mathematics

Article

Stability Estimates for an Arithmetic Functional Equation with Brzdęk Fixed Point Approaches

Heejeong Koh

College of Liberal Arts, Dankook University, 152, Jukjeon, Suji, Yongin 16890, Gyeonggi, Republic of Korea; khjmath@dankook.ac.kr

Abstract: We introduce an arithmetic functional equation $f(x^2+y^2) = f(x^2) + f(y^2)$ and then investigate stability estimates of the functional equation by using the Brzdęk fixed point theorem on a non-Archimedean fuzzy metric space and a non-Archimedean fuzzy normed space. To apply the Brzdęk fixed point theorem, the proof uses the linear relationship between two variables, x and y.

Keywords: stability; arithmetic functional equation; fixed point; Brzdęk fixed point

MSC: 39B82

1. Introduction

In 1940, Ulam [1] proposed the stability problem of a group homomorphism. In other words, the question would be generalized as "Under what conditions a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?". In 1941, Hyers [2] gave the first, affirmative, and partial solution to Ulam's question with an additive function (Cauchy function) in Banach spaces. The Hyers stability result was first generalized in the stability involving p-powers of norm by Aoki [3]. In 1978, Rassias [4] provided a generalization of Hyers' theorem that allows the Cauchy difference to become unbounded. For the last few decades, stability problems of various functional equations have been extensively investigated and generalized by many mathematicians (see [5–9]).

Baker [10] introduced for the first time the Ulam's type stability by using the fixed point method and later applied it in numerous papers (see [11–17]). In fact, fixed point theory is a powerful resource for the research, study and applications of nonlinear functional analysis, optimization theory, and variational inequalities (see [18–22]). Many authors introduced new types of fixed point theorems in different directions. Moreover, Brzdęk and Ciepliński [23] introduced the existence theorem of the fixed point for nonlinear operators in metric spaces:

check for updates

Citation: Koh, H. Stability Estimates for an Arithmetic Functional Equation with Brzdęk Fixed Point Approaches. *Mathematics* **2023**, *11*, 1611. https://doi.org/10.3390/math11071611

Academic Editor: Salvatore Sessa

Received: 8 March 2023
Revised: 23 March 2023
Accepted: 24 March 2023
Published: 27 March 2023

Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

Theorem 1 ([23])**.** *Let X be a non-empty set, (Y, d) be a complete metric space and $\Lambda : Y^X \to Y^X$ be a non-decreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty} \Lambda\delta_n = 0 \text{ for every sequence } \{\delta_n\}_{n\in\mathbb{N}} \text{ with } \lim_{n\to\infty} \delta_n = 0\,.$$

Suppose that $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d(\mathcal{T}\varepsilon(x), \mathcal{T}\mu(x)) \leq \Lambda(\triangle(\varepsilon, \mu))(x),\ \varepsilon, \mu \in Y^X, x \in X$$

where $\triangle : (Y^X)^2 \to \mathbb{R}_+^X$ is a mapping, which is defined by

$$\triangle(\varepsilon, \mu)(x) := d(\varepsilon(x), \mu(x)),\ \varepsilon, \mu \in Y^X, x \in X\,.$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ *and* $\phi : X \to Y$ *such that*

$$d(\mathcal{T}\phi(x), \phi(x)) \le \varepsilon(x)$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} (\Lambda^n \varepsilon)(x) < \infty$$

for all $x \in X$*, then the limit*

$$\lim_{n\to\infty} (\mathcal{T}^n \phi)(x)$$

exists for each $x \in X$*. Moreover, the function* $\phi \in Y^X$ *defined by*

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n \phi)(x)$$

is a fixed point of $\mathcal{T}$ *with*

$$d(\phi(x), \psi(x)) \le \varepsilon^*(x)$$

for all $x \in X$*.*

Brzdęk and Ciepliński [23] used this result to prove the stability problem of functional equations in non-Archimedean metric spaces and obtained the fixed point results in arbitrary metric spaces. In particular, the Brzdęk's fixed point method was also obtained from Theorem 1 (see [24]).

Theorem 2 ([24])**.** *Let* X *be a non-empty set,* (Y, d) *be a complete metric space and* $f_1, f_2 : X \to X$ *be the given mappings. Suppose that* $\mathcal{T} : Y^X \to Y^X$ *and* $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ *are two operators satisfying the following conditions:*

$$d(\mathcal{T}\xi(x), \mathcal{T}\mu(x)) \le d(\xi(f_1(x)), \mu(f_1(x))) + d(\xi(f_2(x)), \mu(f_2(x))) \tag{1}$$

and

$$\Lambda\delta(x) := \delta(f_1(x)) + \delta(f_2(x)) \tag{2}$$

for all $\xi, \mu \in Y^X$*,* $\delta \in \mathbb{R}_+^X$ *and* $x \in X$*. If there exist* $\varepsilon : X \to \mathbb{R}_+$ *and* $\phi : X \to Y$ *such that*

$$d(\mathcal{T}\phi(x), \phi(x)) \le \varepsilon(x) \text{ and } \varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty \tag{3}$$

for all $x \in X$*, then the limit* $\lim_{n\to\infty}(\mathcal{T}^n\phi)(x)$ *exists for each* $x \in X$*. Moreover, the function* $\psi(x) := \lim_{n\to\infty}(\mathcal{T}^n\phi)(x)$ *is a fixed point of* $\mathcal{T}$ *with*

$$d(\phi(x), \psi(x)) \le \varepsilon^*(x)$$

for all $x \in X$*.*

The theory of fuzzy sets was introduced by Zadeh [25] in 1965. George and Veeramani [26] in 1994, introduced a fuzzy metric space by considering points in the crisp set and a fuzzy distance between them. Mirmostafaee and Moslehian [27] introduced a non-Archimedean fuzzy norm on a linear space over a non-Archimedean field. Many mathematicians considered the fuzzy normed spaces in different branches of pure and applied mathematics. In particular, Moslehian and Rassias [28] studied the stability problem of functional equations in non-Archimedean spaces. Moreover, Aiemsomboon and Sintunavarat [29,30] studied the stability problem due to Brzdęk's fixed point theorem.

The purpose of this paper is to introduce an arithmetic functional equation(see [31]) of the following form:

$$f(x^2 + y^2) = f(x^2) + f(y^2) \tag{4}$$

and also to investigate the stability problem by using the Brzdęk's fixed point theorem on a non-Archimedean fuzzy normed space. In fact, for each real number c, a function $f(x) = cx$ satisfies the functional Equation (4). Chung [31] characterized the Equation (4) for all positive integers x and y. In terms of the stability problem, it is also important whether the equation still remains true in the range of real numbers.

In this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of positive integers, the set of non-negative integers, the set of real numbers and the set of non-negative real numbers, respectively.

2. Stability of Arithmetic Functional Equations

In this section, we will investigate the stability problem for the arithmetic functional Equation (4) by using the Brzdęk fixed point method; see Theorem 3. Before proceeding, we will first reproduce the following definitions due to Mirmostafaee and Moslehian [27] and George and Veeramani [26].

Definition 1. *A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is a continuous t-norm if it satisfies the following conditions:*

1. $*$ *is associative and commutative;*
2. $*$ *is continuous;*
3. $a * 1 = a$ *for all* $a \in [0, 1]$*;*
4. $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a, b, c, d \in [0, 1]$.

Obviously, $a * b = a \times b$ and $a * b = \min\{a, b\}$ are common examples of continuous t-norms.

Definition 2. *Let* X *be an arbitrary set. A fuzzy subset* M *of* $M \times M \times [0, \infty]$ *is called a fuzzy metric on* X *if it satisfies the following conditions for all* $x, y \in X$ *and* $t \in \mathbb{R}$.

1. *If* $t \leq 0$, *then* $M(x, y, t) = 0$*;*
2. *For all* $t > 0$, $M(x, y, t) = 1$ *if and only if* $x = y$*;*
3. *For all* $t > 0$, $M(x, y, t) = M(y, x, t)$*;*
4. *For all* $s, t \in \mathbb{R}$, $M(x, z, s + t) \geq M(x, y, s) * M(y, z, t)$*;*
5. $M(x, y, \cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} M(x, y, t) = 1$.

The pair $(X, M, *)$ *is called a fuzzy metric space. If we replace* (4) *by*

6. $M(x, z, max\{s, t\}) \geq M(x, y, s) * M(y, z, t)$,

then we call the triple $(X, M, *)$ *a non-Archimedean fuzzy metric space.*

Definition 3. *Let* V *be a real linear space. A function* $N : X \times [0, \infty] \to [0, 1]$ *is said to be a fuzzy norm on* X *if, for all* $x, y \in X$ *and all* $t, s > 0$, *if it satisfies the following conditions:*

1. $N(x, 0) = 0$*;*
2. $x = 0$ *if and only if* $N(x, t) = 1$ *for all* $t > 0$*;*
3. $N(kx, t) = N(x, \frac{t}{|k|})$ *for all* $k \in \mathbb{R}, k \neq 0$*;*
4. $N(x + y, s + t) \geq N(x, t) * N(y, s)$*;*
5. $N(x, \cdot)$ *is a non decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x, t) = 1$.

The pair $(X, N, *)$ *is called a fuzzy normed space. If we replace* (4) *by*

6. $N(x + y, max\{s, t\}) \geq N(x, s) * N(y, t)$,

then we call the triple $(X, N, *)$ *a non-Archimedean fuzzy normed space.*

Throughout this paper, we assume that

$$a * b = \min\{a, b\}$$

for the continuous t-norm.

Example 1. *Let* $X = [0, \infty)$ *be a metric space with the usual metric* d *and the usual norm* $||\cdot||$. *Let* $a * b \le a \times b$ *for all* $a,\ b \in [0, 1]$. *For each* $t \in (0, \infty)$, *define* $M : X \times X \times [0, \infty] \to [0, 1]$ *by*

$$M(x, y, t) = e^{-\frac{d(x,y)}{t}}$$

for all $x, y \in X$ *and* $t > 0$. *Then,* $(X, M, *)$ *is a non-Archimedean fuzzy metric space.*

Additionally, for each $t \in (0, \infty)$, *define* $N : X \times [0, \infty] \to [0, 1]$ *by*

$$N(x, t) = e^{-\frac{||x||}{t}}$$

for all $x \in X$ *and* $t > 0$. *Then,* $(X, N, *)$ *is a non-Archimedean fuzzy normed space.*

There are some more examples such as: $M(x, y, t) = (1 - e^{-t})^{d(x,y)}$, $M(x, y, t) = (\frac{t}{t+1})^{d(x,y)}$. *It is easy to see that each* $(X, M, *)$ *is a non-Archimedean fuzzy metric space.*

Now, we will investigate stability estimates for an arithmetic functional equation by using Brzdęk's on a non-Archimedean fuzzy metric space and non-Archimedean fuzzy normed space.

Theorem 3. *Let* $(\mathbb{R}_+, M, *)$ *be a non-Archimedean fuzzy metric space, which is invariant (i.e.,* $M(x + z, y + z, t) = M(x, y, t)$ *for* $x, y, z \in \mathbb{R}_+$ *and* $t \in \mathbb{R}$*), and* $(\mathbb{R}, N, *)$ *be a non-Archimedean fuzzy normed space. Let* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function such that*

$$L_0 := \{m \in \mathbb{N} \,|\, s(m^2) + s(1 + m^2) < 1\} \ne \varnothing\,, \tag{5}$$

where

$$s(m) := inf\{t \in \mathbb{R}_+ \,|\, h(mx) \le t\ h(x) \text{ for all } x \in \mathbb{R}_+\}\,.$$

Assume that

$$N(h(nx) + h(my),\ s) \ge N(s(n)h(x) + s(m)h(y),\ s)\,, \tag{6}$$

for all $x, y \in \mathbb{R}_+$, $n, m \in \mathbb{N}$ *and* $s > 0$. *Suppose a function* $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfies the inequality*

$$M(f(x^2 + y^2),\ f(x^2) + f(y^2),\ t) \ge N(h(x) + h(y),\ t) \tag{7}$$

for all $x, y \in \mathbb{R}_+$ *and* $t > 0$. *Then, there exists a unique solution* $T : \mathbb{R}_+ \to \mathbb{R}$ *to the Equation (4) such that*

$$M(f(x), T(x), t) \ge N(s_0 h(x), t) \tag{8}$$

for $x \in \mathbb{R}_+$ *and* $t > 0$, *where*

$$s_0 := inf\{\frac{1 + s(m)}{1 - s(m^2) - s(1 + m^2)} \,|\, m \in L_0\}\,. \tag{9}$$

Proof. Let $m \in L_0$. By letting $y = mx$ in the inequality (7), the inequality (6) implies the following inequality:

$$M\Big(\ f((1 + m^2)x^2),\ f(x^2) + f(m^2x^2),\ t\Big) \ge N(c_m(x),\ t) \tag{10}$$

for $t > 0$, where $c_m(x) := (1 + s(m))h(x)$ for $x \in \mathbb{R}_+$. To apply to the Brzdęk fixed point method, we need to define two operators as in Theorem 2:

1. $\mathcal{T}_m : \mathbb{R}^{\mathbb{R}_+} \to \mathbb{R}^{\mathbb{R}_+}$ by

$$\mathcal{T}_m\xi(x) := \xi((1 + m^2)x) - \xi(m^2x) \tag{11}$$

2. $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_+} \to \mathbb{R}_+^{\mathbb{R}_+}$ by

$$\Lambda_m\mu(x) := \mu((1 + m^2)x) + \mu(m^2x) \tag{12}$$

for all $x \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^{\mathbb{R}_+}$, $\mu \in \mathbb{R}_+^{\mathbb{R}_+}$. Additionally, we have

$$\begin{aligned} & M(\mathcal{T}_m\xi(x), \mathcal{T}_m\mu(x), t) \\ = \ & M(\xi((1+m^2)x) - \xi(m^2x),\, \mu((1+m^2)x) - \mu(m^2x), t) \\ \geq \ & \min\{M(\xi((1+m^2)x) - \xi(m^2x),\, \mu((1+m^2)x) - \xi(m^2x), t), \\ & M(\mu((1+m^2)x) - \xi(m^2x),\, \mu((1+m^2)x) - \mu(m^2x), t)\} \\ = \ & \min\{M(\xi((1+m^2)x),\, \mu((1+m^2)x), t),\, M(\xi(m^2x),\, \mu(m^2x), t)\} \end{aligned}$$

for $x \in \mathbb{R}_+$ and $t > 0$. If we may let $f_1(x) = (1+m^2)x$ and $f_2(x) = m^2x$, then for each $m \in L_0$, two operators $\mathcal{T}_m$ and Λ_m satisfy the inequalities (1) and (2) in Theorem 2. To check the condition (3) in Theorem 2, we note that the inequality (10) implies

$$\begin{aligned} M(\mathcal{T}_m f(x^2), f(x^2), t) &= M(f((1+m^2)x^2) - f(m^2x^2), f(x^2), t) \\ &= M(f((1+m^2)x^2), f(x^2) + f(m^2x^2), t) \\ &\geq N((1+s(m))h(x), t) = N((1+s(m))h(x), t) \end{aligned}$$

for $x \in \mathbb{R}_+$ and $t > 0$. Also, we note that

$$\begin{aligned} \Lambda_m c_m(x) &= (1+s(m))(h((1+m^2)x) + h(m^2x)) \\ &\leq (1+s(m))(s(1+m^2) + s(m^2))h(x) \end{aligned}$$

for all $x \in \mathbb{R}_+$. Using the mathematical induction, for each $k \in \mathbb{N}$, we obtain

$$\Lambda_m^k c_m(x) = (1+s(m))[s(1+m^2) + s(m^2)]^k h(x)$$

for all $x \in \mathbb{R}_+$ and $m \in L_0$. For each $m \in L_0$ and $x \in \mathbb{R}_+$, we have the following condition:

$$c_m^*(x) := \sum_{j=0}^{\infty} \Lambda_m^j c_m(x) = \frac{1+s(m)}{1 - s(1+m^2) - s(m^2)} h(x)$$

where $\Lambda_m^0 c_m(x) = c_m(x)$.

Hence, the Brzdęk fixed point theorem implies that

$$T_m(x) := \lim_{k\to\infty} \mathcal{T}_m^k f(x)$$

exists for each $m \in L_0$ and $x \in \mathbb{R}_+$, and we have

$$M(f(x),\, T_m(x), t) \geq N(c_m^*(x), t)$$

for all $m \in L_0$, $x \in \mathbb{R}_+$ and $t > 0$ (see Theorem 2). Now, we will prove that T_m satisfies the Equation (4) for each $m \in L_0$. Hence, we may conclude that the solution of the Equation (4) is uniquely determined. First, we will inductively check that for $k \in \mathbb{N}_0$

$$\begin{aligned} & M(\mathcal{T}_m^k f(x^2+y^2), \mathcal{T}_m^k f(x^2) + \mathcal{T}_m^k f(y^2), t) \\ & \geq N\Big((s(1+m^2) + s(m^2))^k (h(x) + h(y)), t \Big) \end{aligned}$$

for $x \in \mathbb{R}_+$ and $t > 0$. If $k = 0$, we note that $\mathcal{T}_m^0 f(x^2) = f(x^2)$ and hence the base case follows from the inequality (10). Then,

$$
\begin{aligned}
&M(\mathcal{T}_m^{k+1}f(x^2+y^2), \mathcal{T}_m^{k+1}f(x^2)+\mathcal{T}_m^{k+1}f(y^2), t)\\
&= M(\mathcal{T}_m^k f((1+m^2)(x^2+y^2)) - \mathcal{T}_m^k f(m^2(x^2+y^2)),\\
&\quad \mathcal{T}_m^k f((1+m^2)x^2) - \mathcal{T}_m^k f(m^2x^2) + \mathcal{T}_m^k f((1+m^2)y^2) - \mathcal{T}_m^k f(m^2y^2), t)\\
&\geq \min\{M(\mathcal{T}_m^k f((1+m^2)(x^2+y^2)), \mathcal{T}_m^k f((1+m^2)x^2)+\mathcal{T}_m^k f((1+m^2)y^2), t),\\
&\qquad M(\mathcal{T}_m^k f(m^2(x^2+y^2)), \mathcal{T}_m^k f(m^2x^2)+\mathcal{T}_m^k f(m^2y^2), t)\}\\
&\geq \min\{N(s(1+m^2)^k(h((1+m^2)x)+h((1+m^2)y)), t),\\
&\qquad N(s(m^2)^k(h(m^2x)+h(m^2y)), t)\}\\
&\geq \min\left\{N\left(h(x)+h(y), \frac{t}{s(1+m^2)^{k+1}}\right), N\left(h(x)+h(y), \frac{t}{s(m^2)^{k+1}}\right)\right\}
\end{aligned}
$$

We note that

$$
0 < s(m^2) < s(1+m^2)+s(m^2) < 1 \text{ and } 0 < s(1+m^2) < s(1+m^2)+s(m^2) < 1\,.
$$

Since $N(x,\cdot)$ is a non-decreasing, we have

$$
\begin{aligned}
&M(\mathcal{T}_m^{k+1}f(x^2+y^2), \mathcal{T}_m^{k+1}f(x^2)+\mathcal{T}_m^{k+1}f(y^2), t)\\
&\geq N\left(h(x)+h(y), \frac{1}{(s(1+m^2)+s(m^2))^{k+1}}t\right)
\end{aligned}
$$

for $t>0$. As $k\to\infty$, we have

$$
N\left(h(x)+h(y), \frac{1}{(s(1+m^2)+s(m^2))^{k+1}}t\right) \to 1\,,
$$

where $0 < s(1+m^2)+s(m^2) < 1$.

Hence, we obtain

$$
T_m(x^2+y^2) = T_m(x^2)+T_m(y^2) \tag{13}
$$

for all $x,y\in\mathbb{R}_+$ and $m\in L_0$. That is, for each $m\in L_0$, T_m is a solution of the Equation (4). Now, assume that a mapping $T:\mathbb{R}_+\to\mathbb{R}$ satisfies the Equation (4) such that

$$
M(f(x), T(x), t) \geq N(L\,h(x), t)
$$

for $L>0$ constant. Let

$$
T(x^2+y^2) = T(x^2)+T(y^2) \tag{14}
$$

for all x, $y\in\mathbb{R}_+$. Then, we will show that $T=T_m$ for each $m\in L_0$. By letting $y=mx$ in the inequality (14), we have

$$
T(x^2) = T((1+m^2)x^2) - T(m^2x^2)
$$

for $x\in\mathbb{R}_+$ and $t>0$. Let $m_0\in L_0$. Then,

$$
\begin{aligned}
&M(T(x^2), T_{m_0}(x^2), t) \geq \min\{M(T(x^2), f(x^2), t), M(f(x^2), T_{m_0}(x^2), t)\}\\
&\geq \min\left\{N(L\,h(x), t), N\left(\frac{1+s(m_0)}{1-s(1+m_0^2)-s(m_0^2)}h(x), t\right)\right\}.
\end{aligned}
$$

By letting $S_0 = ((1+s(m_0)) + (1-s(1+m_0^2)-s(m_0^2))L)$, we have

$$
\frac{1+s(m_0)}{1-s(1+m_0^2)-s(m_0^2)} + L = S_0 \cdot \sum_{k=0}^{\infty}(s(1+m_0^2)+s(m_0^2))^k
$$

Since $N(s,\cdot)$ is non-decreasing, we note

$$\begin{aligned}
&\min\left\{N(L\,h(x),t), N\left(\frac{1+s(m_0)}{1-s(1+m_0^2)-s(m_0^2)}h(x),t\right)\right\}\\
&\geq N\left(h(x), \frac{1}{\frac{1+s(m_0)}{1-s(1+m_0^2)-s(m_0^2)}+L}t\right)\\
&= N\left(S_0\cdot\sum_{k=0}^{\infty}(s(1+m_0^2)+s(m_0^2))^k h(x),t\right)
\end{aligned}$$

Hence, we obtain

$$M(T(x^2),T_{m_0}(x^2),t)\geq N\left(S_0\cdot\sum_{k=0}^{\infty}(s(1+m_0^2)+s(m_0^2))^k h(x),\ t\right) \tag{15}$$

for $x\in\mathbb{R}_+$ and $t>0$. For $l\in\mathbb{N}_0$, assume that

$$M(T(x^2),T_{m_0}(x^2),t)\geq N\left(S_0\cdot\sum_{k=l}^{\infty}(s(1+m_0^2)+s(m_0^2))^k h(x),t\right)$$

for $x\in\mathbb{R}_+$ and $t>0$. We will check it by using mathematical induction on l. If $l=0$, it follows from the inequality (15). Then,

$$\begin{aligned}
&M(T(x^2),T_{m_0}(x^2),t)\\
&= M(T((1+m_0^2)x^2)-T(m_0^2x^2),T_{m_0}((1+m_0^2)x^2)-T_{m_0}(m_0^2x^2),t)\\
&\geq\ \min\{M(T((1+m_0^2)x^2),T_{m_0}((1+m_0^2)x^2),t), M(T(m_0^2x^2),T_{m_0}(m_0^2x^2),t)\}\\
&\geq\ \min\{N\left(s(1+m_0^2)S_0h(x)\cdot\sum_{k=l}^{\infty}(s(1+m_0^2)+s(m_0^2))^k,t\right),\\
&\qquad\qquad N\left(s(m_0^2)S_0h(x)\cdot\sum_{k=l}^{\infty}(s(1+m_0^2)+s(m_0^2))^k,t\right)\}\\
&\geq N(S_0h(x)\cdot\sum_{k=l+1}^{\infty}(s(1+m_0^2)+s(m_0^2))^k,t)
\end{aligned}$$

for $x\in\mathbb{R}_+$ and $t>0$. Hence, it holds whenever $l\in\mathbb{N}_0$.

As $l\to\infty$, we have

$$N\left(S_0h(x)\cdot\sum_{k=l+1}^{\infty}(s(1+m_0^2)+s(m_0^2))^k,t\right)\to 1$$

Hence, we have $T=T_{m_0}$, for $m_0\in L_0$. Thus, for each $m_0\in L_0$,

$$T_m=T_{m_0}$$

as desired. □

Example 2. *Let $p<0$ be a real number and a function $h:\mathbb{R}_+\to\mathbb{R}_+$ is defined by*

$$h(x)=A(x)^p\,, x\in\mathbb{R}_+$$

where the map $A:\mathbb{R}_+\to\mathbb{R}_+$ is additive. Then, it is easily seen that the set L_0 is not empty as in Theorem 3. Hence, the Equation (5) is valid.

Corollary 1. *Let $h : \mathbb{R}_+ \to (0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} \inf \sup_{x\in\mathbb{R}_+} \frac{h((1+n^2)x)+h(n^2x)+h(nx)}{h(x)} = 0 \tag{16}$$

Suppose $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$M(f(x^2+y^2),\, f(x^2)+f(y^2), t) \geq N(h(x)+h(y), t) \tag{17}$$

for all $x, y \in \mathbb{R}_+$ and $t > 0$. Then, there exists a unique arithmetic functional equation $T : \mathbb{R}_+ \to \mathbb{R}$ such that

$$M(f(x),\, T(x), t) \geq N(h(x), t) \tag{18}$$

for all $x \in \mathbb{R}_+$ and $t > 0$.

Proof. For each $n \in \mathbb{N}$, let

$$a_n = \sup_{x\in\mathbb{R}_+} \frac{h((1+n^2)x)+h(n^2x)+h(nx)}{h(x)}\,.$$

By the definition $s(n)$ as in Theorem 3, we will see that

$$s(1+n^2) = \sup_{x\in\mathbb{R}_+} \frac{h((1+n^2)x)}{h(x)} \leq a_n\,,$$

$$s(n^2) = \sup_{x\in\mathbb{R}_+} \frac{h(n^2x)}{h(x)} \leq a_n$$

and

$$s(n) = \sup_{x\in\mathbb{R}_+} \frac{h(nx)}{h(x)} \leq a_n$$

These inequalities imply that

$$s(1+n^2)+s(n^2)+s(n) \leq 3a_n \tag{19}$$

for all $x \in \mathbb{R}_+$. By our assumption, the sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = 0$, that is,

$$\lim_{k\to\infty} \sup_{x\in\mathbb{R}_+} \frac{h((1+{n_k}^2)x)+h({n_k}^2x)+h(n_kx)}{h(x)} = 0\,. \tag{20}$$

The inequalities (19) and (20) imply that

$$\lim_{k\to\infty} s(1+{n_k}^2)+s({n_k}^2)+s(n_k) = 0\,,$$

that is, $\lim_{k\to\infty} s(1+{n_k}^2) = 0$, $\lim_{k\to\infty} s({n_k}^2) = 0$ and $\lim_{k\to\infty} s(n_k) = 0$. Thus, we have

$$\lim_{k\to\infty} \frac{1+s(n_k)}{1-s(1+{n_k}^2)-s({n_k}^2)} = 1\,.$$

On letting $s_0 = 1$ as in Theorem 3, the inequality (18) follows from the inequality (8). □

Remark 1. *From the main result of the stability estimates in the Brzdęk fixed point method, the method requires the non-Archimedean fuzzy metric has the invariant property, i.e., $M(x+z,\, y+z,\, t) = M(x,\, y,\, t)$ for all $x,\, y,\, z \in \mathbb{R}_+$ and $t \in \mathbb{R}$. In fact, this property is not required in the different stability methods. Additionally, the use of $y = mx$ in the Brzdęk fixed point method should be remarked. This linear relationship between two variables x and y makes it possible to prove the result and hence obtains the very nice stability approach. Some fixed point approaches required* *strictly contractive mapping and scaling processes. One of the main purposes of this paper is whether the Brzdęk fixed point method can be applied in various spaces such as a non-Archimedean fuzzy normed space. We would like to propose open problems : (1) Can the Brzdęk fixed point method be applied in various fuzzy normed spaces? (2) Can the Brzdęk fixed point method be applied without the linear relationship between two variables x and y?*

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ulam, S.M. *Problems in Morden Mathematics*; Wiley: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
3. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66. [CrossRef]
4. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
5. Cholewa, PW. Remarks on the stability of functional equations. *Aequ. Math.* **1984**, *27*, 76–86. [CrossRef]
6. Czerwik, S. On the stability of the quadratic mapping in normed spaces. *Abh. Aus Dem Math. Semin. Univ. Hamburg*, **1992**, *62*, 59–64. [CrossRef]
7. Gajda, Z. On stability of additive mappings. *Int. J. Math. Math. Sci.* **1991**, *14*, 431–434. [CrossRef]
8. Park, C. Generalized Hyers-Ulam Stability of Quadratic Functional Equations: A Fixed Point Approach. *Fixed Point Theory Appl.* **2008**, *2008* , 493751. [CrossRef]
9. Skof, F. Proprietá locali e approssimazione di operatori. *Rend. Sem. Mat. Fis. Milano.* **1983**, *53*, 113–129. [CrossRef]
10. Baker, J. The stability of certain functional equations. *Proc. Am. Math. Soc.* **1991**, *112*, 729–732. [CrossRef]
11. Cădariu, L.; Radu, V. *On the Stability of the Cauchy Functional Equation: A Fixed Point Approach, Iteration Theory (ECIT '02)*; University of Graz: Graz, Austria, 2004; pp. 43–52.
12. Cădariu, L.; Radu, V. Fixed point methods for the generalized stability of functional equations in a single variable. *Fixed Point Theory Appl.* **2008**, *2008*, 749392. [CrossRef]
13. Ciepliński, K. Applications of fixed point theorems to the Hyers-Ulam stability of functional equations—A survey. *Ann. Funct. Anal.* **2012**, *3*, 151–164. [CrossRef]
14. Jung, S.-M. A fixed point approach to the stability of isometries. *J. Math. Anal. Appl.* **2007**, *329*, 879–890. [CrossRef]
15. Peng, Z.; Wang, J.; Zhao, Y.; Liang, R. Stability on parametric strong symmetric quasi-equilibrium problems via nonlinear scalarization. *J. Nonlinear Var. Anal.* **2022**, *6*, 393–406.
16. Radu, V. The fixed point alternative and the stability of functional equations. *Fixed Point Theory* **2003**, *1*, 91–96.
17. Tian, C.; Gu, H.; Li, N. Existence and stability of solutions for coupled fractional delay q-difference systems. *J. Nonlinear Funct. Anal.* **2022**, *2022*, 20.
18. Ahmad, J.; Al-Rawashdeh, A.; Azam, A. New fxed point theorems for generalized F-contractions in complete metric spaces. *Fixed Point Teory Appl.* **2015**, *1*, 80. [CrossRef]
19. Brzdęk, J.; Cădariu, L.; Ciepliński, K. Fixed point theory and the Ulam stability. *J. Funct. Sp.* **2014**, *2014*, 829419. [CrossRef]
20. Gregori, V.; Sapena, A. On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252. [CrossRef]
21. Piri, H.; Rahrovi, S.; Marasi, H.; Kumam, P. Fixed point theorem for F-Khan-contractions on complete metric spaces and application to the integral equations. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4564–4573. [CrossRef]
22. Zhang, Z.; Wang, K. On fixed point theorems of mixed monotone operators and applications. *Nonlinear Anal. Theory Methods Appl.* **2009**, *70*, 3279–3284. [CrossRef]
23. Brzdęk, J.; Ciepliński, K. A fixed point approach to the stability of functional equations in non-Archimedean metric spaces. *Nonlinear Anal.* **2011**,*74*, 6861–6867. [CrossRef]
24. Brzdęk, J. Stability of additivity and fixed point method. *Fixed Point Theory Appl.* **2013**, *2013*, 265. [CrossRef]
25. Zadeh, L. Fuzzy sets. *Inf. Cont.* **1965**, *8*, 338–353. [CrossRef]
26. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Sys.* **1994**, *64*, 395–399. [CrossRef]
27. Mirmostafaee, A.K.; Moslehian, M.S. Stability of additive mappings in non-Archimedean fuzzy normed spaces. *Fuzzy Sets Sys.* **2009**, *160*, 1643–1652. [CrossRef]
28. Moslehian, M.S.; Rassias, T.M. Stability of functional equations in non Archimedean spaces. *Appl. Anal. Discret. Math.* **2007**, *1*, 325–334. [CrossRef]
29. Aiemsomboon, L.; Sintunavarat, W. On a new stability results for generalized Cauchy functional equation by using Brzdek's fixed point theorem. *J. Fixed Point Theory Appl.* **2016**, *18*, 45–59. [CrossRef]

30. Aiemsomboon, L.; Sintunavarat, W. On a new type of stability of a radical quadratic functional equation using Brzdek's fixed point theorem. *Acta Math. Hungar.* **2017**, *151*, 35–46. [CrossRef]
31. Chung, P.V. Multiplicative funcations satisying the equation $f(m^2+n^2) = f(m^2) + f(n^2)$. *Math. Slovaca* **1996**, *46*, 165–171.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.