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Existence Theoremsfor Solutions of a Nonlinear Fractional-Order Coupled Delayed System via Fixed Point Theory

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Abstract: In this paper, the problem of the existence and uniqueness of solutions for a nonlinear fractional-order coupled delayed system with a new kind of boundary condition is studied. For this reason, we transform the above problem into an equivalent fixed point problem using the integral operator. Moreover, by applying fixed point theorems, a novel set of sufficient conditions that guarantee the existence and uniqueness of solutions of the coupled system is derived. Eventually, an example is presented to illustrate the effectiveness of the obtained results.

Keywords: nonlinear coupled system; fixed point theorem; existence and uniqueness

MSC: 47H09; 47H10

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1. Introduction

In this paper, we discuss a class of nonlinear fractional-order coupled system with time delay:

$$\begin{cases} {}^{ABC}D^{\iota}m(t) = h(t, m(t-\varsigma(t)), q(t-\sigma(t)), \ t \in J := [0,T], T > 0 \\ {}^{ABC}D^{\mu}q(t) = k(t, m(t-\varsigma(t)), q(t-\sigma(t))), \ t \in J := [0,T], \\ (m+q)(0) = -(m+q)(T), \ \int_a^b (m+q)(s)ds = \zeta, \ 0 < a < b < T \end{cases} \quad (1)$$

where ${}^{ABC}D^{\alpha}_{0+}$ is an Atangana–Baleanu fractional derivative operator in Caputo's sense of order $\iota \in \{\iota, \mu\}$, $\iota, \mu \in (0,1]$, h, $k : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are the first-order continuous differentiable functions with respect to t, and $\varsigma(t)$ and $\sigma(t)$ are the time-varying delays satisfying $0 \le \varsigma(t) \le \varsigma$, $0 \le \sigma(t) \le \sigma$, where $\varsigma, \sigma > 0$, ζ is a nonnegative constant.

In the past decades, fractional-order systems have become an active research topic. Because fractional derivatives introduce convolutional integrals with power law memory kernels, fractional-order models are more accurate than integer-order models in practice [1]. Furthermore, fractional derivatives have shown their superiority in describing processes and materials involving memory and genetic property, for example in electromagnetism, mechatronics, and supercapacitors [2–6]. Until now, many scholars have extensively studied the existence, uniqueness, and stability of solutions for fractional-order systems [7–13]. For example, in [14], authors discussed the finite-time stability of fractional-order delayed hopfield neural networks. In [15], the global Mittag-Leffler stability was investigated for fractional-order complex-valued impulsive BAM neural networks.

It should be noted that the fractional operators in the above literature involve singularities in their kernels. However, this singularity presents some difficulties for scientists seeking to best simulate real-world phenomena. In order to overcome the difficulty, some researchers have proposed new fractional operators that do not contain singular kernels. In the work done by Atangana and Baleanu, the most famous fractional derivative that does not contain singularities appears, namely the ABC-fractional derivative [16]. The important applications of the ABC-fractional derivative can be found in [17–22]. For example, in [22], based on ABC-fractional derivative, several fractional masks for image denoising have been proposed. Hasib Khan et al. [23] considered a fractional L-V model involving three different species of ABC-fractional derivatives. Simultaneously, in [24], the existence result and stability criterion of the fuzzy-volterra integro-differential equation in the sense of ABC-fractional derivative was derived.

On the other hand, integral boundary conditions have extensive applications in regularizing ill-posed parabolic backward problems in time partial differential equations [25]. In addition, integral boundary conditions also play an important role in the study of computational fluid dynamics for blood flow problems [26]. Recently, the existence results of fractional-order systems with integral boundary conditions have extensively been studied by many researchers. In [27], some sufficient conditions for the existence theorems for solutions of fractional-order differential equations with nonlocal and average type integral boundary conditions were obtained. Ahmad et al. [28] discussed a coupled system of nonlinear fractional differential equations in the Caputo fractional derivative sense with coupled boundary conditions for the existence and uniqueness of solutions of the type:

$$\begin{cases} {}^{C}D^{\lambda}m(t) = f(t, m(t), q(t)), & t \in [0, T],\ T > 0 \\ {}^{C}D^{\gamma}q(t) = h(t, m(t), q(t)), & t \in [0, T], \\ (m+q)(0) = -(m+q)(T), & \int_c^d (m-q)(s)ds = a,\ 0 < c < d < T \end{cases}$$

where $f, h : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. ${}^{C}D^{u}_{0+}$ is the Caputo fractional derivative with order $u \in \{\lambda, \gamma\}, \lambda, \gamma \in (0, 1]$, and a is nonnegative constant.

To the best of our knowledge, there exists very little work on the existence of the solutions for nonlinear coupled delayed systems involving ABC-fractional derivatives with integral boundary conditions, which is valuable in blood flow problems and regularizing ill-posed parabolic backward problems. In response to the above-mentioned discussions, we study the existence and uniqueness of solutions for a nonlinear ABC-fractional order coupled delayed system with coupled boundary conditions. Different from the existing literature, the salient contributions are summarized as the following two aspects. (1) Based on fixed point theory, a new criterion for ensuring the existence and uniqueness of solutions of the nonlinear ABC-fractional order coupled delayed system is obtained. Furthermore, an example is presented to illustrate the effectiveness of the theoretical results. (2) Considering the universality of delay in real systems, time-varying delays $\varsigma(t)$ and $\sigma(t)$ are considered in system (1). Different from the coupling function considered in [27,28], we study the concept that the coupled systems with time-varying delays and the time-varying delays in the coupling function are different, which is more general.

The paper includes a major update to the theory of fractional order coupled differential equations, and is structured as follows. In Section 2, we introduce some auxiliary lemmas and definitions, which are required for building our theorems. In Section 3, we obtain the main results for the system (1) by utilizing the Contraction Mapping Principle and Schaefer's fixed point theorem. Eventually, in Section 4, one example is given to demonstrate our results.

2. Preliminaries

In this section, we introduce some auxiliary lemmas and definitions. Let $C^1[c, d]$ be the space that consists of all the first-order continuous derivative functions defined on $[c, d]$.

Definition 1 ([29]). *The ABC-fractional derivative of a function $m(t) \in C^1[c,d]$, $0 < \iota < 1$ is*

$$ {}^{ABC}D^{\iota}(u(s)) = \frac{B(\iota)}{1-\iota} \int_0^s u'(s) E_{\iota}\left[\frac{-\iota(t-s)^{\iota}}{1-\iota}\right] ds, \tag{2} $$

where $B(\iota) = (1-\iota) + \frac{\iota}{\Gamma(\iota)}$ satisfies the property $B(0) = B(1) = 1$, and E_{ι} is called the Mittag-Leffler function defined by the series

$$ E_{\iota}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\iota k + 1)}, $$

here $Re(z) > 0$ and $\Gamma(\cdot)$ is the gamma function.

Definition 2 ([30]). *The ABC-fractional integral of a function $u(t) \in L^1[c,d]$, $d > c$, $\lambda \in [0,1)$ is*

$$ {}^{AB}I^{\lambda}(u(s)) = \frac{1-\lambda}{B(\lambda)} u(s) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \int_0^s u(s)(t-s)^{\lambda-1} ds, \tag{3} $$

in which $\Gamma(\cdot)$ is the gamma function.

Lemma 1. *(Schaefer's fixed point theorem (see [31], p. 29)) In the Banach space Ψ, let $\Theta : \Psi \to \Psi$ be a completely continuous operator, and the set $\Lambda = \{x \in \Psi | x = \xi\Theta(x), 0 < \xi < 1\}$ is bounded. Then, Θ has a fixed point in Ψ.*

Lemma 2. *(Arzelá-Ascoli theorem [32]) A subset of $C[a,b]$ is compact if and only if it is closed, bounded and equicontinuous.*

Lemma 3. *(Contraction Mapping Principle [31]) Let T be a contraction operator on a complete metric space Ω; then, there exists a unique point $z \in \Omega$ satisfying $T(z) = z$.*

Lemma 4. *Let H, K, u, $v \in C^1[0,T]$. Then, the solution of the following fractional-order coupled system,*

$$ \begin{cases} {}^{ABC}D^{\iota} m(t) = H(t), \ t \in J := [0,T], \\ {}^{ABC}D^{\mu} q(t) = K(t), \ t \in J := [0,T], \\ (m+q)(0) = -(m+q)(T), \ \int_a^b (m+q)(s) ds = \zeta, \end{cases} \tag{4} $$

is given by

$$ \begin{aligned} m(t) \ = \ & \frac{1-\iota}{B(\iota)} H(t) + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} H(s) ds + \frac{1}{2}\Bigg\{ -\frac{1}{2}\Bigg(\frac{1-\iota}{B(\iota)} H(T) + \frac{\iota}{B(\iota)\Gamma(\iota)} \\ & \times \int_0^T (T-s)^{\iota-1} H(s) ds + \frac{1-\mu}{B(\mu)} K(T) + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^T (T-s)^{\mu-1} K(s) ds \Bigg) \\ & + \frac{\zeta}{b-a} - \frac{1}{b-a} \int_a^b \Bigg(\frac{1-\iota}{B(\iota)} H(s) + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^s (s-\vartheta)^{\iota-1} H(\vartheta) d\vartheta \\ & - \frac{1-\mu}{B(\mu)} K(s) - \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^s (s-\vartheta)^{\mu-1} K(\vartheta) d\vartheta \Bigg) ds \Bigg\}, \end{aligned} \tag{5} $$

and

$$
\begin{aligned}
q(t) &= \frac{1-\mu}{B(\mu)}K(t) + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} K(s)ds + \frac{1}{2}\bigg\{ -\frac{1}{2}\bigg(\frac{1-\iota}{B(\iota)} H(T) + \frac{\iota}{B(\iota)\Gamma(\iota)} \\
&\quad \times \int_0^T (T-s)^{\iota-1} H(s)ds + \frac{1-\mu}{B(\mu)}K(T) + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^T (T-s)^{\mu-1} K(s)ds \bigg) \\
&\quad - \frac{\zeta}{b-a} + \frac{1}{b-a} \int_a^b \bigg(\frac{1-\iota}{B(\iota)} H(s) + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^s (s-\vartheta)^{\iota-1} \\
&\quad \times H(\vartheta)d\vartheta - \frac{1-\mu}{B(\mu)}K(s) - \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^s (s-\vartheta)^{\mu-1} K(\vartheta)d\vartheta \bigg) ds \bigg\}. \qquad (6)
\end{aligned}
$$

Proof. Applying the operators ${}_0^{AB}I^{\iota}$ and ${}_0^{AB}I^{\mu}$ on both sides of the fractional differential equations in (4), respectively, we have

$$
m(t) = \frac{1-\iota}{B(\iota)} H(t) + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} H(s)ds + C_1, \qquad (7)
$$

$$
q(t) = \frac{1-\mu}{B(\mu)} K(t) + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} K(s)ds + C_2, \qquad (8)
$$

where $C_1, C_2 \in \mathbb{R}$.

Furthermore, in light of the boundary conditions of the system (4), we can conclude that

$$
\begin{aligned}
C_1 + C_2 &= -\frac{1}{2}\bigg\{ \frac{1-\iota}{B(\iota)} H(T) + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^T (T-s)^{\iota-1} H(s)ds \\
&\quad + \frac{1-\mu}{B(\mu)}K(T) + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^T (T-s)^{\mu-1} K(s)ds \bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
C_1 - C_2 &= \frac{1}{b-a}\bigg\{ \zeta - \int_a^b \bigg(\frac{1-\iota}{B(\iota)} H(s) + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^s (s-\vartheta)^{\iota-1} H(\vartheta)d\vartheta \\
&\quad - \frac{1-\mu}{B(\mu)}K(s) - \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^s (s-\vartheta)^{\mu-1} K(\vartheta)d\vartheta \bigg) ds \bigg\}.
\end{aligned}
$$

Then, we can deduce that

$$
\begin{aligned}
C_1 &= \frac{1}{2}\bigg\{ -\frac{1}{2}\bigg(\frac{1-\iota}{B(\iota)} H(T) + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^T (T-s)^{\iota-1} H(s)ds + \frac{1-\mu}{B(\mu)}K(T) \\
&\quad + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^T (T-s)^{\mu-1} K(s)ds \bigg) + \frac{\zeta}{b-a} - \frac{1}{b-a} \int_a^b \bigg(\frac{1-\iota}{B(\iota)} H(s) - \frac{1-\mu}{B(\mu)}K(s) \\
&\quad + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^s (s-\vartheta)^{\iota-1} H(\vartheta)d\vartheta - \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^s (s-\vartheta)^{\mu-1} K(\vartheta)d\vartheta \bigg) ds \bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
C_2 &= \frac{1}{2}\bigg\{ -\frac{1}{2}\bigg(\frac{1-\iota}{B(\iota)} H(T) + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^T (T-s)^{\iota-1} H(s)ds + \frac{1-\mu}{B(\mu)}K(T) \\
&\quad + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^T (T-s)^{\mu-1} K(s)ds \bigg) - \frac{\zeta}{b-a} + \frac{1}{b-a} \int_a^b \bigg(\frac{1-\iota}{B(\iota)} H(s) - \frac{1-\mu}{B(\mu)}K(s) \\
&\quad + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^s (s-\vartheta)^{\iota-1} H(\vartheta)d\vartheta - \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^s (s-\vartheta)^{\mu-1} K(\vartheta)d\vartheta \bigg) ds \bigg\}.
\end{aligned}
$$

The proof is completed. □

3. Main Results

In this section, we prove the existence and uniqueness of solutions for the problem (1). We consider the space $\Omega = C^1(E,\mathbb{R}) \times C^1(E,\mathbb{R})$ $(E = [-\varsigma, T] \cap [-\sigma, T])$ equipped with the norm

$$\|(m,q)\| = \|m\|_\infty + \|q\|_\infty = \sup_{t\in[-\varsigma,T]} |m(t)| + \sup_{t\in[-\sigma,T]} |q(t)|,$$

for $(m,q) \in \Omega$. Furthermore, according to Lemma 4, we construct the operator $\Theta : \Omega \to \Omega$ for the problem (1), where $\Theta_i : \Omega \to C^1(E, \mathbb{R})$ $(i = 1,2)$ and

$$\Theta(m,q)(t) := (\Theta_1(m,q)(t), \Theta_2(m,q)(t)), \tag{9}$$

$$\begin{aligned} \Theta_1(m,q)(t) &= \frac{\zeta - C_4}{2(b-a)} - \frac{1}{4}C_3 + \frac{1-\iota}{B(\iota)} h(t, m(t-\varsigma(t)), q(t-\sigma(t))) \\ &\quad + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} h(s, m(s-\varsigma(s)), q(s-\sigma(s)))ds, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \Theta_2(m,q)(t) &= \frac{-\zeta + C_4}{2(b-a)} - \frac{1}{4}C_3 + \frac{1-\mu}{B(\mu)} k(t, m(t-\varsigma(t)), q(t-\sigma(t))) \\ &\quad + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^t (t-s)^{\iota-1} k(s, m(s-\varsigma(s)), q(s-\sigma(s)))ds. \end{aligned} \tag{11}$$

in which

$$\begin{aligned} C_3 &= \frac{1-\iota}{B(\iota)} h(T, m(T-\varsigma(T)), q(T-\sigma(T))) + \frac{1-\mu}{B(\mu)} k(T, m(T-\varsigma(T)), q(T-\sigma(T))) \\ &\quad + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^T (T-s)^{\iota-1} h(s, m(s-\varsigma(s)), q(s-\sigma(s)))ds \\ &\quad + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^T (T-s)^{\mu-1} k(s, m(s-\varsigma(s)), q(s-\sigma(s)))ds, \end{aligned}$$

$$\begin{aligned} C_4 &= \int_a^b \Bigg(\frac{1-\iota}{B(\iota)} h(s, m(s-\varsigma(s)), q(s-\sigma(s))) - \frac{1-\mu}{B(\mu)} k(s, m(s-\varsigma(s)), q(s-\sigma(s))) \\ &\quad + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^s (s-\vartheta)^{\iota-1} h(\vartheta, m(\vartheta-\varsigma(\vartheta)), q(\vartheta-\sigma(\vartheta)))d\vartheta \\ &\quad - \frac{\mu}{B(\mu)\Gamma(\mu)} \int_0^s (s-\vartheta)^{\mu-1} k(\vartheta, m(\vartheta-\varsigma(\vartheta)), q(\vartheta-\sigma(\vartheta)))d\vartheta \Bigg) ds. \end{aligned}$$

In this paper, the following conditions are assumed to be true:

(A_1) For any $(t,\ m,\ q) \in J \times \mathbb{R}^2$, there exist continuous positive functions $\alpha_i,\ \beta_i \in C^1([0,T], R)$, $i = 1,\ 2,\ 3$, such that

$$\begin{aligned} |h(t,\ m(t-\varsigma(t)),\ q(t-\sigma(t)))| &\leq \alpha_1(t) + \alpha_2(t)\|m\|_\infty + \alpha_3(t)\|q\|_\infty, \\ |k(t,\ m(t-\varsigma(t)),\ q(t-\sigma(t)))| &\leq \beta_1(t) + \beta_2(t)\|m\|_\infty + \beta_3(t)\|q\|_\infty. \end{aligned}$$

(A_2) For any $(t,\ m,\ q) \in J \times \mathbb{R}^2$, there exist positive constants $\gamma_i,\ \eta_i,\ i = 1,\ 2$, such that

$$\begin{aligned} & |h(t, m_1(t-\varsigma(t)), q_1(t-\sigma(t))) - h(t, m_2(t-\varsigma(t)), q_2(t-\sigma(t)))| \\ \leq\ & \gamma_1\|m_1 - m_2\|_\infty + \gamma_2\|q_1 - q_2\|_\infty, \\ & |k(t, m_1(t-\varsigma(t)), q_1(t-\sigma(t))) - k(t, m_2(t-\varsigma(t)), q_2(t-\sigma(t)))| \\ \leq\ & \eta_1\|m_1 - m_2\|_\infty + \eta_2\|q_1 - q_2\|_\infty. \end{aligned}$$

For computational convenience, we define

$$v_1 = \frac{5(1-\iota)}{4B(\iota)} + \frac{T^\iota}{4B(\iota)\Gamma(\iota)} + \frac{\iota(b^{\iota+1} - a^{\iota+1})}{2(b-a)B(\iota)\Gamma(\iota+2)}, \tag{12}$$

$$v_2 = \frac{5(1-\mu)}{4B(\mu)} + \frac{T^\mu}{4B(\mu)\Gamma(\mu)} + \frac{\mu(b^{\mu+1} - a^{\mu+1})}{2(b-a)B(\mu)\Gamma(\mu+2)}. \tag{13}$$

Theorem 1. *Assume that (A_1) holds, v_1 and v_2 are defined by (12) and (13), and if the following condition holds,*

$$0 \leq \eta_1,\ \eta_2 \leq 1, \tag{14}$$

where

$$\eta_1 = \|\alpha_2\|_\infty\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + \|\beta_2\|_\infty\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right),$$

$$\eta_2 = \|\alpha_3\|_\infty\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + \|\beta_3\|_\infty\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right),$$

then the problem (1) has at least one solution.

Proof. We complete the proof in three steps.

Step 1. We declare that the operator $\Theta : \Omega \to \Omega$ is completely continuous, which means that Θ is continuous and maps any bounded subset of Ω to a relatively compact subset of Ω. Due to the continuity of the functions h and k, the operator $\Theta : \Omega \to \Omega$ is continuous. Let $\Upsilon \subseteq \Omega$ be bounded. Then, for any $(m, q) \in \Upsilon$, $t \in J$, there exist positive constants $a_1,\ a_2,\ b_1$ and b_2 such that

$$|h(t, m(t-\varsigma(t)), q(t-\sigma(t)))| \leq a_1\|m\|_\infty + a_2\|q\|_\infty, \tag{15}$$

$$|k(t, m(t-\varsigma(t)), q(t-\sigma(t)))| \leq b_1\|m\|_\infty + b_2\|q\|_\infty. \tag{16}$$

Firstly, by using (10), it is not difficult to obtain that

$$\begin{aligned}
&\|\Theta_1(m,q)\|_\infty \\
\leq\ & \frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^t (t-s)^{\iota-1}|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))| + \frac{1-\iota}{B(\iota)}ds \\
&\times|h(t,m(t-\varsigma(t)),q(t-\sigma(t)))| + \frac{1}{2}\bigg\{\frac{1}{2}\bigg(\frac{1-\iota}{B(\iota)}|h(T,m(T-\varsigma(T)),q(T-\sigma(T)))| \\
&+\frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^T (T-s)^{\iota-1}|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))|ds \\
&+\frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^T (T-s)^{\mu-1}|k(s,m(s-\varsigma(s)),q(s-\sigma(s)))|ds \\
&+\frac{1-\mu}{B(\mu)}|k(T,m(T-\varsigma(T)),q(T-\sigma(T)))|\bigg) + \frac{\zeta}{b-a} + \frac{1}{b-a} \\
&\times\int_a^b\bigg(\frac{1-\iota}{B(\iota)}|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))| + \frac{1-\mu}{B(\mu)}|k(s,m(s-\varsigma(s)),q(s-\sigma(s)))| \\
&+\frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^s (s-\vartheta)^{\iota-1}|h(\vartheta,m(\vartheta-\varsigma(\vartheta)),q(\vartheta-\sigma(\vartheta)))|d\vartheta \\
&+\frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^s (s-\vartheta)^{\mu-1}|k(\vartheta,m(\vartheta-\varsigma(\vartheta)),q(\vartheta-\sigma(\vartheta)))|d\vartheta\bigg)ds\bigg\}. \qquad (17)
\end{aligned}$$

Next, by virtue of the inequality (15), we deduce

$$\begin{aligned}
&\frac{1-\iota}{B(\iota)}|h(t,m(t-\varsigma(t)),q(t-\sigma(t)))| \leq \frac{1-\iota}{B(\iota)}(a_1\|m\|_\infty + a_2\|q\|_\infty), \\
&\frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^t (t-s)^{\iota-1}|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))|ds \\
&\leq \frac{T^\iota}{B(\iota)\Gamma(\iota+1)}(a_1\|m\|_\infty + a_2\|q\|_\infty). \qquad (18)
\end{aligned}$$

Similar to the procedures of (18), we can get

$$\begin{aligned}
&\frac{1}{2}\bigg\{\frac{1}{2}\bigg(\frac{1-\iota}{B(\iota)}|h(T,m(T-\varsigma(T)),q(T-\sigma(T)))| + \frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^T (T-s)^{\iota-1} \\
&\times|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))|ds\bigg) + \frac{1}{b-a}\int_a^b\bigg(\frac{1-\iota}{B(\iota)}|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))| \\
&+\frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^s (s-\vartheta)^{\iota-1}|h(\vartheta,m(\vartheta-\varsigma(\vartheta)),q(\vartheta-\sigma(\vartheta)))|d\vartheta\bigg)ds\bigg\} \\
&\leq (a_1\|m\|_\infty + a_2\|q\|_\infty)\bigg(\frac{5(1-\iota)}{4B(\iota)} + \frac{T^\iota}{4B(\iota)\Gamma(\iota)} + \frac{\iota(b^{\iota+1}-a^{\iota+1})}{2(b-a)B(\iota)\Gamma(\iota+2)}\bigg). \qquad (19)
\end{aligned}$$

Finally, by means of the inequality (16), we can also derive

$$\begin{aligned}
&\frac{1}{2}\bigg\{\frac{1}{2}\bigg(\frac{1-\mu}{B(\mu)}|k(T,m(T-\varsigma(T)),q(T-\sigma(T)))| + \frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^T (T-s)^{\mu-1} \\
&\times|k(s,m(s-\varsigma(s)),q(s-\sigma(s)))|ds\bigg) + \frac{1}{b-a}\int_a^b\bigg(\frac{1-\mu}{B(\mu)}|k(s,m(s-\varsigma(s)),q(s-\sigma(s)))| \\
&+\frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^s (s-\vartheta)^{\mu-1}|k(\vartheta,m(\vartheta-\varsigma(\vartheta)),q(\vartheta-\sigma(\vartheta)))|d\vartheta\bigg)ds\bigg\} \\
\leq\ & (b_1\|m\|_\infty + b_2\|q\|_\infty)\bigg(\frac{5(1-\mu)}{4B(\mu)} + \frac{T^\mu}{4B(\mu)\Gamma(\mu)} + \frac{\mu(b^{\mu+1}-a^{\mu+1})}{2(b-a)B(\mu)\Gamma(\mu+2)}\bigg). \qquad (20)
\end{aligned}$$

Therefore, combining with (17)–(20), we can obtain

$$\begin{aligned}\|\Theta_1(m,q)\|_\infty \ \le\ & (a_1\|m\|_\infty + a_2\|q\|_\infty)\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + v_1\right) \\ & + (b_1\|m\|_\infty + b_2\|q\|_\infty)v_2 + \frac{\zeta}{2(b-a)}.\end{aligned}$$

Similarly, we can also get the above inequality for $\|\Theta_2(m,q)(t)\|_\infty$, that is

$$\begin{aligned}\|\Theta_2(m,q)\|_\infty \ \le\ & (a_1\|m\|_\infty + a_2\|q\|_\infty)v_1 + \frac{\zeta}{2(b-a)} \\ & + (b_1\|m\|_\infty + b_2\|q\|_\infty)\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + v_2\right).\end{aligned}$$

Then,

$$\begin{aligned} & \|\Theta(m,q)\| = \|\Theta_1(m,q)\|_\infty + \|\Theta_2(m,q)\|_\infty \\ \le\ & (a_1\|m\|_\infty + a_2\|q\|_\infty)\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) \\ & + (b_1\|m\|_\infty + b_2\|q\|_\infty)\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right) + \frac{\zeta}{b-a},\end{aligned}$$

which shows that the operator Θ is uniformly bounded.

Step 2. We declare that Θ maps the bounded set into the equicontinuous set of Ω. For any $(m,q) \in \Upsilon$, and $t_1 > t_2$, $t_1, t_2 \in [0,T]$, one can derive

$$\begin{aligned} & |\Theta_1(m,q)(t_1) - \Theta_1(m,q)(t_2)| \\ \le\ & \Bigg|\frac{1-\iota}{B(\iota)}h(t_1, m(t_1-\varsigma(t_1)), q(t_1-\sigma(t_1))) - \frac{1-\iota}{B(\iota)}h(t_2, m(t_2-\varsigma(t_2)), q(t_2-\sigma(t_2))) \\ & + \frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^{t_1}(t_1-s)^{\iota-1}h(s, m(s-\varsigma(s)), q(s-\sigma(s)))ds \\ & - \frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^{t_2}(t_2-s)^{\iota-1}h(s, m(s-\varsigma(s)), q(s-\sigma(s)))ds\Bigg| \\ \le\ & \Bigg|\frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^{t_2}[(t_1-s)^{\iota-1} - (t_2-s)^{\iota-1}]h(s, m(s-\varsigma(s)), q(s-\sigma(s)))ds \\ & + \frac{\iota}{B(\iota)\Gamma(\iota)}\int_{t_2}^{t_1}(t_1-s)^{\iota-1}h(s, m(s-\varsigma(s)), q(s-\sigma(s)))ds\Bigg| \\ \le\ & (a_1\|m\|_\infty + a_2\|q\|_\infty)\left(\frac{2(t_1-t_2)^\iota + t_1^\iota - t_2^\iota}{B(\iota)\Gamma(\iota+1)}\right) \to 0, \ as \ \ t_2 \to t_1.\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} & |\Theta_2(m,q)(t_1) - \Theta_2(m,q)(t_2)| \\ \le\ & (b_1\|m\|_\infty + b_2\|q\|_\infty)\left(\frac{2(t_1-t_2)^\mu + t_1^\mu - t_2^\mu}{B(\mu)\Gamma(\mu+1)}\right) \to 0, \ as \ \ t_2 \to t_1.\end{aligned}$$

Therefore, by the Arzelá-Ascoli theorem, the operator $\Theta : \Omega \to \Omega$ is completely continuous.

Step 3. We claim that the set $K = \{(m,q) \in \Omega | (m,q) = \xi\Theta(m,q), 0 < \xi < 1\}$ is bounded. Let $(m,q) \in K$, then $(m,q) = \xi\Theta(m,q)$, $0 < \xi < 1$. For any $t \in J$, we get

$$m(t) = \xi\Theta_1(m,q)(t), \quad q(t) = \xi\Theta_2(m,q)(t).$$

In virtue of the assumption (A_1), we deduce

$$
\begin{aligned}
\|m\|_\infty &= \xi\|\Theta_1(m,q)\|_\infty \\
&\leq (\|\alpha_1\|_\infty + \|\alpha_2\|_\infty\|m\|_\infty + \|\alpha_3\|_\infty\|q\|_\infty)\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + v_1\right) \\
&\quad + (\|\beta_1\|_\infty + \|\beta_2\|_\infty\|m\|_\infty + \|\beta_3\|_\infty\|q\|_\infty)v_2 + \frac{\zeta}{2(b-a)}, \\
\|q\|_\infty &= \xi\|\Theta_2(m,q)\|_\infty \\
&\leq (\|\alpha_1\|_\infty + \|\alpha_2\|_\infty\|m\|_\infty + \|\alpha_3\|_\infty\|q\|_\infty)v_1 + (\|\beta_1\|_\infty \\
&\quad + \|\beta_2\|_\infty\|m\|_\infty + \|\beta_3\|_\infty\|q\|_\infty)\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + v_2\right) + \frac{\zeta}{2(b-a)}.
\end{aligned}
$$

As a result, we obtain that

$$
\begin{aligned}
&\|m\|_\infty + \|q\|_\infty \\
\leq\ & \|\alpha_1\|_\infty\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + \|\beta_1\|_\infty\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right) \\
& + \frac{\zeta}{b-a} + \left[\|\alpha_2\|_\infty\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + \|\beta_2\|_\infty\left(\frac{1-\mu}{B(\mu)} + 2v_2\right.\right. \\
& \left.\left. + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)}\right)\right]\|m\|_\infty + \left[\|\alpha_3\|_\infty\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right)\right. \\
& \left. + \|\beta_3\|_\infty\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right)\right]\|q\|_\infty.
\end{aligned}
$$

Hence, by applying the condition (14), we can get

$$
\|(m,q)\| \leq \frac{\|\alpha_1\|_\infty\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + \|\beta_1\|_\infty\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right) + \frac{\zeta}{b-a}}{\delta},
$$

where

$$
\begin{aligned}
\delta &= \min\left\{1 - \left[\|\alpha_2\|_\infty\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + \|\beta_2\|_\infty\left(\frac{1-\mu}{B(\mu)}\right.\right.\right. \\
&\quad \left.\left. + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right)\right], 1 - \left[\|\alpha_3\|_\infty\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right)\right. \\
&\quad \left.\left. + \|\beta_3\|_\infty\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right)\right]\right\}.
\end{aligned}
$$

Then, the set K is bounded. Hence, in view of Lemma 1, the operator Θ has at least one fixed point; that is, the system (1) has at least one solution. □

Theorem 2. *Assume that the hypothesis (A_2) holds. Then, the system (1) has a unique solution if the following condition holds:*

$$
\gamma\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + \eta\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right) < 1, \tag{21}
$$

where $\gamma = \max\{\gamma_1, \gamma_2\}, \eta = \max\{\eta_1, \eta_2\}$, v_1 *and* v_2 *are defined by (12) and (13).*

Proof. Consider the operator $\Theta : \Omega \to \Omega$ defined by (1) and let

$$
\rho = \frac{M_1\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + M_2\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right) + \frac{\zeta}{b-a}}{1 - \iota\left(\frac{1-\iota}{B(\iota)} + \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} + 2v_1\right) + \eta\left(\frac{1-\mu}{B(\mu)} + \frac{T^\mu}{B(\mu)\Gamma(\mu+1)} + 2v_2\right)},
$$

where $M_1 = \sup_{t\in[0,T]} |h(t,0,0)|$, and $M_2 = \sup_{t\in[0,T]} |k(t,0,0)|$. Then, we declare that $\Theta B_\rho \subset B_\rho$, where $B_\rho = \{(m,q) \in \Omega : \|(m,q)\| \le \rho\}$. For any $(m,q) \in B_\rho$, we get

$$\begin{aligned}
&\|\Theta_1(m,q)\|_\infty \\
\le\ & \frac{1-\iota}{B(\iota)}[|h(t,m(t-\varsigma(t)),q(t-\varsigma(t)))-h(t,0,0)|+M_1] \\
&+\frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^t (t-s)^{\iota-1}[|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))-h(s,0,0)|+M_1]ds \\
&+\frac{1}{2}\left\{\frac{\zeta}{b-a}-\frac{1}{2}\left(\frac{1-\iota}{B(\iota)}[|h(T,m(t-\varsigma(t)),q(t-\varsigma(t)))-h(T,0,0)|+M_1]\right.\right. \\
&+\frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^T (T-s)^{\iota-1}[|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))-h(s,0,0)|+M_1]ds \\
&+\frac{1-\mu}{B(\mu)}[|k(T,m(t-\varsigma(t)),q(t-\varsigma(t)))-k(T,0,0)|+M_2] \\
&\left.+\frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^T (T-s)^{\mu-1}[|k(s,m(s-\varsigma(s)),q(s-\sigma(s)))-k(s,0,0)|+M_2]ds\right) \\
&-\frac{1}{b-a}\int_a^b\left(\frac{1-\iota}{B(\iota)}[|h(s,m(s-\varsigma(s)),q(s-\sigma(s)))-h(s,0,0)|+M_1]ds\right. \\
&+\frac{\iota}{B(\iota)\Gamma(\iota)}\int_0^s (s-\vartheta)^{\iota-1}[|h(\vartheta,m(\vartheta-\varsigma(\vartheta)),q(\vartheta-\varsigma(\vartheta)))-h(\mu,0,0)|+M_1]d\vartheta \\
&-\frac{1-\mu}{B(\mu)}[|k(s,m(s-\varsigma(s)),q(s-\sigma(s)))-k(s,0,0)|+M_2]-\frac{\mu}{B(\mu)\Gamma(\mu)} \\
&\left.\left.\times\int_0^s (s-\vartheta)^{\mu-1}[|k(\vartheta,m(\vartheta-\varsigma(\vartheta)),q(\vartheta-\varsigma(\vartheta)))-k(\mu,0,0)|+M_2]d\vartheta\right)ds\right\} \\
\le\ & \left(\gamma\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+v_1\right)+\eta v_2\right)(\|m\|_\infty+\|q\|_\infty) \\
&+M_1\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+v_1\right)+M_2v_2+\frac{\zeta}{2(b-a)}.
\end{aligned}$$

Similarly, for any $(m,q) \in B_\rho$, we can also deduce

$$\begin{aligned}
\|\Theta_2(m,q)\|_\infty \le\ & \left[\eta\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+v_2\right)+\gamma v_1\right](\|m\|_\infty+\|q\|_\infty) \\
&+M_2\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+v_2\right)+M_1v_1+\frac{\zeta}{2(b-a)}.
\end{aligned}$$

Hence, for $(m,q) \in B_\rho$, we can obtain

$$\begin{aligned}
&\|\Theta(m,q)\| = \|\Theta_1(m,q)\|_\infty + \|\Theta_2(m,q)\|_\infty \\
\le\ & \left[\gamma\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+2v_1\right)+\eta\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+2v_2\right)\right] \\
&\times(\|m\|_\infty+\|q\|_\infty)M_1\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+2v_1\right) \\
&+M_2\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+2v_2\right)+\frac{\zeta}{b-a}<\epsilon,
\end{aligned}$$

which proves that Θ maps B_ρ into itself.

Now, we claim that the operator Θ is a contraction mapping. Let $(m_1, q_1), (m_2, q_2) \in \Omega$, $t \in [0, T]$. We can derive

$$\begin{aligned}
& \|\Theta_1(m_1, q_1) - \Theta_1(m_2, q_2)\|_\infty \\
\leq\ & \frac{1-\iota}{B(\iota)} |h(t, m_1(t - \varsigma(t)), q_1(t - \sigma(t))) - h(t, m_2(t - \varsigma(t)), q_2(t - \sigma(t)))| \\
& + \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} |h(s, m_1(s - \varsigma(s)), q_1(s - \sigma(s))) \\
& - h(s, m_2(s - \varsigma(s)), q_2(s - \sigma(s)))| ds + C_5 + C_6 + C_7 + C_8,
\end{aligned} \tag{22}$$

in which

$$\begin{aligned}
C_5 =\ & \frac{1-\iota}{4B(\iota)} |h(T, m_1(T - \varsigma(T)), q_1(T - \sigma(T))) - h(T, m_2(T - \varsigma(T)), q_2(T - \sigma(T)))| \\
& + \frac{\iota}{4B(\iota)\Gamma(\iota)} \int_0^T (T-s)^{\iota-1} |h(s, m_1(s - \varsigma(s)), q_1(s - \sigma(s))) \\
& - h(s, m_2(s - \varsigma(s)), q_2(s - \sigma(s)))| ds, \\
C_6 =\ & \frac{1}{2(b-a)} \frac{\iota}{B(\iota)\Gamma(\iota)} \int_a^b \int_0^s (s-\vartheta)^{\iota-1} |h(\vartheta, m_1(\vartheta - \varsigma(\vartheta)), q_1(\vartheta - \sigma(\vartheta))) \\
& - h(\vartheta, m_2(\vartheta - \varsigma(\vartheta)), q_2(\vartheta - \sigma(\vartheta)))| d\vartheta ds + \frac{1}{2(b-a)} \frac{1-\iota}{B(\iota)} \\
& \times \int_a^b |h(s, m_1(s - \varsigma(s)), q_1(s - \sigma(s))) - h(s, m_1(s - \varsigma(s)), q_1(s - \sigma(s)))| ds, \\
C_7 =\ & \frac{1-\mu}{4B(\mu)} |k(T, m_1(T - \varsigma(T)), q_1(T - \sigma(T))) - k(T, m_2(T - \varsigma(T)), q_2(T - \sigma(T)))| \\
& + \frac{\mu}{4B(\mu)\Gamma(\mu)} \int_0^T (T-s)^{\mu-1} |k(s, m_1(s - \varsigma(s)), q_1(s - \sigma(s))) \\
& - k(s, m_2(s - \varsigma(s)), q_2(s - \sigma(s)))| ds, \\
C_8 =\ & \frac{1}{2(b-a)} \frac{\mu}{B(\mu)\Gamma(\mu)} \int_a^b \int_0^s (s-\vartheta)^{\mu-1} |k(\vartheta, m_1(\vartheta - \varsigma(\vartheta)), q_1(\vartheta - \sigma(\vartheta))) \\
& - k(\vartheta, m_2(\vartheta - \varsigma(\vartheta)), q_2(\vartheta - \sigma(\vartheta)))| d\vartheta ds + \frac{1}{2(b-a)} \frac{1-\mu}{B(\mu)} \\
& \times \int_a^b |k(s, m_1(s - \varsigma(s)), q_1(s - \sigma(s))) - k(s, m_2(s - \varsigma(s)), q_2(s - \sigma(s)))| ds.
\end{aligned}$$

Firstly, by applying (A_2), we can get

$$\begin{aligned}
& \frac{1-\iota}{B(\iota)} |h(t, m_1(t - \varsigma(t)), q_1(t - \sigma(t))) - h(t, m_2(t - \varsigma(t)), q_2(t - \sigma(t)))| \\
\leq\ & \frac{1-\iota}{B(\iota)} (\gamma_1 \|m_1 - m_2\|_\infty + \gamma_2 \|q_1 - q_2\|_\infty),
\end{aligned} \tag{23}$$

$$\begin{aligned}
& \frac{\iota}{B(\iota)\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} |h(s, m_1(s - \varsigma(s)), q_1(s - \sigma(s))) \\
& - h(s, m_2(s - \varsigma(s)), q_2(s - \sigma(s)))| ds \\
\leq\ & \frac{T^\iota}{B(\iota)\Gamma(\iota+1)} (\gamma_1 \|m_1 - m_2\|_\infty + \gamma_2 \|q_1 - q_2\|_\infty).
\end{aligned} \tag{24}$$

Next, similar to (23), it follows from the assumption (A_2) that

$$\begin{aligned} &\frac{1-\iota}{4B(\iota)}|h(T,m_1(T-\varsigma(T)),q_1(T-\sigma(T)))-h(T,m_2(T-\varsigma(T)),q_2(T-\sigma(T)))| \\ &+\frac{\iota}{4B(\iota)\Gamma(\iota)}\int_0^T(T-s)^{\iota-1}|h(s,m_1(s-\varsigma(s)),q_1(s-\sigma(s))) \\ &-h(s,m_2(s-\varsigma(s)),q_2(s-\sigma(s)))|ds \\ \leq\ &\left(\frac{5(1-\iota)}{4B(\iota)}+\frac{T^\iota}{4B(\iota)\Gamma(\iota)}\right)(\gamma_1\|m_1-m_2\|_\infty+\gamma_2\|q_1-q_2\|_\infty), \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\frac{1}{2(b-a)}\frac{\iota}{B(\iota)\Gamma(\iota)}\int_a^b\int_0^s(s-\vartheta)^{\iota-1}|h(\vartheta,m_1(\vartheta-\varsigma(\vartheta)),q_1(\vartheta-\sigma(\vartheta))) \\ &-h(\vartheta,m_2(\vartheta-\varsigma(\vartheta)),q_2(\vartheta-\sigma(\vartheta)))|d\vartheta ds+\frac{1}{2(b-a)}\frac{1-\iota}{B(\iota)} \\ &\times\int_a^b|h(s,m_1(s-\varsigma(s)),q_1(s-\sigma(s)))-h(s,m_1(s-\varsigma(s)),q_1(s-\sigma(s)))|ds \\ \leq\ &\frac{\iota(b^{\iota+1}-a^{\iota+1})}{2(b-a)B(\iota)\Gamma(\iota+2)}(\gamma_1\|m_1-m_2\|_\infty+\gamma_2\|q_1-q_2\|_\infty). \end{aligned} \tag{26}$$

Furthermore, we can also obtain

$$\begin{aligned} &\frac{1-\mu}{4B(\mu)}|k(T,m_1(T-\varsigma(T)),q_1(T-\sigma(T)))-k(T,m_2(T-\varsigma(T)),q_2(T-\sigma(T)))| \\ &+\frac{\mu}{4B(\mu)\Gamma(\mu)}\int_0^T(T-s)^{\mu-1}|k(s,m_1(s-\varsigma(s)),q_1(s-\sigma(s))) \\ &-k(s,m_2(s-\varsigma(s)),q_2(s-\sigma(s)))|ds \\ \leq\ &\left(\frac{5(1-\mu)}{4B(\mu)}+\frac{T^\mu}{4B(\mu)\Gamma(\mu)}\right)(\eta_1\|m_1-m_2\|_\infty+\eta_2\|q_1-q_2\|_\infty), \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\frac{1}{2(b-a)}\frac{\mu}{B(\mu)\Gamma(\mu)}\int_a^b\int_0^s(s-\vartheta)^{\mu-1}|k(\vartheta,m_1(\vartheta-\varsigma(\vartheta)),q_1(\vartheta-\sigma(\vartheta))) \\ &-k(\vartheta,m_2(\vartheta-\varsigma(\vartheta)),q_2(\vartheta-\sigma(\vartheta)))|d\vartheta ds+\frac{1}{2(b-a)}\frac{1-\mu}{B(\mu)} \\ &\times\int_a^b|k(s,m_1(s-\varsigma(s)),q_1(s-\sigma(s)))-k(s,m_2(s-\varsigma(s)),q_2(s-\sigma(s)))|ds \\ \leq\ &\frac{\mu(b^{\mu+1}-a^{\mu+1})}{2(b-a)B(\mu)\Gamma(\mu+2)}(\eta_1\|m_1-m_2\|_\infty+\eta_2\|q_1-q_2\|_\infty). \end{aligned} \tag{28}$$

Finally, combining with (22)–(28), we can conclude

$$\begin{aligned} &\|\Theta_1(m_1,q_1)-\Theta_1(m_2,q_2)\|_\infty \\ \leq\ &\left(\gamma\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+v_1\right)+\eta v_2\right)(\|m_1-m_2\|_\infty+\|q_1-q_2\|_\infty), \end{aligned}$$

where $\gamma=\max\{\gamma_1,\gamma_2\},\eta=\max\{\eta_1,\eta_2\}$, v_1 and v_2 are defined by (12) and (13). Similarly, we can also get

$$\begin{aligned} &\|\Theta_2(m_1,q_1)-\Theta_2(m_2,q_2)\|_\infty \\ \leq\ &\left(\eta\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+v_2\right)+\gamma v_1\right)(\|m_1-m_2\|_\infty+\|q_1-q_2\|_\infty). \end{aligned}$$

Hence, from the above inequalities, it follows that

$$\begin{aligned}
& \|\Theta(m_1,q_1)-\Theta(m_2,q_2)\| \\
= & \|\Theta_1(m_1,q_1)-\Theta_1(m_2,q_2)\|_\infty+\|\Theta_2(m_1,q_1)-\Theta_2(m_2,q_2)\|_\infty \\
\leq & \left[\gamma\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+2v_1\right)+\eta\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+2v_2\right)\right] \\
& \times\|(m_1-m_2),(q_1-q_2)\|.
\end{aligned}$$

In view of (21) and Lemma 3, we can find that Θ has a unique fixed point, which is a unique solution of the coupled system (1). □

4. Numerical Example

Example 1. *Consider the following nonlinear fractional-order coupled system:*

$$\begin{cases}
{}^{ABC}D^{\frac{1}{3}}m(t)=h(t,m(t-\varsigma(t)),q(t-\sigma(t)),\ t\in J:=[0,3], \\
{}^{ABC}D^{\frac{1}{4}}q(t)=k(t,m(t-\varsigma(t)),q(t-\sigma(t))),\ t\in J:=[0,3], \\
(m+q)(0)=-(m+q)(3),\ \int_{\frac{1}{2}}^{\frac{3}{2}}(m+q)(s)ds=2,
\end{cases} \tag{29}$$

where $\varsigma(t)=\sin t,\sigma(t)=\cos t,\ \iota=\frac{1}{3},\mu=\frac{1}{4},a=\frac{1}{2},b=\frac{3}{2},\zeta=2,T=3.$

Using the given data, we find that $v_1=0.97813256, v_2=0.99470521$*, where* v_1 *and* v_2 *are respectively given by (12) and (13). In order to verify Theorem 1, let*

$$\begin{aligned}
h(t,m,q) &= \frac{1}{(3+t)^2}\left(e^{-t}+\frac{q}{10}+\sin m\right), \\
k(t,m,q) &= \frac{e^{-t}}{\sqrt{200+t^2}}\left(\cos t+\frac{\tan^{-1}m}{2}+\sin q\right).
\end{aligned}$$

Obviously, $\alpha_1(t)=\frac{e^{-t}}{(3+t)^2}$, $\alpha_2(t)=\frac{1}{(3+t)^2}$, $\alpha_3(t)=\frac{1}{10(3+t)^2}$ *and* $\beta_1(t)=\frac{e^{-2t}}{\sqrt{200+t^2}}$, $\beta_2(t)=\frac{e^{-t}}{2\sqrt{200+t^2}}$, $\beta_3(t)=\frac{e^{-t}}{\sqrt{200+t^2}}$. *Clearly, h and k are the first-order continuous differentiable functions with respect to t and satisfy the assumption (A1). Finally, by Matlab software, we can obtain*

$$\begin{aligned}
& \|\alpha_2\|_\infty\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+2v_1\right)+\|\beta_2\|_\infty\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+2v_2\right) \\
& \approx 0.76620146<1, \\
& \|\alpha_3\|_\infty\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+2v_1\right)+\|\beta_3\|_\infty\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+2v_2\right) \\
& \approx 0.84265419<1.
\end{aligned}$$

Therefore, all the conditions of Theorem 1 are satisfied; that is, there exists at least one solution for the system (29).

Next, in order to demonstrate the application of Theorem 2, we take

$$h(t,m,q)=\frac{1}{10(1+t)^2}\left(\frac{|m|}{1+|m|}+tan^{-1}q\right),\ k(t,m,q)=\frac{1}{\sqrt{900+t^2}}\left(2\tan^{-1}m+3\sin q\right).$$

Observe that h and k are continuous and satisfy the condition (A_2) *with* $\gamma_1=\gamma_2=\frac{1}{10}=\gamma$ *and* $\eta_1=\frac{1}{15}$, $\gamma_2=\frac{1}{10}$, *Hence* $\eta=\frac{1}{10}$. *Through calculation, we can derive that*

$$\gamma\left(\frac{1-\iota}{B(\iota)}+\frac{T^\iota}{B(\iota)\Gamma(\iota+1)}+2v_1\right)+\eta\left(\frac{1-\mu}{B(\mu)}+\frac{T^\mu}{B(\mu)\Gamma(\mu+1)}+2v_2\right)\approx 0.83238566<1.$$

Thus, all the conditions of Theorem 2 are satisfied, and consequently there exists a unique solution for the system (29).

5. Conclusions

In this work, we investigated the existence and uniqueness of solutions for a nonlinear ABC-fractional order coupled delayed system with a new kind of coupled delayed boundary condition. Based on fixed point theorems, a novel set of sufficient conditions to guarantee the existence and uniqueness of solutions of the nonlinear ABC-fractional order coupled delayed system are derived. Eventually, an example was presented to illustrate the effectiveness of the obtained results. In real life, because the systems may be affected by external random disturbances, many interesting results on the stability of stochastic coupled system with time-varying delays have been published in recent years [33,34].When studying the existence of solutions for fractional-order coupled systems with time-varying delays, it is necessary to add random disturbances terms, which will be a part of our future work. In addition, future work will focus on the case of order of $1 < \lambda < 2$.

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