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New Solitary Wave Patterns of the Fokas System in Fiber Optics

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Abstract: The Fokas system, which models wave dynamics using a single model of fiber optics, is the design under discussion in this study. Different types of solitary wave solutions are obtained by utilizing generalized Kudryashov (GKP) and modified Kudryashov procedures (MKP). These novel concepts make use of symbolic computations to come up with a dynamic and powerful mathematical approach for dealing with a variety of nonlinear wave situations. The results obtained in this paper are original and have the potential to be useful in mathematical physics.

Keywords: exact solutions; modified Kudryashov procedure; generalized Kudryashov method; symbolic computation

MSC: 68W30; 26A33; 00A69

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1. Introduction

The essential issue in mathematical physics is nonlinear partial differential equations (NPDEs), and numerous advanced techniques have been published for investigating the exact solutions of NPDEs. Some examples of such works include the discovery of soliton solutions for the Radhakrishnan–Kundu–Lakshmanan equation using the solitary wave ansatz approach by Zahran et al. [1], the demonstration of the existence of three-wave lump solutions for a (3+1)-dimensional generalized CBS (gCBS) equation by Zhou et al. [2], the study of the fractional-type system of nonlinear Chen–Lee–Liu (CLL) equations using the classical and non-classical Lie group analysis by Hashemi et al. [3], the implementation of the $\exp(-\phi(\xi))$-expansion by Raza et al. to obtain analytical soliton solutions of the Biswas–Milovic equation in Kerr and non-Kerr law media [4], the discovery of new explicit wave solutions for the fractional Kudryashov–Sinelshchikov (KS) equation using liquid-gas bubbles under thermodynamic assumptions via sinh-Gordon equation expansion by Abdel-Aty [5], the implementation of the modified Kudryashov technique to the unstable nonlinear Schrödinger equation by Hosseini et al. [6], and the application of the MKP to the generalized Schrödinger–Boussinesq equation by Kumar and Kaplan [7].

The Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equations are integrable expansions of the higher-dimensional Fokas equation [8]. Because it represents the physical processes of waves on the surface and beneath the water, the Fokas model is crucial in wave theory. The Fokas system simulates the nonlinear pulse transmission along monomode optical fibers [9].

$$\begin{aligned} i\Omega_t + \varepsilon_1 \Omega_{xx} + \varepsilon_2 \Omega \iota &= 0 \\ \varepsilon_3 \iota_y - \varepsilon_4 \left(|\Omega|^2 \right)_x &= 0 \end{aligned} \tag{1}$$

In Equation (1), $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 are nonzero constants, while Ω and ι are complex functions. For analyzing the NLSE in (2 + 1) dimensions, Fokas [10] and Shulman [11] devised the Fokas system. Moreover, the system Equation (1) was transformed into an NLS equation that allowed y to approach x. Moreover, Chakravarty et al. [12] looked into this concept. The suggested method was in fact carefully explored in [13,14] for new rogue wave and lump soliton solutions in focused zones. The governing model and a breather with hybrid solutions were both investigated in [15,16]. Sinh-cosh and sine-cosine approaches were both used to resolve the model in [17]. Alotaibi et al. established periodic, kink, dark, and bright optical solitons, chaotic patterns, and bifurcations of the Fokas system [18]. Atas et al. used the novel modified generalized exp rational procedure for the Fokas system [19]. Alrebdi et al. considered the sine-Gordon expansion technique for the Fokas system [20].

In this article, we will explore new solitary wave solutions of the Fokas system. Solitary waves are localized traveling waves that approach zero at long distances. In wave theory, kink waves and periodic waves are also considered as important types of waves that drop or ascend from one asymptotic state to another. All of these wave types are solitons in the completely integrable case, which result from an eigenvalue problem's inverse scattering (IS) solution and depend on a free parameter. However, the existence of these solutions does not depend on the integrability of the model or its compatibility with an IS transform approach [21].

The current study looks into the applicability and effectiveness of the Kudryashov approaches on the Fokas system, which models wave dynamics using single-mode fiber optics. Section 2 contains the preliminary information and the main steps of the procedures being considered. Section 3 marks out the implementation of the generalized Kudryashov and modified Kudryashov methods. Section 4 gives some results and discussions, and lastly, Section 5 offers some conclusions.

2. Results

The processes will be used on the provided model in this section. The supplied model will be reduced to the NPDE. If we apply the wave transformation described in Equation (17) to Equation (1),

$$\begin{aligned} \Omega(x,y,t) =& \quad v(\epsilon)\exp(\eta_1 x + \eta_2 y + \eta_3 t), \\ \iota(x,y,t) =& \quad \Lambda(\epsilon), \end{aligned} \tag{2}$$

where $\epsilon = x + y - \theta t$, then separate real and imaginary parts, we obtain the system as follows:

$$\left[\varepsilon_1 v'' - iv'(\theta - 2\eta_1\varepsilon_1) - \left(\eta_1^2\varepsilon_1 - \varepsilon_2\Lambda + \eta_3\right)v\right]e^{i(\eta_1 x+\eta_2 y+\eta_3 t)} = 0, \tag{3}$$

$$\varepsilon_3\Lambda^{'} - 2\varepsilon_4 v v^{'} = 0. \tag{4}$$

According to (4), we obtain the following relation:

$$\Lambda = \frac{\varepsilon_4 v^2}{\varepsilon_3}. \tag{5}$$

If we substitute (5) in (3), we obtain the following reduced ODE:

$$\varepsilon_1 v'' - \left(\eta_1^2\varepsilon_1 + \eta_3\right)v + \frac{\varepsilon_2\varepsilon_4}{\varepsilon_3}v^3 = 0, \tag{6}$$

where the imaginary part provides soliton velocity;

$$\theta = 2\eta_1\varepsilon_1. \tag{7}$$

If we balance v'' and v^3 in Equation (6), the balance number is obtained 1.

2.1. Application of GKP

The generalized Kudryashov approach will be used to solve Equation (6) in this paragraph. We assume the following in light of the method:

$$v(\epsilon) = \frac{a_0 + a_1\psi + a_2\psi^2}{b_0 + b_1\psi}. \tag{8}$$

If we substitute Equation (6) for Equation (8) without neglecting Equation (20), we have an overdetermining equation system. Four solution families are found if the resulting system is solved by using symbolic computation packages.

First Family: The following steps are used to determine the arbitrary constants:

$$a_0 = 0, a_1 = \pm b_1\sqrt{-\frac{\varepsilon_1\varepsilon_3}{2\varepsilon_2\varepsilon_4}}, a_2 = \pm\frac{b_1\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4\sqrt{-\frac{\varepsilon_1\varepsilon_3}{2\varepsilon_2\varepsilon_4}}}, b_0 = 0, b_1 = b_1,$$

$$\eta_3 = -\frac{\varepsilon_1(2\eta_1^2 + 1)}{2}.$$

Hence, the resulting solutions for the given model are as follows:

$$\begin{aligned} \Omega(x,y,t) = & \left(\mp\frac{\varepsilon_1\varepsilon_3\sqrt{2}(\Gamma e^{\epsilon}-1)}{2\varepsilon_2\varepsilon_4\sqrt{-\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}(1+\Gamma e^{\epsilon})}\right)e^{i\left(\eta_1 x+\eta_2 y-\frac{\varepsilon_1\left(2\eta_1^2+1\right)}{2}t\right)}, \\ \iota(x,y,t) = & -\frac{\varepsilon_1(\Gamma e^{\epsilon}-1)^2}{2\varepsilon_2(1+\Gamma e^{\epsilon})^2}, \end{aligned} \tag{9}$$

where $\epsilon = x + y - 2\eta_1\varepsilon_1 t$.

Second Family: These steps are used to determine the arbitrary constants:

$$a_0 = 0, a_1 = \pm 2b_0\sqrt{-\frac{2\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}, a_2 = \mp 2b_0\sqrt{-\frac{2\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}, b_0 = b_0, b_1 = -2b_0,$$

$$\eta_3 = -\eta_1^2\varepsilon_1 + \varepsilon_1,$$

and the solutions are given by

$$\begin{aligned} \Omega(x,y,t) = & \left(\pm\frac{2\sqrt{2}\sqrt{-\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}\Gamma e^{\epsilon}}{2\varepsilon_2\varepsilon_4\sqrt{-\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}(1+\Gamma e^{\epsilon})}\right)e^{i\left(\eta_1 x+\eta_2 y+\left(-\eta_1^2\varepsilon_1+\varepsilon_1\right)t\right)}, \\ \iota(x,y,t) = & -\frac{8\varepsilon_1\Gamma^2 e^{2\epsilon}}{\varepsilon_2(\Gamma^2 e^{2\epsilon}-1)^2}, \end{aligned} \tag{10}$$

where $\epsilon = x + y - 2\eta_1\varepsilon_1 t$.

Third Family: These steps are used to determine the arbitrary constants:

$$a_0 = \mp\frac{b_1\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4\sqrt{-\frac{2\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}}, a_1 = \pm\frac{2b_1\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4\sqrt{-\frac{2\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}}, a_2 = \pm b_1\sqrt{-\frac{2\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}, b_0 = -\frac{b_1}{2}, b_1 = b_1,$$

$$\eta_3 = -\eta_1^2\varepsilon_1 - 2\varepsilon_1,$$

and the solutions are given as follows:

$$\begin{aligned} \Omega(x,y,t) = & \left(\pm\frac{\varepsilon_1\varepsilon_3\sqrt{2}\left(\Gamma^2 e^{2\epsilon}+1\right)}{\varepsilon_2\varepsilon_4\sqrt{-\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}(\Gamma^2 e^{2\epsilon}-1)}\right)e^{i\left(\eta_1 x+\eta_2 y+\left(-\eta_1^2\varepsilon_1-2\varepsilon_1\right)t\right)}, \\ \iota(x,y,t) = & -\frac{2\varepsilon_1\left(\Gamma^2 e^{2\epsilon}+1\right)^2}{\varepsilon_2(\Gamma^2 e^{2\epsilon}-1)^2}. \end{aligned} \tag{11}$$

Fourth Family: These steps are used to determine the values of the arbitrary constants:

$$a_0 = \mp \frac{b_1 \varepsilon_1 \varepsilon_3}{2\varepsilon_2 \varepsilon_4 \sqrt{-\frac{2\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}}, a_1 = \pm \frac{2b_1 \varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4 \sqrt{-\frac{2\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}}, a_2 = \pm b_1 \sqrt{-\frac{2\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}, b_0 = -\frac{b_1}{2}, b_1 = b_1,$$

$$\eta_3 = -\eta_1^2 \varepsilon_1 - \frac{1}{2}\varepsilon_1,$$

and the following are the solutions:

$$\begin{aligned} \Omega(x,y,t) = & \left(\pm \frac{\varepsilon_1 \varepsilon_3 \sqrt{2}(\Gamma e^{\epsilon}-1)}{2\varepsilon_2 \varepsilon_4 \sqrt{-\frac{\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}(\Gamma e^{\epsilon}+1)}\right) e^{i\left(\eta_1 x + \eta_2 y - \left(\eta_1^2 \varepsilon_1 + \frac{1}{2}\varepsilon_1\right)t\right)}, \\ \iota(x,y,t) = & -\frac{\varepsilon_1 (\Gamma e^{\epsilon}-1)^2}{2\varepsilon_2 (\Gamma e^{\epsilon}+1)^2}. \end{aligned} \tag{12}$$

Fifth Family: These steps are used to determine the values of the arbitrary constants:

$$a_0 = \pm \frac{b_0 \varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4 \sqrt{-\frac{2\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}}, a_1 = \mp \frac{\varepsilon_1 \varepsilon_3 (2b_0 - b_1)}{\varepsilon_2 \varepsilon_4 \sqrt{-\frac{2\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}}, a_2 = \pm b_1 \sqrt{-\frac{2\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}, b_0 = b_0, b_1 = b_1,$$

$$\eta_3 = -\frac{\varepsilon_1 \left(2\eta_1^2 + 1\right)}{2},$$

and the solutions are given as follows:

$$\begin{aligned} \Omega(x,y,t) = & \left(\pm \frac{\varepsilon_1 \varepsilon_3 \sqrt{2}(\Gamma e^{\epsilon}-1)}{2\varepsilon_2 \varepsilon_4 \sqrt{-\frac{\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}(\Gamma e^{\epsilon}+1)}\right) e^{i\left(\eta_1 x + \eta_2 y - \frac{\varepsilon_1 \left(2\eta_1^2+1\right)}{2} t\right)}, \\ \iota(x,y,t) = & -\frac{\varepsilon_1 (\Gamma e^{\epsilon}-1)^2}{2\varepsilon_2 (\Gamma e^{\epsilon}+1)^2}. \end{aligned} \tag{13}$$

2.2. Application of MKP

Equation (6) will be subjected to the modified Kudryashov technique in this paragraph. We assume the following in light of the method:

$$v(\epsilon) = \varpi_0 + \varpi_1 \psi(\epsilon). \tag{14}$$

Without disregarding Equation (24) in Equation (6), we may substitute the solution Equation (14) and collect the polynomial of $\psi(\epsilon)$, yielding the overdetermining equation system shown below:

$$\psi^3 : 2(\ln(a))^2 \varpi_1 \varepsilon_1 + \frac{\varepsilon_2 \varepsilon_4 \varpi_1^3}{\varepsilon_3},$$

$$\psi^2 : -3(\ln(a))^2 \varpi_1 \varepsilon_1 + \frac{3\varepsilon_2 \varepsilon_4 \varpi_0 \varpi_1^2}{\varepsilon_3},$$

$$\psi^1 : (\ln(a))^2 \varpi_1 \varepsilon_1 - \varpi_1 \eta_1^2 \varepsilon_1 - \varpi_1 \eta_3 + \frac{3\varepsilon_2 \varepsilon_4 \varpi_0^2 \varpi_1}{\varepsilon_3},$$

$$\psi^0 : -\varpi_0 \eta_1^2 \varepsilon_1 - \varpi_0 \eta_3 + \frac{\varepsilon_2 \varepsilon_4 \varpi_0^3}{\varepsilon_3}.$$

If the aforementioned system is solved, the constants can be specified as follows:

$$\varpi_0 = \pm \frac{\ln(a) \varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4 \sqrt{-\frac{2\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}}, \varpi_1 = \pm \ln(a) \sqrt{-\frac{2\varepsilon_1 \varepsilon_3}{\varepsilon_2 \varepsilon_4}}, \eta_3 = -\frac{\varepsilon_1 \left(2\eta_1^2 + (\ln(a))^2\right)}{2}.$$

The solitary wave solutions are provided as follows:

$$\begin{aligned}\Omega(x,y,t) = & \pm\left(\frac{\ln(a)\varepsilon_1\varepsilon_3(\Gamma a^{\epsilon}-1)\sqrt{2}}{2\varepsilon_2\varepsilon_4(1+\Gamma a^{\epsilon})\sqrt{-\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2\varepsilon_4}}}\right)e^{i\left(\eta_1 x+\eta_2 y-\frac{\varepsilon_1\left(2\eta_1^2+(\ln(a))^2\right)}{2}t\right)},\\ \iota(x,y,t) = & -\frac{(\ln(a))^2\varepsilon_1(\Gamma a^{\epsilon}-1)^2}{2\varepsilon_2(1+\Gamma a^{\epsilon})^2},\end{aligned} \tag{15}$$

where $\epsilon = x+y-2\eta_1\varepsilon_1 t$.

Remark 1. *If we set a as exp function in solution* (15)*, we obtain the same solution with* (9)*.*

3. Discussion

In this article, five different solution families have been founded via the generalized Kudryashov technique, and one type of solitary wave solution has been founded via the modified Kudryashov technique. The obtained solutions are different from the outcomes attained by other researchers using earlier techniques [17–20]. Equations (8)–(13) and (15) present a variety of different types of solutions by providing various parameter values. The solutions include seven arbitrary parameters, and different solutions can be constructed by setting the parameters as different values. The obtained solutions have been classified. Additionally, the drawings of two- and three-dimensional graphics have been made. The following information can be given for these graphics: Figures 1–4 depict solitary waves in different structures, namely Figures 1 and 2 represent bright soliton solutions and Figures 3 and 4 represent dark soliton solutions. Figures 1 and 2 represent the plot of the $|(11)|$ for $\varepsilon_1 = -0.9, \varepsilon_2 = 0.5, \varepsilon_3 = -0.2, \varepsilon_4 = 0.1, \eta_1 = 0.5, \eta_2 = 0.5, \Gamma = 2, y = 0$. Figures 3 and 4 represent the plot of the $|(15)|$ for $\varepsilon_1 = -0.12, \varepsilon_2 = 0.55, \varepsilon_3 = -1.2, \varepsilon_4 = 0.5, \eta_1 = 0.9, \eta_2 = 0.01, \Gamma = 2, a = 2, y = 0$. In Figures 1 and 2, the 3D and contour plots for Equation (11) are shown. Figures 1 and 2 illustrate the bright soliton solution and its 2D plot. In Figures 3 and 4, the 3D and contour plots are shown which represent the dark soliton wave for Equation (15). The proposed approaches are practical and effective. We employed the Maple software program to carry out the simulations and examine the results.

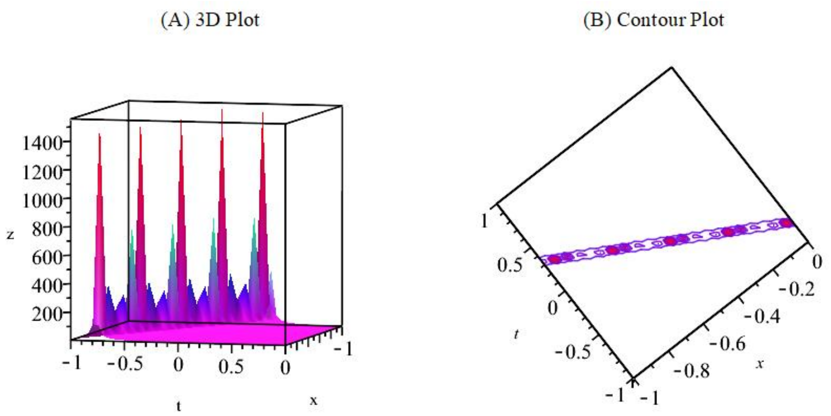

Figure 1. *Cont.*

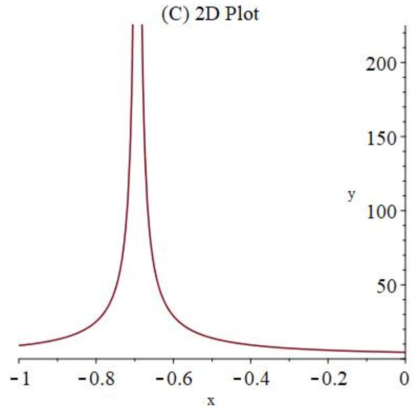

Figure 1. Two-dimensional and three-dimensional plots of the bright soliton solution |Equation (11)| for $\varepsilon_1 = -0.9, \varepsilon_2 = 0.5, \varepsilon_3 = -0.2, \varepsilon_4 = 0.1, \eta_1 = 0.5, \eta_2 = 0.5, \Gamma = 2, y = 0$.

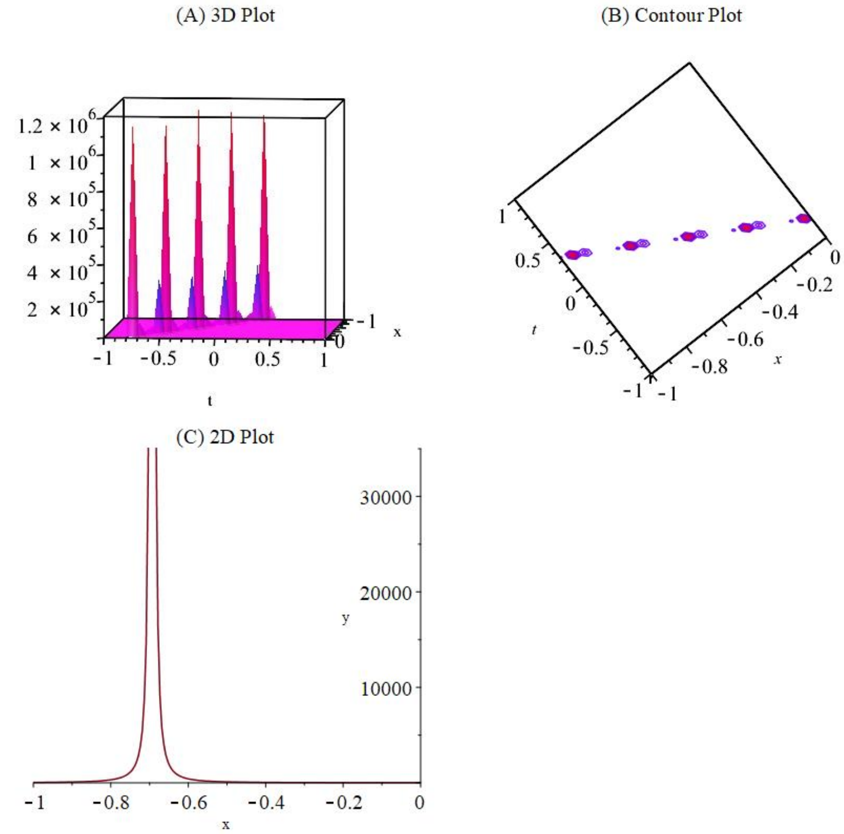

Figure 2. Two-dimensional and three-dimensional plots of the bright soliton solution |Equation (11)| for $\varepsilon_1 = -0.9, \varepsilon_2 = 0.5, \varepsilon_3 = -0.2, \varepsilon_4 = 0.1, \eta_1 = 0.5, \eta_2 = 0.5, \Gamma = 2, y = 0$.

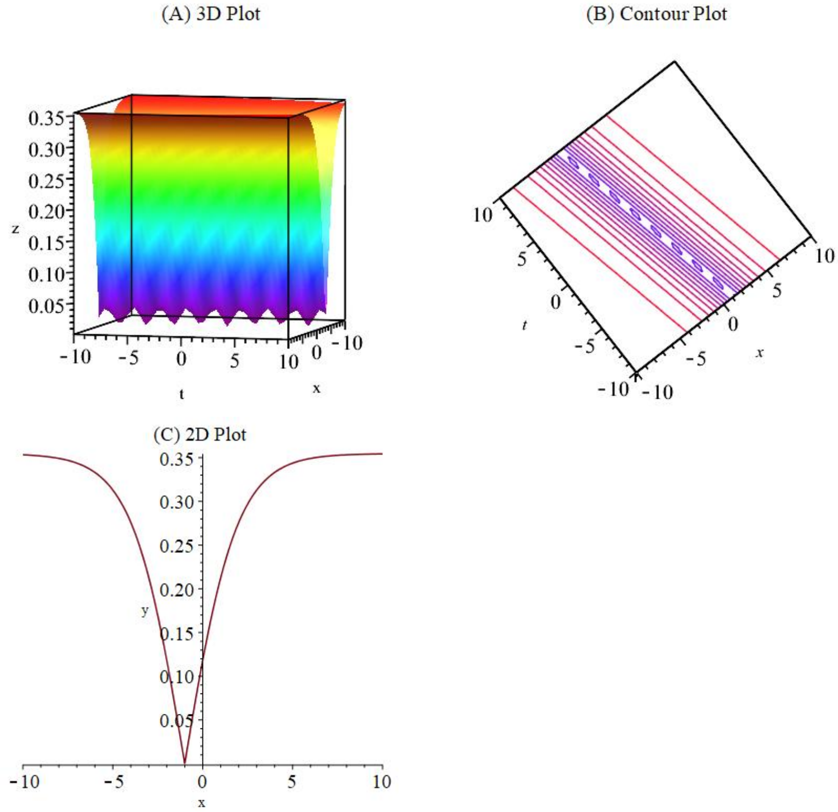

Figure 3. Two-dimensional and three-dimensional plots of the dark soliton solution |Equation (15)| for $\varepsilon_1 = -0.12, \varepsilon_2 = 0.55, \varepsilon_3 = -1.2, \varepsilon_4 = 0.5, \eta_1 = 0.9, \eta_2 = 0.01, \Gamma = 2, a = 2, y = 0$.

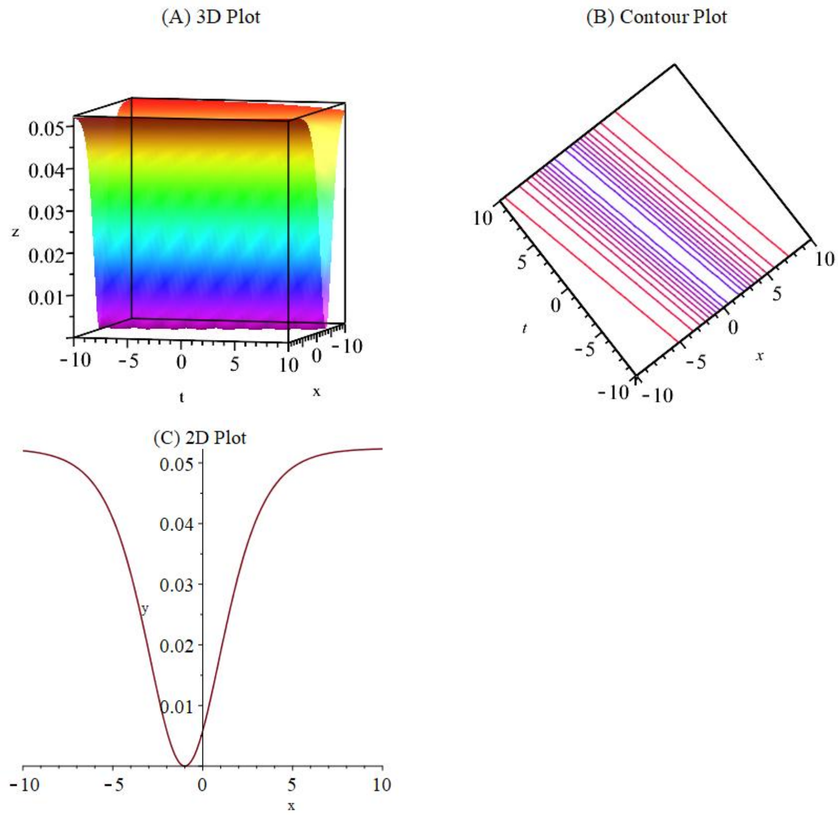

Figure 4. Two-dimensional and three-dimensional plots of the dark soliton solution |Equation (15)| for $\varepsilon_1 = -0.12, \varepsilon_2 = 0.55, \varepsilon_3 = -1.2, \varepsilon_4 = 0.5, \eta_1 = 0.9, \eta_2 = 0.01, \Gamma = 2, a = 2, y = 0$.

4. Conclusions

This work examines the exact solutions of the Fokas system, which is used to simulate the dynamics of the waves in a monomode fiber communication system. The modified Kudryashov process and the generalized Kudryashov procedure were both intensively utilized in this study to obtain intriguing Fokas system solutions. Both methods can be considered as advanced methods for identifying innovative exact solutions to NPDEs. With the help of the corresponding 2D line plots and 3D plots, the behaviors of the retrieved solutions are shown. The results show that the suggested procedures are effective instruments for investigating nonlinear evolution equations and give useful model-related data. The results of the dynamics analysis approach and their representation are significant and useful for future research on the Fokas system's reliability control. In the future, looking for additional exact solutions to Fokas' system will be seen as a crucial task. The reliability control of the Fokas system may be further studied with the help of the dynamics analysis approach.

5. Materials and Methods

5.1. Preliminary Information

The procedures that are employed will be listed in this section. We assume a system of general NPDEs with the following structure:

$$\Phi^{\alpha}\left(\Omega, \frac{\partial\Omega}{\partial t}, \frac{\partial\Omega}{\partial x}, \frac{\partial\Omega}{\partial y}, \frac{\partial^2\Omega}{\partial t^2}, \frac{\partial^2\Omega}{\partial x^2}, \frac{\partial^2\Omega}{\partial y^2}, \dots\right) = 0. \tag{16}$$

where $\Omega = \Omega(x, y, t)$ is a complex-valued function. If we transform Equation (16) using the wave transformation described below,

$$\Omega(x, y, t) = v(\epsilon)e^{i\varphi}, \tag{17}$$

where $\epsilon = x + y - \theta t$ and $\varphi = \eta_1 x + \eta_2 y + \eta_3 t$, it results in the following ordinary differential equation (ODE):

$$\phi(v, v', v'', \dots) = 0, \tag{18}$$

and here prime stands for the derivative of v with respect to ϵ.

5.2. The Generalized Kudryashov Procedure

We make the following assumption about $v(\epsilon)$ according to the GKP:

$$v(\epsilon) = \frac{\sum\limits_{m=0}^{M} a_m \psi^m(\epsilon)}{\sum\limits_{n=0}^{N} b_n \psi^n(\epsilon)}, (a_M \neq 0, b_N \neq 0) \tag{19}$$

where $a_m, b_n (m = 0, 1, \dots M, n = 0, 1, \dots N)$ are constants and the following ODE is satisfied by $\psi(\epsilon)$:

$$\frac{d\psi}{d\epsilon} = \psi^2(\epsilon) - \psi(\epsilon), \tag{20}$$

and $\psi(\epsilon)$ is given as follows:

$$\psi(\epsilon) = \frac{1}{1 + \Gamma e^{\epsilon}}, \tag{21}$$

where Γ is an integration constant, N and M can be computed by operating the homogeneous balance principle for Equation (18). We can calculate a polynomial of ψ by substituting Equation (19) into Equation (18) without ignoring Equation (20). Then, all the coefficients of polynomial ψ will be set as zero. If the obtained system is solved, the values of the $a_m, b_n, \eta_i (i = 1, 2, 3), \theta$ are obtained. At last, the given model's soliton-type solutions are discovered [22,23].

5.3. The Modified Kudryashov Procedure

We transform Equation (18) using the wave transformation described below:

$$u(\epsilon) = \sum_{n=0}^{N} \varpi_n(\psi(\zeta))^n, \varpi_N \neq 0, \tag{22}$$

where $\varpi_n(n = 0, 1, \ldots, N)$ are constants that will be determined later, N is calculated by the homogeneous balance principle, and the function $\psi(\epsilon)$ is given by

$$\psi(\epsilon) = \frac{1}{1 + \Gamma a^{\epsilon}}, \tag{23}$$

where (23) satisfies the following ODE:

$$\psi'(\epsilon) = \left(\psi^2(\epsilon) - \psi(\epsilon)\right) \ln a. \tag{24}$$

Without ignoring Equation (24), Equation (22) is substituted into Equation (24) to produce a set of algebraic equations for $\varpi_n, a, \eta_i(i = 1, 2, 3), \theta$. Eventually, after solving the resulting system, the exact solutions to equation Equation (16) are determined [24,25].

Remark 2. *The Comparison of the Considered Procedures.*

The Kudryashov procedure, modified Kudryashov procedure, generalized Kudryashov procedure, and extended Kudryashov procedure are just a few examples of a variety of Kudryashov procedures that are described in the literature. These procedures vary in terms of their auxiliary equations and the desired shape of the solution. Below is a comparison of two of these approaches.

(i) The solution to the auxiliary equation given by the traditional Kudryashov approach is

$$R(\epsilon) = \frac{1}{1 + e^{\epsilon}}.$$

(ii) The solution to the auxiliary equation for the modified Kudryashov approach is

$$R(\epsilon) = \frac{1}{1 \pm a^{\epsilon}},$$

and

$$\frac{dR}{d\epsilon} = (R^2(\epsilon) - R(\epsilon)) \ln a.$$

(iii) The solution to the auxiliary equation for the extended Kudryashov approach is

$$\frac{dR}{d\epsilon} = R^3(\epsilon) - R(\epsilon).$$

We consider the solutions for these techniques to be

$$u(\epsilon) = \sum_{n=0}^{M} a_n R^n(\epsilon), a_n(n = 0, 1, 2, \ldots, M), (a_M \neq 0),$$

$a_1, a_2 \ldots a_n$ are constants to be calculated.

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