**b**)

Figure 3. Inter-layer synchronization (RLS) errors. (**a**) The two−layered network without control. (**b**) The two−layered network under event−triggered control (10).

Remark 6. *In this paper, the coupling strength is regarded as the parameters of the network itself, and it is not taken as the control means of network synchronization. Therefore,* c_1*,* c_2*, and* ε *are arbitrarily assumed. Moreover, in order to prove the effectiveness of the control strategy proposed in this paper, the network required by simulation experiment should not be able to realize ALS or RLS without controllers.*

We take the unified chaotic system as network nodal self-dynamics.

$$\begin{cases} \dot{\theta}_{i1} =(25\lambda + 10)(\theta_{i2} - \theta_{i1}) \\ \dot{\theta}_{i2} =(28 - 35\lambda)\theta_{i1} - \theta_{i1}\theta_{i3} + (29\lambda - 1)\theta_{i2} \\ \dot{\theta}_{i3} =\theta_{i1}\theta_{i2} - \dfrac{1}{3}(\lambda + 8)\theta_{i3} \end{cases} \tag{12}$$

When $\lambda = 0$ (resp., $\lambda = 1$), the system (12) is a Lorenz system (resp., Chen system) [32]. The functions corresponding to a Lorenz system and a Chen system are represented by $f(x)$ and $g(x)$, respectively. Therefore, the functions $f(x)$ and $g(x)$ satisfy the following conditions [22,32].

For a Lorenz system,

$$(\alpha_1 - \alpha_2)^{\mathrm{T}}[f(\nu_1) - f(\nu_2)] \leq 39(\nu_1 - \nu_2)^{\mathrm{T}}(\nu_1 - \nu_2).$$

For a Chen system,

$$(\nu_1 - \nu_2)^{\mathrm{T}}[g(\nu_1) - g(\nu_2)] \leq 53(\nu_1 - \nu_2)^{\mathrm{T}}(\nu_1 - \nu_2).$$

where ν_1, ν_2 are any two vectors in $\mathbb{R}^n$.

Under the one-to-one inter-layer connections and the intra-layer connections as shown in A and B, the network structure is shown in Figure 4. Nodes of the same color indicate that they can eventually reach the same state, that is, achieve synchronization.

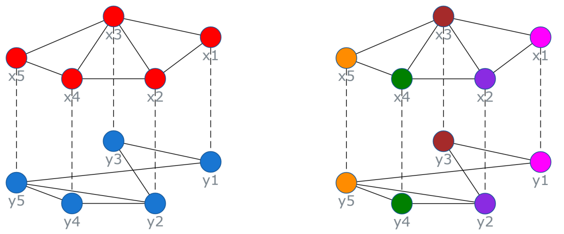


Figure 4. The topological diagram of the two−layer coupled network for ALS (**left**) and RLS (**right**).

Due to the limitations of numerical simulation, we can only judge whether the event will be triggered at some sampling points. In order to more truly simulate the system state, the sampling interval is taken as 0.001. The event-triggered instants of nodes are shown in Figures 5, 6b, 7, and 8b. In these diagrams, black dots represent the sampling instants, and other marks represent the event-triggered instants of the corresponding node. In addition, the triggering rates and average triggering rate (ATR) of all nodes are given in Tables 1–4.

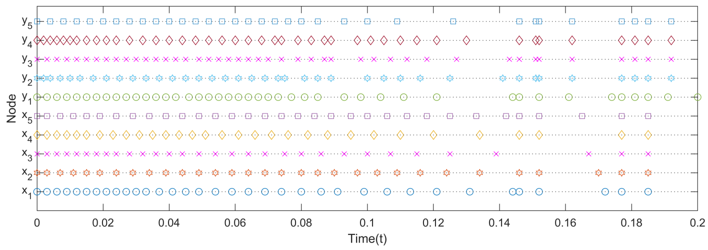
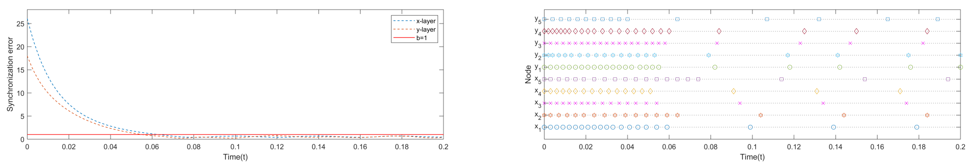


Figure 5. Event−triggered instants on nodes x_i and y_i under condition (3).



(**a**) (**b**)

Figure 6. (**a**) ALS errors in the two−layered network. (**b**) Event−triggered instants on nodes x_i and y_i under condition (7).

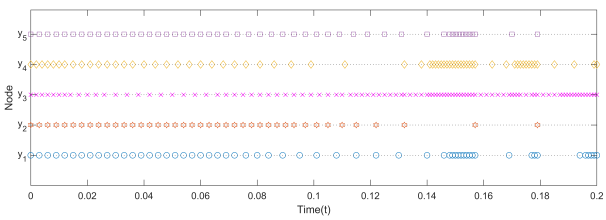
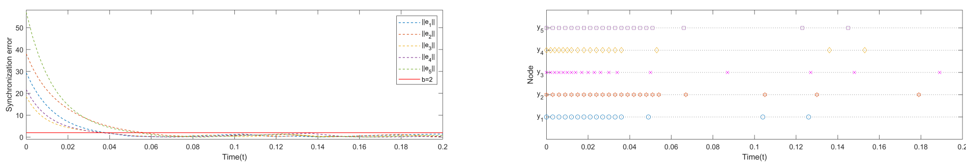


Figure 7. Event−triggered instants on node y_i under condition (9).



(**a**) (**b**)

Figure 8. (**a**) RLS errors in the two−layered network. (**b**) Event−triggered instants on node y_i under condition (11).

Table 1. ALS: Triggering rates for nodes under condition (3).

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	**ATR**
17.5%	15.5%	15.5%	15.5%	16.5%	21%	19.5%	22%	21.5%	17%	18.15%

Table 2. Intra-layer quasi-synchronization (ALQS): Triggering rates for nodes under condition (7).

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	**ATR**
10.5%	9%	9.5%	9%	10%	11.5%	10%	12%	12%	8%	10.15%

Table 3. RLS: Triggering rates for nodes under condition (9).

y_1	y_2	y_3	y_4	y_5	**ATR**
28%	20.5%	49.5%	31.5%	26%	37.7%

Table 4. Inter-layer quasi-synchronization (RLQS): Triggering rates for nodes under condition (11).

y_1	y_2	y_3	y_4	y_5	**ATR**
8%	11.5%	9.5%	9%	10.5%	9.7%

4.1. Intra-Layer Synchronization and Quasi-Synchronization

A Lorenz system and a Chen system are selected as the node self-dynamics of the x-layer and y-layer, respectively.

The initial values of each node in the x-layer are chosen as $x_{10} = (29.2, -21.9, 7)^{\mathrm{T}}$, $x_{20} = (14, 2, 0.9)^{\mathrm{T}}$, $x_{30} = (-24, 8, -5)^{\mathrm{T}}$, $x_{40} = (-10, 16, 9)^{\mathrm{T}}$, $x_{50} = (24, 4, -3.2)^{\mathrm{T}}$, respectively. The initial values of each node in the y-layer are chosen as $y_{10} = (0.1, 18, 23.5)^{\mathrm{T}}$, $y_{20} = (9, -6, 8.2)^{\mathrm{T}}$, $y_{30} = (-6, -12, 2)^{\mathrm{T}}$, $y_{40} = (26, -9.8, 1)^{\mathrm{T}}$, $y_{50} = (6, 13, 9.5)^{\mathrm{T}}$, respectively. These initial values are arbitrarily chosen. In order to meet the synchronization requirement in Theorem 1, we take $K_1 = 50E_3$, $K_2 = 64E_3$. Obviously, these values are not unique.

The ALS errors of the x-layer and y-layer are defined as

$$e_x = \frac{1}{5}\sum_{i=1}^{5}\|e_{xi}(t)\|, e_y = \frac{1}{5}\sum_{i=1}^{5}\|e_{yi}(t)\|.$$

Under the event-triggered controller given by Theorem 1, the numerical simulation results are shown in Figures 2 and 5 and Table 1. And under the event-triggered controller given by Theorem 2, b and C_0 are taken as 1 and 100, respectively. The numerical simulation results are shown in Figure 6 and Table 2.

Figures 2 and 6a show the evolution of ALS errors. The comparison between Figure 2a,b shows that the control strategy given by Theorem 1 can make the two-layer coupled network realize ALS. The red horizontal line in Figure 6a indicates the synchronization error bound $b = 1$. Figure 6a shows that the control strategy in Theorem 2 can achieve ALQS of this network. Figures 5 and 6b record the event-triggered time of each node. Obviously, there is no Zeno phenomenon. It can be intuitively seen from the comparison that the new event-triggered rule (7) can significantly reduce triggering frequency. The comparison of the data in Tables 1 and 2 reflects this more rigorously.

4.2. Inter-Layer Synchronization and Quasi-Synchronization

A Lorenz system is selected as the node self-dynamics of the x-layer and y-layer.

The initial values of each node in the x-layer are chosen as $x_{10} = (-1, 12.8, 0)^{\mathrm{T}}$, $x_{20} = (23, 19, -9.7)^{\mathrm{T}}$, $x_{30} = (17.6, -13, -19.4)^{\mathrm{T}}$, $x_{40} = (5, -4.7, -6)^{\mathrm{T}}$, $x_{50} = (27, -26, 14.4)^{\mathrm{T}}$, respectively. The initial values of each node in the y-layer are chosen as $y_{10} = (6.3, -6.3, 21)^{\mathrm{T}}$, $y_{20} = (-9, 3.7, 4)^{\mathrm{T}}$, $y_{30} = (1.3, -5, -16.3)^{\mathrm{T}}$, $y_{40} = (24.8, -10, 1)^{\mathrm{T}}$, $y_{50} = (21, 16.6, -23)^{\mathrm{T}}$, respectively. In order to meet the synchronization requirement in Theorem 3, we take $K_3 = 60E_3$. Obviously, this value is not unique. Under the event-triggered controller given by Theorem 3, the numerical simulation results are shown in Figures 3 and 7 and Table 3. And under the event-triggered controller given by Theorem 4, b and C_0 are taken as 2 and 100, respectively. The numerical simulation results are shown in Figure 8 and Table 4.

Figures 3 and 8a show the evolution of RLS errors. The comparison between Figure 3a,b shows that the control strategy given by Theorem 3 can make the two-layer coupled network realize RLS. The red horizontal line in Figure 8a indicates the synchronization error bound $b = 2$. Figure 8a shows that the control strategy in Theorem 4 can achieve RLQS of this network. Figures 7 and 8b record the event-triggered time of each node. Apparently, there is no Zeno phenomenon. It can be intuitively seen from the comparison that the new event-triggered rule (11) can significantly reduce the trigger frequency. The comparison of the data in Tables 3 and 4 reflects this more rigorously.

The above numerical simulations further confirm our theoretical results. According to the requirements of the final synchronization error threshold and event-triggered frequency, the control parameters can be flexibly changed. For example, if only a larger synchronization error threshold and a lower event-triggered frequency are required, b and C_0 can be adjusted to larger values.

5. Conclusions

In this paper, the synchronization of two-layer coupled networks under event-triggered control is studied. Appropriate event-triggered controllers are designed to realize ALS and RLS. Moreover, a novel event-triggered rule is proposed, which can realize ALQS and RLQS. Under this rule, event-triggered times can be greatly reduced, and the synchronization error threshold can be adjusted flexibly. Finally, a simulation experiment is given to further verify the effectiveness of the proposed control scheme. Of course, the control scheme in this paper has some limitations. This scheme needs to exert control over all nodes, and it is impossible to predict when the network system will reach synchronization. That is a problem we want to explore in the future.

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