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Structures of Critical Nontree Graphs with Cutwidth Four

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Abstract: The cutwidth of a graph G is the smallest integer k ($k \geq 1$) such that the vertices of G are arranged in a linear layout $[v_1, v_2, \ldots, v_n]$, in such a way that for each $i = 1, 2, \ldots, n-1$, there are at most k edges with one endpoint in $\{v_1, v_2, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_n\}$. The cutwidth problem for G is to determine the cutwidth k of G. A graph G with cutwidth k is k-cutwidth critical if every proper subgraph of G has a cutwidth less than k and G is homeomorphically minimal. In this paper, except five irregular graphs, other 4-cutwidth critical graphs were resonably classified into two classes, which are graph class with a central vertex v_0, and graph class with a central cycle C_q of length $q \leq 6$, respectively, and any member of two graph classes can skillfuly achieve a subgraph decomposition $\mathcal{S}$ with cardinality 2, 3 or 4, where each member of $\mathcal{S}$ is either a 2-cutwith graph or a 3-cutwidth graph.

Keywords: graph labeling; cutwidth; critical graph; graph decomposition

MSC: 05C75; 05C78; 90C27

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1. Introduction

The graphs under consideration in this paper are finite, simple and connected, and for the undefined graph-theoretic terminologies, we refer the reader to the book by Bondy and Murty [1]. The cutwidth of a graph G is the smallest integer k ($k \geq 1$), such that the vertices of G are arranged in a linear layout $[v_1, v_2, \ldots, v_n]$, in such a way that for each $i = 1, 2, \ldots, n-1$, there are at most k edges with one endpoint in $\{v_1, v_2, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_n\}$. The method used to compute the optimum cutwidth of a graph G is usually referred to as the cutwidth minimization problem, and has received an enormous amount of interest in graph theory literature [2] since the 1950s. From [3–6], for a graph G and a nonnegative integer k, deciding whether the cutwidth value of graph G is less than k is an NP-complete problem for general graphs except for trees, and it remains to be NP-complete even though G is planar with a maximum vertex degree of 3, by [7]. Therefore, most of previous investigations of the cutwidth problem have been mainly concentrated on polynomial time approximation algorithms for general graphs, and on polynomial time algorithms for special graphs for solving their cutwidth [2,4,5]. Despite these theoretical algorithms of the cutwidth minimization problem, research on studying the structural properties of the extreme (or critical) graph classes whose cutwidth is a given integer value $k > 1$ have been paid little attention. As far as we know, the 2-cutwidth graph class has five forbidden subgraphs τ_1–τ_5 [8] (see Figure 1 below), the family of 3-cutwidth trees possesses 18 forbidden subtrees [9], and 50 forbidden subgraphs of unicyclic graphs with cutwidth 3 were also found by [10]. As for the inner structures of the critical graphs with cutwidth k, ref. [11] found that any critical tree with cutwidth value k can be decomposed into three $(k-1)$-cutwidth subtrees which are either edge-joint or edge-disjoint. Recently, the decomposability of a class of special k-cutwidth critical graphs with a central vertex v_0 and at least two cut edges v_0v_1 and v_0v_2 was also characterized by [12]. However, for general critical graphs with cutwidth $k \geq 4$, their inner structural properties are unfortunately not yet known. The cutwidth minimization problem for graphs has many significant applications. In the early 1970s, Adolphson and Hu used it to model the number of channels in the optimum layout of a circuit [13]. Other applications of this problem include $VLSI$ circuits' layout [14,15], automatic graph drawing [16], network reliability [17], information retrieval [18], urban drainage network design [19] and others. In particular, the cutwidth is closely connected to a basic parameter called the congestion, in designing microchip circuits and micro communication element system [2,20,21]. Herein, a graph G is considered to be a mathematical model of the wiring diagram of an electronic circuit, in which the vertices of G mean components and the edges of G represent wires connecting these vertices. When a circuit is embedded into a certain architecture (say, a path P_n or a cycle C_n), the largest number of overlapping wires is referred to as the congestion, which is one of the key parameters determining the electronic performance. These are of great interest to scholars investigating the cutwidth problem in graph theory practically. Theoretically, the cutwidth problem is also closely bound up with other graph parameters such as bandwidth, modified bandwidth, pathwidth and treewidth [2,22,23]. For example, this is the case for any graph G with vertices of a degree bound by an integer $r \geq 1$, $pw(G) \leq c(G) \leq r \cdot pw(G)$, where $c(G)$ and $pw(G)$ are cutwidth value and pathwidth value, respectively. In this paper, by virtue of classifying 4-cutwidth critical graphs reasonably, we shall attempt to characterize the inner structural features of the critical graphs with cutwidth-4 in detail.

Let $\mathcal{S}_n = \{1, 2, \cdots, n\}$ for an integer $n > 0$. The labeling of a graph $G = (V(G), E(G))$ with $|V(G)| = n$ is a bijection $\pi : V(G) \to \mathcal{S}_n$, viewed as an embedding of G into a path P_n with vertices in $\mathcal{S}_n$, where consecutive integers are the adjacent vertices. The cutwidth of G with respect to π is

$$c(G, \pi) = \max_{1 \leq j < n} |\{uv \in E(G) : \pi(u) \leq \pi < \pi(v)\}|, \tag{1}$$

which is also the congestion of the embedding. The cutwidth of G is defined to be

$$c(G) = \min_{\pi} c(G, \pi), \tag{2}$$

where the minimum is taken over all labelings π. If $k = c(G, \pi)$, then π, as well as the embedding induced by π, is called a k-cutwidth embedding of G. A labeling π attaining the minimum in (2) is an optimal labeling. For each i with $1 \leq i \leq n$, let $u_i = \pi^{-1}(i)$ and $S_j = \{u_1, u_2, \cdots, u_j\}$. Define $\nabla_\pi(S_j) = \{u_i u_h \in E : i \leq j < h\}$, which is called the (edge) cut at $[j, j+1]$ with respect to π. Using (2), we have

$$c(G, \pi) = \max_{1 \leq j < n} |\nabla_\pi(S_j)|. \tag{3}$$

A π-max-cut of G is $\nabla_\pi(S_j)$, achieving the maximum in (3). For an optimal labeling π of G with a π-max-cut $\nabla_\pi(S_{j_0})$, if vertex $v_0 = \pi^{-1}(j_0)$ and $|\nabla_\pi(S_j)| \leq k - 2$ for every $1 \leq j \leq j_0 - 1$ (or $j_0 + 1 \leq j < n$), then v_0 is called the small-cut vertex with respect to π.

For graph G and integer $i > 0$, let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ in which $d_G(v)$ is the degree of vertex $v \in V(G)$. Any vertex in $D_1(G)$ is called a pendant vertex in G. Any edge incident with a vertex in $D_1(G)$ is a pendant edge of G, and $E_p(G) = \{v_i v_j : v_i v_j \in E(G)$ and $v_i v_j$ is pendant$\}$ is a set of all pendant edges of G. For each $v \in V(G)$, let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. If G possesses a vertex $v \in D_2(G)$ with $N_G(v) = \{v_1, v_2\}$ and $v_1 v_2 \notin E(G)$, then $G - v + v_1 v_2$, the graph obtained from $G - v$ by adding a new edge $v_1 v_2$, is called a series reduction of G. A graph H is a minor of G if H is obtained by deleting vertices, edges or carrying out series reductions in G and $c(H) = c(G)$. If H, H' are subgraphs of G, and $X \subseteq E(G)$, then, as in [1], $G[X]$ is an edge subgraph of G induced by X, $H \cup H' = G[E(H) \cup E(H')]$ and $H \cup X = G[E(H) \cup X]$. Specifically, if $X = \{e\}$, then we write $G + e$ instead of $G \cup \{e\}$. Let G and G' be two disjoint graphs with $u \in V(G), v \in V(G')$; then, to identify u and v, denoted as $G \oplus_{u,v} G'$, is to replace u, v with a single vertex z (i.e.,$u = v = z$) incident to all the edges which were incident to u and v, where z is called the identified vertex. Clearly, if $G' = K_2$ with $K_2 = u_0u_1$, then $G \oplus_{u,u_0} K_2 = G \oplus_{u,u_0} u_0u_1 = G + u_1 + u_0u_1$. If graph G is 2-connected, then any two vertices of G lie on a common cycle. A subgraph decomposition $\mathcal{S}$ of G is a set of proper connected subgraphs $H_1, H_2, \ldots, H_r$ of G whose union $\bigcup_{i=1}^{r} H_i$ is G, where $H_i, H_j \in \mathcal{S}$ are not necessarily edge-disjoint. A graph G is homeomorphically minimal if G does not have any series reductions. Two graphs G and H are homeomorphic if they can both be obtained from the same graph $\mathcal{G}$ by inserting new vertices of degree two into its edges. A graph G is said to be k-cutwidth critical if G is homeomorphically minimal with $c(G) = k$, such that every proper subgraph H of G satisfies $c(H) < k$. From definition, three properties of cutwidth below can be obtained immediately.

Lemma 1. *For graphs G and H, each of the following holds.*

(1) *If H is a subgraph of G, then $c(H) \leq c(G)$.*
(2) *If H is homeomorphic to G, then $c(H) = c(G)$.*
(3) *For a cut edge e in G, if V_1, V_2 are the vertex sets of two components of $G - e$, then there exists an optimal labeling f^*, such that the vertices in each of V_1 and V_2 are labeled consecutively.*

Lemma 2 ([8])**.** *The unique 1-cutwidth critical graph is K_2. The only 2-cutwidth critical graphs are K_3 and $K_{1,3}$. All 3-cutwidth critical graphs are $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 in Figure 1.*

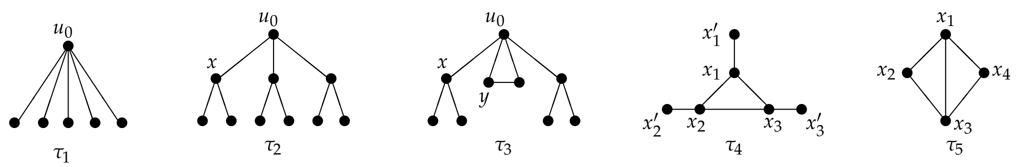

Figure 1. Five 3-cutwidth critical graphs.

Lemma 3 ([11])**.** *For $k \geq 4$, a tree T is k-cutwidth critical if and only if T can be decomposed into three $(k-1)$-cutwidth subtrees, each of which is either a $(k-1)$-cutwidth critical tree or a sum of a $(k-1)$-cutwidth critical tree and a pendant edge.*

Lemma 4 ([12])**.** *Let G be a k-cutwidth graph with a central vertex v_0 of $d_G(v_0) \geq 4$ and at least two cut edges v_0v_1 and v_0v_2. If G can be decomposed into three $(k-1)$-cutwidth graphs G_1, G_2 and G_3, then G is k-cutwidth critical if and only if each element of $\{G_i : 1 \leq i \leq 3\}$ is $(k-1)$-cutwidth critical.*

The rest of this paper is organized as follows. Section 2 presents some preliminary results. Section 3 is focused on investigating 4-cutwidth critical graphs with a central vertex v_0. The characterizations of 4-cutwidth critical graphs with a central cycle C_q ($q \geq 3$) are given in Section 4. Five 4-cutwidth critical graphs without a central vertex and a central cycle are discussed in Section 5. Furthermore, we give short concluding remarks in Section 6.

2. Preliminary Results

From [1], if $\mathcal{S}$ is a decomposition of a graph G, then $E(H_i) \cap E(H_j) = \emptyset$ for arbitrary $H_i, H_j \in \mathcal{S}$ $(i \neq j)$, that is to say H_i, H_j are edge-disjoint in G. In this article, for graph G and an integer $r > 1$, if $G = \bigcup_{i=1}^{r} H_i$ and there are at least two subgraphs H_i, H_j such that H_i, H_j $(i \neq j)$ are edge-joint, then $\{H_i : 1 \leq i \leq r\}$ is also called a decomposition of G, also denoted by $\mathcal{S}$. For example, $\{\tau_4[\{x_1x_1', x_1x_2, x_1x_3\}], \tau_4[\{x_2x_2', x_2x_1, x_2x_3\}], \tau_4[\{x_3x_3', x_3x_2, x_3x_1\}]\}$ is an edge-joint decomposition of τ_4, each of which is $K_{1,3}$ (see τ_4 in Figure 1). Let $P_n = u_1u_2...u_n$ be a path with n vertices, such that for $1 \leq i < n$, u_i and u_{i+1} are adjacent vertices in P_n. By [9], $K_{1,2k-1}$ is k-cutwidth critical, so we let $d_G(v) \leq 2k-2$ for each $v \in V(G)$. For G and G' which are homeomorphic, when no confusion occurs, if G is k-cutwidth critical after the series reductions are carried out, then we shall say that G' is also k-cutwidth critical. The following is immediate from Lemma 1:

$$\text{if } v \in V(G), \text{then } c(G - v) \leq c(G). \tag{4}$$

Definition 1. *(i) For graph G and integer $r > 0$, let $v \in V(G)$ with $d_G(v) > r$. For $v_1, v_2, \ldots, v_r \in N_G(v)$, define $G(v; v_1, v_2, \ldots, v_r)$ to be the component of $G - \{vv_1, vv_2, \ldots, vv_r\}$ that contains v.*

(ii) Let G_1, G_2 be two disjoint graphs with $u \in V(G_1)$ and $v \in V(G_2)$. To identify u and v, denoted as $G_1 \oplus_{u,v} G_2$, is to replace u, v by a single vertex z (i.e., $u = v = z$) incident to all the edges which were incident to u and v, where z is called the identified vertex.

(iii) Let G_1, G_2 and G_3 be three disjoint graphs, $D_3(K_{1,3}) = \{u_0\}$ and $D_1(K_{1,3}) = \{u_1, u_2, u_3\}$, $v_j \in V(G_j)$ for each $j \in \mathcal{S}_3$. Define $K_{1,3} \circ (G_1, G_2, G_3)$ as the graph obtained from the disjoint union G_1, G_2, G_3 and $K_{1,3}$ by identifying u_j with v_j (again denoted as v_j) for each $j \in \mathcal{S}_3$ (see Figure 3d in Section 3.1 below).

(iv) Let G_1, G_2 and G_3 be three disjoint graphs, $P_3 = u_1u_2u_3$ with $d_{P_3}(u_2) = 2$ and $v_j \in V(G_j)$ for each $j \in \mathcal{S}_3$. Define $P_3 \circ (G_1, G_2, G_3)$ as the graph obtained from the disjoint union G_1, G_2, G_3 and P_3 by identifying u_j with v_j (again denoted as v_j) for each $j \in \mathcal{S}_3$.

(v) For $i \in \{1, 2, \ldots, t\}$ with $t \geq 3$, let G_i be a graph with $D_1(G_i) \neq \emptyset$ and $z_i \in D_1(G_i)$. Define $G = \oplus_{z_0}(G_1, G_2, \ldots, G_t)$ to be a graph obtained from disjoint union of $G_1, G_2, \ldots, G_t$ by identifying $z_1, z_2, \ldots, z_t$ into a single vertex z_0 in G. As $z_0 = z_i$ in G_i, z_0 is viewed as the vertex z_i in G_i.

(vi) If $|V(G)| \geq 3$, then define $\mathcal{M}(G) = \{G - uv : uv \in E(G)$ and uv is not a cut edge$\} \cup \{G - v : v \in D_1(G)\}$ to be the family of all proper maximal subgraphs of G.

Definition 2. *Suppose that vertex $v_0 \in V(G)$ with $N_G(v_0) = \{v_1, v_2, \cdots, v_p\}$, v_0v_1, v_0v_2 are two cut edges of G, $G'_1 = G(v_0; v_2, v_3, \cdots, v_p) - v_0$, $G'_2 = G(v_0; v_1, v_2)$ and $G'_3 = G(v_0; v_1, v_3, \cdots, v_p) - v_0$. For $i \in \mathcal{S}_3$, let $\pi_i : V(G'_i) \to \mathcal{S}_{|V(G'_i)|}$ be an optimal labeling of G'_i, and let the labeling $\pi : V(G) \to \mathcal{S}_n$ of G be as follows: for $v \in V(G)$,*

$$\pi(v) = \begin{cases} \pi_1(v) & \text{if } v \in V(G'_1), \\ \pi_2(v) + |V(G'_1)| & \text{if } v \in V(G'_2), \\ \pi_3(v) + |V(G'_1)| + |V(G'_2)| & \text{if } v \in V(G'_3). \end{cases} \tag{5}$$

Then, the labeling π is called a labeling by the order (π_1, π_2, π_3) or $(V(G'_1), V(G'_2), V(G'_3))$.

Theorem 1 ([12])**.** *For any $v \in D_{\geq 3}(G)$, if there always are two vertices v_1, v_2 in $N_G(v)$ such that vv_1, vv_2 are cut edges in G, then $c(G) \leq k$ if and only if $c(G(v; v_1, v_2)) \leq k - 1$.*

Corollary 1. *For graph G, if there is a vertex $v \in D_{\geq 3}(G)$ such that $c(G(v; v_i, v_j)) \geq k - 1$ holds for any $v_i, v_j \in N_G(v)$, then $c(G) \geq k$, where vv_i, vv_j are both cut edges in G.*

Lemma 5 ([10])**.** *Let graph G be k-cutwidth critical and $K_2 = u_0u_1$. Then $c(G \oplus_{v_0,u_0} K_2) = k$ for $v_0 \in V(G)$.*

Theorem 2 ([12])**.** *With the notation of Definition 1(iii), let at least one of $\{G_1, G_2, G_3\}$, say G_2, be $(k-1)$-cutwidth critical with $D_1(G_2) \neq \emptyset$. Then $c(K_{1,3} \circ (G_1, G_2, G_3)) = k$.*

Corollary 2 ([12])**.** *With the notation of Definition 1(iii), for each $j \in \mathcal{S}_3$, if G_j is $(k-1)$-cutwidth critical with $v_j \in D_1(G_j)$, then $K_{1,3} \circ (G_1, G_2, G_3)$ is k-cutwidth critical.*

Theorem 3. *With notation of Definition 1(iv), if $c(G_j) = k - 1$ for each $j \in \mathcal{S}_3$, then $c(P_3 \circ (G_1, G_2, G_3)) = k$.*

Proof. Let $G = P_3 \circ (G_1, G_2, G_3)$. If $d_G(v_j) = 2$ for $j = 1$ or 3 then the series reductions are first carried out without effecting $c(G) = k$. As $G - \{v_2v_1, v_2v_3\}$ has three components G_1, G_2 and G_3 with cutwidth $k - 1$, similar to that of (5), an optimal labeling $\pi : V(G) \to \mathcal{S}_n$ obtained by the order $(V(G_1), V(G_2), V(G_3))$ satisfies $c(G, \pi) \leq (k - 1) + 1 = k$. Therefore,

$c(G) \leq k$ by (2). Additionally, it is not hard to verify that $c(G) \geq k$ by Corollary 1; this is because $c(G(v_2; v_i, v_j)) = k-1$ for any $v_i, v_j \in N_G(v_2)$. Hence $c(G) = k$, i.e., $c(P_3 \circ (G_1, G_2, G_3)) = k$. □

Corollary 3. *With notation of Definition 1 (iv), if the following hold:*

(1) *G_1, G_3 are 2-connected;*
(2) *v_j is a small-cut vertex corresponding to an optimal labeling π_j of G_j for each $j \in \mathcal{S}_3$;*
(3) *G_1, G_2, G_3 are $(k-1)$-cutwidth critical, then $P_3 \circ (G_1, G_2, G_3)$ is k-cutwidth critical, where G_1, G_2, G_3 are not necessarily distinct.*

Proof. Let $G = P_3 \circ (G_1, G_2, G_3)$. Since $N_G(v_2) = \{v_1, v_3\}$, $G(v_1; v_2) = G_1, G(v_2; v_1, v_3) = G_2$ and $G(v_3; v_2) = G_3$. First, $c(G) = k$ by Theorem 3. Second, we show $c(G') \leq k-1$ for any $G' \in \mathcal{M}(G)$, that is, G is k-cutwidth critical. Because any G' can be obtained by deleting a pendant edge xy or an non-pendant edge $xy \in E(C_t)$ in G, $xy \notin \{v_2v_1, v_2v_3\}$, where C_t is a cycle with length $t \geq 3$. There are two cases to consider: (1) $xy \in E(G_2)$; (2) $xy \in E(G_1)$ or $E(G_3)$. For Case (1), since G_2 is $(k-1)$-cutwidth critical, there is an optimal labeling π'_2 such that $c(G_2 - xy, \pi'_2) \leq k-2$. Now, by Lemma 5, let π'_j be a labeling of G_j such that $c(G_j \oplus_{u_j, v_j} v_jv_2) = k-1$ for $j = 1, 3$. Thus, a labeling π of G by the order (π'_1, π'_2, π'_3) is obtained with $c(G - xy, \pi) \leq k-1$ implying $c(G - xy) \leq k-1$. For Case (2), let $xy \in E(G_3)$. By assumption, $c(G_3 - xy) \leq k-2$. Since v_j is a small-cut vertex corresponding to an optimal labeling π_j of G_j for each $j \in \mathcal{S}_3$, a labeling π of G by the order (π_2, π_3, π_1) is obtained with $c(G - xy, \pi) \leq k-1$ implying $c(G - xy) \leq k-1$. Likewise, if $xy \in E(G_1)$ then $c(G - xy) \leq k-1$ also. To sum up, G is k-cutwidth critical. □

Lemma 6. *Each graph in Figure 2 is* 4*-cutwidth critical.*

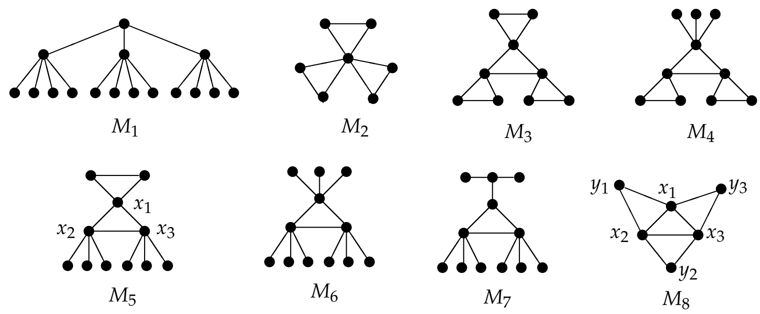

Figure 2. Eight special 4-cutwidth critical graphs.

Proof. Two steps can be used to finish the proof. For each M_i $(1 \leq i \leq 8)$, Step 1 is used to show $c(M_i) = 4$. This can be accomplished by two operations: (1) $c(M_i, \pi) \geq 4$ for any labeling π of M_i, which implies $c(M_i) \geq 4$; (2) M_i has an optimal labeling π_0 with $c(M_i, \pi_0) = 4$. In Step 2, for any $M'_i \in \mathcal{M}(M_i)$, $c(M'_i) \leq 3$ must be shown. Since operation of each of the two steps is easy, we omitted it here.

Let v be a cut-vertex with $d_G(v) \geq 3$ in G and $G_1, G_2, \ldots, G_q$ be q connected components of $G - v$. Then, $G[V(G_i) \cup \{v\}]$ $(1 \leq i \leq q)$, denoted by H_i, is called the ith v-component of $G - v$. A vertex $v_0 \in V(G)$ is called the central vertex of a k-cutwidth graph G if v_0 is a cut-vertex in G, such that all v_0-components of $G - v_0$ can form a decomposition $\mathcal{S}$ of G in which each element has equal cutwidth ρ with $\rho < k$. For example, for graph τ_1 in Figure 1, $H_i = K_2$ $(1 \leq i \leq 5)$ with edge v_0v_i is the ith v_0-component of $\tau_1 - v_0$; we can see that $\{H_1 \cup H_2 \cup H_3, H_1 \cup H_2 \cup H_4, H_1 \cup H_2 \cup H_5\}$ is a decomposition of τ_1, each of which is a 2-cutwidth critical tree $K_{1,3}$, so v_0 is the central vertex of τ_1. Likewise, each of $\{\tau_2, \tau_3\}$ has a decomposition of equal cutwidth-2 and a central vertex v_0 also, respectively.

For a cycle $C_q = x_1x_2 \ldots x_qx_1$ of G with $q \geq 3$ and $d_G(x_i) \geq 3$ for $1 \leq i \leq q$, let $V(C_q)$ be a vertex-cut set of G. If $E(C_q)$ is also an edge-cut set of G and G_i is the ith connected component of $G - E(C_q)$ leading from x_i, then $G[E(G_i) \cup \tilde{E}_i]$, denoted by H_i, is called the

ith C_q-component leading from x_i of $G - E(C_q)$, where $\tilde{E}_i \subseteq E(C_q)$ and at least an $\tilde{E}_i \neq \emptyset$. A cycle C_q with $q \geq 3$ is called a central cycle of a k-cutwidth graph G if $E(C_q)$ is an edge-cut set, such that one of the following is a decomposition $\mathcal{S}$ of G, each element of which has equal cutwidth ρ with $\rho < k$,

(1) $\{H_i : 1 \leq i \leq q\}$, or
(2) $\{H'_i : 1 \leq i < q\}$ in which H'_i may be one of $\{H_i, H_{i-1} \cup H_i \cup H_{i+1}\}$ with $H_0 = H_q, H_{q+1} = H_1$, and there exists at least $H'_i \neq H_i$, or
(3) $\{H''_i : 1 \leq i < q\}$, each of which is either H_i or $H_{i-1}[E'] \cup H_i \cup H_{i+1}[E'']$ with $H_0 = H_q, H_{q+1} = H_1$, and there exists at least $H''_i \neq H_i$, where $H_{i-1}[E'] \subset H_{i-1}$ and $H_{i+1}[E''] \subset H_{i+1}$.

For example, in Figure 1, τ_4 has a cycle $C_3 = x_1x_2x_3x_1$, and $\tau_4 - E(C_3)$ has three components G_1, G_2, G_3, each of which equals K_2. Let $\tilde{E}_1 = \{x_1x_2, x_1x_3\}$, $\tilde{E}_2 = \{x_2x_1, x_2x_3\}$ and $\tilde{E}_3 = \{x_3x_1, x_3x_2\}$, and let $H_i = G[E(G_i) \cup \tilde{E}_i]$ for $1 \leq i \leq 3$. Then, $\{H_1, H_2, H_3\}$ is a decomposition $\mathcal{S}$ of τ_4, in which each member is a 2-cutwidth critical subgraph $K_{1,3}$, and C_3 is the central cycle in τ_4. For Case (3), we take M_5 with a central cycle $C_3 = x_1x_2x_3x_1$ in Figure 2 as an example. $M_5 - E(C_3)$ also has three connected components G_1, G_2, G_3, which are $K_{1,3}, K_{1,3}, C_3$ and three C_3-components $H_1 = G_1 + x_1x_2 + x_1x_3$ with $d_{G_1}(x'_1) = d_{G_1}(x''_1) = 1$, $H_2 = G_2 + x_2x_1 + x_2x_3$ with $d_{G_2}(x'_2) = d_{G_2}(x''_2) = 1$ and $H_3 = G_3 + x_3x_1 + x_3x_2$, respectively. Let $E' = \{x_1x'_1, x_1x''_1\}, E'' = \{x_2x'_2, x_2x''_2\}$, then $H_1[E'] \subset H_1, H_2[E''] \subset H_2$ and $\{H_1, H_2, H_1[E'] \cup H_3 \cup H_2[E'']\}$ is a decomposition of equal cutwidth 3 of M_5, each member of which is also 3-cutwidth critical.

In the case that G is 2-connected and $E(C_q)$ is not an edge-cut set of G, suppose that $G - V(C_q)$ has q connected components $G_1, G_2, \ldots, G_q$, with $V(G_i) \neq \emptyset$ for each $1 \leq i \leq q$, and let $G[V(G_i) \cup \{x_i, x_{i+1}\}]$ be the ith 2-connected subgraph that contains edge $x_ix_{i+1} \in E(C_q)$. If $\{G[V(C_q) \cup V(G_i)] : 1 \leq i \leq q\}$ is a subgraph decomposition of equal cutwidth $\rho \leq k-1$, then C_q is also called the central cycle of G. For example, let $G = M_8$ with $C_3 = x_1x_2x_3x_1$ in Figure 2. Clearly, $G - \{x_1, x_2, x_3\}$ has three components y_1, y_2, y_3, and $\{G[\{x_1, x_2, x_3, y_i\}] : 1 \leq i \leq 3\} = \{\tau_5, \tau_5, \tau_5\}$ is an edge-joint subgraph decomposition of equal cutwidth 3 of G. Hence, $C_3 = x_1x_2x_3x_1$ is the central cycle of M_8. □

From Lemma 2, we have

Theorem 4. *For a 2-cutwidth critical graph $G \in \{K_{1,3}, C_3\}$, one of the following holds:*

(1) G has a central vertex v_0, and v_0-components of $G - v_0$ constitute a decomposition $\mathcal{S}$ with $|\mathcal{S}| = 3$, each of which is K_2 with cutwidth 1;
(2) G is a cycle C_3, whose three edges constitute a decomposition $\mathcal{S}$ with $|\mathcal{S}| = 3$, each element of which is K_2 with cutwidth 1.

Theorem 5. *For a 3-cutwidth critical graph $G \in \{\tau_i : 1 \leq i \leq 5\}$, one of the following holds:*

(1) has a central vertex v_0, and v_0-components of $G - v_0$ constitute a decomposition $\mathcal{S}$ with $|\mathcal{S}| = 3$, each of which equals $K_{1,3}$ or C_3 with cutwidth 2; or
(2) G has a central cycle $C_3 = x_1x_2x_3x_1$ with $d_G(x_i) = 3$ for $x_i \in V(C_3)$, and C_3-components of $G - E(C_3)$ constitute a decomposition $\mathcal{S}$ with $|\mathcal{S}| = 3$, each member of which equals $K_{1,3}$ with cutwidth 2; or
(3) G equals $C_4 + x_1x_3$ or $C_4 + x_2x_4$, where $C_4 = x_1x_2x_3x_4x_1$ is a cycle of length 4.

3. 4-Cutwidth Critical Graphs with a Central Vertex

In this section, we shall verify the decomposability of the 4-cutwidth critical graphs with a central vertex. Since a k-cutwidth critical graph G is homeomorphically minimal, for the central cycle C_q $(q \geq 3)$ of G, we can let

$$d_G(v_i) \geq 3 \text{ for every } v_i \in V(C_q). \tag{6}$$

3.1. 4-Cutwidth Critical Trees with a Central Vertex

Definition 3. *For a cut-vertex v_0 with $N_T(v_0) = \{v_i : 1 \le i \le q$ and $q \ge 4\}$ in a tree T, let H_i be a v_0-component of $T - v_0$ with $c(H_1) \ge c(H_2) \ge ... \ge c(H_q)$ and $c(\bigcup_{i=4}^{q} H_i) < k-1$, then define*

$$T_i = \begin{cases} K_{1,2k-3} & \text{if } i < 3 \text{ and } H_i = K_{1,2k-3}, \\ H_i \cup (\bigcup_{i=4}^{q} H_i) & \text{if } i < 3 \text{ and } H_i \neq K_{1,2k-3}, \\ H_3 \cup (\bigcup_{i=4}^{q} H_i) & \text{if } i = 3. \end{cases} \quad (7)$$

If $c(T_i) = k-1$ for $1 \le i \le 3$, then $\{T_1, T_2, T_3\}$ is called a subtree decomposition of equal cutwidth $k-1$ of T.

In Definition 3, for a decomposition $\{T_1, T_2, T_3\}$ of equal cutwidth $k-1$ of a k-cutwidth critical tree T, $E(T_{i_1}) \cap E(T_{i_2}) = E(\bigcup_{i=4}^{q} H_i)$ ($1 \le i_1 \neq i_2 \le 3$). If $E(T_{i_1}) \cap E(T_{i_2}) \neq \emptyset$, then $\{T_1, T_2, T_3\}$ is edge-joint; Otherwise $\{T_1, T_2, T_3\}$ is edge-disjoint.

There are eighteen 4-cutwidth critical trees in total by [9], each of which can be decomposed into three 3-cutwidth subtrees by Lemma 3. In fact, among these eighteen 4-cutwidth critical trees, each possesses one of the structures listed in Figure 3, in which $H_i \cup (\bigcup_{i=4}^{q} H_i)$ is either one of τ_1 and τ_2 or homeomorphic to τ_2 for $i = 1,2,3$ in Figure 3a. $H_i \cup (\bigcup_{i=4}^{q} H_i)$ is either τ_2 or homeomorphic to τ_2 for $i = 2,3$ in Figure 3b, $H_i \cup (\bigcup_{i=4}^{q} H_i)$ is either τ_2 or homeomorphic to τ_2 for $i = 3$ in Figure 3c, either H_i or $H_i - v_0v_i$ with $v_i \in N_{H_i}(v_0)$ is in $\{\tau_1, \tau_2\}$ for $i = 1,2,3$ in Figure 3d. Thus, based on this, M_1 (see Figure 2) is 4-cutwidth critical, and again we have the following:

Theorem 6. *For a 4-cutwidth critical tree T, one of the following holds:*

(1) T possesses a configuration $K_{1,3} \circ (T_1, T_2, T_3)$ which can be decomposed into three edge-disjoint 3-cutwidth trees T_1, T_2 and T_3 (not necessarily distinct), and the 3-degree vertex of $K_{1,3}$ is the central vertex of T, where T_i is a v_0-component of $T - v_0$ with either $T_i \in \{\tau_1, \tau_2\}$ or $T_i - v_0 \in \{\tau_1, \tau_2\}$ for each $1 \le i \le 3$ (see Figure 3d); or

(2) T is a tree with a central vertex v_0 with $d_T(v_0) \ge 4$ and with an edge-joint decomposition $\{T_1, T_2, T_3\}$ of equal cutwidth 3, where T_1, T_2 and T_3 (not necessarily distinct), which are defined by (7), are either in $\{\tau_1, \tau_2\}$ or homeomorphic to τ_2, and at least one of them, say T_3, is not τ_1 (see Figure 3a–c, respectively).

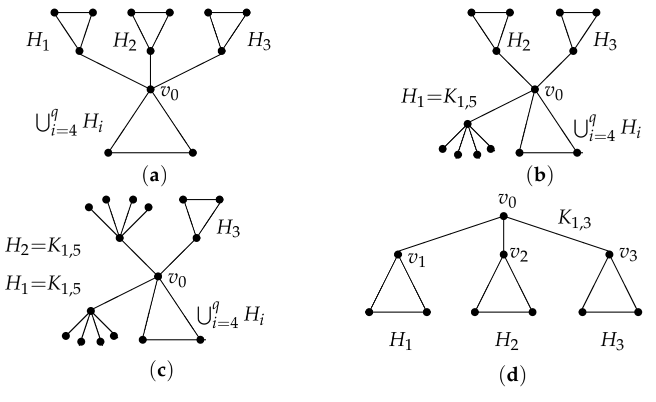

Figure 3. Four structures of 4-cutwidth critical trees.

3.2. 4-Cutwidth Critical Nontrees with a Central Vertex

We shall focus primarily on the structures of 4-cutwidth critical non-trees with a central vertex in this subsection.

Suppose now that G_1, G_2 and G_3 (not necessarily distinct) are mutually disjoint graphs, and at least one of them is not a tree. Let $K_{1,3} \circ (G'_1, G'_2, G'_3)$ be a graph obtained from the disjoint graphs G'_1, G'_2, G'_3 and $K_{1,3}$ by identifying u_i with v_i (again denoted as v_i) for $i \in \mathcal{S}_3$, where

$$G_i' = \begin{cases} G_i & \text{if } v_i \notin V(E_p(G_i)) \setminus D_1(G_i), \\ G_i - v_i' & \text{if } v_i \in V(E_p(G_i)) \setminus D_1(G_i) \text{ and } v_i v_i' \in E(E_p(G_i)). \end{cases} \quad (8)$$

u_i is a pendant vertex of $K_{1,3}$ and $v_i \in V(G_i)$ for $1 \le i \le 3$. Obviously, if $c(G_1) = c(G_2) = c(G_3)$ and $D_3(K_{1,3}) = \{u_0\}$ then u_0 is the central vertex of $K_{1,3} \circ (G_1', G_2', G_3')$.

Lemma 7. *Suppose that G_i is $(k-1)$-cutwidth critical for $1 \le i \le 3$, then $K_{1,3} \circ (G_1', G_2', G_3')$ is a k-cutwidt critical graph, where G_1, G_2, G_3 are not necessarily distinct.*

Proof. Let $G = K_{1,3} \circ (G_1', G_2', G_3')$. If there exists at least a vertex $v_i \in N_G(u_0)$ such that $d_G(v_i) = 2$, then the series reductions can be implemented first. Two cases need to be considered as follows.
Case 1. For $i \in \mathcal{S}_3$, $v_i \notin V(E_p(G_i)) \setminus D_1(G_i)$.

By (8), $G_i' = G_i$ for $i \in \mathcal{S}_3$. So $c(G_i') = c(G_i) = k-1$ by assumption, and $c(G_2' + u_0v_2) = k-1$ by Lemma 5. Now, let π_1, π_2, π_3 be the labelings such that $c(G_1', \pi_1) = k-1$, $c(G_2' + u_0v_2, \pi_2) = k-1$ and $c(G_3', \pi_3) = k-1$, respectively. Then, a labeling π of G by the order (π_1, π_2, π_3) is obtained, and $c(G, \pi) = \max\{c(G_1', \pi_1), c(G_2' + u_0v_2, \pi_2), c(G_3', \pi_3)\}$ $+1 = (k-1)+1 = k$, implying $c(G) \le k$. Since u_0v_1, u_0v_2 and u_0v_3 are cut-edges in G, $c(u_0; v_i, v_j) \ge k-1$ for any $v_i, v_j \in \{v_1, v_2, v_3\}$, leading to $c(G) \ge k$ by Corollary 1. Hence $c(G) = k$.

On the other hand, any $G' \in \mathcal{M}(G)$ can be obtained by deleting a vertex y with degree one of a pendant edge $xy \notin E(C_t)$ or a nonpendant edge $xy \in E(C_t)$ in G, so $xy \ne u_0v_1, u_0v_2$ or u_0v_3, where $C_t = x_1x_2 \dots x_tx_1$ is a cycle with length $t \ge 3$ in G. Without loss of generality, let $xy \in E(G_2')$. If xy is pendant with $y \in D_1(G)$, then by the criticality of G_2', $c(G_2' - y) \le k-2$ with a labeling π_2' such that $c(G_2' - y, \pi_2') \le k-2$. Since G_1' and G_3' are $(k-1)$-cutwidth critical, by (6) in Lemma 5, two labelings π_1', π_3' can be obtained such that $c(G_1' \oplus_{u_1,v_1} v_1u_0, \pi_1') = k-1$ with $\pi_1'(u_0) = 1$ and $c(G_3' \oplus_{u_3,v_3} v_3u_0, \pi_1') = k-1$ with $f_3'(u_0) = |V(G_3')| + 1$, respectively. Now, define $\pi : V(G') \to \{1, 2, \dots, |V(G')| - 1\}$ to be a labeling of G' by the order (π_1', π_2', π_3'), then $c(G', \pi) \le (k-2)+1 = k-1$, i.e., $c(G') \le k-1$, meaning that G is k-cutwidth critical. Likewise, if xy is not pendant with $xy \in E(C_t)$, then $c(G_2' - xy) \le k-2$, and a labeling $\pi : V(G') \to \{1, 2, \dots, |V(G')|\}$ by the order $(G_1' \oplus_{u_1,v_1} v_1u_0, G_2' - xy, G_3' \oplus_{u_3,v_3} v_3u_0)$ is also obtained, under which $c(G', \pi) \le (k-2)+1 = k-1$, i.e., $c(G') \le k-1$, meaning that G is also k-cutwidth critical. The cases of $xy \in E(G_1')$ or $E(G_3')$ are the same as that of $xy \in E(G_2')$, omitted here.
Case 2. There are at least a v_{i_0}, such that $v_{i_0} \in V(E_p(G_{i_0})) \setminus D_1(G_{i_0})$ $(1 \le i_0 \le 3)$.

Three subcases need to be considered: (1) there is unique v_i (say v_2), such that $v_2 \in V(E_p(G_2)) \setminus D_1(G_2)$; (2) there are two $v_i's$ (say v_1, v_3), such that $v_1 \in V(E_p(G_1)) \setminus D_1(G_1)$ and $v_3 \in V(E_p(G_3)) \setminus D_1(G_3)$; (3) $v_i \in V(E_p(G_i)) \setminus D_1(G_i)$ for each $1 \le i \le 3$. For Subcase (1), since $v_2 \in V(E_p(G_2)) \setminus D_1(G_2)$, $G_2' = G_2 - v_2v_2'$ with $v_2' \in D_1(G_2)$. In this case, $G_2' \oplus_{u_2,v_2} u_2u_0 = G_2$, i.e., $G(u_0; v_1, v_3) = G_2$, $G_1' = G_1$ and $G_3' = G_3$ by (9). Similarly, for Subcase (2), $G_1' \oplus_{u_1,v_1} u_1u_0 = G_1$, $G_2' = G_2$ and $G_3' \oplus_{u_3,v_3} u_3u_0 = G_3$; for Subcase (3), $G_1' \oplus_{u_1,v_1} u_1u_0 = G_1$, $G_2' \oplus_{u_2,v_2} u_2u_0 = G_2$ and $G_3' \oplus_{u_3,v_3} u_3u_0 = G_3$. The remaining argument of any Subcase (j) $(j = 1, 2, 3)$ is similar to that of Case 1, omitted here. To sum up, G is k-cutwidth critical. □

Corollary 4. *Suppose that $G_i \in \{\tau_i : 1 \le i \le 5\}$ for $1 \le i \le 3$, then $K_{1,3} \circ (G_1', G_2', G_3')$ is a 4-cutwidt critical graph, where at least a $G_i \in \{\tau_3, \tau_4, \tau_5\}$, and G_1, G_2, G_3 are not necessarily distinct.*

Corollary 5. *Suppose that $G_i \in \{\tau_i : 1 \le i \le 4\}$ for $1 \le i \le 3$, then $\oplus_{u_0}(G_1, G_2, G_3)$ is a 4-cutwidt critical graph, where at least a $G_i \in \{\tau_3, \tau_4\}$, and G_1, G_2, G_3 are not necessarily distinct.*

Lemma 8. *Let $P_3 = u_1u_2u_3$, G_i be 3-cutwidth critical with $v_i \in V(G_i)$ for $1 \le i \le 3$ and satisfy the following:*

(i) *each non cut-edge of G_2 may be subdivided once, and v_2 may possibly be the subdivision vertex;*
(ii) *$G_2 \ne \tau_1$;*

(iii) *if $G_2 \in \{\tau_2, \tau_3\}$, then v_2 is not either the central vertex or the pendant vertex of it;*
(iv) *$G_i \notin \{\tau_2, \tau_3\}$ for $i = 1$ or 3 if $G_2 \in \{\tau_2, \tau_3\}$.*

Then, $P_3 \circ (G_1', G_2, G_3')$ is a 4-cutwidt critical graph, where G_1, G_2, G_3 are not necessarily distinct, and at least one of them is not in $\{\tau_1, \tau_2\}$.

Proof. Let $G = P_3 \circ (G_1', G_2, G_3')$. By assumption, for $i = 1, 3$, $G_i' = G_i$ with $v_i \notin V(E_p(G_i)) \setminus D_1(G_i)$ or $G_i - v_i'$ with $v_i v_i' \in E(E_p(G_i))$ and $v_i' \in D_1(G_i)$. So, $H_i = G_i$ or $G_i + v_i v_2$ for $i = 1, 3$ and $H_i = G_2$ for $i = 2$. Thus, with an argument similar to that of Lemma 7, G is 4-cutwidth critical. □

Suppose that $G_1 \in \{\tau_2, \tau_3\}$ with the central vertex $u_1(=v_0)$ and two cut edge u_1v_1, u_1v_2, such that any u_1-component of $G_1 - u_1$ is 2-cutwidth critical (see $\tau_1 - \tau_5$ in Figure 1). For any 3-cutwidth nontree graph $G_2 \in \{\tau_3, \tau_4, \tau_5\}$ with cycle $C_3 = x_1x_2x_3x_1$, if there is a vertex (say x_1) in C_3 such that (1) $x_1 \neq u_2$ when $G_2 = \tau_3$ with the central vertex $u_2(=v_0)$; or (2) if F_1 is a component of $G_2 - E(C_3)$ leading from x_1, then either $F_1 = x_1x_1'$ with $d_{G_2}(x_1') = 1$ when $G_2 = \tau_4$ or $F_1 = x_1$ only when $G_2 = \tau_5$; or (3) G_1 and G_2 are not necessarily distinct; or (4) if $G_1 = \tau_3$ and $G_2 = \tau_5$, then $d_{G_2}(x_1) = 2$. Only then, by (8), do we have

Lemma 9. *Graph $G_1 \oplus_{u_1,x_1} G_2'$ is a 4-cutwidth critical graph.*

Proof. Let $G = G_1 \oplus_{u_1,x_1} G_2'$ with optimal labeling π, and π_1 be a sublabeling of π restricted on G_1. By assumption, $G_1 - u_1$ has three u_1-components H_1, H_2 and H_3, each of which is either $K_{1,3}$ or C_3 by Theorem 3. Suppose that π_1 is obtained by the order (π_1', π_2', π_3') with $\max\{\pi_1'(v) : v \in V(H_1 - u_1)\} < \pi_2'(v) < \min\{\pi_3'(v) : v \in V(H_3 - u_1)\}$ for $v \in V(H_2)$ if π_1', π_2', π_3' are optimal labelings of $H_1 - u_1, H_2$ and $H_3 - u_1$, respectively. Without loss of generality, let $H_1 = K_{1,3}$ with cutwidth 2. Since G_2 is 3-cuwidth critical and $x_1 \in V(C_3)$ in G_2, whether $G_2 = \tau_4$ or $G_2 = \tau_i$ with $i = 3, 5$, if $\nabla_\pi(S_j)$ is a π-max-cut of G, then $j < \pi(u_1)$ and $|\nabla_\pi(S_j)| = 4$. Hence, $c(G) \leq 4$. On the other hand, assuming that u_1v_1, u_1v_2 are cut edges in G, $\pi(u_1; v_1, v_2) = K_{1,3} \oplus_{u_1,x_1} G_2'$ when $G_1 = \tau_2$ or $C_3 \oplus_{u_1,x_1} G_2'$ when $G_1 = \tau_3$, so $c(\pi(u_1; v_1, v_2)) \geq 3$, resulting in $c(G) \geq 4$ by Corollary 1. Thus, $c(G) = 4$.

We now verify that G is 4-cutwidth critical. For any edge $e \in E(G)$, e is in either $E(G_1)$ or $E(G_2)$. Since $G_1 \in \{\tau_2, \tau_3\}$ which is 3-cutwidth critical, if $e \in E(G_1)$ then we can always find a labeling $\bar{\pi}_1$ of $G_1 - e$ such that $c(G_1 - e) = 3$ and $\bar{\pi}_1(u_1) = |V(G_1)|$. For $G_2 \in \{\tau_3, \tau_4, \tau_5\}$, we can always find an optimal labeling $\bar{\pi}_2$ of G_2' such that $\bar{\pi}_2(x_1) = 1$. Thus, a labeling $\bar{\pi}$ of $G - e$ by the order $(\bar{\pi}_1, \bar{\pi}_2)$ is obtained with $c(G - e, \bar{\pi}) = 3$ leading to $c(G - e) \leq 3$. Similarly, if $e \in E(G_2)$ then $c(G - e) \leq 3$ also. This completes the proof. □

From Lemma 9, we can see that if a critical non-tree G with cutwidth 4 can be decomposed into two 3-cutwidth critical subgraphs $G_1 \in \{\tau_2, \tau_3\}$ and $G_2 \in \{\tau_3, \tau_4, \tau_5\}$, then (1) G has a central vertex u_1; (2) u_1 is also the central vertex of at least one of G_1 and G_2. For example, let $K_{1,3}^{(j)}$ with a pendant vertex $y^{(j)}$ $(1 \leq j \leq 4)$ be the copy of $K_{1,3}$ with a pendant vertex y, and $y^{(j)}$ be the copy of y, $y^{(0)}$ be a vertex of a 3-cycle C_3'. Then, graph $\oplus_{y_0}(K_{1,3}^{(1)}, \ldots, K_{1,3}^{(4)}, C_3')$, obtained by identifying $y^{(0)}, y^{(1)}, \ldots, y^{(4)}$ into a vertex y_0 (*i.e.*, $y_0 = y^{(0)} = ... = y^{(4)}$), is a 4-cutwidth critical graph with the central vertex y_0, and this graph can be decomposed into two 3-cutwidth critical subgraphs $\oplus_{y_0}(K_{1,3}^{(1)}, K_{1,3}^{(2)}, K_{1,3}^{(3)})$ $(= \tau_2)$ with the central vertex y_0, and $\oplus_{y_0}(K_{1,3}^{(3)}, K_{1,3}^{(4)}, C_3')$ $(= \tau_3)$ with the central vertex y_0; (3) there are at least two cut edges u_1v_1, u_1v_2.

Lemma 10. *Let G be a 4-cutwidth critical nontree graph with the central vertex v_0 and at least two cut edges v_0v_1 and v_0v_2. If G can be decomposed into two 3-cutwidth graphs G_1, G_2 (not necessarily distinct), then the following hold:*

(1) *G_1, G_2 are in $\{\tau_i : 2 \leq i \leq 5\}$;*
(2) *at least one of G_1 and G_2, say G_1, is in $\{\tau_2, \tau_3\}$, while $G_2 \neq \tau_2$;*

(3) *v_0 is the central vertex of G_1, but v_0 is only a vertex of any 3-cycle C_3 of G_2.*

Proof. Since G is a non-tree graph, we do not consider the cases that G_1 and G_2 are both τ_1 or τ_2. We first show that G_1, G_2 are in $\{\tau_i : 2 \le i \le 5\}$ by contradiction. Suppose that there is some G_i (say G_2) such that $c(G_2) = 3$ but G_2 is not 3-cutwidth critical, then there is at least a pendant edge $xy \in E(G_2)$ with $y \in D_1(G)$ or a non-pendant edge $xy \in E(C_t)$ such that $c(G_2 - y) = 3$ or $c(G_2 - xy) = 3$, respectively, where $C_t = x_1x_2 \dots x_tx_1$ is a cycle with length $t \ge 3$. For the former, because $c(G_1) = 3$ by assumption, $c(G - y) = 4$ by Lemma 9. Likewise, for the latter, $c(G - xy) = 4$ also by Lemma 9. All are contrary to the criticality of G. Hence G_1, G_2 are both in $\{\tau_i : 2 \le i \le 5\}$.

Next, by the assumption that v_0 is the central vertex and v_0v_1 and v_0v_2 are both cut edges in G, we claim that at least one of G_1 and G_2 (say G_1) must be τ_2 or τ_3. This is because otherwise, there is at most a vertex $v_1 \in N_G(v_0)$ such that v_0v_1 is a cut edge in G if G_1 and G_2 are both in $\{\tau_4, \tau_5\}$, which is a contradiction. So (2) holds and $G_1 \in \{\tau_2, \tau_3\}$.

Third, assume that v_0 is neither the central vertex of G_1 nor a vertex of a 3-cycle C_3 of G_2 if $G_2 = \tau_3$. Without loss of generality, let $G_1 = \tau_2$. Then G_2 is either τ_3 or one of $\{\tau_4, \tau_5\}$. For $G_2 = \tau_3$, by assumption, v_0 is not also the central vertex of G_2. Thus, except three vertices of 3-cycle C_3 of G_2, three cases need to be considered: (a) v_0 is not only a subdivision vertex of some non-pendant edge in G_1 but also a subdivision vertex of some non-pendant cut edge in G_2; (b) v_0 is a subdivision vertex of some non-pendant edge in G_1, but v_0 is a nonpendent vertex of G_2; (c) v_0 is not only a non-pendant vertex of G_1 but also a non-pendant vertex of G_2. For any case of Cases (a)–(c), we can easily verify that $c(G) = 3$ by Lemma 1(3) and Theorem 5, contrary to the assumption of $c(G) = 4$. Likewise, for $G_2 \in \{\tau_4, \tau_5\}$, there are only two cases to consider: $(a)'$ v_0 is a subdivision vertex of some non-pendant edge in G_1, but v_0 is a arbitrary vertex of G_2; $(b)'$ v_0 is a nonpendant vertex of G_1, but v_0 is a arbitrary vertex of G_2. Furthermore, in any case, $c(G) = 3$, also a contradiction. This completes the proof. □

For a cut-vertex $v_0 \in D_{\ge 4}(G)$ graph G and all v_0-components $H_i = G[V(G_i) \cup \{v_0\}]$ $(1 \le i \le m)$ of $G - v_0$, we define a decomposition $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ each of which has cutwidth 3 below. Let E_0 be an edge subset taken from $\bar{G}_3$ such that the cutwidth of the connected subgraph $H_i \cup G[E_0]$ is 3 if $c(H_i) < 3$, for $1 \le i < 3$. Then, we obtain the following:

Definition 4. *For a cut-vertex $v_0 \in D_{\ge 4}(G)$ of G and the v_0-component H_i $(1 \le i \le q)$ of $G - v_0$, $\min\{c(H_i) : 1 \le i \le 3\} \ge \max\{c(H_i) : 4 \le i \le q\}$ and the cutwidth of $\bigcup_{i=3}^{q} H_i$ is three. For $1 \le i \le 3$, define*

$$\bar{G}_i = \begin{cases} H_i & \text{if } i < 3 \text{ and } c(H_i) = 3, \\ H_i \cup G[E_0] & \text{if } i < 3 \text{ and } c(H_i) < 3, \\ \bigcup_{i=3}^{q} H_i & \text{if } i = 3. \end{cases} \tag{9}$$

If $c(\bar{G}_i) = 3$ for $i = 1,2$, then $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ is called a decomposition of equal cutwidth 3 of G, and G is called a graph with a central vertex v_0, where E_0 is an edge subset taken from $\bar{G}_3$ such that $c(H_i \cup G[E_0]) = 3$ if $c(H_i) < 3$ for $i = 1,2$

Lemma 11. *Let G be a 4-cutwidth critical graph with the central vertex v_0 and at least two cut edges v_0v_1 and v_0v_2. If G can be decomposed into three 3-cutwidth graphs $\bar{G}_1, \bar{G}_2$ and $\bar{G}_3$, then G is 4-cutwidth critical if and only if each of $\{\bar{G}_i : 1 \le i \le 3\}$ is either a 3-cutwidth critical graph or homeomorphic to a 3-cutwidth critical nontree graph, and v_0 is not the central vertex of $\bar{G}_i$ if $\bar{G}_i \in \{\tau_1, \tau_2, \tau_3\}$.*

Proof. The proof is straightforward using Lemma 4, omitted here. □

Lemma 12. *For a 4-cutwidth graph G with a central vertex $v_0 \in V(G)$, if $G - v_0$ has at least three v_0-component H_i's and each H_i is 2-connected in G, then G is 4-cutwidth critical if and only if $G = M_2$ (see M_2 in Figure 2).*

Proof. Sufficiency: this is obvious using Lemma 6.

Necessity: By assumption, for any vertex $v_i \in N_G(v_0)$, $v_i \in V(C_{t_i})$, where C_{t_i} $(t_i \geq 3)$ is a cycle of H_i and $V(C_{t_i}) \cap V(C_{t_j}) = \{v_0\}$ for any $i \neq j$ only. Since C_3 is a minor of any C_{t_i} and $c(G) = 4$, M_2 with cutwidth 4 is a minor. Hence $G = M_2$ by the criticality of G. □

Lemma 13. *For a 4-cutwidth critical non-tree graph G with a central vertex $v_0 \in V(G)$, G has a subgraph decomposition $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$, in which $\bar{G}_i$ is 2-cutwidth critical for $i = 1$, one of whose pendant vertices is v_0, and 3-cutwidth critical for $i = 2, 3$ if and only if G is one of graphs M_9–M_{17} in Figure 4, where $\bar{G}_1 = K_{1,3}$, $\bar{G}_i \in \{\tau_3, \tau_4, \tau_5\}$ for $i = 2, 3$ with $\tau_4 = H_2 + v_0x$ (see Figure 4).*

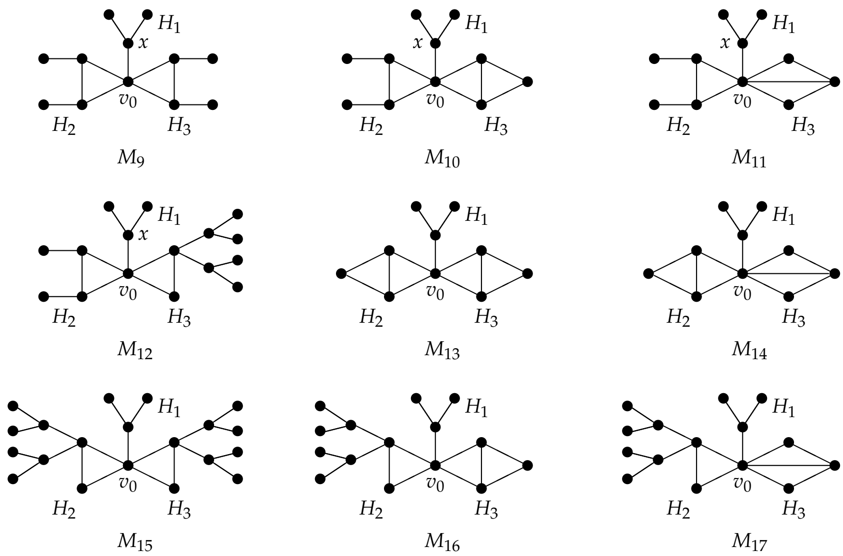

Figure 4. Nine special 4-cutwidth critical graphs.

Proof. Similar to that of Lemma 6, we can show that graphs M_9–M_{17} in Figure 4 are all 4-cutwidth critical.

Sufficiency: For graph M_9, let $\bar{G}_1 = K_{1,3}$, $\bar{G}_2 = H_2 + v_0x = \tau_4$ and $\bar{G}_3 = H_3 + v_0x = \tau_4$, $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ is the subgraph decomposition desired. Likewise, for graphs M_{10}–M_{12}, let $\bar{G}_1 = K_{1,3}$, $\bar{G}_2 = H_2 + v_0x = \tau_4$, $\bar{G}_3 = H_3 \in \{\tau_3, \tau_5\}$ with $d_{\tau_3}(v_0) = 2$ and $d_{\tau_5}(v_0) = 2$ or 3, respectively; for graphs M_{13}–M_{17}, let $\bar{G}_1 = K_{1,3}$, $\bar{G}_2 = H_2 \in \{\tau_3, \tau_5\}$ with $d_{\tau_3}(v_0) = d_{\tau_5}(v_0) = 2$, $\bar{G}_3 = H_3 \in \{\tau_3, \tau_5\}$ with $d_{\tau_3}(v_0) = 2$ and $d_{\tau_5}(v_0) = 2$ or 3, respectively, $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ is the subgraph decomposition desired.

Necessity: Suppose by contradiction that $G \notin \{M_i : 9 \leq i \leq 17\}$. By assumption, $\bar{G}_i \in \{K_{1,3}, C_3\}$ for $i = 1$, and $\{\tau_i : 1 \leq i \leq 5\}$ for $i = 2, 3$. Three cases, which are at least a $\bar{G}_i = \tau_1$ for $i = 2, 3$, $\bar{G}_2 = \bar{G}_3 = \tau_2$ and $\bar{G}_2 = \bar{G}_3 = \tau_5$ with $d_{\bar{G}_2}(v_0) = d_{\bar{G}_3}(v_0) = 3$, respectively, can be first excluded; this is because that G either is a tree or is not 4-cutwidth critical in these cases, which is a contradiction. Thus, noting that 3-cutwidth critical subgraphs $\bar{G}_2$, $\bar{G}_3$ are symmetrical in G and $c(G) = 4$ is sufficient to verify two cases:

(1) $\bar{G}_1 = K_{1,3}$, one of whose three pendant vertices is v_0, $\bar{G}_2 \in \{\tau_2, \tau_3\}$ and $\bar{G}_3 \in \{\tau_3, \tau_4, \tau_5\}$;
(2) $\bar{G}_1 = C_3$, one of whose three 2-degree vertices is v_0, $\bar{G}_2 \in \{\tau_2, \tau_3\}$ and $\bar{G}_3 \in \{\tau_3, \tau_4, \tau_5\}$.

By assumption, we do not consider the following five subcases contained in cases (1) and (2), respectively:

(a1) $d_G(v_0) \geq 7$ because of $c(K_{1,7}) = 4$;
(a2) M_2 is a subgraph of G because of $c(M_2) = 4$;
(a3) G is a tree because G is a non-tree graph;
(a4) $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ is a decomposition of equal cutwidth 3;
(a5) $c(G) = 3$ because G is 4-cutwidth critical.

Based on this, for cases (1) and (2), we only consider vertices u_0, x of τ_2, vertices u_0, x, y of τ_3, vertex x_1 of τ_4 and vertices x_1, x_2 of τ_5 (see Figure 1), respectively, which may be the central v_0 of G. For convenience, let $\bar{G}_2 \in \{\tau_2^{u_0}, \tau_2^x, \tau_3^{u_0}, \tau_3^x, \tau_3^y\}$, $\bar{G}_3 \in \{\tau_3^{u_0}, \tau_3^x, \tau_3^y, \tau_4^{x_1}, \tau_5^{x_1}, \tau_5^{x_2}\}$, where $\tau_2^{u_0}, \tau_2^x$ are copies of τ_2 corresponding to u_0, x of τ_2, $\tau_3^{u_0}, \tau_3^x, \tau_3^y$ are copies of τ_3 corresponding to u_0, x, y of τ_3; $\tau_4^{x_1}$ is a copy of τ_4 corresponding to x_1 of τ_4, and $\tau_5^{x_1}$ and $\tau_5^{x_2}$ are copies of τ_5 corresponding to x_1, x_2 of τ_5, respectively. In this case, we can see that there are at least a $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$, which is a decomposition of one of $\{M_i : 9 \le i \le 17\}$ not considered here. For example, $\{K_{1,3}, \tau_3^y, \tau_3^y\}$ is a decomposition of M_{15}. So, by at most $2 \times C_5^1 \times C_6^1 - 1$ direct operations and at most $2 \times C_5^1 \times C_6^1 - 1$ computations without considering $G \in \{M_i : 9 \le i \le 17\}$ by assumption, we can see that G is not 4-cutwidth critical, which is a contradiction. Hence, $G \in \{M_i : 9 \le i \le 17\}$. □

From Lemmas 7–13, we have:

Theorem 7. *For a 4-cutwidth non-tree graph G with a central vertex v_0, G is 4-cutwidth critical if and only if G has one of the following six configurations.*

(1) For $1 \le i \le 3$, if G_i is some τ_i $(1 \le i \le 5)$ in Figure 1 and G_i' corresponding to G_i is a graph defined in (8), then $G = K_{1,3} \circ (G_1', G_2', G_3')$, where G_1, G_2 and G_3 are not necessarily different;

(2) $G = P_3 \circ (G_1', G_2, G_3')$, where $G_i \in \{\tau_i : 1 \le i \le 5\}$ with $v_i \in V(G_i)$ for $1 \le i \le 3$ and G_i' corresponding to G_i is a graph defined in (8), $G_i \notin \{\tau_2, \tau_3\}$ for $i = 1, 3$ and $G_i \ne \tau_1$ for $i = 2$ v_2 is not either the central vertex or the pendent vertex when $G_2 \in \{\tau_2, \tau_3\}$ but v_2 is possible to a subdivision vertex of a non cut-edge of G_2 when $G_2 \in \{\tau_3, \tau_4, \tau_5\}$;

(3) $G = G_1 \oplus_{u_1, x_1} G_2'$ with the central vertex u_1 of $d_G(u_1) < 7$, where $G_1 \in \{\tau_2, \tau_3\}$ with the central vertex u_1 ($u_1 = v_0$ of τ_2 or τ_3, respectively, see Figure 1), $G_2 \in \{\tau_3, \tau_4, \tau_5\}$ with a 3-cycle $C_3 \subset G_2$ and $x_1 \in V(C_3)$ with $d_{G_1}(u_1) + d_{G_2}(x_1) \le 6$, G_2' corresponding to G_2 is a graph defined in (8);

(4) G has a subgraph decomposition $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ of equal cutwidth 3, defined in Definition 4, where G is a graph with a central vertex v_0 of $d_G(v_0) \ge 4$ and at least two cut edges v_0v_1, v_0v_2, $\bar{G}_i$ is 3-cutwidth critical for $1 \le i \le 3$;

(5) G has a subgraph decomposition $\{C_3, C_3', C_3''\}$ of equal cutwidth 2, each of which is a v_0-component of $G - v_0$, where v_0 is the central vertex v_0 of degree 6 of G, and C_3' and C_3'' are the copies of a 3-cycle C_3;

(6) G is one member of $\{M_i : 9 \le i \le 17\}$ with a central vertex v_0 (see Figure 4) and a subgraph decomposition $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$, in which $\bar{G}_1 = K_{1,3}$, one of whose pendant vertices is v_0, $\bar{G}_i \in \{\tau_3, \tau_4, \tau_5\}$ for $i = 2, 3$, where $\bar{G}_i$ satisfies:

(i) v_0 is a 2-degree vertex y of C_3 of $\bar{G}_i$ for $\bar{G}_i = \tau_3$;

(ii) if the 3-degree vertex of $\bar{G}_1 (= K_{1,3})$ is x and $\bar{G}_i = \tau_4$, then $\tau_4 = H_2 + v_0x$ and v_0 is a 3-degree vertex of $\bar{G}_i$;

(iii) v_0 is either a 2-degree vertex of $\bar{G}_i$ or a 3-degree vertex of $\bar{G}_i$ for $\bar{G}_i = \tau_5$, but if $\bar{G}_2 = \bar{G}_3 = \tau_5$ and v_0 is a 3-degree vertex of $\bar{G}_2$, then v_0 must not be a 3-degree vertex of $\bar{G}_3$, and vice versa.

4. 4-Cutwidth Critical Graphs with a Central Cycle

In this section, we aim to investigate 4-cutwidth critical graphs with a central cycle $C_q = x_1x_2 \ldots x_qx_1$ with $q \ge 3$.

Lemma 14. *Assume that graph G is 4-cutwidth critical with a central cycle C_q of length q, then $q \le 6$.*

Proof. Assume, contrary to that, that $q \ge 7$ and G_i is the ith connected component leading from x_i of $G - E(C_q)$. Without loss of generality, let $q = 7$, i.e., $C_7 = x_1x_2 \ldots x_7x_1$ with $d_G(x_i) \ge 3$ for $1 \le i \le 7$ (see an example in Figure 5a), and let $\pi : V(G) \to \mathcal{S}_n$ be an optimal 4-cutwidth labeling with $\min\{\pi(v) : v \in V(G_1)\} = \pi(x_1) < \pi(x_7) < \pi(x_2) < \pi(x_6) < \pi(x_3) < \pi(x_5) < \pi(x_4) = \max\{\pi(v) : v \in V(G_4)\}$. By the criticality of G, we may always assume that $G_i \in \{K_2, K_{1,3}, C_3\}$. By direct computations, there are at least three G_i's (say G_1, G_4 and G_6) such that $G_1 \neq K_2, G_4 \neq K_2$ and $G_6 \neq K_2$. Otherwise, $c(G) = 3$, contrary to $c(G) = 4$. Since G is 4-cutwidth critical, we can let $G_1, G_4, G_6 \in \{K_{1,3}, C_3\}$, say $G_1 = C_3$ and $G_4 = G_6 = K_{1,3}$ (see Figure 5a). In this case, $c(G) = 4$ and $c(G - x_i') = 4$ for any $G_i = K_2 = x_i x_i'$ with $i = 2, 3, 5$, contrary to the criticality of G. On the other hand, there is at least a 4-cutwidth critical graph G with a central cycle $C_6 = x_1 x_2 \ldots x_6 x_1$ such that $G_1 = G_3 = G_5 = K_{1,3}$, $G_2 = G_4 = G_6 = K_2$ and $d_G(x_i) = 3$ for $1 \leq i \leq 6$ (see Figure 5b). Hence $q \leq 6$. □

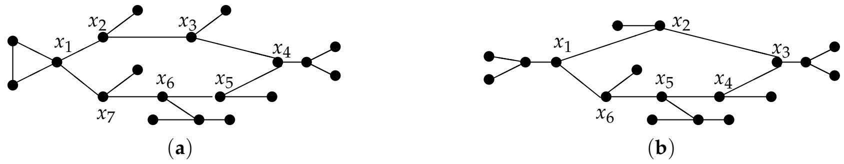

Figure 5. Examples of Lemma 14.

From Lemma 14, in the sequel, we shall characterize the 4-cutwidth critical graphs with a central cycle of lengths 3–6, respectively.

4.1. Graphs with a Central Cycle of Length Three

Definition 5. *Let $C_3 = x_1 x_2 x_3 x_1$ be the central cycle of G, G_i $(1 \leq i \leq 3)$ be the ith connected component leading from x_i of $G - E(C_3)$, and x_1, x_2, x_3 be cut vertices in G. Then, for $1 \leq i \leq 3$, define*

$$H_i = \begin{cases} G_i & \text{if } c(G_i) = 3 \text{ but } c((G_i - xy) \cup G[E']) < 3 \text{ for } E' \subseteq E(C_3) \text{ with } xy \in E(G_i), \\ G_i & \text{if } c(G_i \cup G[E'']) < 3 \text{ for } E'' \subseteq E(C_3) \text{ with } E'' \neq \emptyset, \\ G_i \cup G[E'''] & \text{if } c(G_i \cup G[E''']) = 3 \text{ for } E''' \subseteq E(C_3) \text{ with } E''' \neq \emptyset. \end{cases}$$

If, for each $1 \leq i \leq 3$, $c(H_i) = \rho$ with $\rho = 2$ or 3, then $\{H_1, H_2, H_3\}$ is called a decomposition of equal cutwidth ρ of G; if there are at least two H_i's (say H_1, H_3) such that $c(H_1) = 2$ and $c(H_3) = 3$, then $\{H_1, H_2, H_3\}$ is called a decomposition of nonequal cutwidth ρ with $\rho = 2$ or 3 of G, where E', E'' and E''' are not necessarily distinct, and E' is not necessarily non-empty.

Lemma 15. *With notation in Definition 5, let G be 4-cutwidth critical with the central cycle $C_3 = x_1 x_2 x_3 x_1$, x_1, x_2, x_3 be cut vertices in G, and C_3 has at least two vertices (say x_2, x_3) such that $d_G(x_2) \geq 4$ and $d_G(x_3) \geq 4$. If $\{H_1, H_2, H_3\}$ is a decomposition of nonequal cutwidth ρ with $\rho = 2$ or 3, then H_i is ρ-cutwidth critical for $1 \leq i \leq 3$ except M_4 in Figure 2.*

Proof. Since $\{H_1, H_2, H_3\}$ is a decomposition of nonequal cutwidth ρ with $\rho = 2$ or 3, we can assume that $c(H_2) = c(H_3) = 3$, but $c(H_1) = 2$, implying $c(G_1 \cup G[E(C_3)]) \leq 2$. Since G is 4-cutwidth critical with $d_G(x_2) \geq 4$ and $d_G(x_3) \geq 4$, $H_1 = K_{1,3}$ or C_3, $H_i \neq \tau_2, \tau_4$ or τ_5 for $i = 2, 3$ by Lemma 6, meaning to that $H_i = \tau_1$ or τ_3 for $i = 2, 3$. Thus, for H_2 and H_3, there are three cases to consider: (i) $H_2 = G_2 \cup G[\{x_2 x_3, x_2 x_1\}] = K_{1,5}$, $H_3 = G_3 \cup G[\{x_3 x_2, x_3 x_1\}] = K_{1,5}$; (ii) $H_2 = G_2 \cup G[\{x_2 x_3, x_2 x_1\}] = K_{1,5}$, $H_3 = G_3 \cup C_3 = \tau_3$; (iii) $H_2 = G_2 \cup C_3 = \tau_3$, $H_3 = G_3 \cup C_3 = \tau_3$, where x_2, x_3 are the central vertices of H_2 and H_3, respectively (see M_5, M_6, M_7 in Figures 2 and 6d,e below). In any of Cases (i)–(iii), $H_1 = G_1 = K_{1,3}$ with $d_G(x_1) = 3$ or C_3 with $d_G(x_1) = 4$. Thus, we can see that H_i is 2-cutwidth critical for $i = 1$ and 3-cutwidth critical for $i = 2, 3$. Now let $c(H_3) = 3$ but $c(H_1) = c(H_2) = 2$ with $H_1 = C_3$ and $H_2 = C_3$; then, we can conclude that $G = M_4$ by the 4-cutwidth criticality of G, which has a decomposition $\{K_{1,3}, C_3, C_3', C_3''\}$ of equal cutwidth two in which C_3' and C_3'' are the copies of C_3. This is because in this case, if $\{K_{1,5}, C_3', C_3''\}$ is a decomposition of nonequal cutwidth of 2 and 3 of G, then edge $x_1 x_2 \notin E(K_{1,5}) \cup E(C_3') \cup E(C_3'')$. As $x_2 x_3 \notin E(H_i)$ for each $1 \leq i \leq 3$ in this case, this decomposition of nonequal cutwidth does not hold. Thus, this case is not possible. The proof is complete. □

Lemma 16. *With notation in Definition 5, let $\{H_1, H_2, H_3\}$ be a decomposition of nonequal cutwidth ρ with $\rho = 2$ or 3 of 4-cutwidth graph G with the central cycle $C_3 = x_1x_2x_3x_1$. If H_i is ρ-cutwidth critical for $1 \le i \le 3$, then G is 4-cutwidth critical, where x_1, x_2, x_3 are all cut vertices, and C_3 has at least two vertices (say x_2, x_3) such that $d_G(x_2) \ge 4$ and $d_G(x_3) \ge 4$.*

Proof. Let π be an optimal labeling of G with $\pi(x_1) < \pi(x_2) < \pi(x_3)$ and intervals $I_1 = [1, \pi(x_1)], I_2 = (\pi(x_1), \pi(x_3)), I_3 = [\pi(x_3), n]$ with $n = |V(G)|$, respectively. Then, G_1 is embedded in I_1 with congestion 3, G_2 is embedded in I_2 with congestion 4, G_3 is embedded in I_3 with congestion 3. Herein, G_1 and G_3 are a star $K_{1,3}$ with center x_i or two stars $K_{1,3}$ with an identifying leaf at x_i $(i = 1,3)$. Let H_i denote G_i combining with the two edges in C_3 incident with v_i. Then $H_i \in \{\tau_1, \tau_3\}$ for $i = 1,3$. As for G_2 embedded in I_2 with congestion 4, since the central cycle C_3 yields congestion 2 in I_2, we chose G_2 as a 2-cutwidth critical tree, namely, a $K_{1,3}$, such that either $d_G(x_2) = 3$ or $d_G(x_2) = 5$. For this construction, the maximum congestion is 4, i.e., $c(G) = 4$. Furthermore, for any edge $e \in E(G)$, if $e \in \{x_1x_2, x_1x_3, x_2x_3\}$, then the deletion of e reduces the congestion 2 of cycle-edge in I_2 by one. Hence H_2 embedded in I_2 has congestion 3, and so $c(G - e) < 4$. If $e \notin \{x_1x_2, x_1x_3, x_2x_3\}$, for Case (i) in Proof of Lemma 15, two subcases need to be considered: (a) $G_i = K_{1,3}$ with $d_G(x_i) = 5$ for each $1 \le i \le 3$; (b) $G_i = K_{1,3}$ with $d_G(x_i) = 5$ for $i = 1,3$, but $G_2 = C_3$ with $d_G(x_2) = 4$ for $i = 1,3$. Without loss of generality, we can let $e \in E(G_2)$ with $G_3 = K_{1,3}$. Since $G_2 - e = K_{1,3} - e$ with congestion 1, we can embed G_1 in I_1, $G_2 - e$ in $(\pi(v_1), \pi(v_3) - 1)$ and G_3 in $[\pi(v_3) - 1, n - 1]$, respectively, which results in $c(G - e) = 3$. So does the case of $e \in E(G_1)$ (or $E(G_3)$). Likewise, for Cases (ii) and (iii) in Proof of Lemma 15, $c(G - e) = 3$ for any $e \in E(G)$ also. Therefore, G is 4-cutwidth critical. The lemma holds. □

Lemma 17. *With notation in Definition 5, let G be 4-cutwidth critical with the central cycle $C_3 = x_1x_2x_3x_1$, where x_1, x_2, x_3 are all cut vertices in G, and C_3 has at most one vertex (say x_1) such that $d_G(x_1) \ge 4$. If $\{H_1, H_2, H_3\}$ is a decomposition of equal cutwidth 3, then H_i (or $H_i - x_ix_i'$ with $x_i' \in N_G(x_i) \cap V(G_i)$) is 3-cutwidth critical for $1 \le i \le 3$.*

Proof. We first give Claim 1 below.
Claim 1. There is at least H_i $(1 \le i \le 3)$ such that H_i is one of $G_i \cup G[\{x_ix_{i-1}, x_ix_{i+1}\}]$ and $G_i \cup C_3$ (say $G_i \cup G[\{x_ix_{i-1}, x_ix_{i+1}\}]$) with $c(H_i) = 3$, where $x_0 = x_3$ and $x_4 = x_1$.

Let $H_i = G_i$ or $G_i + x_ix_{i+1}$ with $c(H_i) = 3$ for each $1 \le i \le 3$. As the arguments are similar, we only consider two cases: (a) $H_1 = G_1$ with $d_G(x_1) \ge 4$, $H_i = G_i + x_ix_{i+1}$ with $d_G(x_i) = 3$ for $i = 2,3$; (b) $H_i = G_i + x_ix_{i+1}$ with $d_G(x_i) = 3$ for each $1 \le i \le 3$. For Case (a), x_ix_{i+1} is a pendent edge of H_i for $i = 2,3$, $d_{H_2}(x_2) = 2$ and $d_{H_3}(x_3) = 2$. So, $H_2 = G_2 + x_2x_1$ also and $c(G_2) = c(G_3) = 3$ by a series reduction in H_2 and H_3, respectively. Thus, $G - x_2x_3 = P_3 \circ (G_2, G_1, G_3)$ which results in that $c(G - x_2x_3) = 4$ by Theorem 3, contrary to the criticality of G. For Case (b), $d_G(x_i) = 3$ and $d_{H_i}(x_i) = 2$ for each $1 \le i \le 3$, so every $c(G_i) = c(H_i) = 3$ by a series reduction in H_i and $d_{G_i}(x_i) = 1$. Thus, there is an edge in C_3, say x_1x_3, such that $G - x_1x_3 = K_{1,3} \circ (G_1, G_2, G_3)$. Hence $c(G - x_1x_3) = 4$ by Theorem 2, also a contradiction. Claim 1 holds.

From Claim 1 and assumption, there are nine cases to consider, as follows (see graphs (a)–(c) in Figure 6 below):

(1) $H_1 = G_1 \cup G[\{x_1x_2, x_1x_3\}]$ with $d_G(x_1) \ge 4$, $H_2 = G_2$ and $H_3 = G_3$;
(2) $H_1 = G_1 \cup G[\{x_1x_2, x_1x_3\}]$ with $d_G(x_1) \ge 4$, $H_2 = G_2 \cup G[\{x_2x_1, x_2x_3\}]$ and $H_3 = G_3$;
(3) $H_1 = G_1 \cup G[\{x_1x_2, x_1x_3\}]$ with $d_G(x_1) \ge 4$, $H_2 = G_2 \cup G[\{x_2x_1, x_2x_3\}]$ and $H_3 = G_3 \cup G[\{x_3x_1, x_3x_2\}]$;
(4) $H_1 = G_1 \cup C_3$ with $d_G(x_1) \ge 4$, $H_2 = G_2$ and $H_3 = G_3$;
(5) $H_1 = G_1 \cup C_3$ with $d_G(x_1) \ge 4$, $H_2 = G_2 \cup G[\{x_2x_1, x_2x_3\}]$ and $H_3 = G_3$;
(6) $H_1 = G_1 \cup C_3$ with $d_G(x_1) \ge 4$, $H_2 = G_2 \cup G[\{x_2x_1, x_2x_3\}]$ and $H_3 = G_3 \cup G[\{x_3x_1, x_3x_2\}]$;
(7) $H_1 = G_1 \cup G[\{x_1x_2, x_1x_3\}]$, $H_2 = G_2$ and $H_3 = G_3$;

(8) $H_1 = G_1 \cup G[\{x_1x_2, x_1x_3\}]$, $H_2 = G_2 \cup G[\{x_2x_1, x_2x_3\}]$ and $H_3 = G_3$;
(9) $H_1 = G_1 \cup G[\{x_1x_2, x_1x_3\}]$, $H_2 = G_2 \cup G[\{x_2x_1, x_2x_3\}]$ and $H_3 = G_3 \cup G[\{x_3x_1, x_3x_2\}]$,

where $d_G(x_i) = 3$ for $i = 2, 3$ in Cases (1)–(6), and $d_G(x_i) = 3$ for each $1 \le i \le 3$ in Cases (7)–(9). We consider Case (1) by contradiction. Assuming that there is at least an edge $xy \in E(H_i)$ such that $c(H_i - xy) = 3$, i.e., H_i is not 3-cutwidth critical. There are three subcases to consider: (i) $c(H_1 - xy) = 3$ with $xy \in E(H_1)$; (ii) $c(H_2 - xy) = 3$ with $xy \in E(H_2)$; (iii) $c(H_3 - xy) = 3$ with $xy \in E(H_3)$. For Subcase (i), by assumption and Definition 3, for $i = 2, 3$, $d_G(x_2) = d_G(x_3) = 3$, G_i is 3-cutwidth critical, and $c((G_i - x'y') \cup G[E']) < 3$ for $x'y' \in E(G_i)$ and $E' \subseteq E(C_3)$ with $E' \neq \emptyset$, so either $G_i \in \{\tau_1, \tau_4\}$ or $G_i = K_2 \cup \tau_5$. Thus, if $xy \in \{x_1x_2, x_1x_3\}$ (say $xy = x_1x_2$), then $G - xy$ is changed to $\oplus_{x_3}(H_1 - xy, G_2, G_3)$ with cutwidth 4 resulting in $c(G - xy) = 4$; if $xy \notin \{x_1x_2, x_1x_3\}$, i.e., $xy \in E(G_1)$ then $G - xy - x_2x_3$ is changed to be $\oplus_{x_2}(H_1 - xy, G_2, G_3)$ with cutwidth 4 resulting in $c(G - xy) \ge c(G - xy - x_2x_3) = 4$. So $c(G - xy) = 4$ by $c(G - xy) \le c(G) = 4$ again, and contrary to that, G is 4-cutwidth critical. For Subcase (ii), we can conclude that $H_1 = K_{1,5}$ and either $G_3 \in \{\tau_1, \tau_4\}$ or $G_i = K_2 \cup \tau_5$ with cutwidth 3. By Lemma 1(3), an optimal labeling f^* by the order $(V(H_2) - xy), V(H_1 + x_2x_3), V(H_3))$ of $G - xy$ can be obtained, and $c(G - xy, f) = 4$, implying $c(G - xy) \le 4$. So, $c(G - xy) = 4$ by the optimality of f^*, also a contradiction. The argument of Subcase (iii) is the same as that of Subcase (ii), omitted here. Thus, for Case (1), $\bar{G}_i$ is 3-cutwidth critical for $1 \le i \le 3$. Similarly, for Cases (2)–(9), H_i is also 3-cutwidth critical for $1 \le i \le 3$. This completes the proof. □

Lemma 18. *With notation in Definition 5, let $\{H_1, H_2, H_3\}$ be a decomposition of equal cutwidth 3 of graph G with the central cycle $C_3 = x_1x_2x_3x_1$, where x_1, x_2, x_3 are all cut vertices of G, and C_3 has at most one vertex (say x_1) such that $d_G(x_1) \ge 4$, and either $\{x_1x_2, x_1x_3\} \subset E(H_1)$ or $E(C_3) \subset E(H_1)$. If H_i is 3-cutwidth critical or there are at least a $H_i = G_i = x_ix_i' + \tau_5$ with $x_i' \in N_G(x_i)$ for $1 \le i \le 3$, then G is 4-cutwidth critical.*

Proof. By Lemmas 1(3), we can show $c(G) = 4$. By assumption again, $H_1 \in \{\tau_1, \tau_3\}$ and $d_G(x_2) = d_G(x_3) = 3$. There are nine cases (1)–(9) listed in Proof of Lemma 17 to consider. For each case (i) $(1 \le i \le 9)$, via using an argument similar to that of Lemma 16, we can show $c(G') \le 3$ for any $G' \in \mathcal{M}(G)$, omitted here. □

Lemma 19. *Let G be a 2-connected graph with a central cycle $C_3 = x_1x_2x_3x_1$. Then G is 4-cutwidth critical with a decomposition $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ of equal cutwidth 3 if and only if $G = M_8$ (see Figure 2).*

Proof. Sufficiency. Since $G = M_8$, G is 4-cutwidth critical by Lemma 6. Clearly, let $\bar{G}_i = G[\{x_1, x_2, x_3, y_i\}]$ for $1 \le i \le 3$, then $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ is a decomposition desired because of $\bar{G}_i = \tau_5$ for each $1 \le i \le 3$.

Necessity. In fact, since G is 2-connected with a central cycle $C_3 = x_1x_2x_3x_1$ and a decomposition $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$ of equal cutwidth 3, the arbitrary two vertices x_i and x_{i+1} $(1 \le i \le 3)$ of C_3 must be in a cycle C_t' $(t \ge 3)$ and $C_t' \neq C_3$, where $x_4 = x_1$. That is to say, by the criticality of G, there must be another vertex $y_i \neq x_i$ in G such that $y_ix_i \in E(G)$ and $y_ix_{i+1} \in E(G)$ for each $1 \le i \le 3$. In this case, $G = M_8$, induced by $\{x_1, x_2, x_3, y_1, y_2, y_3\}$. Hence $G = M_8$. □

Lemma 20. *Assume that G is a 4-cutwidth critical graph with a central cycle $C_3 = x_1x_2x_3x_1$, then G has an edge-disjoint decomposition $\{G_1, G_2, G_3, C_3\}$ of equal cutwidth 2 if and only if $G \in \{M_3, M_4, M_5, M_6, M_7\}$ (see Figure 2), where x_i is a cut vertex and G_i is the connected component of $G - E(C_3)$ leading from x_i for $1 \le i \le 3$.*

Proof. Sufficiency is obvious in Lemma 6, omitted here.

Necessity: Let π be an optimal labeling of G with $\pi(x_1) < \pi(x_2) < \pi(x_3)$ and $|V(G)| = n$. Then, the number set $\mathcal{S}_n$ is divided into three intervals $I_1 = [1, \pi(x_1)]$,

$I_2 = (\pi(x_1), \pi(x_2))$ and $I_3 = [\pi(x_3), n]$ and G_1, G_2, G_3 are embedded into I_1, I_2, I_3 in different manners, respectively. As G_1, G_2, G_3 are all 2-cutwidth graphs and x_i is a cut vertex in G for $1 \le i \le 3$, G_1 is embedded into I_1 with congestion 2, G_2 is embedded into I_2 with congestion 4, and G_3 is embedded into I_3 with congestion 2. By the criticality of G and $c(K_{1,3}) = c(C_3) = 2$, G_i is either a star $K_{1,3}$ with the 3-degree vertex x_i or a cycle $C_3^{(i)}$, which is a copy of C_3 for $i = 1, 3$. As for G_2 embedded in I_2 with a congestion of 4, the central cycle C_3 leads to a congestion of 2 in I_2, so G_2 must be either a $K_{1,3}$ or a copy $C_3^{(2)}$ of C_3 such that $d_G(x_2) = 3, 4$ or 5. Thus, G must be one member of $\{M_3, M_4, M_5, M_6, M_7\}$, each element of which has a edge-disjoint decomposition $\{G_1, G_2, G_3, C_3\}$ of equal cutwidth 2, where G_i is either $K_{1,3}$ or C_3 for $1 \le i \le 3$. □

Theorem 8. *For a 4-cutwidth nontree graph G with a central cycle $C_3 = x_1x_2x_3x_1$, G is 4-cutwidth critical if and only if G has one of the following configurations.*

(1) G has a decomposition $\{H_1, H_2, H_3\}$ of nonequal cutwidth ρ with ρ=2 or 3, each of which is ρ-cutwidth critical, where x_i is a cut vertex for each $1 \le i \le 3$ and there are at least two vertices (say x_2, x_3) such that $d_G(x_2) \ge 4$ and $d_G(x_3) \ge 4$ (see M_5–M_7 in Figure 2 and Illustration in Figure 6d,e);

(2) G has a decomposition $\{H_1, H_2, H_3\}$ of equal cutwidth 3 in which H_i or $H_i - x_ix_i'$ with $x_i' \in N_G(x_i) \cap V(G_i)$ is 3-cutwidth critical, and at least a H_i (say H_1) contains at least two edges x_1x_2 and x_1x_3 of C_3, where x_i is a cut vertex for each $1 \le i \le 3$ and there is at most a vertex (say x_3) such that $d_G(x_3) \ge 4$ (see Illustration in Figure 6a–c);

(3) G is 2-connected and $G = M_8$ (see Figure 2) with a decomposition $\{H_1, H_2, H_3\}$ of equal cutwidth 3 in which $H_i = G[\{x_1, x_2, x_3, y_i\}] = \tau_5$ for $1 \le i \le 3$;

(4) $G \in \{M_3, M_4, M_5, M_6, M_7\}$ with an edge-disjoint decomposition $\{G_1, G_2, G_3, C_3\}$ of equal cutwidth 2, in which G_i is either $K_{1,3}$ or a copy C_3' of C_3 for $1 \le i \le 3$ (see M_3–M_7 in Figure 2).

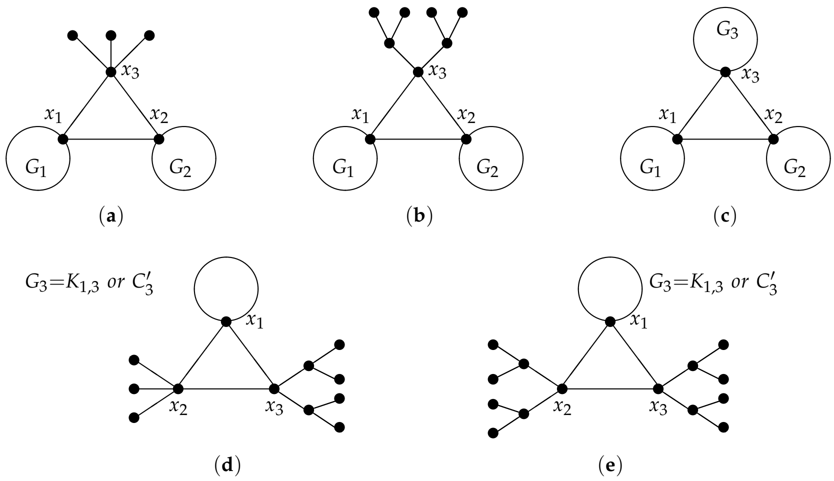

Figure 6. Illustrations of Theorem 8.

In Figure 6a–c, for $i = 1, 2, 3$, $G_i = \tau_1, \tau_4, \tau_5 + x_ix_i'$ with $x_i' \in V(\tau_5)$ or $G_i + x_ix_{i-1} + x_ix_{i+1} = \tau_2, \tau_3$ with $x_0 = x_3$, but in Figure 6c, there is at least a G_i, say G_3, such that $G_3 + x_3x_1 + x_3x_2 = \tau_2$ or τ_3. In Figure 6d,e, if $G_1 = K_{1,3}$ then $d_G(x_1) = 3$, i.e., x_1 is a pendant vertex of $K_{1,3}$ in this case, C_3' is a copy of C_3.

4.2. Graphs with a Central Cycle of Length Four

For a graph G with a central cycle $C_4 = x_1x_2x_3x_4x_1$ with length 4, suppose that G_i $(1 \le i \le 4)$ is the ith connected component leading from x_i of $G - E(C_4)$, $c(G_1) \ge c(G_2) \ge c(G_4) \ge c(G_3)$ and $d_G(x_1) \ge 4$, and $G - E(C_4)$ has no G_i, such that $c(G_i) = 3$ but $c(G_i' \cup \{x_ix_{i-1}, x_ix_{i+1}\}) = 3$ for any proper subgraph $G_i' \subset G_i$. Let $\bar{G}_1 = G_1$ when $c(G_1) = 3$ or $G_1 \cup G[E']$ with $E' \subseteq E(C_4)$ and $E' \neq \emptyset$ when $c(G_1) < 3$, $\bar{G}_2 = G_2 \cup G_3 \cup (C_4 - x_1x_4 +$

$\{x_1x_1', x_1x_1''\})$ and $\bar{G}_4 = G_4 \cup G_3 \cup (C_4 - x_1x_2 + \{x_1x_1', x_1x_1''\})$ for $x_1', x_1'' \in V(G_1)$ with $x_1' \neq x_1''$, where x_1, x_2, x_3, x_4 are all cut vertices in G, and there is at least a vertex between x_2 and x_4 (say x_4) such that $d_G(x_2) \geq 4$. Then, we have the following:

Lemma 21. *For a graph G with the central cycle $C_4 = x_1x_2x_3x_4x_1$, if $\{\bar{G}_1, \bar{G}_2, \bar{G}_4\}$ is a decomposition of equal cutwidth 3 of G and $\bar{G}_i$ is 3-cutwidth critical for each $i \in \{1,2,4\}$, then G is 4-cutwidth critical (see Illustrations in Figure 7).*

Proof. By assumption, $d_G(x_2) = d_G(x_3) = 3$, and since $\bar{G}_i$ is 3-cutwidth critical for $i \in \{1,2,4\}$, $\bar{G}_1 \in \{\tau_1, \tau_3, \tau_4, \tau_5\}$, $\bar{G}_2 = \tau_2$ with $\bar{G}_2 = K_{1,3}$ and $\bar{G}_4 = \tau_2$ with $G_2 = K_{1,3}$ or τ_3 with $G_3 = C_3$ resulting in $G_3 = K_2$. Suppose that $\pi : V(G) \to \mathcal{S}_n$ is a labeling of G with $\pi(x_1) < \pi(x_2) < \pi(x_3) < \pi(x_4)$, then $\mathcal{S}_n$ is partitioned into three intervals $I_1 = [1, \pi(x_1)]$, $I_2 = (\pi(x_2), \pi(x_4)]$ and $I_3 = (\pi(x_4), n]$. Now, we embed G_1 in I_1 with congestion 3, $\bar{G}_2 - \{x_1x_1', x_1x_1''\}$ in I_2 and connect x_1x_4 with congestion 4, $G_4 - x_4$ in I_3 with congestion 2. Thus, $c(G, \pi) = 4$, implying $c(G) \leq 4$. On the other hand, $c(G) \geq 4$. Hence $c(G) = 4$.

The remaining is to show $c(G - e) < 4$ for any $e \in E(G)$. There are three cases to consider: (1) $e \in E(G_1)$; (2) $e \in E(C_4)$; (3) e is a pendant edge of G_i for $i = 2,3,4$. For Case (1), $c(G_1 - e) \leq 2$. Since $d_G(x_2) = 3$, by Lemma 1(3), if $e = v_1v_2$ is a pendant edge of G_1 with $d_G(v_2) = 1$, then we can find an optimal labeling $\pi' : V(G - v_2) \to \mathcal{S}_{n-1}$ with $c(G - v_2, \pi') = 3$, under which $G_2 - x_2$ is embedded in interval $[1, \min\{\pi(v) : v \in V(G_1 - v_2)\})$ with congestion 3. If $e \in E(C_3)$ (note that $\bar{G}_1 = \tau_4$ or τ_5 in this subcase), then we can find an optimal labeling $\pi'' : V(G - e) \to \mathcal{S}_n$ with $c(G - e, \pi'') = 3$, under which $G_2 - x_2$ is embedded in interval $[1, \min\{\pi(v) : v \in V(G_1)\}]$ with congestion 3. So $c(G - e) = 3$. Similarly, for Cases (2) and (3), $c(G - e) = 3$ also. Hence, G is 4-cutwidth critical. □

Lemma 22. *Let G be a 4-cutwidth critical graph with the central cycle $C_4 = x_1x_2x_3x_4x_1$. If G has a decomposition $\{\bar{G}_1, \bar{G}_2, \bar{G}_4\}$ of equal cutwidth 3, then $\bar{G}_i$ is 3-cutwidth critical for each $i \in \{1,2,4\}$ (see Illustrations in Figure 7).*

Proof. By contradiction, suppose that there is at least a $\bar{G}_i$ (say $\bar{G}_2$) such that $\bar{G}_2$ is not 3-cutwidth critical, then there exists an edge $e \in E(\bar{G}_2)$ such that $c(\bar{G}_2 - e) = 3$ also. Two cases need to be considered: (1) $e = vv'$ is a pendant edge with $d_G(v') = 1$ in $\bar{G}_2$; (2) $e \in E(C')$ if $\bar{G}_2$ contains a cycle C' which does not equal the central cycle C_4. Using an argument similar to that of Lemma 21, for Case (1), we can find a labeling $\pi : V(G - v') \to \mathcal{S}_{n-1}$ with $c(G - v') = 4$, thereby contradicting that G is 4-cutwidth critical. Furthermore, likewise, for Case (2), we can find a labeling $\pi : V(G - e) \to \mathcal{S}_n$ with $c(G - e) = 4$, also contradicting that G is 4-cutwidth critical. Similarly, if $e \in E(\bar{G}_i)$ for $i = 1$ or 4 then we can also find a contradiction to the assertion that G is 4-cutwidth critical. Therefore, $\bar{G}_i$ is 3-cutwidth critical for each $i \in \{1,2,4\}$. □

From Lemmas 21 and 22, the structure of a 4-cutwidth critical graph G with a central cycle $C_4 = x_1x_2x_3x_4x_1$ can be obtained below.

Theorem 9. *Assume that G is a 4-cutwidth graph with a central cycle $C_4 = x_1x_2x_3x_4x_1$, and x_i is a cut vertex for $1 \leq i \leq 4$, then G is 4-cutwidth critical if and only if G has a decomposition $\{\bar{G}_1, \bar{G}_2, \bar{G}_4\}$ of equal cutwidth 3, each of which is 3-cutwidth critical, where $\bar{G}_1, \bar{G}_2, \bar{G}_4$ are one of the following:*

(1) $\bar{G}_1 = K_{1,5}$ with the central vertex x_1 of $d_G(x_1) = 5$ or τ_5 with $d_G(x_1) = 4$, and $\bar{G}_2$ and $\bar{G}_4$ are both in $\{\tau_2, \tau_3\}$, but $\bar{G}_2$ and $\bar{G}_4$ do not equal τ_3 simultaneously (see Illustration in Figure 7a);

(2) $\bar{G}_1$ is homeomorphic to τ_3 with the central vertex x_1 of $d_G(x_1) = 4$ and $C_4 \subset \bar{G}_1$, $\bar{G}_2$ and $\bar{G}_4$ are both in $\{\tau_2, \tau_3\}$. $\bar{G}_2, \bar{G}_4$ are not necessarily different (see Illustration in Figure 7b).

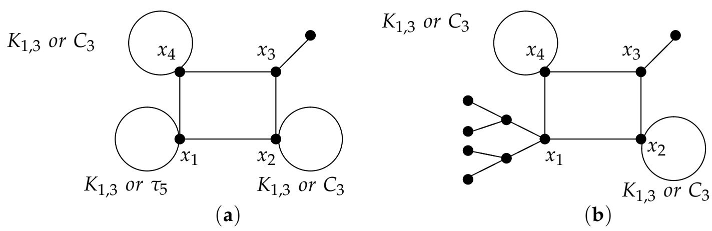

Figure 7. Illustrations of Theorem 9.

4.3. Graphs with a Central Cycle of Length at Least Five

Suppose that G is a graph with the central cycle $C_5 = x_1x_2x_3x_4x_5x_1$, and for $1 \leq i \leq 5$, $G - E(C_5)$ has no component G_i leading from x_i, such that $c(G_i) = 3$, but $c(G_i' \cup \{x_ix_{i-1}, x_ix_{i+1}\}) = 3$ for any proper subgraph $G_i' \subset G_i$, C_5 has at most two $x_i's$ with $d_G(x_i) \geq 4$, where $x_0 = x_5, x_6 = x_1$. Let one of the following hold:

(1) $\bar{G}_1 = G_1 \cup G_2 \cup G_5 \cup (C_5 - x_3x_4)$, $\bar{G}_i = G_i$ or $G_i + x_3x_4$ if $c(G_i) = 3$ or $G_i + x_ix_{i-1} + x_ix_{i+1}$ if $c(G_i) < 3$ for $i = 3, 4$ with $d_G(x_3) = d_G(x_4) = 3$;

(2) $\bar{G}_1 = G_1 \cup G_2 \cup G_5 \cup (C_5 - x_3x_4)$, $\bar{G}_3 = G_3 \cup (C_5 - x_1x_5 + x_2x_2' + x_4x_4')$, $\bar{G}_4 = G_4 \cup (C_5 - x_1x_2 + x_3x_3' + x_5x_5')$ with $d_G(x_3) = d_G(x_4) = 4$ and $c(G_3) = c(G_4) = 2$, $x_i' \in N_G(x_i) \cap V(G_i - x_i)$ for $2 \leq i \leq 5$;

(3) $\bar{G}_1$ is homeomorphic to subgraph $(G_1 + x_1x_2 + x_1x_5) \cup G_2 \cup G_5$, $\bar{G}_3 = G_3 \cup (C_5 - x_1x_5 + x_2x_2' + x_4x_4')$, $\bar{G}_4 = G_4 \cup (C_5 - x_1x_2 + x_3x_3' + x_5x_5')$ with $c(G_3) = c(G_4) = 2$, where C_5 has at most two 4-degree vertices (say, x_1 and x_4) which are nonadjacent.

Then, we have the following:

Lemma 23. *For a graph G with the central cycle $C_5 = x_1x_2x_3x_4x_5x_1$, if G is 4-cutwidth critical and $\{\bar{G}_1, \bar{G}_3, \bar{G}_4\}$ is a subgraph decomposition of equal cutwidth 3 of G, then $\bar{G}_i$ (or $\bar{G}_i - x_i$) is 3-cutwidth critical for $i \in \{1, 3, 4\}$ (see Illustrations in Figure 8).*

Proof. By contradiction, we first consider Case (1) above. Suppose that there exists some $\bar{G}_i$, say $\bar{G}_1$ first, such that $\bar{G}_1$ is not 3-cutwidth critical. There are two subcases to consider: (i) $\bar{G}_1$ contains no cycle; (ii) $\bar{G}_1$ contains at least a cycle. For (i), $\bar{G}_1$ has at least a pendant vertex v such that $c(\bar{G}_1 - v) = 3$. By $d_G(x_3) = d_G(x_4) = 3$, let x_3x_3', x_4x_4' be cut edges in G with $x_3' \in V(G_3 - x_3) \cap N_G(x_3)$ and $x_4' \in V(G_4 - x_4) \cap N_G(x_4)$. Then $d_{G-v}(x_3) = d_{G-v}(x_4) = 3$, and x_3x_3', x_4x_4' are both cut edges in $G - v$ clearly. So, by Lemma 1(3), $G - v$ has an optimal labeling π such that the vertices in each of $V(G_3 - x_3)$, $V(\bar{G}_1 - v + x_3x_4)$ and $V(G_4 - x_4)$ are labeled consecutively. Without loss of generality, let $\max\{\pi(v) : v \in V(G_3 - x_3)\} < \min\{\pi(v) : v \in V(\bar{G}_1 - v + x_3x_4)\}$ and $\max\{\pi(v) : v \in V(\bar{G}_1 - v + x_3x_4)\} < \min\{\pi(v) : v \in V(G_4 - x_4)\}$. Then $c(G - v, \pi) = c(\bar{G}_1 - v) + 1 = 4$. Since π is optimal, $c(G - v) = c(G - v, \pi) = 4$, contradicting that G is 4-cutwidth critical. For (ii), two subcases need to be considered: (a) $\bar{G}_1$ has at least a pendant vertex v such that $c(\bar{G}_1 - v) = 3$; (b) $\bar{G}_1$ has at least a non-pendant edge e such that $c(\bar{G}_1 - e) = 3$. Subcase (a) is the same as case (i), omitted here; For subcase (b), using a similar method to that of case (i), we can show $c(G - e) = 4$, also a contradiction. Now, we consider $\bar{G}_3$ or $\bar{G}_4$, and without loss of generality; let $c(\bar{G}_3 - x_3x_3') = c(G_3 - x_3x_3') = 3$ with $x_3' \in N_G(x_3) \cap V(G_3 - x_3)$ and $\bar{G}_4 = G_4 + x_4x_3 + x_4x_5$. Assume that there is an edge e such that $c(\bar{G}_3 - x_3x_3' - e) = 3$, i.e., $\bar{G}_3 - x_3x_3' - e$ is not 3-cutwidth critical. Similar to Case (i), $d_{G-e}(x_3) = d_{G-e}(x_4) = 3$, and x_3x_3', x_4x_4' are both cut edges in $G - e$. By Lemma 1(3), $G - e$ has an optimal labeling π' such that the vertices in each of $V(\bar{G}_1 + x_3x_4)$, $V(G_3 - x_3 - e)$ and $V(G_4 - x_4)$ are labeled consecutively with $\max\{\pi'(v) : v \in V(\bar{G}_1 + x_3x_4)\} < \min\{\pi'(v) : v \in V(G_3 - x_3 - e)\}$ and $\max\{\pi'(v) : v \in V(G_3 - x_3 - e)\} < \min\{\pi'(v) : v \in V(G_4 - x_4)\}$. Thus $c(G - e) = c(G - e, \pi') = c(G_3 - x_3x_3' - e) + 1 = 4$, contradicting that G is 4-cutwidth critical. Likewise, let $\bar{G}_3$ and $\bar{G}_4$ be one of the followings, and one of $\{\bar{G}_3, \bar{G}_4\}$ be not 3-cutwidth critical: (A1) each $\bar{G}_i = G_i$ with $c(\bar{G}_i - x_ix_i') = c(G_i - x_ix_i') = 3$ for $i = 3, 4$;
(A2) each $\bar{G}_i = G_i + x_ix_{i-1} + x_ix_{i+1}$ with $c(G_i) < 3$ for $i = 3, 4$;

(A3) $\bar{G}_3 = G_3$ with $c(G_3) = 3$ but $c(G_3 - x_3x_3') < 3$, $\bar{G}_4 = G_4 + x_4x_3 + x_4x_5$ with $c(G_4) < 3$;
(A4) $\bar{G}_3 = G_3$ with $c(G_3) = 3$, $\bar{G}_4 = G_4$ with $c(G_4) = 3$ but $c(G_4 - x_4x_4') < 3$.
Then we can also obtain a contradiction to the assertion that G is 4-cutwidth critical. Hence, each $\bar{G}_i$ (or $\bar{G}_i - x_i$) is 3-cutwidth critical.

Similarly, for Cases (2) and (3) above, $\bar{G}_i$ (or $\bar{G}_i - x_i$) is also 3-cutwidth critical for $i \in \{1,3,4\}$. This completes the proof. □

Lemma 24. *For a 4-cutwidth graph G with the central cycle $C_5 = x_1x_2x_3x_4x_5x_1$, if $\{\bar{G}_1, \bar{G}_3, \bar{G}_4\}$ is a decomposition of equal cutwidth 3 of G, $\bar{G}_i$ (or $\bar{G}_i - x_i$) is 3-cutwidth critical for $i \in \{1,3,4\}$, then G is 4-cutwidth critical.*

Proof. Three cases similar to those of Lemma 23 need to be considered. We first consider Case (1) by contradiction. Suppose that G is not 4-cutwidth critical, i.e., there exists a pendant vertex v (or a non-pendant edge e) such that $c(G - v) = 4$ (or $c(G - e) = 4$). There are three subcases to consider: (i) $v \in V(\bar{G}_1)$ (or $e \in E(\bar{G}_1)$); (ii) $v \in V(\bar{G}_2)$ (or $e \in E(\bar{G}_2)$); (iii) $v \in V(\bar{G}_3)$ (or $e \in E(\bar{G}_3)$). For Case (i), by assumption, $c(\bar{G}_1 - v) < 3$ (or $c(\bar{G}_1 - e) < 3$). Since $d_G(x_3) = d_G(x_4) = 3$, using a similar method to that of Lemma 22, we can verify that $c(G - v) < 4$ (or $c(G - e) < 4$) contrary to $c(G - v) = 4$ (or $c(G - e) = 4$). So, G is 4-cutwidth critical. Likewise, for Subcases (ii) and (iii), G is 4-cutwidth critical also.

Similarly, for Cases (2) and (3), G is 4-cutwidth critical also. This proof is completed. □

From Lemmas 23 and 24:

Theorem 10. *Assume that G is a 4-cutwidth graph with a central cycle $C_4 = x_1x_2x_3x_4x_5x_1$, and x_i is a cut vertex for $1 \le i \le 5$, then G is 4-cutwidth critical if and only if G has a decomposition $\{\bar{G}_1, \bar{G}_3, \bar{G}_4\}$ (or $\{\bar{G}_1, \bar{G}_3 + x_3x_4, \bar{G}_4\}$, $\{\bar{G}_1, \bar{G}_3, \bar{G}_4 + x_4x_3\}$, $\{\bar{G}_1, \bar{G}_3 + x_3x_4, \bar{G}_4 + x_4x_3\}$) of equal cutwidth 3, where $\bar{G}_1, \bar{G}_3, \bar{G}_4$ are one of the following:*

(1) $\bar{G}_1 \in \{\tau_2, \tau_3\}$ with the central vertex x_1 of degree three or four, for $i = 3,4$, $\bar{G}_i$ (or $\bar{G}_i - x_i$) is one of $\{\tau_i : 1 \le i \le 5\}$ and x_i satisfies: (i) $d_G(x_i) = 3$, (ii) x_i is not the central vertex of $\bar{G}_i$ when $\bar{G}_i \in \{\tau_1, \tau_2, \tau_3\}$, and (iii) x_ix_{i-1}, x_ix_{i+1} are the pendant edges of $\bar{G}_i$ when $\bar{G}_i$ is τ_2 or τ_3 (see Illustration in Figure 8a);

(2) $\bar{G}_1$ is homeomorphic to τ_2 with the central vertex x_1 of degree three, for $i = 3,4$, $\bar{G}_i$ is homeomorphic to τ_2 or τ_3 with $G_i \in \{K_{1,3}, C_3\}$, where G_3, G_4 are not necessarily different (see Illustration in Figure 8b);

(3) $\bar{G}_1$ is homeomorphic to τ_3 with the central vertex x_1 of degree four, for $i = 3,4$, $\bar{G}_i$ is homeomorphic to τ_2 or τ_3 with $G_i \in \{K_{1,3}, C_3\}$, but if $G_3 = C_3$, then $G_4 \neq C_3$ and vice versa (see Illustration in Figure 8b).

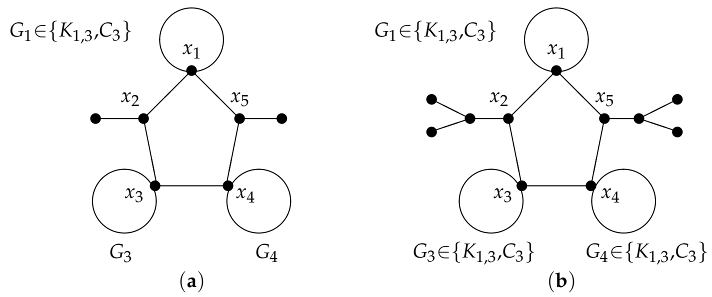

Figure 8. Illustrations of Theorem 10.

In Figure 8a, $\bar{G}_i$ is either G_i which is τ_1 or τ_4 or $\tau_5 + x_ix_i'$ with $x_i' \in V(\tau_5)$ or $G_i + x_ix_{i-1} + x_ix_{i+1}$ which is in $\{\tau_2, \tau_3\}$ for $i = 3,4$; Additionally, if $G_1 = K_{1,3}$ then G_3, G_4 can be 3-cycle C_3 simultaneously.

Lemma 25. *For a graph G with a central cycle $C_6 = x_1x_2x_3x_4x_5x_6x_1$, G is 4-cutwidth critical if and only if $G \in \{M_{18}, M_{19}, M_{20}\}$ in Figure 9.*

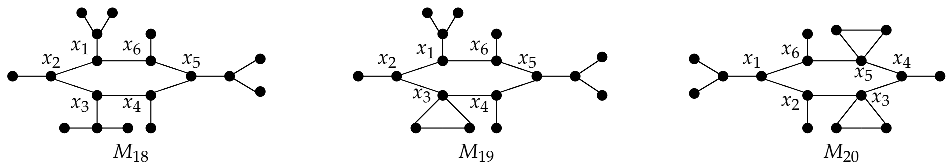

Figure 9. Three 4-cutwidth critical graphs with a C_6.

Proof. Sufficiency. For any $G \in \{M_{18}, M_{19}, M_{20}\}$, G can be easily shown to be 4-cutwidth critical by proving two conclusions: (1) $c(G) = 4$; (2) $c(G') = 3$ for any $G' \in \mathcal{M}(G)$, omitted here.

Necessity. Let G be a 4-cutwidth critical graph with the central cycle $C_6 = x_1x_2x_3x_4x_5x_6x_1$.

Observation. For any $18 \le i \le 20$, M_i has a decomposition $\{\bar{G}_1, \bar{G}_3, \bar{G}_5\}$ of equal cutwidth 3, where $\bar{G}_l = H_{l-1} \cup H_l \cup H_{l+1} = \tau_2$ or τ_3 with $H_l = G_l + x_lx_{l-1} + x_lx_{l+1}$ for $l \in \{1,3,5\}$, $H_0 = H_6$ and $H_7 = H_1$.

By observation, suppose by contradiction that $G \notin \{M_{18}, M_{19}, M_{20}\}$, then two cases need to be considered as follows.

Case 1. G has a decomposition $\{\bar{G}_1, \bar{G}_3, \bar{G}_5\}$ of equal cutwidth 3, but there is at least an element in $\{\bar{G}_1, \bar{G}_3, \bar{G}_5\}$, say $\bar{G}_3 (= H_2 \cup H_3 \cup H_4)$, such that $\bar{G}_3$ does not equal τ_2 (or τ_3). In this case, $G - E(C_r)$ has at least a connected component G_i leading from x_i, say G_3, such that $G_3 \supset K_{1,3}$ (or K_3); this is because the connected component leading from x_3 in M_{18} is $K_{1,3}$ (or in any of $\{M_{19}, M_{20}\}$ is K_3. Without loss of generality, let G_3 be a minimum graph such that $K_{1,3} \subset G_3$ (or $K_3 \subset G_3$), i.e., $|E(G_3) \setminus E(K_{1,3})| = 1$ (or $|E(G_3) \setminus E(K_3)| = 1$). Then, by direct computations, $c(G) = 4$ and $c(\bar{G}_3) = 3$, but G is not 4-cutwidth critical. Similarly, if $G_2 \neq K_2$ or $G_4 \neq K_2$ in $\bar{G}_3$ then G is not 4-cutwidth critical also. So this case is not possible.

Case 2. G has not a decomposition $\{\bar{G}_1, \bar{G}_3, \bar{G}_5\}$ of equal cutwidth 3. In this case, there are at least an element in $\{\bar{G}_1, \bar{G}_3, \bar{G}_5\}$, say $\bar{G}_1$, such that $c(\bar{G}_1)$ is either at most 2 or at least 4, i.e., either $c(\bar{G}_1) \le 2$ or $c(\bar{G}_1) \ge 4$. Since G is 4-cutwidth critical, the subcase of $c(\bar{G}_1) \ge 4$ is impossible. For the subcase of $c(\bar{G}_1) \le 2$, we claim that G_1 must be a path P_2 with length 2 in which either $d_{G_1}(x_2) = 1$ or $d_{G_1}(x_2) = 2$. By direct computations, we can easily show that $c(G) = 3$, contrary to $c(G) = 4$. Therefore, this case is also impossible. The proof is complete. □

By Lemma 25, we have

Theorem 11. *Let G be a 4-cutwidth graph with a central cycle $C_6 = x_1x_2x_3x_4x_5x_6x_1$. Then G is 4-cutwidth critical if and only if G is one of $\{M_{18}, M_{19}, M_{20}\}$ in Figure 9, which has a subgraph decomposition $\{\bar{G}_1, \bar{G}_3, \bar{G}_5\}$ of equal cutwidth 3, in which $\bar{G}_i = \tau_2$ or τ_3 with central vertex x_i for $i \in \{1,3,5\}$ and there is at least a $\bar{G}_{i_0}$ such that $\bar{G}_{i_0} = \tau_2$ with $i_0 \in \{1,3,5\}$.*

5. 4-Cutwidth Critical Graphs without a Central Vertex and Central Cycle

We now consider the 4-cutwidth critical graphs with neither a central vertex nor a central cycle (see five graphs M_{21}–M_{25} in Figure 10).

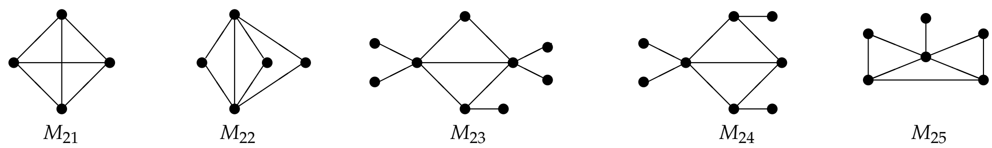

Figure 10. 4-cutwidth critical graphs without a central vertex and central cycle.

Theorem 12. *A graph G is 4-cutwidth critical with neither a central vertex nor a central cycle if and only if $G \in \{M_{21}, M_{22}, M_{23}, M_{24}, M_{25}\}$.*

Proof. Sufficiency. For any $G \in \{M_{21}, M_{22}, M_{23}, M_{24}, M_{25}\}$, G is needed to show (1) $c(G) = 4$; (2) $c(G') = 3$ for any $G' \in \mathcal{M}(G)$. These can be done easily, omitted here. On the other hand, we can see that G has neither the central vertex nor the central cycle.

Necessity. Suppose that G is a 4-cutwidth critical graph without central vertex and central cycle, then G has at least two cycles C_3, sharing a common edge. This is because otherwise, G can be thought of as having either a central vertex or a central cycle. So, we have the following:

Claim 2. τ_5 in Figure 1 is an edge-induced proper subgraph with cutwidth 3 of G.

By Claim 2, we have

Claim 3. Suppose that H is a 1-connected and minimum 3-cutwidth graph with $\tau_5 \subset H$, $d_H(x_1) \leq 4$ and $d_H(x_3) \leq 4$, in which $d_H(x_i)$ is maximum for each $x_i \in V(\tau_5)$, then H is graph (a) in Figure 11.

Claim 4. Suppose that H is a 2-connected and minimum noncritical 3-cutwidth graph with $\tau_5 \subset H$, then H is graph (b) in Figure 11.

Figure 11. Two 3-cutwidth graphs containing τ_5.

By Claims 3 and 4 and the minimality of G, if G is 1-connected, then G must be one of $\{M_{23}, M_{24}, M_{25}\}$ by direct computations and comparisons. Now, we consider the case that G is 2-connected. Since 4-cutwidth critical graph M_8 can be thought of as having a central cycle C_3, we can exclude M_8 here. Thus, by direct computations and comparisons, G must be one member of $\{M_{21}, M_{22}\}$. So, $G \in \{M_{21}, M_{22}, M_{23}, M_{24}, M_{25}\}$. □

6. Concluding Remarks

In this paper, we have completely characterized the structural properties of 4-cutwidth critical graphs, from which we can see that except for a handful of irregular critical graphs M_{21}–M_{25} in Figure 10, the other 4-cutwidth critical graphs can be classified into two classes: graph class with a central vertex v_0, and graph class with a central cycle C_q of length $q \leq 6$. By means of some ingenious combination, any member of two classes can achieve a subgraph decomposition $\{H_1, H_2, H_3\}$ (or $\{\bar{G}_1, \bar{G}_2, \bar{G}_3\}$), in which H_i (or $\bar{G}_i$) is either a 2-cutwith graph or a 3-cutwidth graph for each $1 \leq i \leq 3$, or a subgraph decomposition $\{H_1, H_2, H_3, H_4\}$ of equal cutwidth 2. For a given integer $k > 4$, although it seems difficult to characterize the detailed structures of k-cutwidth critical graphs, some structural properties of some special graph classes can be found. For instance, using [11], any k-cutwidth critical tree with a central vertex v_0 has a subtree decomposition $\{T_1, T_2, T_3\}$ of equal cutwidth $k-1$, where, for $1 \leq i \leq 3$, T_i (or $T_i - v_0$) is either a $(k-1)$-cutwidth critical tree or homeomorphic to a $(k-1)$-cutwidth critical tree. Similarly, a k-cutwidth critical non-tree graph $G = \oplus_{z_0}(G_1, G_2, G_3)$ also has a subgraph decomposition $\{G_1, G_2, G_3\}$ of equal cutwidth $k-1$, and G_1, G_2, G_3 are all $(k-1)$-cutwidth critical. In the k-cutwidth critical graphs G with a central cycle C_q of length $q \geq 3$, the structural properties are not yet known. Additionally, for a fixed integer $k_0 > 4$, finding all the k_0-cutwidth critical graph Gs with neither a central vertex nor a central cycle is also a difficult task. All of these are the further objectives to investigate in future works.

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