mathematics

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Geodesics and Translation Curves in Sol_0^4

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Abstract: A translation curve in a Thurston space is a curve such that for given unit vector at the origin, the translation of this vector is tangent to the curve in every point of the curve. In most Thurston spaces, translation curves coincide with geodesic lines. However, this does not hold for Thurston spaces equipped with twisted product. In these spaces, translation curves seem more intuitive and simpler than geodesics. In this paper, geodesics and translation curves in Sol_0^4 space are classified and the curvature properties of translation curves are investigated.

Keywords: geodesic; translation curve; solvable Lie group; Sol_0^4 space

MSC: 53C30; 53B20; 53C22

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Citation: Erjavec, Z. Geodesics and Translation Curves in Sol_0^4. *Mathematics* **2023**, *11*, 1533. https://doi.org/10.3390/math11061533

Academic Editor: Adolfo Ballester-Bolinches

Received: 22 February 2023
Revised: 16 March 2023
Accepted: 20 March 2023
Published: 22 March 2023

1. Introduction

A homogeneous geometry is a pair (G, X) consisting of a smooth manifold X, equipped with the transitive action of a Lie group G. The manifold X defines the underlying homogeneous space, and the group G defines the set of allowable motions.

In dimension two, the uniformization theorem states that every two-dimensional manifold can be equipped with a geometric structure modeled on one of the three homogeneous spaces $\mathbb{H}^2$, $\mathbb{E}^2$, or $\mathbb{S}^2$.

In the 1980s, Thurston realized that a similar (but more complicated) result might hold in three dimensions. Thurston's geometrization conjecture stated that every compact orientable three-manifold has a canonical decomposition into parts, each of which admits a canonical geometric structure from among the eight maximal simply connected homogeneous Riemannian three-dimensional geometries: $\mathbb{H}^3$, $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, Nil, Sol, $\widetilde{SL(2,\mathbb{R})}$ (see [1,2]). The proof of geometrization conjecture was completed by Perelman in 2003 [3–5]. The mentioned three-dimensional geometries can be defined abstractly as follows.

A *Thurston model geometry* (G, X) is a manifold X with a Lie group G of diffeomorphisms of X such that X is connected and simply connected; G acts transitively on X with compact point stabilizers; G is not contained in any larger group of diffeomorphisms of X, and there is at least one compact manifold modeled on (G, X).

The model space Sol_0^4 is one of the four-dimensional Thurston geometries. According to Filipkiewicz [6], there are 19 homogeneous model spaces in dimension four.

Complex space forms	Direct Product Spaces	Direct Product Spaces	Warped Product Spaces
$\mathbb{E}^4$, $\mathbb{H}^4$, $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{C}H^2$	$\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{E}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$	$\mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$ $\mathrm{Nil}_3 \times \mathbb{E}^1$, $\widetilde{\mathrm{SL}_2\mathbb{R}} \times \mathbb{E}^1$	Sol_0^4, Sol_1^4, F^4, Nil^4, $\mathrm{Sol}_{m,n}^4$

According to Wall [7], among these model spaces, the space Sol_0^4 belongs to 14 spaces which admit complex structure compatible with the geometric structure. Moreover, it is known that Sol_0^4 possesses a locally conformal Kahler (LCK) structure. This structure is used in [8], where minimal invariant, totally real, and CR-submanifolds of Sol_0^4 are considered. In addition, in our previous work [9], J-trajectories, which represent an analog of magnetic curves in LCK spaces, and hence generalization of geodesics, are studied. The first and the second curvature of a non-geodesic J-trajectory in an arbitrary LCK manifold whose anti-Lee field has constant length are examined, too.

In a homogeneous space there are postulated isometries, mapping each point to any other point. Moreover, in some homogeneous spaces, it is possible to introduce a specific translation different from geodesic translation. This new translation will carry the unit vector given at the origin to any point by its tangent mapping. The corresponding curve is called the *translation curve*. The study of translation curves was initiated by Molnár and Szilágyi in [10] where authors studied translation curves and translation spheres in three-dimensional product and twisted product Thurston geometries.

Motivated by the fact that there are no results about translation curves in four-dimensional Thurston geometries, we examine translation curves and geodesic lines in Sol_0^4 space, one of five four-dimensional Thurston spaces which can be represented as warped product space.

The purpose of the present paper is to classify geodesics and translation curves in Sol_0^4 space.

In the next section, we recall the basic properties of Sol_0^4 space, and then we examine geodesics and classify translation curves in Sol_0^4 space. Finally, we discuss the curvature properties of translation curves and present translation spheres.

2. The Model Space Sol_0^4

2.1. Lie Group and Lie Algebra

The underlying manifold of the model space Sol_0^4 is $\mathbb{R}^4(x, y, z, t)$ with the group operation

$$(x_1, y_1, z_1, t_1) * (x_2, y_2, z_2, t_2) = (x_1 + e^{t_1} x_2, y_1 + e^{t_1} y_2, z_1 + e^{-2t_1} z_2, t_1 + t_2). \tag{1}$$

This operation is derived from the matrix multiplication by the following identification

$$(x, y, z, t) := \begin{pmatrix} e^t & 0 & 0 & 0 & x \\ 0 & e^t & 0 & 0 & y \\ 0 & 0 & e^{-2t} & 0 & z \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, the underlying manifold of the model space Sol_0^4 is the connected solvable Lie group $G_6(1)$ described in [6] (p. 98).

The neutral element is $(0, 0, 0, 0)$. The inverse element of (x, y, z, t) is given by

$$(x, y, z, t)^{-1} = (-e^{-t} x, -e^{-t} y, -e^{2t} z, -t).$$

The Lie algebra $\mathfrak{g}_6(1)$ of $G_6(1)$ is spanned by the basis $\{e_1, e_2, e_3, e_4\}$, given by

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 1. *Note that we could use the reduced matrix representation*

$$(x,y,z,t) = \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & e^t & 0 & y \\ 0 & 0 & e^{-2t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

However, we must note that translation of arbitrary vectors, or coordinate differentials by the inverse which we address in the next subsection, becomes less elegant.

2.2. Metric and Basis

Using the inverse translation T^{-1}, by pullback of coordinate differentials,

$$\begin{pmatrix} e^{-t} & 0 & 0 & 0 & -e^{-t}x \\ 0 & e^{-t} & 0 & 0 & -e^{-t}y \\ 0 & 0 & e^{2t} & 0 & -e^{2t}z \\ 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-t}dx \\ e^{-t}dy \\ e^{2t}dz \\ dt \\ 0 \end{pmatrix} \tag{2}$$

we obtain the left invariant Riemannian metric g of Sol_0^4

$$g = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2. \tag{3}$$

Hence, the orthonormal coframe $\{\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4\}$ is given by

$$\vartheta^1 = e^{-t}dx, \quad \vartheta^2 = e^{-t}dy, \quad \vartheta^3 = e^{2t}dz, \quad \vartheta^4 = dt.$$

Thus, the metrically dual left invariant basis vector fields are

$$e_1 = e^t\frac{\partial}{\partial x}, \quad e_2 = e^t\frac{\partial}{\partial y}, \quad e_3 = e^{-2t}\frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}. \tag{4}$$

2.3. Levi–Civita Connection

The Levi–Civita connection is given by

$$\begin{array}{llll} \nabla_{e_1}e_1 = e_4, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = -e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = e_4, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = -e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = -2e_4, & \nabla_{e_3}e_4 = 2e_3, \\ \nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = 0. \end{array} \tag{5}$$

The basis vector fields satisfy the following commutation relations:

$$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0, \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = -2e_3.$$

2.4. Riemannian and Sectional Curvatures

If the Riemannian curvature tensor is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

then its non-vanishing components are

$$\begin{array}{lll} R(e_1,e_2)e_2 = -e_1, & R(e_1,e_3)e_3 = 2e_1, & R(e_1,e_4)e_4 = -e_1, \\ R(e_2,e_3)e_3 = 2e_2, & R(e_2,e_4)e_4 = -e_2, & R(e_3,e_4)e_4 = -4e_3. \end{array}$$

Hence, we obtain sectional curvatures

$$K_{12} = -1, \quad K_{13} = 2, \quad K_{14} = -1, \quad K_{23} = 2, \quad K_{24} = -1, \quad K_{34} = -4,$$

and the scalar curvature $K = -6$.

3. Geodesics in Sol_0^4

Local existence, uniqueness, and smoothness of a geodesic through any point with initial velocity vector follow from the classical ODE theory on a smooth Riemannan manifold. Given any two points in a complete Riemannan manifold, standard limiting arguments show that there is a smooth curve of minimal length between these points. Any such curve is *a geodesic*.

As is known, J-trajectories are analogs of magnetic curves, and magnetic curves represent a generalization of geodesics. As we mentioned, some types of J-trajectories in Sol_0^4 are determined in [9] (Theorem 1). However, they are not classified and corresponding geodesics are not easy to recognize. Thus, here we consider geodesics in Sol_0^4.

Let $\gamma(s) = (x(s), y(s), z(s), t(s))$ be an arc length parameterized curve in Sol_0^4. Then its unit tangent vector field is expressed as

$$\begin{aligned}\dot{\gamma}(s) =& \dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} + \dot{z}(s)\frac{\partial}{\partial z} + \dot{t}(s)\frac{\partial}{\partial t}\\ =& e^{-t(s)}\dot{x}(s)e_1 + e^{-t(s)}\dot{y}(s)e_2 + e^{2t(s)}\dot{z}(s)e_3 + \dot{t}(s)e_4.\end{aligned}$$

The arc length condition is

$$e^{-2t(s)}\dot{x}(s)^2 + e^{-2t(s)}\dot{y}(s)^2 + e^{4t(s)}\dot{z}(s)^2 + \dot{t}(s)^2 = 1. \tag{6}$$

Using (5), we have

$$\begin{aligned}\nabla_{\dot{\gamma}}\dot{\gamma} = \quad & e^{-t(s)}(\ddot{x}(s) - 2\dot{x}(s)\dot{t}(s))e_1 + e^{-t(s)}(\ddot{y}(s) - 2\dot{y}(s)\dot{t}(s))e_2 +\\ & + e^{2t(s)}(\ddot{z}(s) + 4\dot{z}(s)\dot{t}(s))e_3 + \left(\ddot{t}(s) + e^{-2t(s)}\left(\dot{x}(s)^2 + \dot{y}(s)^2\right) - 2e^{4t(s)}\dot{z}(s)^2\right)e_4.\end{aligned}$$

From geodesic equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, we obtain the following system

$$\begin{aligned}\ddot{x}(s) - 2\dot{x}(s)\dot{t}(s) &= 0,\\ \ddot{y}(s) - 2\dot{y}(s)\dot{t}(s) &= 0,\\ \ddot{z}(s) + 4\dot{z}(s)\dot{t}(s) &= 0,\\ \ddot{t}(s) + e^{-2t(s)}\left(\dot{x}(s)^2 + \dot{y}(s)^2\right) &= 2e^{4t(s)}\dot{z}(s)^2.\end{aligned} \tag{7}$$

By homogeneity, we can assume that the initial conditions are given by

$$(x(0), y(0), z(0), t(0)) = (0,0,0,0) \quad \text{and} \quad (\dot{x}(0), \dot{y}(0), \dot{z}(0), \dot{t}(0)) = (\alpha, \beta, \gamma, \delta), \tag{8}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Next, we solve the system (7), with respect to (6) and (8).

We multiply the first and the second equation of the system (7) by $\dot{y}$ and $(-\dot{x})$, respectively, and then add them. It follows

$$\ddot{x}(s)\dot{y}(s) - \ddot{y}(s)\dot{x}(s) = 0.$$

Hence,

$$\dot{x}(s) = c_1\,\dot{y}(s),\ c_1 \in \mathbb{R}. \tag{9}$$

From the second equation of (7) and (8), we get

$$\dot{y}(s) = \beta\, e^{2t(s)}. \tag{10}$$

From (9) and (8), it follows

$$\dot{x}(s) = \alpha\, e^{2t(s)}. \tag{11}$$

By integration, from the third equation of (7), it follows

$$\dot{z}(s) = \gamma\, e^{-4t(s)}. \tag{12}$$

Substituting (10)–(12) in the fourth equation of (7), we have

$$\ddot{t}(s) + (\alpha^2 + \beta^2)e^{2t(s)} - 2\gamma^2 e^{-4t(s)} = 0. \tag{13}$$

Next, we consider the arc length condition (6). Substituting (10)–(12) in (6), we obtain the differential equation

$$\dot{t}(s)^2 + (\alpha^2 + \beta^2)e^{2t(s)} + \gamma^2 e^{-4t(s)} - 1 = 0. \tag{14}$$

If we differentiate (14), we get

$$2\dot{t}(s)\Big(\ddot{t}(s) + (\alpha^2 + \beta^2)e^{2t(s)} - 2\gamma^2 e^{-4t(s)}\Big) = 0.$$

Notice that this equation coincides with the Equation (13) when $\dot{t}(s) \neq 0$. Hence, we only need to consider (14).

After the separation of variables, the solution of this equation is given by the following elliptic integral

$$\frac{dt}{\pm\sqrt{1 - (\alpha^2 + \beta^2)e^{2t} - \gamma^2 e^{-4t}}} = ds \tag{15}$$

Thus, the following theorem is proven.

Theorem 1. *The geodesics in* Sol_0^4 *space, parameterized by the arc length and starting at the origin, are given by the following equations*

$$x(s) = \alpha \int_0^s e^{2t(\sigma)} d\sigma, \qquad y(s) = \beta \int_0^s e^{2t(\sigma)} d\sigma,$$
$$z(s) = \gamma \int_0^s e^{-4t(\sigma)} d\sigma, \quad ds = \frac{dt}{\pm\sqrt{1 - (\alpha^2 + \beta^2)e^{2t} - \gamma^2 e^{-4t}}},$$

where $\alpha, \beta, \gamma, \delta = \dot{t}(0) \in \mathbb{R}$ *such that* $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

Next, we consider geodesic lines in the characteristic hypersurfaces of Sol_0^4 space.

3.1. Geodesics in Hypersurface t = const

A hypersurface $M(1,2,3,t_0)$ defined by $M(1,2,3,t_0) = \{(x, y, z, t_0) \in \mathrm{Sol}_0^4 : t_0 \in \mathbb{R}\}$ and equipped by the metric

$$g = e^{-2t_0}(dx^2 + dy^2) + e^{4t_0} dz^2$$

is isometric to the Euclidean 3-space. This submanifold is minimal and non-totally geodesic in Sol_0^4 (see [8]). From (7), geodesics are determined by the system

$$\begin{aligned} \ddot{x}(s) &= 0, \qquad \ddot{y}(s) = 0, \\ \ddot{z}(s) &= 0, \qquad \dot{x}(s)^2 + \dot{y}(s)^2 = 2e^{6t_0}\dot{z}(s)^2. \end{aligned}$$

Hence, geodesics in hypersurface $t = t_0$ parameterized by the arc length are given by

$$\begin{aligned} x(s) &= \alpha s + x_0, \qquad y(s) = \beta s + y_0, \\ z(s) &= \pm\frac{\sqrt{3}}{3}e^{-2t_0}s + z_0, \qquad t(s) = t_0, \end{aligned}$$

where $\alpha, \beta, x_0, y_0, z_0, t_0 \in \mathbb{R}$, and $\alpha^2 + \beta^2 = \frac{2}{3}e^{2t_0}$.

3.2. Geodesics in Hypersurface z = const

A hypersurface $M(1,2,z_0,4)$ defined by $M(1,2,z_0,4) = \{(x,y,z_0,t) \in \mathrm{Sol}_0^4 : z_0 \in \mathbb{R}\}$ and equipped by the metric

$$g = e^{-2t}(dx^2 + dy^2) + dt^2,$$

is isometric to the hyperbolic 3-space of curvature -1. The hypersurface $M(1,2,z_0,4)$ is totally geodesic in Sol_0^4 and represents a leaf of the warped product representation $\mathrm{Sol}_0^4 = \mathbb{H}^3(-1) \times_{e^{2t}} \mathbb{E}^1$. From (7), geodesics are determined by the system

$$\begin{aligned} \ddot{x}(s) - 2\dot{x}(s)\dot{t}(s) &= 0, \\ \ddot{y}(s) - 2\dot{y}(s)\dot{t}(s) &= 0, \\ \ddot{t}(s) + e^{-2t(s)}\left(\dot{x}(s)^2 + \dot{y}(s)^2\right) &= 0. \end{aligned}$$

Hence, geodesics in hypersurface $z = z_0$ parameterized by the arc length are given by

$$\begin{aligned} x(s) &= \frac{\alpha \sinh s}{\cosh s - (\alpha^2 + \beta^2)\sinh s} + x_0, \qquad y(s) = \frac{\beta \sinh s}{\cosh s - (\alpha^2 + \beta^2)\sinh s} + y_0, \\ z(s) &= z_0, \qquad t(s) = -\log\left(\cosh s - (\alpha^2 + \beta^2)\sinh s\right) + t_0, \end{aligned}$$

where $\alpha, \beta, x_0, y_0, z_0, t_0 \in \mathbb{R}$, and $(\alpha^2 + \beta^2)^2 + e^{-2t_0}(\alpha^2 + \beta^2) = 1$.

3.3. Geodesics in Hypersurface y = const

A hypersurface $M(1,y_0,3,4)$ defined by $M(1,y_0,3,4) = \{(x,y_0,z,t) \in \mathrm{Sol}_0^4 : y_0 \in \mathbb{R}\}$ and equipped by the metric

$$g = e^{-2t}dx^2 + e^{4t}dz^2 + dt^2,$$

although it looks similar, it is not isometric to the "standard" Sol 3-space. Namely, the metric of the Sol 3-space, described in [11], is given by $g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$. From (7), geodesics are determined by the system

$$\begin{aligned} \ddot{x}(s) - 2\dot{x}(s)\dot{t}(s) &= 0, \\ \ddot{z}(s) + 4\dot{z}(s)\dot{t}(s) &= 0, \\ \ddot{t}(s) + e^{-2t(s)}\dot{x}(s)^2 &= 2e^{4t(s)}\dot{z}(s)^2. \end{aligned}$$

Hence, geodesics in hypersurface $y = y_0$ parameterized by the arc length are given by

$$\begin{aligned} x(s) &= \alpha \int_0^s e^{2t(\sigma)}d\sigma, \qquad y(s) = y_0, \\ z(s) &= \gamma \int_0^s e^{-4t(\sigma)}d\sigma, \qquad ds = \frac{dt}{\pm\sqrt{1 - \alpha^2 e^{2t} - \gamma^2 e^{-4t}}}, \end{aligned}$$

where $\alpha, \gamma, \delta = \dot{t}(0), y_0 \in \mathbb{R}$, $t(0) = 0$, and $\alpha^2 + \gamma^2 + \delta^2 = 1$.

More details on geodesic in three-dimensional Sol space can be found in [12].

Remark 2. *Note that study of geodesics in hypersurfaces $x = const$ is analog to the study of geodesics in hypersurfaces $y = const$.*

4. Translation Curves in $\mathbf{Sol}_0^4$

4.1. Translation Curves in Sol_0^4

As explained before, we are interested in such curves that for a given unit vector at the origin, this unit vector after translation coincides with the tangent on the curve in each point of this curve.

Hence, for a given starting unit vector $(\alpha, \beta, \gamma, \delta) = (\dot{x}(0), \dot{y}(0), \dot{z}(0), \dot{t}(0))$ at the origin $(x(0), y(0), z(0), t(0)) = (0, 0, 0, 0)$, we define its image in a point $(x(s), y(s), z(s), t(s))$ by the translation T such that

$$\begin{pmatrix} e^{t(s)} & 0 & 0 & 0 & x(s) \\ 0 & e^{t(s)} & 0 & 0 & y(s) \\ 0 & 0 & e^{-2t(s)} & 0 & z(s) \\ 0 & 0 & 0 & 1 & t(s) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{x}(s) \\ \dot{y}(s) \\ \dot{z}(s) \\ \dot{t}(s) \\ 0 \end{pmatrix}. \tag{16}$$

This yields a curve starting at the origin, in direction $(\alpha, \beta, \gamma, \delta)$, determined by the following differential equations

$$\begin{aligned} \dot{x}(s) &= \alpha\, e^{t(s)}, \\ \dot{y}(s) &= \beta\, e^{t(s)}, \\ \dot{z}(s) &= \gamma\, e^{-2t(s)}, \\ \dot{t}(s) &= \delta. \end{aligned} \tag{17}$$

Solving this system is a much easier task than solving the system for geodesics. From the fourth equation, we have $t(s) = \delta s$. Substituting $t(s) = \delta s$ in remaining equations of (17), after integration, we obtain the following result.

Theorem 2. *Translation curves in* Sol_0^4 *space, starting at the origin, are given by the following equations*

$$\begin{aligned} x(s) &= \frac{\alpha}{\delta}\left(e^{\delta s} - 1\right), \quad y(s) = \frac{\beta}{\delta}\left(e^{\delta s} - 1\right), \\ z(s) &= -\frac{\gamma}{2\delta}\left(e^{-2\delta s} - 1\right), \quad t(s) = \delta s, \end{aligned} \tag{18}$$

where $\alpha, \beta, \gamma, \delta \neq 0 \in \mathbb{R}$, *such that* $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

Remark 3. *Observe that if* $\delta = 0$, *then* $t = const$ *and we consider translation curves in Euclidean 3-space. From* (17), *it follows that translation curves are straight lines which coincide with geodesics. If* $\gamma = 0$, *then we consider translation curves in hyperbolic 3-space and translation curves coincide with geodesics, too. If* $\alpha = 0$ *or* $\beta = 0$, *then corresponding space is "similar" to the* Sol_3 *space and translation curves differ from geodesic. In this case, obtained translation curves are comparable with translation curves described in [10].*

4.2. Curvature Properties of Translation Curves

The definition of the Frenet curve of osculating order r in a Riemannian manifold (e.g., see [13]) implies the following definition.

Definition 1. *If c is a curve in* Sol_0^4 *space parameterized by arc length s, we say that c is a* Frenet curve of osculating order r *($r = 1, \ldots, 4$) if there exist orthonormal vector fields* E_1, E_2, E_3 *and* E_4 *along c, such that*

$$\begin{aligned} \dot{c} = E_1, \quad \nabla_{\dot{c}} E_1 = \kappa E_2, \quad \nabla_{\dot{c}} E_2 = -\kappa E_1 + \tau E_3, \\ \nabla_{\dot{c}} E_3 = -\tau E_2 + \sigma E_4, \quad \nabla_{\dot{c}} E_4 = -\sigma E_3, \end{aligned} \tag{19}$$

where κ, τ, σ *are positive* C^∞ *functions of s.*

Vector fields E_1, E_2, E_3 and E_4 are called the tangent, *the* normal, *the* binormal, *and the* trinormal *vector field of the curve c, respectively. Functions $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ are called the first, the second, and the third curvature of c, respectively.*

A *geodesic* is regarded as a Frenet curve of osculating order 1.

A *helix of order* 2 is a Frenet curve of osculating order 2 with constant curvature κ, i.e., it is a *circle*.

A *helix of order* 3 is a Frenet curve of osculating order 3 with constant curvatures κ and τ, i.e., it is a *circular helix*.

A *helix of order* 4 is a Frenet curve of osculating order 4 such that all curvatures κ, τ, σ are constant.

Next we determine curvatures of translation curves.

We start with the unit velocity vector $E_1 = \dot{c} = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4$. Using (5), we have

$$\nabla_{\dot{c}} E_1 = -\alpha\delta e_1 - \beta\delta e_2 + 2\gamma\delta e_3 + (\alpha^2 + \beta^2 - 2\gamma^2) e_4.$$

From $\nabla_{\dot{c}} E_1 = \kappa E_2$, we obtain

$$\kappa^2 = (\alpha^2 + \beta^2 - 2\gamma^2)^2 + \delta^2 \Big(\alpha^2 + \beta^2 + 4\gamma^2\Big). \tag{20}$$

For easier reading, we introduce substitutions $A = \alpha^2 + \beta^2 - 2\gamma^2$, and $B = \alpha^2 + \beta^2 + 4\gamma^2$. Thus, we have

$$\kappa^2 = A^2 + \delta^2 B = const.$$

Notice that κ can be zero in two cases. The first case is if $\alpha = \beta = \gamma = 0$, i.e., $\delta = 1$ (vertical geodesics) and the second case is if $\alpha = \beta = \gamma = \frac{\sqrt{3}}{3}$ and $\delta = 0$ (geodesic line in Euclidean 3D space). Next, we find

$$\nabla_{\dot{c}} E_2 = \frac{1}{\kappa}\Big(- \alpha A e_1 - \beta A e_2 + 2\gamma A e_3 - \delta B e_4 \Big),$$

and then from $\nabla_{\dot{c}} E_2 = -\kappa E_1 + \tau E_3$, we have

$$\tau E_3 = \frac{1}{\kappa}\Big(\alpha\Big(\kappa^2 - A\Big) e_1 + \beta\Big(\kappa^2 - A\Big) e_2 + \gamma\Big(\kappa^2 + 2A\Big) e_3 + \delta\Big(\kappa^2 - B\Big) e_4 \Big).$$

Hence, we obtain the second curvature

$$\tau^2 = \frac{1}{\kappa^2}\Big(\alpha^2\Big(\kappa^2 - A\Big)^2 + \beta^2\Big(\kappa^2 - A\Big)^2 + \gamma^2\Big(\kappa^2 + 2A\Big)^2 + \delta^2\Big(\kappa^2 - B\Big)^2 \Big) = const. \tag{21}$$

Although it is not obvious, it is not hard to prove that

$$\tau^2 = B - \kappa^2, \quad \text{i.e.,} \quad \kappa^2 + \tau^2 = B. \tag{22}$$

Next, we find

$$\nabla_{\dot{c}} E_3 = \frac{1}{\kappa\tau}\Big(- \alpha\delta\Big(\kappa^2 - B\Big) e_1 - \beta\delta\Big(\kappa^2 - B\Big) e_2 + 2\gamma\delta\Big(\kappa^2 - B\Big) e_3 + A\Big(\kappa^2 - B\Big) e_4 \Big).$$

Finally, from $\nabla_{\dot{c}} E_3 = -\tau E_2 + \sigma E_4$, after long but straightforward computation, we obtain

$$\sigma = 0. \tag{23}$$

Therefore, we conclude with the following theorem.

Theorem 3. *Translation curves in* Sol_0^4 *space are helices of order 3, i.e., circular helices.*

4.3. Translation Spheres in Sol_0^4

Let us assume that initial unit vector of translation curve (18) is given by

$$\alpha = \sin\vartheta\cos\varphi\cos\psi, \quad \beta = \sin\vartheta\cos\varphi\sin\psi, \quad \gamma = \sin\vartheta\sin\varphi, \quad \delta = \cos\vartheta.$$

Then, we can define the sphere of radius R centered at the origin. Namely, the unit velocity translation curve ending in parameter R describes the translation sphere.

Proposition 1. *Translation sphere of radius R in Sol_0^4 space is given by the following equations*

$$\begin{aligned} x(\vartheta,\varphi,\psi) &= \tan\vartheta\cos\varphi\cos\psi\left(e^{R\cos\vartheta}-1\right), \\ y(\vartheta,\varphi,\psi) &= \tan\vartheta\cos\varphi\sin\psi\left(e^{R\cos\vartheta}-1\right), \\ z(\vartheta,\varphi,\psi) &= -\frac{1}{2}\tan\vartheta\sin\varphi\left(e^{-2R\cos\vartheta}-1\right), \\ t(\vartheta,\varphi,\psi) &= R\cos\vartheta, \end{aligned} \tag{24}$$

where $\vartheta,\varphi,\psi \in [0,2\pi)$ and $R \in \mathbb{R}^+$.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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