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On AP–Henstock–Kurzweil Integrals and Non-Atomic Radon Measure

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Abstract: The AP–Henstock–Kurzweil-type integral is defined on $\mathtt{X}$, where $\mathtt{X}$ is a complete measure metric space. We present some properties of the integral, continuing the study's use of a Radon measure μ. Finally, using locally finite measures, we extend the AP–Henstock–Kurzweil integral theory to second countable Hausdorff spaces that are locally compact. A Saks–Henstock-type Lemma is proved here.

Keywords: AP–Henstock–Kurzweil integral; second countable locally compact Hausdorff spaces

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1. Introduction

J. Kurzweil first proposed a solution to the primitives problem in 1957, and R. Henstock did the same in 1963. It is commonly known that the Henstock–Kurzweil integral (HK-integral) is a generalized form of the Riemann integral. J. Kurzweil and R. Henstock each separately produced this generalization. The Henstock–Kurzweil integral has a construction that is comparable to that of the Riemann integral but stronger than the Lebesgue integral. Additionally, it is well known that the HK-integral can resolve the issue of primitives in the real line. The Riemann sums limit over the appropriate integration domain partitions is referred to as the HK-valued integrals. The HK-integral seems to have a constructive definition. Within the HK-integral, a gauge-like positive function is employed to assess a partition's fineness rather than a constant as in the Riemann integral, which is the fundamental distinction between the two definitions. Cao in [1] introduced the Banach valued Henstock–Kurzweil integral, Boccuto et al. in [2] defined the Henstock–Kurzweil-type integral for functions defined on a (possibly unbounded) subinterval on the extended real line and with values in Banach spaces. A Fubini-type theorem was proved, for the Kurzweil–Henstock integral of Riesz-space-valued functions defined on (not necessarily bounded) subrectangles of the extended real plane (see [3]). The integral of the function close to singular points is better approximated as a result. The problem turns out to be more challenging for the integration of approximative derivatives. Finding relationships between the Denjoy–Khintchine integral and its roughly continuous generalizations and approximate Perron-type integrals was the focus of most studies in this area (see [4] for Denjoy integral). John Burkill [5] was credited with the invention of the roughly continuous Perron integral (AP-integral). According to the uniformly AP integrable in short UAP and element-wise boundedness conditions, Jae Myung Park et al. [6] investigated convergence theorems for the AP-integral. By demonstrating that the AP–Denjoy integral and the AP–Henstock–Kurzweil integral are equal and identical, Jae Myung Park et al. [7] defined the AP-Denjoy integral. In [8], Skvortsov et al. draw attention to results that are powerful than those shown in Jae's work. They demonstrate how some of them are amenable to formulation in perspective of a derivation basis specified by a local system, of which it is known that the approximate basis is a specific case. They also consider how the σ-finiteness of the variational measure generated by a function interacts with the conventional idea of generalized bounded variation. The Riemann-type integral is equal to the properly defined Perron-type integral on a large class of bases (see [9]). Skvortsov et al. [10] say only that Burkill's AP-integral is covered by AP–Henstock–Kurzweil integral. Shin and Yoon [11] introduced the concept of approximately negligible variation and give a necessary and sufficient condition that a function $\mathcal{F}$ be an indefinite integral of an AP–Henstock–Kurzweil integrable function $\mathtt{f}$ on $[a, b]$. The concept of bounded variation is then used to describe the characterization of AP–Henstock–Kurzweil integrable functions. Bongiorno et al. mention in their research [12] a type of Henstock–Kurzweil integral defined on a complete metric space $\mathtt{X}$, using a Radon measure μ and a family of sets $\mathtt{F}$ which fulfill the covering theorem of Vitali for μ. In particular, the traditional Henstock–Kurzweil integral on $\mathtt{R}$, where $\mathtt{R}$ is the set of reals, is enclosed by this integral; for more information, see [13]. The construction of the μ–Henstock–Kurzweil integral motivated us to construct μ_{AP}–Henstock–Kurzweil integrals on complete metric spaces with a non-atomic Radon measure. In this paper, we analyze the AP-integral of Henstock–Kurzweil type, described on $\mathtt{X}$, possessed by μ and a class of "sets" $\mathtt{F}$ fulfilling the Vitali's covering theorem. We finally enlarge this notion in the setting of locally compact second countable Hausdorff spaces with bounded or locally finite measures.

The paper is organized as follows: in Section 2, the basic concepts and terminology are introduced together with some definitions and results. In Section 3, we introduce the AP–Henstock–Kurzweil-type integral called μ_{AP}–Henstock–Kurzweil integral of a set-valued function with respect to a Radon measure. Simple properties of μ_{AP}–Henstock–Kurzweil integrals are discussed in Section 3.1. The relationships between μ_{AP}–Henstock–Kurzweil integrable functions and Lebesgue integrable functions are discussed in Proposition 4. In Section 4, we extend the theory of the μ_{AP}–Henstock–Kurzweil integral to locally finite measures on locally compact second countable Hausdorff spaces. A few fundamental results are discussed in this section. The main result in this section is the Saks–Henstock-type Lemma on Theorem 12.

2. Preliminaries

Let us fix $\mathtt{X} = (\mathtt{X}, d)$ as a Cauchy metric space with a non-atomic Radon measure μ. Throughout the paper, the complete metric spaces or the Cauchy metric spaces will be termed as Cauchy spaces. A σ-algebra is a collection $\mathtt{M}$ of subsets of $\mathtt{X}$ satisfying the conditions:

1. $\mathtt{X}$ is in $\mathtt{M}$.
2. $\mathtt{A}$ is in $\mathtt{M}$ implies $\mathtt{X} \setminus \mathtt{A} \in \mathtt{M}$.
3. $\mathtt{A}_n$ is in $\mathtt{M}$, $n = 1, 2, \ldots$, implies $\bigcup\limits_{n=1}^{\infty} \mathtt{A}_n \in \mathtt{M}$.

Let $\mathcal{C}$ be an arbitrary collection of subsets of $\mathtt{X}$. The smallest σ-algebra $\sigma(\mathtt{C})$ containing $\mathtt{C}$, called the σ-algebra generated by $\mathtt{C}$, is the intersection of all σ-algebras in $\mathtt{X}$ which contain $\mathtt{C}$.

Let $\mathtt{M}$ be a σ-algebra of subsets of a set $\mathtt{X}$. A positive function $\mu : \mathtt{M} \to [0, +\infty]$ is called a measure if

1. $\mu(\emptyset) = 0$;
2. $\mu(\bigcup\limits_{j=1}^{\infty} \mathtt{A}_j) = \sum\limits_{j=1}^{\infty} \mu(\mathtt{A}_j)$ for every sequence $\{\mathtt{A}_j\}_j$ of pointwise disjoint sets from $\mathtt{M}$.

Then, $(\mathtt{X}, \mathtt{M}, \mu)$ is termed as a measure space. Suppose U is the Borel σ-algebra of $\mathtt{X}$. Recall that a measure μ defined on U is called locally finite if for every $x \in \mathtt{X}$, there is $r > 0$ such that $\mu(B(x, r)) < \infty$, where $B(x, r)$ is the open ball of center x and radius r. μ is called a Radon measure if μ is a Borel measure with the followings:

1. $\mu(\mathtt{K}) < \infty$ for every compact set $\mathtt{K} \subset \mathtt{X}$.
2. $\mu(\mathtt{V}) = \sup\{\mu(\mathtt{K}) : \mathtt{K} \subset \mathtt{V}, \mathtt{K} \text{ is compact}\}$ for every open set $\mathtt{V} \subset \mathtt{X}$.

3. $\mu(\mathtt{A}) = \inf\{\mu(\mathtt{V}) : \mathtt{A} \subset \mathtt{V}, \mathtt{V} \text{ is open}\}$ for every $\mathtt{A} \subset \mathtt{X}$.

In the entire work, we consider μ a non-atomic Radon measure, $\mathtt{F}$ is a family of μ-measurable sets in short μ sets and $Q \in \mathtt{F}$. For $\mathtt{E} \subset \mathtt{X}$, we denote the indicator function, diameter, interior and the boundary of $\mathtt{E}$ by $\chi_{\mathtt{E}}$, $\mathrm{diam}(\mathtt{E})$, $\mathtt{E}^0$ and $\partial\mathtt{E}$, respectively. Throughout the article, we denote $\mathfrak{d}(x,\mathtt{E})$ as the distance from x to $\mathtt{E}$. Let us define $\mathtt{F}$ as a family of non-empty closed subsets of $\mathtt{X}$.

If $\bigcup_{i=1}^{m} Qi = Q$, then a partition of Q is a finite gathering of $Q_1, Q_2, \ldots, Q_n$ pairwise non-overlapping elements of $\mathtt{F}$.

Let $\mathtt{E} \subset \mathtt{X}$ and let $\mathfrak{F}$ be a subfamily of $\mathtt{F}$. We say that $\mathfrak{F}$ is a fine cover of $\mathtt{E}$ if

$$\inf\left\{\mathrm{diam}(Q) : Q \in \mathfrak{F},\ x \in Q\right\} = 0$$

for each $x \in \mathtt{E}$.

Definition 1 ([13], Definition 2.14)**.** *We say that $\mathtt{F}$ is a μ-Vitali family if for each subset $\mathtt{E}$ of $\mathtt{X}$ and for each subfamily $\mathfrak{F}$ of $\mathtt{F}$ that is a fine cover of $\mathtt{E}$, there exists a countable system $\{Q_1, Q_2, .., Q_j, ..\}$ of pairwise non-overlapping elements of $\mathfrak{F}$ such that $\mu(\mathtt{E} \setminus \bigcup Qj) = 0$.*

Consider a fine cover $\mathfrak{F}$ of $\mathtt{E} \subset \mathtt{X}$. Recall that a family $\mathtt{F}$ of non-void closed subsets of $\mathtt{X}$ is a μ Vitali family if the following Vitali covering theorem is fulfilled:

Theorem 1 ([13], Theorem 2.1)**.** *For each subset $\mathtt{E}$ of $\mathtt{X}$ and for each subfamily $\mathfrak{F}$ of $\mathtt{F}$ that is a fine cover of $\mathtt{E}$, there exists a countable system $\{Q_1, Q_2, \ldots, Q_j, \ldots\} \subset \mathfrak{F}$ such that Q_i and Q_j are non-overlapping (i.e., the interiors of Q_i and Q_j are disjoint), for each $i \neq j$, and such that $\mu(\mathtt{E} \setminus \cup Q_j) = 0$.*

A μ-Vitali family $\mathtt{F}$ is said to be a family of μ sets if it satisfies the following conditions:

(a) Given $Q \in \mathtt{F}$ and a constant $\delta > 0$, there exist $Q_1, Q_2, \ldots, Q_m, \ldots$ subsets of Q, such that Q_i and Q_j are non-overlapping for each $i \neq j$, $\bigcup_{i=1}^{m} Q_i = Q$, and $\mathrm{diam}(Q_i) < \delta$, for $i = 1, 2, \ldots, m$;
(b) Given $A,\ Q \in \mathtt{F}$ with $A \subset Q$, there exist $Q_1, Q_2, \ldots, Q_m, \ldots$ subsets of Q, such that Q_i and Q_j are non-overlapping for each $i \neq j$ and $A = Q_1$;
(c) $\mu(\partial Q) = 0$ for each $Q \in \mathtt{F}$.

The Vitali covering theorem is one of the most useful tools of measure theory. Given a collection of sets that cover some set A, the Vitali theorem selects a disjoint subcollection that covers almost all of A. Here, we recall Vitali's covering theorem for Radon measures as follows:

Theorem 2 ([14], Page 34)**.** *Let μ be a Radon measure on $\mathtt{R}^n$, $A \subset \mathtt{R}^n$ and $\mathtt{B}$ a family of closed balls such that each point of A is the center of arbitrarily small balls of $\mathtt{B}$, that is*

$$\inf\left\{r : \mathtt{B}(x,r) \in \mathtt{B}\right\} = 0,$$

for each $x \in A$. Then, there are disjoint balls $\mathtt{B}_i \in \mathtt{B}, i = 1, 2, \ldots,$ such that $\mu\left(A \setminus \bigcup_i \mathtt{B}_i\right) = 0$.

Definition 2 ([15])**.** *Consider a measurable set $\mathtt{E}$ included in $\mathtt{R}$ and c is a real number. At c, the density of $\mathtt{E}$ equals*

$$d_c(\mathtt{E}) = \lim_{h \to 0+} \frac{\mu(\mathtt{E} \cap (c-h, c+h))}{2h}$$

provided the limit exists. Clearly, $0 \leq d_c(\mathtt{E}) \leq 1$ *when it exists. The point c is a point of density of* $\mathtt{E}$ *if* $d_c\mathtt{E} = 1$ *and a point of dispersion of* $\mathtt{E}$ *if* $d_c(\mathtt{E}) = 0$.

An approximate neighborhood of $x \in [a,b]$ is a measurable set $N_x \subset [a,b]$ containing x as a point of density. Let $\mathtt{E} \subset [a,b]$. For every $x \in \mathtt{E} \subset [a,b]$, choose an approximate neighborhood $N_x \subset [a,b]$ of x. Then, $N = \{N_x : x \in E\}$ is a choice on $\mathtt{E}$. If each point of N_x is a point of density of N_x, then a tagged interval $([c,d],x)$ is said to be fine to the choice $N = \{N_x\}$ if $c,d \in N_x$ and $x \in [c,d]$. A tagged subpartition $P = \Big\{([c_i,d_i],t_i) : 1 \leq i \leq n\Big\}$ of $[a,b]$ is a finite collection of non-overlapping tagged intervals in $[a,b]$ such that $t_i \in [c_i,d_i]$ for $i = 1,2,\ldots,n$. If $([c_i,d_i],t_i)$ is fine to the choice N for each $i = 1,2,\ldots,n$; then, P is N-fine. If P is N-fine and $t_i \in \mathtt{E}$ for each $1 \leq i \leq n$, then we say that P is $(N,\mathtt{E})$-fine. If P is N-fine and $t_i \in \mathtt{E}$ for each $1 \leq i \leq n$, then P is $(N,\mathtt{E})$-fine. If P is N-fine and $[a,b] = \bigcup_{i=1}^{n}[c_i,d_i]$,, then P is an N-fine tagged partition of $[a,b]$. For a tagged partition $P = \Big\{(x_i,[c_i,d_i]) : 1 \leq i \leq n\Big\}$ of $[a,b]$, the Riemann sum is $\mathbb{S}(f,P) = \sum_{i=1}^{n} f(x_i)(d_i - c_i)$.

Consider a metric measure space $(\mathtt{X},\mathfrak{d},\mu)$ in the sense that the metric induces a topology $\mathfrak{T}$, and the measure is the Borel measure arising from the sigma field induced by the metric $\mathfrak{d}$. If $\mathfrak{T}$ is second countable locally compact and Hausdorff, we can consider a basis consisting of relatively compact open sets. Let $\mathfrak{T}_1$ be a basis for $\mathfrak{T}$ consisting of relatively compact open sets. If the topology $\mathfrak{T}$ of $\mathtt{X}$ is induced by a metric $\mathfrak{d}$ on $\mathtt{X}$, then $\mathfrak{T}_1$ is the set of all $\mathfrak{d}$-open balls say $\mathtt{B}(x,r)$ are $\mathfrak{d}-$open balls with $\mu(\mathtt{B}(x,r)) < \epsilon$.

Let $\mathfrak{u} \in \mathfrak{T}_1$ and $\mathfrak{u} \neq \emptyset$; then, $\mu(\mathfrak{u}) > 0$. In addition, if $\overline{\mathfrak{u}}$ is a closure of $\mathfrak{u}$, then $\mu(\mathfrak{u}) = \mu(\overline{\mathfrak{u}})$. This means $\mu(\partial\mathfrak{u}) = 0$ for all $\mathfrak{u} \in \mathfrak{T}_1$. Consider

$$Q_0 = \Big\{\overline{\mathfrak{u}_1} \setminus \overline{\mathfrak{u}_2},\ \mathfrak{u}_1,\ \mathfrak{u}_2 \in \mathfrak{T}_1 \text{ with } \mathfrak{u}_1 \not\subseteq \mathfrak{u}_2 \text{ and } \mathfrak{u}_2 \not\subseteq \mathfrak{u}_1\Big\}$$

and

$$Q_1 = \Big\{\bigcap_{i\in\mathbb{N}} \mathfrak{v}_i \neq \emptyset;\ \mathfrak{v}_i \in Q_0\Big\},$$

then Q_0 includes all sets of the form $\overline{\mathfrak{u}}$ where $\mathfrak{u} \in \mathfrak{T}_1$ and $Q_0 \subseteq Q_1$. Clearly, Q_1 is closed under finite intersections. If the intersection is non-empty and since μ is a σ-algebra, members of Q_1 are μ-measurable. A set Q is called elementary if Q is a finite union of (possibly just one) mutually disjoint sets. We say that a set $P = \{(Q_i,x_i) : i = 1,2,\ldots,n\}$ is a partial partition of Q if $Q_1,Q_2,\ldots,Q_n$ are mutually disjoint subsets of Q such that $Q \setminus \bigcup_{i=1}^{n} Q_i = \emptyset$ or elementary subset of Q for each i, with $x_i \in \overline{Q_i}$. Throughout the article, the closure of Q is denoted by $\overline{Q}$.

Let $\Gamma : \overline{Q} \to \mathfrak{T}_1$ be a function such that for every $x \in \overline{Q}$, we have $x \in \Gamma(x) \in \mathfrak{T}_1$. We call Γ as a gauge on Q. If $\mathfrak{T}$ of $\mathtt{X}$ is induced by the metric $\mathfrak{d}$, then $\mathfrak{T}_1$, the collection of all $\mathfrak{d}$-open balls and a gauge Γ on Q be as $\Gamma(x) = \mathcal{B}(x,\delta(x))$ for all $x \in \overline{Q}$ with certain $\delta(x) > 0$. If Γ is a gauge on Q, then (Q_i,x_i) is N-fine if $Q \subseteq \Gamma(x)$. If for all $i = 1,2,\ldots,n$, (Q_i,x_i) is N-fine, then $P = \{(Q_i,x_i) : i = 1,2,\ldots,n\}$ of Q is also N-fine. Given that partitions of Q are only partial partitions of Q, an N-fine partition of Q can be defined similarly. Let Γ_1 and Γ_2 be two gauges on Q. Since $\mathfrak{T}_1$ is a basis of $\mathfrak{T}$, for each $x \in Q$, there exists $\Gamma(x) \in \mathfrak{T}_1$ such that $\Gamma(x) \subseteq \Gamma_1(x) \cap \Gamma_2(x)$. We can then define a gauge Γ on Q which is finer than both Γ_1 and Γ_2. As a result, P is both Γ_1-fine and Γ_2-fine if P is a N-fine partition of Q.

Proposition 1 ([13], Lemma 2.2.1)**.** *(Cousin's-type lemma) if δ is a gauge on Q, then there exists a δ-fine partition of Q.*

Recalling the AP–Henstock–Kurzweil integral as follows:

Definition 3 ([16], Definition 16.4). *A mapping* $\mathtt{f} : [a, b] \to \mathtt{R}$ *is called an AP–Henstock–Kurzweil integrable if a real number* A *exists such that for each* $\epsilon > 0$*, there is a choice* N *on* $[a, b]$ *such that* $|\mathbb{S}(f, P) - A| < \epsilon$ *whenever* P *is an* N*-fine tagged partition of* $[a, b]$*. In this case,* A *is called the AP–Henstock–Kurzweil integral of* f *on* $[a, b]$*, and is denoted by* $A = \int_a^b f$*.*

Given a set function $\mathbb{F}$ defined on $\mathtt{F}$ and given $x \in \mathtt{X}$, the upper derivative of $\mathbb{F}$ at x, with respect to μ, is defined as

$$UD\mathbb{F}(x) = \lim_{\mathtt{B} \to x \in \mathtt{F}} \sup \frac{\mathbb{F}(\mathtt{B})}{\mu(\mathtt{B})}.$$

Here, $\mathtt{B} \to x$ implies $\mu(\mathtt{B}) \neq 0$, $\operatorname{diam}(\mathtt{B}) \to 0$ and $x \in \mathtt{B}$.

The lower derivative $LD\mathbb{F}$ is defined similarly. $UD\mathbb{F}$ and $LD\mathbb{F}$ are studied in [12].

Let $\phi : \mathtt{F} \to \mathtt{R}$ be a function. We say that ϕ is an additive function of set if for each $Q_i \in \mathtt{F}$, for all $i = 1, 2, \ldots, m$ and for each division $\{Q_1, Q_2, \ldots, Q_m\}$ of Q, we have

$$\phi(Q) = \sum_{i=1}^{m} \phi(Q_i).$$

Proposition 2 ([17], page 5). *For every measurable set* $\mathtt{W}$ *and every* $\epsilon > 0$*, there exist an open set* $\mathtt{U}$ *and a closed set* $\mathtt{Y}$ *such that* $\mathtt{Y} \subseteq \mathtt{W} \subseteq \mathtt{U}$ *and* $\mu(\mathtt{U} \setminus \mathtt{Y}) < \epsilon$*.*

3. AP–Henstock–Kurzweil Integral in Regard to a Radon Measure

In this section, we discuss the AP–Henstock–Kurzweil integral with respect to a Radon measure. We consider μ a non-atomic Radon measure and $Q \in \mathtt{F}$, where $\mathtt{F}$ is a family of μ sets. An approximate neighbourhood $x \in Q$ is a measurable set $N_x \subset Q$ containing x as a point of density. Suppose $\mathtt{E} \subset Q$. For every $x \in \mathtt{E} \subset Q$, choose an approximate neighborhood $N_x \subset Q$ of x. Then, $N = \{N_x : x \in \mathtt{E}\}$ is a choice on $\mathtt{E}$. If each point of N_x is a point of density of N_x, then a tagged $(\mathtt{E}_i, x)$ is said to be fine to the choice $N = \{N_x\}$ if $E_i \in N_x$ and $x \in E_i$.

Definition 4 ([15]). *Given a* μ*-measurable set* $\mathtt{E} \subset Q$*, a set-valued function* $N : \mathtt{E} \to 2^Q$ *is called an AP-neighborhood function (ANF) on* $\mathtt{E}$ *if for every* $x \in \mathtt{E}$*, there exists an ap-neighbourhood* $N_x \subset Q$ *of* x *such that* $N(x) = N_x$*.*

A tagged subpartition, denoted by the symbol $P = \{(\mathtt{E}_i, t_i) : 1 \le i \le n\}$, consists of a finite set of non-overlapping tagged subsets in Q that way $t_i \in \mathtt{E}_i$ for $i = 1, 2, 3, \ldots, n$. We say that P is N-fine if $(\mathtt{E}_i, t_i)$ is acceptable to the selection N for $i = 1, 2, \ldots, n$. If P is N-fine and $t_i \in \mathtt{E}$ for each $1 \le i \le n$, we may say that P is $(N, \mathtt{E})$ fine. We refer to P as the tagged partition of Q if P is N-fine and $Q = \bigcup_{i=1}^{n} \mathtt{E}_i$.

Definition 5. *Let* $\mathtt{f}$ *be a function defined on* μ *set* Q*. We say* $\mathtt{f} : Q \subset \mathtt{F} \to \mathtt{R}$ *is approximately continuous at* $c \in Q$ *if there exist a measurable set* $\mathtt{E} \subset Q$ *with density* 1 *at* c *such that*

$$\lim_{x \to c} \mathtt{f}(x) = A \text{ for } x \in \mathtt{E}.$$

We say $\mathtt{f}$ *is approximately differentiable at* c *if there exists a real number* A *and a measurable set* $\mathtt{E} \subset Q$ *such that the density of* $\mathtt{E}$ *at* c *is* 1 *and*

$$\lim_{x \to c} \frac{\mathtt{f}(x) - \mathtt{f}(c)}{x - c} = A \text{ for } x \in \mathtt{E}.$$

We denote $A = \mathtt{f}'_{ap}(c)$*. For a tagged partition* $P = \{(\mathtt{E}_i, t_i) : 1 \le i \le n\}$ *of* Q*, we denote* $\mathbb{S}(\mathtt{f}, P) = \sum_{i=1}^{n} \mathtt{f}(t_i)\mu(\mathtt{E}_i)$ *and* $\mathtt{f}(P) = \mathbb{S}(\mathtt{f}, P)$*. For a function* $F_1 : Q \to \mathtt{R}$*,* F_1 *can be utilized as a* *set-based function by specifying $F_1(Q) = \mu(Q)$. We will denote $F_1(P) = \sum_{i=1}^{n} F_1(Q_i)$ for an N-fine tagged partition $P = \{(Q_i, t_i),\ 1 \le i \le n\}$.*

Definition 6. *A function $\mathtt{f} : Q \to \mathtt{R}$ is called μ_{AP}–Henstock–Kurzweil integrable if there exists a real number A such that for each $\epsilon > 0$, there exists a choice N on Q such that $|\mathbb{S}(\mathtt{f}, P) - A| < \epsilon$.*

In this instance, A is referred to as the μ_{AP}–Henstock–Kurzweil integral of $\mathtt{f}$ on Q, and we write

$$A = (AP) \int_Q \mathtt{f} d\mu. \tag{1}$$

The collection of all integrable μ_{AP}–Henstock–Kurzweil functions on Q (in regard to μ) shall be written as $\mu_{AP}HK(Q)$. It is observe that A of (1) is unique. The $\mu_{AP}HK$ integral includes the AP-real line Henstock–Kurzweil integral.

Example 1. *Using Euclidean distance in $\mathtt{R}$ along with the Lebesgue measure λ, let $\mathtt{X}$ be the interval of the real line $[0, 1]$. If we consider*

$$\mathtt{F} = \{I :\ I\ \text{is non empty closed subinterval of}\ \mathtt{X}\}$$

Clearly, $\mathtt{F}$ is a μ set in $[0, 1]$ and the μ_{AP}–Henstock–Kurzweil integral is the usual AP–Henstock–Kurzweil integral on $[0, 1]$.

Remark 1. *Since the intersection of two approximate full covers of Q is another approximate full cover of Q, also the number A of (1) is unique. Every μ–Henstock–Kurzweil integrable function is certainly μ_{AP}–Henstock–Kurzweil integrable and the integrals are equal.*

3.1. Simple Properties

Here, we will study some fundamental characteristics of $\mu_{AP}HK(Q)$ integrable functions.

Theorem 3. *1. Suppose $\mathtt{f}_1, \mathtt{f}_2 \in \mu_{AP}HK(Q)$ on this occasion $\mathtt{f}_1 + \mathtt{f}_2 \in \mu_{AP}HK(Q)$ along with*

$$(AP) \int_Q (\mathtt{f}_1 + \mathtt{f}_2) d\mu = (AP) \int_Q \mathtt{f}_1 d\mu + (AP) \int_Q \mathtt{f}_2 d\mu.$$

2. Suppose $\mathtt{f}_1 \in \mu_{AP}HK(Q)$ α is a scalar, then $\alpha\mathtt{f}_1 \in \mu_{AP}HK(Q)$ with $(AP) \int_Q \alpha\mathtt{f}_1 d\mu = (AP)\alpha \int_Q \mathtt{f}_1 d\mu$.

Proof. For (1) Let $\epsilon > 0$ be given and suppose $A_1,\ A_2$ are μ_{AP}–Henstock–Kurzweil integrals of $\mathtt{f}_1, \mathtt{f}_2$, respectively. Since $\mathtt{f}_1 \in \mu_{AP}HK(Q)$, consider a gauge δ_1 on Q such that

$$|\mathbb{S}(\mathtt{f}_1, P_1) - A_1| < \frac{\epsilon}{2},$$

for each δ_1-fine partition P_1 of Q. Similarly, there exists a positive function (gauge) δ_2 on Q so that for every δ_2-fine partition P_2 of Q, we have

$$|\mathbb{S}(\mathtt{f}_2, P_2) - A_2| < \frac{\epsilon}{2}.$$

Assuming a gauge on Q, with $\delta = \min\{\delta_1, \delta_2\}$. Being aware of the fact that if δ is a gauge on Q, then there is a δ-fine partition of Q. Since P is both δ_1, δ_2-fine, we can find

$$|\mathbb{S}(\mathtt{f}_1 + \mathtt{f}_2, P) - (A_1 + A_2)| < \epsilon.$$

Therefore, $\mathtt{f}_1 + \mathtt{f}_2 \in \mu_{AP}HK(Q)$.

The proof of (2) is similar to the proof of (1). □

Theorem 4. *Suppose* $\mathtt{f} \in \mu_{AP}HK(Q)$ *and suppose for every* $x \in Q$, $\mathtt{f}(x) \geq 0$. *This will give* $(AP)\int_Q \mathtt{f} d\mu \geq 0$.

Proof. Let $\epsilon > 0$ be given. Since $\mathtt{f} \in \mu_{AP}HK(Q)$, there exists a gauge δ on Q such that

$$\left|\mathbb{S}(\mathtt{f}, P) - (AP)\int_Q \mathtt{f} d\mu\right| < \epsilon,$$

for each δ-fine partition $P = \{(x_i, Q_i)\}_{i=1}^m$ of Q. Since $\mathtt{f}(x) \geq 0$ for each $x \in Q$, we have

$$\mathbb{S}(\mathtt{f}, P) = \sum_{i=1}^{m} \mathtt{f}(x_i)\mu(Q_i) \geq 0.$$

Therefore

$$\mathbb{S}(\mathtt{f}, P) - \epsilon < (AP)\int_Q \mathtt{f} d\mu < \mathbb{S}(\mathtt{f}, P) + \epsilon.$$

Since ϵ is arbitrary, we obtain $(AP)\int_Q \mathtt{f} d\mu \geq 0$. □

Theorem 5. *Suppose* $\mathtt{f}_1, \mathtt{f}_2 \in \mu_{AP}HK(Q)$. *In addition, if* $\mathtt{f}_1 \leq \mathtt{f}_2$ *for every* $x \in Q$, *then* $(AP)\int_Q \mathtt{f}_1 d\mu \leq (AP)\int_Q \mathtt{f}_2 d\mu$.

Proof. Let $\mathtt{f} = \mathtt{f}_2 - \mathtt{f}_1$. By Theorem 3, we have $\mathtt{f} \in \mu_{AP}HK(Q)$ and

$$(AP)\int_Q \mathtt{f} d\mu = (AP)\int_Q \mathtt{f}_2 d\mu - (AP)\int_Q \mathtt{f}_1 d\mu.$$

Since $\mathtt{f}_1 \leq \mathtt{f}_2$ then $\mathtt{f} \geq 0$ for each $x \in Q$ and by Theorem 4, we obtain that $(AP)\int_Q \mathtt{f} d\mu \geq 0$. Therefore $(AP)\int_Q \mathtt{f}_1 d\mu \leq (AP)\int_Q \mathtt{f}_2 d\mu$. □

Theorem 6 (The Cauchy Criterion)**.** *A mapping* $\mathtt{f} : Q \to \mathtt{R}$ *is* $\mu_{AP}HK$ *integrable on* Q *if and only if for every* $\epsilon > 0$, *there exists a positive function (gauge)* δ *on* Q *such that*

$$|\mathbb{S}(\mathtt{f}, P_1) - \mathbb{S}(\mathtt{f}, P_2)| < \epsilon,$$

for each pair N-fine partitions P_1 *and* P_2 *of* Q.

Proof. Let us consider $\mathtt{f} : Q \to \mathbb{R}$ is μ_{AP}HK integable on Q. Given $\epsilon > 0$, there exists a gauge δ on Q such that

$$\left|\mathbb{S}(\mathtt{f}, P) - (AP)\int_Q \mathtt{f} d\mu\right| < \frac{\epsilon}{2},$$

for each N-fine partition P of Q. If P_1 and P_2 are two N-fine partitions of Q, we have

$$\begin{aligned}|\mathbb{S}(\mathtt{f}, P_1) - \mathbb{S}(\mathtt{f}, P_2)| &\leq \left|\mathbb{S}(\mathtt{f}, P_1) - (AP)\int_Q \mathtt{f} d\mu\right| + \left|\mathbb{S}(\mathtt{f}, P_2) - (AP)\int_Q \mathtt{f} d\mu\right| \\ &< \epsilon.\end{aligned}$$

Conversely, for each $n \in \mathbb{N}$, let δ_n be a gauge on Q such that

$$|\mathbb{S}(\mathtt{f}, P_n^{'}) - \mathbb{S}(\mathtt{f}, P_n^{''})| < \frac{1}{n}$$

for each pair N-fine partitions $P_n^{'}$ and $P_n^{''}$ of Q. Let $\mathfrak{d}_n(x) = \min\{\delta_1(x), \delta_2(x), \ldots, \delta_n(x)\}$ be a gauge on Q. By the Proposition 1, there exists a $\mathfrak{d}$-fine partition (respectively, N-fine) of P_n

of Q, for each $n \in \mathbb{N}$. Let $\epsilon > 0$ be given and choose a positive natural N such that $\frac{1}{N} < \frac{\epsilon}{2}$. If m and n are positive natural ($n < m$) such that $n \geq N$, then P_n and P_m are N-fine partitions on Q; hence

$$|\mathbb{S}(\mathtt{f}, P_n) - \mathbb{S}(\mathtt{f}, P_m)| < \frac{1}{n} < \frac{\epsilon}{2}.$$

Consequently, $\left\{\mathbb{S}(\mathtt{f}, P_n)\right\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers and hence converges.

If $A = \lim_{n\to\infty} \mathbb{S}(\mathtt{f}, P_n)$, then

$$|\mathbb{S}(\mathtt{f}, P_n) - A| < \frac{\epsilon}{2},$$

for each $n \geq N$. Let P be an N-fine partition on Q, then

$$\begin{aligned} |\mathbb{S}(\mathtt{f}, P) - A| &\leq |\mathbb{S}(\mathtt{f}, P) - \mathbb{S}(\mathtt{f}, P_N)| + |\mathbb{S}(\mathtt{f}, P_N) - A| \\ &< \epsilon. \end{aligned}$$

Hence, f is μ_{AP}HK integrable on Q. □

Corollary 1. *Corollary Suppose $Q \in \mathtt{F}$ and $\mathtt{f} \in \mu_{AP}HK(Q)$. Assume there exists a partition D of Q with $A \in D$. Then, $\mathtt{f}$ is in $\mu_{AP}HK(A)$ with $(AP)\int_A \mathtt{f}d\mu = (AP)\int_Q \mathtt{f}\chi_A d\mu$.*

Remark 2. *1. Let $\mathtt{f} : Q \to \mathtt{R}$ be μ_{AP}–Henstock–Kurzweil integrable on Q; then, $\mathtt{f}$ is μ_{AP}–Henstock–Kurzweil integrable on every subset $E_i \subset Q \in \mathtt{F}$.*
2. If $\mathtt{f}$ is μ_{AP}–Henstock–Kurzweil integrable on each subset $E_i \subset Q$, then $\mathtt{f}$ is μ_{AP}–Henstock–Kurzweil integrable on Q.

Proposition 3. *Consider $\mathtt{f} : Q \to \mathtt{R}$ in $\mu_{AP}(HK)(Q)$ and a partition $\{Q_1, Q_2, \ldots, Q_m\}$ of Q. Then*

$$\mathtt{f} \in \mu_{AP}(HK)(Q_1) \cap \mu_{AP}(HK)(Q_2) \cap .. \cap \mu_{AP}(HK)(Q_m),$$

with $(AP)\int_Q \mathtt{f}d\mu = \sum_{i=1}^{m}(AP)\int_{Q_i} \mathtt{f}d\mu$.

Proof. Since $\mathtt{f} : Q \to \mathtt{R}$ is μ_{AP}–HK integrable on Q, for given $\epsilon > 0$ and for a gauge δ, there will be a choice N on Q so that

$$\left|\mathbb{S}(\mathtt{f}, P) - (AP)\int_Q \mathtt{f}d\mu\right| < \epsilon.$$

Now, according to Corollary 1, $\mathtt{f}$ is in $\mu_{AP}(HK)(Q_i)$ for $i = 1, 2, \ldots; m$; then, there exists a choice N_i for gauge δ_i on Q_i for $i = 1, 2, \ldots, m$ such that $\delta_i(x) < \delta(x)$ for each x, and

$$\left|\mathbb{S}(\mathtt{f}, P_i) - (AP)\int_{Q_i} \mathtt{f}d\mu\right| < \frac{\epsilon}{m},$$

for each δ_i-fine partition P_i of Q. If $\delta = \max\{\delta_i : 1, 2, \ldots\}$, then $P = P_1 \cup P_2 \cup \cdots \cup P_m$ is a δ-fine partition on Q for the choice $\max\{N_i\} = N$. Consequently,

$$\begin{aligned} \left|\mathbb{S}(\mathtt{f}, P) - \sum_{i=1}^{m}(AP)\int_{Q_i} \mathtt{f}d\mu\right| &\leq \left|\mathbb{S}(\mathtt{f}, P_1) - (AP)\int_{Q_1} \mathtt{f}d\mu\right| + \cdots + \left|\mathbb{S}(\mathtt{f}, P_m) - (AP)\int_{Q_m} \mathtt{f}d\mu\right| \\ &< \epsilon. \end{aligned}$$

So, $(AP)\int_Q \mathtt{f}d\mu = \sum_{i=1}^{m}(AP)\int_{Q_i} \mathtt{f}d\mu$. □

Definition 7. *We call the map* $\overline{F} : \mathtt{E} \rightsquigarrow (AP)\int_{\mathtt{E}} f d\mu$ *defined on each subset* $\mathtt{E}$ *of* Q *the indefinite* $\mu_{AP}HK$ *integral of* $\mathtt{f}$ *on* Q.

Theorem 7. *The map* $\overline{F} \to \mathtt{E} \rightsquigarrow (AP)\int_{\mathtt{E}} \mathtt{f} d\mu$ *of Definition 7 is an additive set function.*

Proof. The proof follows from Proposition 3. □

Definition 8. *A collection* S *of N-tagged in* $\mathfrak{F}$ *is an approximate full cover of* $Q \subset \mathfrak{F}$ *if for each* $x \in Q$ *there exists a measurable set* $S_x \subset \mathfrak{F}$ *such that for a compact subset* $\mathtt{E}$, $x \in S_x$ *and* $(x, \mathtt{E}) \subset S$ *if and only if* $x_{\mathtt{E}} \in \mathtt{E} \in S_x$. *The collection* $S = \{S_x : x \in Q\}$ *is called the collection of sets generated by* S. *If* $Q_1 \subset Q$, *then* $S_x = \{(x, \mathtt{E}) \in S : \ x \in Q_1\}$.

Theorem 8. *Let* ϕ *be a set function that is defined on the class of all subsets of* Q. *Suppose* ϕ *is additive. A mapping* $\mathtt{f} : Q \to \mathtt{R}$ *is* $\mu_{AP}HK$ *integrable on* Q *if and only if for each* $\epsilon > 0$, *there exists an N-fine gauge* δ *on* Q *with*

$$\sum_{(x_i, Q_i) \in P} |\phi(Q_i) - \mathtt{f}(x_i)\mu(Q_i)| < \epsilon$$

for every P *partial partition of* Q *that is N-fine.*

Proof. Proof is similar to the ([13], Lemma 2.4.1), so we omit the proof. □

Now, we discuss the properties of the μ_{AP}–Henstock–Kurzweil integral with approximate differentiation with respect to the Radon measure μ. Recall that a Radon measure μ is a Borel regular measure if it is a Borel measure. Additionally, a Borel regular measure μ becomes a Borel measure if for each $\mathtt{E} \subset \mathtt{X}$, there exists a Borel subset B of $\mathtt{X}$ such that $\mathtt{E} \subset \mathtt{B}$ and $\mu(\mathtt{B}) = \mu(\mathtt{E})$. In addition, see [13,14] for details.

In order to consider the upper approximate differentiation at the point of density, with respect to μ, we define the following notions:

Definition 9. *Let us consider* $\overline{F}$ *a set function defined on* $\mathtt{F}$. *For a given* $x \in \mathtt{X}$, *the upper approximate differentiation at* x *with regard to* μ *is determined by*

$$U_{AD}\overline{F}(x) = \inf\left\{\alpha \in \mathtt{R} : x \textit{ is a point of dispersion of } \left\{x \in Q : \lim_{\mathtt{B} \to x \in \mathtt{F}} \frac{\overline{F}(\mathtt{B})}{\mu(\mathtt{B})} \geq \alpha\right\}\right\},$$

where $\mathtt{B} \to x$ *means* $\mu(\mathtt{B}) \neq 0$, $diam(\mathtt{B}) \to 0$ *and* $x \in \mathtt{B}$.

Similarly, we define the lower approximate derivative of $\overline{F}$ at x with respect to μ as

$$L_{AD}\overline{F}(x) = \sup\left\{\beta \in \mathtt{R} : \ \text{x is a point of dispersion of } \left\{x \in Q \ : \lim_{\mathtt{B} \to x \in \mathtt{F}} \frac{\overline{F}(\mathtt{B})}{\mu(\mathtt{B})} \leq \beta\right\}\right\},$$

where $\mathtt{B} \to x$ means $\mu(\mathtt{B}) \neq 0$, $\mathrm{diam}(\mathtt{B}) \to 0$ and $x \in \mathtt{B}$.

When $U_{AD}\overline{F}(c)$ and $L_{AD}\overline{F}(c)$ are equal but different from ∞ and $-\infty$, then $\overline{F}$ is called approximate differentiable at c. The common value is known as the approximate derivative of $\overline{F}$ at c. It is described as $\overline{F}'_{AP}(x)$.

Clearly, $LD\overline{F}(x) \leq L_{AP}\overline{F}(x) \leq U_{AP}\overline{F}(x) \leq UD\overline{F}(x)$. The properties of the approximate derivatives are similar to those for ordinary derivatives with respect to μ.

Theorem 9. *1.* *If* $\mathtt{f}$ *is a non-negative* $\mu_{AP}HK$ *integrable function on a set* Q *and* $\overline{F}$ *is its indefinite* $\mu_{AP}HK$*-integral, then* $\overline{F}$ *is approximate differentiable* μ*-almost everywhere on* Q *and* $\overline{F}'_{AP} = f$ μ*-a.e.*
2. *The function* $\overline{F}$ *is* μ*-measurable.*

Proof. We prove (1) by contradiction. Suppose $\overline{F}'_{AP} \neq \mathtt{f}$ a.e on Q, then at least one of the sets

$$\left\{x \in Q : \ \mathtt{f}(x) - L_{AD}\overline{F}(x) > 0\right\} \tag{2}$$

$$\left\{x \in Q : -\mathtt{f}(x) + L_{AD}\overline{F}(x) > 0\right\} \tag{3}$$

has a positive outer measure. Due to the positive outer measure of Equation (2), there exists positive numbers r_0, r_1 so that $\mu^*(A) > r_0$ where $A = \left\{x \in Q : \mathtt{f}(x) - L_{AD}\overline{F}(x) > r_1\right\}$. Let $\mathtt{S}$ be the system of all Borel sets $\mathtt{B} \subset Q$ such that $\overline{F}(\mathtt{B}) > \mu(\mathtt{B})$ and there exists $x \in \mathtt{B} \cap A$ with $\operatorname{diam}(\mathtt{B}) < \delta(x)$. It is easy to see that $\mathtt{S}$ is a fine cover of A. Therefore, there exists a system of pairwise non-overlapping sets $\{\mathtt{B}_i\}_{i=1}^{m} \subset \mathtt{S}$ such that $\mu(A) \leq \sum_{i=1}^{m} \mu(\mathtt{B}_i) + \epsilon$. Now, $\mathtt{f}$ being μ_{AP}–Henstock–Kurzweil integrable on Q, there exists an approximate full cover $\mathtt{S}$ of Q so that $|\mathtt{S}(\mathtt{f}, P) - \overline{F}(P)| < r_0 r_1, \forall\ P \subseteq \mathtt{S}$. Let $\{\mathtt{S}_x : x \in Q\}$ be the collection of sets generated by $\mathtt{S}$. Since $L_{AD}\overline{F}(x) < \mathtt{f}(x) - r_1$ for each $x \in A$, x is not a point of dispersion of the set

$$\mathtt{B}_x = \left\{x \in Q : \lim_{\mathtt{B} \to x \in \mathtt{F}} \frac{\overline{F}(\mathtt{B})}{\mu(\mathtt{B})} \leq \mathtt{f}(x) - r_1\right\}.$$

Since x is a point of dispersion of $(x - \delta, x + \delta) \setminus \mathtt{S}_x$ so, $\mathtt{B}_x \cap (x - \delta, x + \delta) \cap \mathtt{S}_x \neq \emptyset$ for each $\delta > 0$. Let us choose a strictly monotone sequence $\{y_n^x\} \subseteq \mathtt{B}_x \cap \mathtt{S}_x \to x$. Since $\mu^*(A) > r_0$, so the collection $J = \bigcup_{x \in A} \{[y_n^x, x] : n \in Z^+\}$ is a μ-Vitali cover of A. Now, by the Vitali Covering Lemma, there exists a finite collection $\{Q_i : 1 \leq i \leq q\}$ of disjoints subsets of J such that $\sum_{i=1}^{q} \mu(Q_i) > r_0$. Let $P = \{(Q_i, x_i) : 1 \leq i \leq q\}$ and $P \subset \mathtt{S}$. Now,

$$\begin{aligned}
\mathbb{S}(\mathtt{f}, P) - \overline{F}(P) &= \sum_{i=1}^{q} \{\mathtt{f}(x_i)\mu(\mathtt{Q}_i) - \overline{F}(\mathtt{Q})\} \\
&= \sum_{i=1}^{q} \{\mathtt{f}(x_i) - \frac{\overline{F}(\mathtt{Q})}{\mu(\mathtt{Q}_i)}(\mu(\mathtt{Q}_i))\} \\
&\geq \sum_{i=1}^{q} r_1 \mu(\mathtt{Q}_i) \\
&> r_0 r_1
\end{aligned}$$

which is a contradiction. So, $\overline{F}' = f$ a.e. on Q.

For (2): To prove the function $\overline{F}$ is μ-measurable, let P_k be a $\frac{1}{k}N$-fine partial partition of Q and let $\mathtt{f}_k$ be the μ-simple functions as

$$\mathtt{f}_k(x) = \sum_{(x, \mathtt{B}) \in P_k} \frac{\overline{F}(\mathtt{B})}{\mu(\mathtt{B})},$$

where $\mathtt{B}$ is a Borel subset of $\mathtt{X}$ such that $\mathtt{E} \subset \mathtt{B}$ and $\mu(\mathtt{B}) = \mu(\mathtt{E})$. Let $\mathcal{C} = \bigcup_{k=1}^{\infty} \bigcup_{\mathtt{B} \in P_k} \partial \mathtt{B}$ also when $D = \{x \in Q : \overline{F}'_{AP}$ does not exist, or $\overline{F}'_{AP}$ exists and $\overline{F}'_{AP}(x) \neq \mathtt{f}(x)\}$. Since $\mu(\partial \mathtt{B}) = 0$ for each $\mathtt{B} \in Q$, and $\mathtt{f}$ is a non-negative $\mu_{AP}HK$ integrable function on a set Q with $\overline{F}'_{AP} = f$ μ-a.e, we obtain that $Q_1 = \mathcal{C} \cup D$ is μ-null. Let $x \in Q \setminus Q_1$. For each $k \in \mathbb{N}$, there exists $Q_{k,x} \in \mathtt{F}$ such that $(x, Q_{k,x}) \in P_k$, $\operatorname{diam}(Q_k, x) < \frac{1}{k}$ and $\mathtt{f}_k(x) = \frac{\overline{F}(Q_{k,x})}{\mu(Q_k,x)}$. Then, by $\overline{F}'_{AP} = \mathtt{f}$ μ-a.e., we obtain $\mathtt{f}_k(x) \to \mathtt{f}(x)$. Therefore, $\overline{F}$ is μ-measurable. □

Theorem 10. *Every non-negative μ_{AP}–Henstock–Kurzweil integrable function $\mathtt{f}$ on a set Q is μ-measurable if its indefinite integral $\overline{F}$ is μ_{AP}–Henstock–Kurzweil integrable.*

Proof. The proof is similar to the ([12], Theorem 5.3), so we omit the proof. □

Recall that on a set Q, each Lebesgue integrable function coincides with the μ–Henstock–Kurzweil integrable function. We recall the Vitali–Carateódory Theorem below.

Theorem 11 ((Vitali–Carateódory Theorem) [13], Theorem 2.5.1)**.** *Let $\mathtt{f}$ be a real function defined on a set Q. If f is Lebesgue integrable on Q with respect to μ and $\epsilon > 0$, then there exist functions $\mathtt{g}$ and $\mathtt{h}$ on Q such that $\mathtt{g} \le \mathtt{f} \le \mathtt{h}$, $\mathtt{g}$ is upper semicontinuous and bounded above, $\mathtt{h}$ is lower semicontinuous and bounded below, and $(L)\int_Q (\mathtt{h} - \mathtt{g})d\mu < \epsilon$.*

We find the relation between μ_{AP}–Henstock–Kurzweil integrable functions and Lebesgue integrable functions on Q with the Lebesgue integral as follows:

Proposition 4. *Every Lebesgue integrable function $\mathtt{f}_1 : Q \to \mathtt{R}$ on Q with regard to μ is μ_{AP}–Henstock–Kurzweil integrable on Q. Consequently, $(L)\int_Q \mathtt{f}_1 d\mu = (AP)\int_Q \mathtt{f}_1 d\mu$.*

Proof. Suppose $\mathtt{f}_1 : Q \to \mathtt{R}$ is Lebesgue integrable on Q. Using the Vitali–Carateódory Theorem, for $\epsilon > 0$, there exist the functions $\mathtt{f}_2$ and $\mathtt{f}_3$ that are upper and lower semicontinuous, respectively, on Q such that $-\infty \le \mathtt{f}_2 \le \mathtt{f}_1 \le \mathtt{f}_3 \le +\infty$ and $(L)\int_Q (\mathtt{f}_3 - \mathtt{f}_2)d\mu < \epsilon$. Let δ be an N-fine gauge on Q so that $\mathtt{f}_2(t) \le \mathtt{f}_1(x) + \epsilon$ and $\mathtt{f}_3(t) \ge \mathtt{f}_1(x) - \epsilon$ for every $t \in Q$ along with $d(x,t) < \delta(x)$. For an N-fine partition $P = \{(Q_i, x_i)\}_{i=1}^n; i = 1, 2, \ldots, n$ of Q, we have

$$(L)\int_{Q_i} \mathtt{f}_2 d\mu \le (L)\int_{Q_i} \mathtt{f}_1 d\mu \le (L)\int_{Q_i} \mathtt{f}_3 d\mu. \tag{4}$$

Therefore,

$$(L)\int_{Q_i} (\mathtt{f}_2 - \epsilon) d\mu \le (L)\int_{Q_i} \mathtt{f}_1(x_i) d\mu.$$

Hence, $(L)\int_{Q_i} \mathtt{f}_2 d\mu - \epsilon\ \mu(Q_i) \le \mathtt{f}_1(x_i)\mu(Q_i)$. In addition, $\mathtt{f}_1(x_i)\mu(Q_i) \le (L)\int_{Q_i} \mathtt{f}_3 d\mu + \epsilon\ \mu(Q_i)$. Therefore, for $i = 1, 2, \ldots, n$,

$$(L)\int_{Q_i} \mathtt{f}_2 d\mu - \epsilon\ \mu(Q_i) \le \mathtt{f}_1(x_i)\mu(Q_i) \le (L)\int_{Q_i} \mathtt{f}_3 d\mu + \epsilon\ \mu(Q_i).$$

This gives

$$(L)\int_{Q} \mathtt{f}_2 d\mu - \epsilon \le \mathbb{S}(\mathtt{f}_1, P) \le (L)\int_{Q} \mathtt{f}_3 d\mu + \epsilon. \tag{5}$$

From (4) and (5), $|\mathbb{S}(\mathtt{f}_1, P) - (L)\int_Q \mathtt{f}_1 d\mu| < \epsilon$. Hence, $\mathtt{f}_1$ is a μ_{AP}–Henstock–Kurzweil integrable on Q with respect to μ with $(L)\int_Q \mathtt{f}_1 d\mu = (AP)\int_Q \mathtt{f}_1 d\mu$. □

4. AP–Henstock–Kurzweil Integral with Respect to Locally Finite Measures on Locally Compact Second Countable Hausdorff Spaces

We apply the fundamental findings from the theory of the μ_{AP}–Henstock–Kurzweil integral to the case of bounded or locally finite measures on second countable Hausdorff spaces that are locally compact. A second countable locally compact Hausdorff space will now be represented by $\mathtt{X}$. Consider $\mathtt{X} = \bigcup_{i\in\mathbb{N}} \mathtt{X}_i$, where $(\mathtt{X}_i)_{i\in\mathbb{N}}$ is an increasing sequence of relatively compact open subsets of $\mathtt{X}$ so that $\mathtt{X}_i \subseteq \mathtt{X}_{i+1}$. Radon measures are always Borel regular by definition, but in general, the converse is not true. Indeed, for example, the counting measure n on $\mathtt{X}$, defined by letting $n(A)$ be the number of elements in A, where $A \subset \mathtt{X}$ is Borel regular on any metric space $\mathtt{X}$, but it is a Radon measure only if every compact subset of $\mathtt{X}$ is finite, that is, $\mathtt{X}$ is discrete. A Borel measure μ on a locally compact Hausdorff space is regular if for all Borel subsets $\mathtt{B}$ of $\mathtt{X}$, we have

$$\mu(\mathtt{B}) = \inf\{\mu(O) : \mathtt{B} \subseteq O,\ O \text{ is open}\}.$$

That is,

$$\mu(\mathtt{B}) = \sup\{\mu(\mathtt{K}) :\ \mathtt{K} \subseteq \mathtt{B},\ \mathtt{K} \text{ is compact}\}.$$

In addition, any bounded Borel measure on a second countable locally compact Hausdorff space is regular. So, the Borel measure is equivalent to the Radon measure on a second countable locally compact Hausdorff space. Now, onwards, consider the Radon measure μ on a σ-algebra $\mathtt{M}$ of $\mathtt{X}$ endowed with a second countable locally compact Hausdorff topology $\mathfrak{T} \subseteq \mathtt{M}$. We define the μ_{AP}–Henstock–Kurzweil integrable relative to μ on Q as follows:

Definition 10. *Let $\mathtt{f} : \overline{Q} \to \mathtt{R}$ be a function. $\mathtt{f}$ is said to be μ_{AP}–Henstock–Kurzweil integrable relative to μ on Q if there exists a real number A such that for each $\epsilon > 0$, there exists a choice N-fine gauge Γ on Q such that*

$$\left|\sum \mathtt{f}(x)\mu(Q_i) - A\right| < \epsilon.$$

We denote the real number $A = (\overline{AP}) \int_Q \mathtt{f}$. A is called the $\mu_{AP}HK$ integral of $\mathtt{f}$ relative to μ on Q.

It is easy to see that the integral A is unique. An N-fine gauge on Q has to be outlined on $\overline{Q}$ not simply on Q since for each pair of sets for each (Q_i, x_i) during a partition of Q, the relevant point $x \in \overline{Q}$ while this is not true for Q. The set of all functions that are μ_{AP}–Henstock–Kurzweil integrable relative to μ on Q shall be denoted by $\mu_{rAP}HK(Q)$.

We call a finite union of mutually disjoint μ sets as a μ-elementary set (in short, an elementary set). Any subset of an elementary set is called a μ-elementary subset (in short elementary subset). Say Q be an elementary set and Q_0 be an elementary subset of Q. We call Q_0 a μ-fundamental set (in short fundamental set) if Q_0 and $Q \setminus Q_0$ are μ-elementary sets.

Remark 3. *If $P = \{(Q_i, x_i)\}_{i=1}^{n}$ is an N-fine partial division of Q which is not a division of Q, then $Q \setminus \bigcup_{i=1}^{n} Q_i$ is necessarily an elementary set. This means for each $i = 1, 2, \ldots, n$ the set $Q \setminus Q_i$ is an elementary subset of Q and thus each Q_i is a fundamental subset of Q.*

Fundamental Characteristics

Within this subsection, we lay out a few basic characteristics of the AP_{HK} integral. The main result here is a Saks–Henstock-type Lemma. In the sequel, Q is a μ set and $Q \in \mathtt{F}$. Within this subsection, almost all x in $\overline{Q}$ means almost everywhere in $\overline{Q}$. We consider a property is said to hold almost everywhere in $\overline{Q}$ if it holds everywhere except perhaps in a set of measure zero, that is that property holds for all $x \in \overline{Q} \setminus Y$ where $\mu(Y) = 0$.

Proposition 5. *Let Q be a μ set and $\mathtt{f} : \overline{Q} \to \mathtt{R}$. If $\mathtt{f}(x) = 0$ for almost all x in $\overline{Q}$, then $\mathtt{f}$ is μ_{rAP}Henstock–Kurzweil integrable with the value 0 on Q.*

Proof. Let $\mathtt{f}(x) = 0$ for all $x \in \overline{Q} \setminus Y$, where $\mu(Y) = 0$ and Y is the union of $\mathtt{X}_i, i = 1, 2, \ldots$, where $\mathtt{X}_i$ is a subset of Y such that $i - 1 \leq |\mathtt{f}(x)| < i$ for $x \in \mathtt{X}_i$. Each $\mu(\mathtt{X}_i) = 0$ as $0 \leq \mu(\mathtt{X}_i) \leq \mu(Y) = 0$. Given $\epsilon > 0$ and for each i, using Proposition 2, we can choose an open set U_i such that $\mu(U_i) < \frac{\epsilon}{2^i \times i}$ and $\mathtt{X}_i \subseteq U_i$. Let us define an N-fine gauge Γ on Q such that $\Gamma(x) \subseteq U_i$ for $x \in \mathtt{X}_i, i = 1, 2, \ldots$. Then, for any N-fine partition $P = \{(Q_i, x_i)\}_{i=1}^{n}$, we have

$$
\begin{aligned}
|\mathbb{S}(\mathtt{f}, P) - 0| &= \left|\sum \mathtt{f}(x_i)\mu(Q_i) - 0\right| \\
&= \left| \sum_{x_i \in \overline{Q}_i \setminus Y} \mathtt{f}(x_i)\mu(Q_i) + \sum_{x_i \in Y} \mathtt{f}(x_i)\mu(Q_i) \right| \\
&\leq \sum_{i=1}^{\infty} \sum_{x_i \in \mathtt{X}_i} |\mathtt{f}(x_i)|\mu(Q_i) \\
&< \sum_{i=1}^{\infty} \left(i \times \frac{\epsilon}{2^i \times i} \right) \\
&= \epsilon.
\end{aligned}
$$

□

Proposition 6. *If $\mathtt{f}_1, \mathtt{f}_2 \in \mu_{rAP}HK(Q)$, then for any scalars α and β, $\alpha\mathtt{f}_1 + \beta\mathtt{f}_2 \in \mu_{rAP}HK(Q)$ and*

$$
(\overline{AP}) \int_Q (\alpha\mathtt{f}_1 + \beta\mathtt{f}_2) d\mu = \alpha(\overline{AP}) \int_Q \mathtt{f}_1 d\mu + \beta(\overline{AP}) \int_Q \mathtt{f}_2 d\mu.
$$

Proposition 7. *Let the functions $\mathtt{f}_1$, $\mathtt{f}_2 \in \mu_{rAP}HK(Q)$ such that $\mathtt{f}_1(x) = \mathtt{f}_2(x)$ almost everywhere in $\overline{Q}$, then $\mathtt{f}_1 \in \mu_{rAP}HK(Q)$ if and only if $\mathtt{f}_2 \in \mu_{rAP}HK(Q)$, and $(\overline{AP}) \int_Q \mathtt{f}_1 d\mu = (\overline{AP}) \int_Q \mathtt{f}_2 d\mu$ μ-a.e.*

Proposition 8 (Cauchy's criterion)**.** *Let $\mathtt{f}$ be a real-valued function on $\overline{Q}$. Then, $\mathtt{f} \in \mu_{rAP}HK(Q)$ if and only if for every $\epsilon > 0$, there exists an N-fine gauge Γ on Q such that for all N-fine partitions P_1, P_2 that are Γ-fine of Q, it holds*

$$
|\mathbb{S}(\mathtt{f}, P_1) - \mathbb{S}(\mathtt{f}, P_2)| < \epsilon.
$$

Proof. The necessity follows from Definition 10. To prove the sufficiency, let Γ_n be an N-fine gauge on Q such that for each $n \in \mathbb{N}$ with each pair N-fine partitions $P_n^{'}$ and $P_n^{''}$ of Q,

$$
|\mathbb{S}(\mathtt{f}, P_n') - \mathbb{S}(\mathtt{f}, P_n^{''})| < \epsilon.
$$

Let us consider an N-fine gauge $\Gamma_\Delta = \min\{\Gamma_1(x), \Gamma_2(x), \ldots, \Gamma_n(x)\}$ on Q. Then, there is a Γ_Δ-fine partition P_n which is an N-fine of Q for each $n \in \mathbb{N}$. If $m, n \in \mathbb{N}$, $n < m : n \geq N$. This implies: P_n and P_m are N-fine partitions of Q. Hence,

$$
|\mathbb{S}(\mathtt{f}, P_n) - \mathbb{S}(\mathtt{f}, P_m)| < \frac{\epsilon}{2}.
$$

This implies $\left\{\mathbb{S}(\mathtt{f}, P_n)\right\}_{n=1}^{\infty}$ is a Cauchy sequence that converges to real number $A = \lim_{n \to \infty} \mathbb{S}(\mathtt{f}, P_n)$. Then, $|\mathbb{S}(\mathtt{f}, P_n) - A| < \frac{\epsilon}{2}$ for each $n \geq N$. On the condition that P is an N-fine partition on Q, then $|\mathbb{S}(\mathtt{f}, P) - A| < \epsilon$. Hence, $\mathtt{f} \in \mu_{rAP}HK(Q)$ and $A = (\overline{AP}) \int_Q \mathtt{f} d\mu$. □

Proposition 9. *Let $\mathtt{f}$ be a real-valued function defined on $\overline{Q}$. If $\mathtt{f} \in \mu_{rAP}HK(Q)$, then $\mathtt{f} \in \mu_{rAP}HK(Q_0)$ for every subset Q_0 of Q.*

Proof. Let $Q_0 \subset Q$. Let $P_1 = \{(Q_i, x_i)\}_{i=1}^{n}$ and $P_2 = \{(Q_j, x_j)\}_{j=1}^{m}$ be N-fine partitions of Q_0 and let P_3 be an N-fine partition of $Q \setminus Q_0$. It is very clear that $P_1 \cap P_3 = \emptyset$, $P_2 \cap P_3 = \emptyset$ and $P_1 \cup P_3$, $P_2 \cup P_3$ are N-fine partitions of Q. Then, by Cauchy's criterion

$$|\mathbb{S}(\mathtt{f}, P_1 \cup P_3) - \mathbb{S}(\mathtt{f}, P_2 \cup P_3)| < \epsilon.$$

Consequently, we obtain $|\mathbb{S}(\mathtt{f}, P_1) - \mathbb{S}(\mathtt{f}, P_2)| < \epsilon$ so, $\mathtt{f} \in \mu_{rAP}HK(Q_0)$. □

Proposition 10. *Let Q be a disjoint union of subsets $Q_1, Q_2, \dots, Q_m$. If $\mathtt{f} \in \mu_{rAP}HK(Q_i)$ for each $i = 1, 2, \dots, m$, then $\mathtt{f} \in \mu_{rAP}HK(Q)$ and $(\overline{AP}) \int_Q \mathtt{f} d\mu = \sum_{i=1}^{m} (\overline{AP}) \int_{Q_i} \mathtt{f} d\mu$.*

Proof. The proof is very straightforward by using the Cousin's lemma. □

Proposition 11. *Let Y be a closed subset of $\overline{Q}$. Then, χ_Y is μ_{rAP}–Henstock–Kurzweil integrable on Q with the value $\mu(Y)$.*

Proof. Let $\epsilon > 0$; then, from Proposition 2, there exists an open set U such that $Y \subseteq U$ and $\mu(U \setminus Y) < \epsilon$. Let Γ be an N-fine gauge on Q such that $\Gamma(x) \subseteq U$ if $x \in Y$ and $\Gamma(x) \subseteq \overline{Q} \setminus Y$ if $x \in \overline{Q} \setminus Y$. Let $P = \{(Q_i, x_i)\}_{i=1}^{n}$ be an N-fine partition of Q. Then

$$\sum_{x \in Y} \chi_Y(x)\mu(Q_i) = \sum_{x \in Y} \mu(Q_i)$$

and

$$\sum_{x \notin Y} \chi_Y(x, \mu(Q_i)) = 0.$$

Let Q_i be the union of subsets of Q such that $(Q_i, x_i) \in P$ and $x \in Y$, then $Y \subseteq Q_i \subseteq U$. So,

$$\begin{aligned} \left|\sum \chi_Y(x, \mu(Q_i)) - \mu(Y)\right| &= \left|\sum_{x \in Y} \mu(Q_i) - \mu(Y)\right| \\ &= \mu(Q_i \setminus Y) \\ &\leq \epsilon. \end{aligned}$$

Hence the proof. □

Theorem 12 (Saks–Henstock-type Lemma). *Let $\mathtt{f} \in \mu_{rAP}HK(Q)$. For every $\epsilon > 0$, there exists an N-fine gauge Γ on Q such that for any N-fine partition $P = \{(x_i, Q_i)\}_{i=1}^{n}$ of Q, we have*

$$\sum_{i=1}^{n} \left| \mathtt{f}(x_i)\mu(Q_i) - (\overline{AP}) \int_{Q_i} \mathtt{f} d\mu \right| < \epsilon.$$

Proof. Let $\epsilon > 0$ be given and let Γ be an N-fine gauge on Q such that for any N-fine partition $P = \{(Q_i, x_i)\}$ on Q, we have

$$\left| \sum \mathtt{f}(x_i)\mu(Q_i) - (\overline{AP}) \int_{Q} \mathtt{f} d\mu \right| < \epsilon.$$

Now, by Proposition 9 and Remark 3, the integral $(\overline{AP}) \int_{Q_i} \mathtt{f} d\mu$ exists for $i = 1, 2, \dots$ such that $\mathtt{f}(x_i)\mu(Q_i) - (\overline{AP}) \int_{Q_i} \mathtt{f} d\mu \geq 0$.

By the $\mu_{rAP}HK$ integrability of $\mathtt{f}$ on each Q_i, there exists an N-fine gauge Γ^* finer than Γ, and N-fine (Γ^*-fine) partitions $P_1, P_2, \ldots, P_n$ on $Q_1, Q_2, \ldots, Q_n$, respectively, such that for $i = 1, 2, \ldots,$ we have

$$\left| \sum \mathtt{f}(x_i)\mu(Q_i) - (\overline{AP}) \int_{Q_i} \mathtt{f} d\mu \right| < \frac{\epsilon}{n}. \tag{6}$$

Now, using (6), it results

$$\begin{aligned} \sum_{i=1}^{n} \left| \mathtt{f}(x_i)\mu(Q_i) - (\overline{AP}) \int_{Q_i} \mathtt{f} d\mu \right| &\leq \left| \sum \mathtt{f}(x_i)\mu(Q_1) - (\overline{AP}) \int_{Q_1} \mathtt{f} d\mu \right| + \ldots \\ &+ \left| \sum \mathtt{f}(x_i)\mu(Q_n) - (\overline{AP}) \int_{Q_n} \mathtt{f} d\mu \right| \\ &< \epsilon. \end{aligned}$$

□

5. Conclusions

The concept of an AP–Henstock–Kurzweil-type integral is given on a Cauchy metric measure space $\mathtt{X}$ with a Radon measure μ and a family of "sets" $\mathtt{F}$ that satisfy the Vitali covering theorem with respect to μ. The classical Henstock–Kurzweil integral on the real line is specifically enclosed by this integral. In this setting, Cauchy's criterion of an AP–Henstock–Kurzweil integral is discussed. Finally, we extend this idea to second countable, locally compact Hausdorff spaces having bounded or locally finite measures. In this approach, the Saks–Henstock-type lemma is discussed. As a future research topic, we will investigate the validity of the converse of Proposition 4.

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