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Two Generalizations of the Core Inverse in Rings with Some Applications

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Abstract: In this paper, we introduce two new generalized core inverses, namely, the (p, q, m)-core inverse and the $\langle p, q, n\rangle$-core inverse; both extend the inverses of the $\langle \mathrm{i}, \mathrm{m}\rangle$-core inverse, the (j, m)-core inverse, the core inverse, the core-EP inverse and the DMP-inverse.

Keywords: (p, q, m)-core inverse; $\langle p, q, n\rangle$-core inverse; $\langle \mathrm{i}, \mathrm{m}\rangle$-core inverse; (j, m)-core inverse; core inverse; DMP-inverse; core-EP inverse

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1. Introduction

Throughout this paper, R denotes a unital ring with involution, i.e., a ring with unity 1, and a mapping $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$, for all $a, b \in R$. Let $a, x \in R$, if $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$, then x is called a *Moore–Penrose inverse* of a. If such an element x exists, then it is unique and denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements will be denoted by $R^\dagger$.

An element $a \in R$ is said to be *Drazin invertible* if there exists $b \in R$ such that $ab = ba$, $bab = b$ and $a^m = a^{m+1}b$ for some integer m. The element b above is unique if it exists and denoted by a^D. The smallest positive integer m is called the Drazin index of a, denoted by $\mathrm{ind}(a)$. The set of all Drazin invertible elements in R will denoted by R^D. The DMP-inverse for a complex matrix was introduced by Malik and Thome [1]. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = m$, where $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ matrices over the field of complex numbers. A matrix $X \in \mathbb{C}^{n\times n}$ is called a *DMP-inverse* of A if it satisfies $XAX = X$, $XA = A^DA$ and $A^mX = A^mA^\dagger$. It is unique (and denoted by $A^{d,\dagger}$). Malik and Thome gave several characterizations of the core inverse by using the decomposition of Hartwig and Spindelböck [2].

The notion of the core-EP inverse for a complex matrix was introduced by Manjunatha Prasad and Mohana [3]. A matrix $X \in \mathbb{C}^{n\times n}$ is a *core-EP inverse* of $A \in \mathbb{C}^{n\times n}$ if X is an outer inverse of A satisfying $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^m)$, where m is the index of A and $\mathcal{R}(A)$ stands for the range (column space) of $A \in \mathbb{C}^{n\times n}$. It is unique and denoted by $A^{\textcircled{\dagger}}$. The core-EP inverse for a complex matrix can be investigated by the Core-EP decomposition of a complex matrix by Wang [4]. The notion of the core-EP inverse is extended from the complex matrix to an element in a ring with involution. We will also use the following notations: $aR = \{ax : x \in R\}$, $Ra = \{xa : x \in R\}$, ${}^\circ a = \{x \in R : xa = 0\}$ and $a^\circ = \{x \in R : ax = 0\}$. Let $a \in R$ with $\mathrm{ind}(a) = k$. An element $b \in R$ is called the core-EP inverse of a if it is an outer inverse of a and b is a $*$-EP element satisfies $bR = a^kR$.

The notion of the core inverse for a complex matrix was introduced by Baksalary and Trenkler [5]. In [6], Rakić et al. generalized the core inverse of a complex matrix to the case of an element in R. More precisely, let $a, x \in R$, if $axa = a$, $xR = aR$ and $Rx = Ra^*$, then x is called a *core inverse* of a. The core inverse can be investigated by three equations by Xu,

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Chen and Zhang [7]. If such an element x exists, then it is unique and denoted by $a^{\textcircled{\#}}$. The set of all core invertible elements in R will be denoted by $R^{\textcircled{\#}}$.

In addition, $\mathbf{1}_n$ and $\mathbf{0}_n$ will denote the $n \times 1$ column vectors all of whose components are 1 and 0, respectively. The zero matrix of size $m \times n$ is denoted by $0_{m\times n}$ (abbr. 0). If $\mathcal{S}$ is a subspace of $\mathbb{C}^n$, then $P_{\mathcal{S}}$ stands for the *orthogonal projector* onto the subspace $\mathcal{S}$. A matrix $A \in \mathbb{C}^{n\times n}$ is *unitary* if $AA^* = I_n$, where I_n denotes the *identity matrix* of size n. Let $a \in R$, a is called idempotent if $a^2 = a$. The symbol $\mathbb{N}$ denotes the set of all positive integers.

2. Preliminaries

A related decomposition of the matrix decomposition of Hartwig and Spindelböck [2] was given in ([8], Theorem 2.1) by Benítez; in [9] a simpler proof of this decomposition can be found. Let us start this section with the concept of principal angles.

Definition 1 ([10])**.** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two nontrivial subspaces of $\mathbb{C}^n$. We define the principal angles $\theta_1, \dots, \theta_r \in [0, \pi/2]$ between $\mathcal{S}_1$ and $\mathcal{S}_2$ by*

$$\cos\theta_i = \sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}),$$

for $i = 1, \dots, r$, where $r = \min\{\dim \mathcal{S}_1, \dim \mathcal{S}_2\}$. The real numbers $\sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}) \geq 0$ are the singular values of $P_{\mathcal{S}_1} P_{\mathcal{S}_2}$.

The following theorem can be found in ([8], Theorem 2.1).

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$, $r = \mathrm{rk}(A)$, and let $\theta_1, \dots, \theta_p$ be the principal angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ belonging to $]0, \pi/2[$. Denote by x and y the multiplicities of the angles 0 and $\pi/2$ as a canonical angle between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. There exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*, \tag{1}$$

where $M \in \mathbb{C}^{r\times r}$ is nonsingular,

$$C = \mathrm{diag}(\mathbf{0}_y, \cos\theta_1, \dots, \cos\theta_p, \mathbf{1}_x),$$

$$S = \begin{bmatrix} \mathrm{diag}(\mathbf{1}_y, \sin\theta_1, \dots, \sin\theta_p) & 0_{p+y,n-(r+p+y)} \\ 0_{x,p+y} & 0_{x,n-(r+p+y)} \end{bmatrix},$$

and $r = y + p + x$. Furthermore, x and $y + n - r$ are the multiplicities of the singular values 1 and 0 in $P_{\mathcal{R}(A)} P_{\mathcal{R}(A^*)}$, respectively. We call (1) as the CS decomposition of A.

In this decomposition, one has $C^2 + SS^* = I_r$ and $C^* = C$. This decomposition can answer the question "how far is a matrix from being EP". Moreover, it can be applied to some partial matrix ordering, such as star ordering and sharp ordering.

3. (p, q, m)-Core Inverse

Let us start this section by introducing the notation of the (p, q, m)-core inverse.

Definition 2. *Let $a, p, q \in R$ and $m \in \mathbb{N}$. If $pa = ap$ and pa is idempotent, then $x \in R$ is called a* (p, q, m)-core inverse *of a, if it satisfies*

$$x = pax \ \text{ and } \ a^m x = q. \tag{2}$$

It will be proved that if x exists, then it is unique and denoted by $a^{\textcircled{*}}_{p,q,m}$.

Remark 1. *If $a \in R$ is (p,q,m)-core invertible, then we have $pa = ap$ and pa is idempotent. Since this property of the (p,q,m)-core inverse is used many times in the sequel, thus we emphasize it here.*

Theorem 2. *If equations in (2) have a solution, then it is unique.*

Proof. Let x_1 and x_2 be two candidates (p,q,m)-core inverse of a, that is $x_1 = pax_1$, $a^m x_1 = q$, $x_2 = pax_2$ and $a^m x_2 = q$. Thus by $pa = ap$ and pa is idempotent, we have

$$x_1 = p^m q = p^m a^m x_2 = pax_2 = x_2.$$

□

In the following lemma, we will show that $q = paq$ if a is (p,q,m)-core invertible.

Lemma 1. *Let $a, p, q \in R$ and $m, n \in \mathbb{N}$. If a is (p,q,m)-core invertible, then*

(1) $q = pa^{m+1}a^{\circledast}_{p,q,m}$;
(2) $q = paq$;
(3) $a^n a^{\circledast}_{p,q,m} = p^{m-n}q$, *where* $m \geq n$.

Proof. (1) and (2). If a is (p,q,m)-core invertible, then we have $a^{\circledast}_{p,q,m} = paa^{\circledast}_{p,q,m}$ and $a^m a^{\circledast}_{p,q,m} = q$. Having in mind that $ap = pa$ and the idempotency of pa, we obtain

$$q = a^m a^{\circledast}_{p,q,m} = a^m (paa^{\circledast}_{p,q,m}) = pa^{m+1}a^{\circledast}_{p,q,m}; \tag{3}$$
$$a^{\circledast}_{p,q,m} = paa^{\circledast}_{p,q,m} = p^m a^m a^{\circledast}_{p,q,m} = p^m q. \tag{4}$$

Thus, by (3) and (4), we have

$$q = pa^{m+1}a^{\circledast}_{p,q,m} = pa^{m+1}(p^m q) = p^{m+1}a^{m+1}q = paq. \tag{5}$$

(3). If $m \geq n$, then $a^n a^{\circledast}_{p,q,m} = a^n(paa^{\circledast}_{p,q,m}) = a^n p^m a^m a^{\circledast}_{p,q,m} = a^n p^m q = p^{m-n}p^n a^n q = p^{m-n}paq = p^{m-n}q$ by the definition of the (p,q,m)-core inverse and (2). □

Theorem 3. *If the solution of the equations in (2) exists, then the unique solution is $x = p^m q$.*

Proof. By Lemma 1, we have $q = paq$. Having in mind that $ap = pa$ and the idempotency of pa, we obtain

$$\begin{aligned} pax &= pap^m q = p^m(paq) = p^m q = x \\ a^m x &= a^m p^m q = paq = q. \end{aligned}$$

□

Remark 2. *If $a \in R^D$ and $a^i, a^j \in R^{\dagger}$, then the (p,q,m)-core inverse is the generalizations of the $\langle \mathrm{i}, \mathrm{m} \rangle$-core inverse and the (j,m)-core inverse [11], respectively. More precisely, we have the following statements:*

(1) *If $p = a^D$ and $q = a^i(a^i)^{\dagger}$, then the (p,q,m)-core inverse coincides with the $\langle \mathrm{i}, \mathrm{m} \rangle$-core inverse;*
(2) *If $p = a^D$ and $q = a^m(a^j)^{\dagger}$, then the (p,q,m)-core inverse coincides with the (j,m)-core inverse.*

By Remarks 3.5, 4.7 and 4.8 in [11], we have the $\langle \mathrm{m}, \mathrm{j} \rangle$-core inverse for a complex matrix, which extends the notions of the core inverse defined by Baksalary and Trenkler [5] and the core-EP inverse defined by Manjunatha Prasad and Mohana [3], respectively. The (m, k)-core inverse for a complex matrix, which extends the notions of the core inverse and the DMP-inverse defined by Malik and Thome [1], respectively. Therefore, we have the following remark by Remark 2. We can use generalized inverses to study the system of constrained matrix equations and Toeplitz matrix, etc. [12,13].

Remark 3. *If $a \in R^D$ and $a^j \in R^\dagger$, then the (p,q,m)-core inverse is a generalization of the core inverse, the DMP inverse and the core-EP inverse. More precisely, we have the following statements:*

(1) *If $p = a^\#$, $m = 1$ and $q = aa^\dagger$, then the (p,q,m)-core inverse coincides with the core inverse;*
(2) *If $p = a^D$, $m = \mathrm{ind}(a)$ and $q = a^m a^\dagger$, then the (p,q,m)-core inverse coincides with the DMP inverse;*
(3) *If $p = a^D$, $m = 1$, $j = \mathrm{ind}(a)$ and $q = a^j (a^j)^\dagger$, then the (p,q,m)-core inverse coincides with the core-EP inverse.*

Example 1. *The (p,q,m)-core inverse is different from the group inverse and the Moore–Penrose inverse. Let $A = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$. Then $A^\# = A$ by $A^2 = A$, but A is not Moore–Penrose invertible by $AA^* = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Note that if A is Moore–Penrose invertible, then $A = AA^\dagger A = A(A^\dagger A)^* = AA^*(A^\dagger)^* = 0$, but $A \neq 0$. In fact, AA^* implies A is not $\{1,4\}$-invertible. If we let $p = a^\#$, $q = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $a^{\circledast}_{p,q,m} = \begin{bmatrix} 1+3i & 2+4i \\ 0 & 0 \end{bmatrix}$.*

Theorem 4. *Let $a, p, q \in R$ and $m \in \mathbb{N}$. If $pa = ap$ and pa is idempotent, then the following are equivalent:*

(1) *a is (p,q,m)-core invertible with $a^{\circledast}_{p,q,m} = x$;*
(2) *$x = pax$ and $q = pa^{m+1}x$;*
(3) *$x = pax$, $aq = a^{m+1}x$ and $q = paq$.*

Proof. $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ are trivial by Lemma 1 and the definition of the (p,q,m)-core inverse.

$(2) \Rightarrow (1)$. From $a^m x = a^m(pax) = pa^{m+1}x = q$ we have that x is the (p,q,m)-core inverse of a.

$(3) \Rightarrow (2)$. It is sufficient to prove $q = pa^{m+1}x$. We have $q = paq = pa^{m+1}x$. □

Remark 4. *Note that $x = pax$ iff $xR \subseteq paR$ iff ${}^\circ(pa) \subseteq {}^\circ x$. Moreover, $q = paq$ iff $qR \subseteq paR$ iff ${}^\circ(pa) \subseteq {}^\circ q$. Thus, we can obtain more conditions such that a is (p,q,m)-core invertible in Theorem 4.*

If $p = a^\#$, $m = 1$ and $q = aa^\dagger$, then the (p,q,m)-core inverse coincides with the core inverse, thus we have the following corollary by Theorem 4.

Corollary 1. *Let $a \in R$ with $a \in R^\# \cap R^\dagger$. Then the following are equivalent:*

(1) *a is core invertible with $a^{\#\!\!\!\bigcirc} = x$;*
(2) *$x = a^\# ax$ and $aa^\dagger = ax$;*
(3) *$x = a^\# ax$ and $a^2 a^\dagger = a^2 x$.*

Since the (p,q,m)-core inverse is a generalization of the core inverse, the core-EP inverse, the DMP-inverse, $\langle i,m\rangle$-core inverse and (j,m)-core inverse, we can obtain some analogous corollaries as Corollary 1.

Recall that for $e = e^2 \in R$, we can represent any $a \in R$ as a matrix

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{e\times e},$$

where $a_{11} = eae$, $a_{12} = ea(1-e)$, $a_{21} = (1-e)ae$ and $a_{22} = (1-e)a(1-e)$.

Now we present the result concerning the matrix form of (p,q,m)-core invertible element $a \in R$.

Theorem 5. *Let $a, p, q \in R$ and $m \in \mathbb{N}$. Then a is (p,q,m)-core invertible if and only if there exists $e \in R$ such that $e = e^2$,*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{e\times e}, \quad p = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}_{e\times e} \quad \text{and} \quad q = \begin{bmatrix} q_1 & q_2 \\ 0 & 0 \end{bmatrix}_{e\times e},$$

where $p_1a_1 = a_1p_1 = (p_1a_1)^2$, $p_2a_2 = a_2p_2 = 0$, a_1 is (p_1, q_1, m)-core invertible and (p_1, q_2, m)-core invertible. The (p,q,m)-core inverse of a is given by

$$a^{\circledast}_{p,q,m} = \begin{bmatrix} (a_1)^{\circledast}_{p_1,q_1,m} & (a_1)^{\circledast}_{p_1,q_2,m} \\ 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} p_1^m q_1 & p_1^m q_2 \\ 0 & 0 \end{bmatrix}_{e\times e}.$$

Proof. Suppose that a is (p,q,m)-core invertible and let $e = pa$. Then $e^2 = (pa)^2 = pa = e$, $ea(1-e) = apa(1-pa) = 0$ and $(1-e)ae = 0$. Hence,

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{e\times e},$$

where $a_1 = pa^2$ and $a_2 = (1-pa)a$. Similarly, we obtain, for $p_1 = p^2a$ and $p_2 = (1-pa)p$,

$$p = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}_{e\times e}.$$

The equalities $pa = ap$ and $(pa)^2 = pa$ give $p_1a_1 = a_1p_1 = (p_1a_1)^2$ and $p_2a_2 = a_2p_2 = (1-pa)ap(1-pa) = 0$. Set

$$a^{\circledast}_{p,q,m} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{e\times e} \quad \text{and} \quad q = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}_{e\times e}.$$

From $a^{\circledast}_{p,q,m} = paa^{\circledast}_{p,q,m} = \begin{bmatrix} p_1a_1 & 0 \\ 0 & 0 \end{bmatrix}_{e\times e} a^{\circledast}_{p,q,m}$, we obtain $x_1 = p_1a_1x_1$, $x_2 = p_1a_1x_2$ and $x_3 = x_4 = 0$. Since $q = paq$, then $q_3 = q_4 = 0$. Now, by

$$\begin{bmatrix} a_1^m x_1 & a_1^m x_2 \\ 0 & 0 \end{bmatrix}_{e\times e} = a^m a^{\circledast}_{p,q,m} = q = \begin{bmatrix} q_1 & q_2 \\ 0 & 0 \end{bmatrix}_{e\times e},$$

we conclude that $a_1^m x_1 = q_1$ and $a_1^m x_2 = q_2$. Hence, a_1 is (p_1, q_1, m)-core invertible and (p_1, q_2, m)-core invertible with $x_1 = (a_1)^{\circledast}_{p_1,q_1,m}$ and $x_2 = (a_1)^{\circledast}_{p_1,q_2,m}$.

Conversely, by the assumption $p_1a_1 = a_1p_1 = (p_1a_1)^2$ and $p_2a_2 = a_2p_2 = 0$, we check that $pa = ap = (pa)^2$. Since a_1 is (p_1, q_1, m)-core invertible and (p_1, q_2, m)-core invertible, if we let

$$x = \begin{bmatrix} (a_1)^{\circledast}_{p_1,q_1,m} & (a_1)^{\circledast}_{p_1,q_2,m} \\ 0 & 0 \end{bmatrix}_{e\times e},$$

we get $x = pax$ and $a^m x = q$. So, a is (p,q,m)-core invertible and $x = a^{\circledast}_{p,q,m}$. □

Under some conditions, we obtain that the (p,q,m)-core inverse of a and the (p,r,m)-core inverse of b commute.

Lemma 2. *Let $a, b, p, q, r \in R$ and $m \in \mathbb{N}$. If a is (p,q,m)-core invertible, b is (p,r,m)-core invertible and $qp^m r = rp^m q$ (or equivalently $qb^{\circledast}_{p,r,m} = ra^{\circledast}_{p,q,m}$), then $a^{\circledast}_{p,q,m} b^{\circledast}_{p,r,m} = b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m}$.*

Proof. Because $a^{\circledast}_{p,q,m} = p^m q$ and $b^{\circledast}_{p,r,m} = p^m r$ by Theorem 3, we get $a^{\circledast}_{p,q,m} b^{\circledast}_{p,r,m} = p^m q p^m r = p^m r p^m q = b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m}$. □

Now, we study when the product of one (p,q,m)-core invertible element and one (p,r,m)-core invertible element is (p^2, rq, m)-core invertible.

Theorem 6. *Let $a,b,p,q,r \in R$ and $m \in \mathbb{N}$ such that a is (p,q,m)-core invertible, b is (p,r,m)-core invertible, $ab = ba$ and $a^m r = r a^m$. We have the following statements:*

(1) *If $pap^m r = p^m rap$ (or equivalently $pab^{\circledast}_{p,r,m} = b^{\circledast}_{p,r,m} ap$), then ab is (p^2, rq, m)-core invertible and $(ab)^{\circledast}_{p^2,rq,m} = b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m}$;*

(2) *If $qb^{\circledast}_{p,r,m} = ra^{\circledast}_{p,q,m}$, then ab is (p^2, rq, m)-core invertible and $(ab)^{\circledast}_{p^2,rq,m} = b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m} = a^{\circledast}_{p,q,m} b^{\circledast}_{p,r,m}$.*

Proof. Since $pa = ap$ and $pb = bp$ are idempotents and $ab = ba$, notice that $p^2 ab = abp^2$ and $(p^2 ab)^2 = (pa)^2 (pb)^2 = p^2 ab$. The assumptions $ab = ba$ and $a^m r = ra^m$ imply $(ab)^m b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m} = a^m (b^m b^{\circledast}_{p,r,m}) a^{\circledast}_{p,q,m} = (a^m r) a^{\circledast}_{p,q,m} = r(a^m a^{\circledast}_{p,q,m}) = rq$.

(1). Since $pap^m r = p^m rap$, $ap = pa$ and $bp = pb$, we have $p^2 abb^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m} = pa(pbb^{\circledast}_{p,r,m}) a^{\circledast}_{p,q,m} = (pab^{\circledast}_{p,r,m}) a^{\circledast}_{p,q,m} = b^{\circledast}_{p,r,m}(paa^{\circledast}_{p,q,m}) = b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m}$. Therefore, ab is (p^2, rq, m)-core invertible and $(ab)^{\circledast}_{p^2,rq,m} = b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m}$.

(2). From $qp^m r = rp^m q$ we can get that $b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m} = a^{\circledast}_{p,q,m} b^{\circledast}_{p,r,m}$ by Lemma 2. By $p^2 abb^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m} = pa(b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m}) = (paa^{\circledast}_{p,q,m}) b^{\circledast}_{p,r,m} = a^{\circledast}_{p,q,m} b^{\circledast}_{p,r,m} = b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m}$, we deduce that $(ab)^{\circledast}_{p^2,rq,m} = b^{\circledast}_{p,r,m} a^{\circledast}_{p,q,m} = a^{\circledast}_{p,q,m} b^{\circledast}_{p,r,m}$. □

In the case that $ab = ba = 0$, the sum of (p,q,m)-core invertible element a and (p,r,m)-core invertible element b is $(p, q+r, m)$-core invertible.

Theorem 7. *Let $a,b,p,q,r \in R$ and $m \in \mathbb{N}$ such that a is (p,q,m)-core invertible, b is (p,r,m)-core invertible and $ab = ba = 0$. Then $a+b$ is $(p, q+r, m)$-core invertible and $(a+b)^{\circledast}_{p,q+r,m} = a^{\circledast}_{p,q,m} + b^{\circledast}_{p,r,m}$.*

Proof. First, observe that $p(a+b) = (a+b)p$ and $[p(a+b)]^2 = p^2(a^2+b^2) = pa + pb = p(a+b)$. Further,

$$a^m b^{\circledast}_{p,r,m} = a^m pbb^{\circledast}_{p,r,m} = pa^m bb^{\circledast}_{p,r,m} = 0$$

and $pab^{\circledast}_{p,r,m} = p^m(a^m b^{\circledast}_{p,r,m}) = 0$. Analogously, $b^m a^{\circledast}_{p,q,m} = 0 = pba^{\circledast}_{p,q,m}$. Thus,

$$p(a+b)(a^{\circledast}_{p,q,m} + b^{\circledast}_{p,r,m}) = (pa+pb)(a^{\circledast}_{p,q,m} + b^{\circledast}_{p,r,m}) = paa^{\circledast}_{p,q,m} + pbb^{\circledast}_{p,r,m} = a^{\circledast}_{p,q,m} + b^{\circledast}_{p,r,m}$$

and

$$(a+b)^m (a^{\circledast}_{p,q,m} + b^{\circledast}_{p,r,m}) = (a^m + b^m)(a^{\circledast}_{p,q,m} + b^{\circledast}_{p,r,m}) = a^m a^{\circledast}_{p,q,m} + b^m b^{\circledast}_{p,r,m} = q + r,$$

that is, $a+b$ is $(p, q+r, m)$-core invertible and $(a+b)^{\circledast}_{p,q+r,m} = a^{\circledast}_{p,q,m} + b^{\circledast}_{p,r,m}$. □

Lemma 3. *Let $a,p,q \in R$ and $m \in \mathbb{N}$ and a is (p,q,m)-core invertible. Then $aa^{\circledast}_{p,q,m} = a^{\circledast}_{p,q,m} a$ if and only if $p^{m-1} q = p^m qa$.*

Proof. By Lemma 1, we have $q = paq$. If $aa^{\circledast}_{p,q,m} = a^{\circledast}_{p,q,m} a$, then $p^m qa = a^{\circledast}_{p,q,m} a = aa^{\circledast}_{p,q,m} = ap^m q = p^{m-1}(paq) = p^{m-1} q$. For the opposite implication, we have $aa^{\circledast}_{p,q,m} = ap^m q = app^{m-1} q = app^m qa = p^m (paq) a = p^m qa = a^{\circledast}_{p,q,m} a$. □

Proposition 1. *Let $a,p,q \in R$ and $m \in \mathbb{N}$. If a is (p,q,m)-core invertible, then*

(1) *If $qa^m = a^m$, then $a^{\circledast}_{p,q,m}$ is an inner inverse of a^m and q is idempotent;*

(2) *If $aq = qa$ (or equivalently $a^{m+1}a^{\circledast}_{p,q,m} = a^m a^{\circledast}_{p,q,m}a$), then $aa^{\circledast}_{p,q,m} = a^{\circledast}_{p,q,m}a$;*
(3) *If q is idempotent, then $a^{\circledast}_{p,q,m}$ is an outer inverse of a^m;*
(4) *If $q = q^*$, then $a^m a^{\circledast}_{p,q,m} = (a^m a^{\circledast}_{p,q,m})^*$;*
(5) *If $aq = qa$ and $q = q^*$, then $a^{\circledast}_{p,q,m}a^m = (a^{\circledast}_{p,q,m}a^m)^*$.*

Proof. (1). Since $qa^m = a^m$ and $q = a^m a^{\circledast}_{p,q,m}$, we have that $a^m = qa^m = a^m a^{\circledast}_{p,q,m}a^m$ and $q = a^m a^{\circledast}_{p,q,m} = qa^m a^{\circledast}_{p,q,m} = q^2$.

(2). It is easy to check that $p^{m-1}q = p^m qa$ by $aq = qa$ and $q = paq$. Thus, we have $aa^{\circledast}_{p,q,m} = a^{\circledast}_{p,q,m}a$ by Lemma 3.

(3). The condition $q = q^2$ gives $a^{\circledast}_{p,q,m}a^m a^{\circledast}_{p,q,m} = a^{\circledast}_{p,q,m}a^m p^m q = a^{\circledast}_{p,q,m}paq = p^m q^2 = p^m q = a^{\circledast}_{p,q,m}$.

(4). By definition of the (p,q,m)-core inverse.

(5). It follows from (2) and (4). □

Applying Proposition 1, we obtain the next result.

Corollary 2. *Let $a, p, q \in R$ and $m \in \mathbb{N}$. If a is (p,q,m)-core invertible, then*
(1) *If $qa^m = a^m$ and $aq = qa$, then $a^m \in R^\#$ and $(a^m)^\# = a^{\circledast}_{p,q,m}$;*
(2) *If $qa^m = a^m$, $q = q^*$ and $aq = qa$, then $a^m \in R^\# \cap R^\dagger$ and $(a^m)^\dagger = (a^m)^\# = a^{\circledast}_{p,q,m}$ (that is, a^m is EP).*

4. $\langle p,q,n\rangle$-Core Inverse

Definition 3. *Let $a, p, q \in R$ and $n \in \mathbb{N}$. We say that $x \in R$ is a $\langle p,q,n\rangle$-*core inverse *of a, if it satisfies*
$$x = pa^n x \ \text{ and } \ a^n x = q. \tag{6}$$

It will be proved that if x exists, then it is unique and denoted by $a^{\circledcirc}_{p,q,n}$.

Theorem 8. *If equations in (6) have a solution, then it is unique and the unique solution is $x = pq$.*

Proof. Let x satisfy (6). Then $x = pa^n q = pq$". Observe that this implies the uniqueness of the equations (6): the unique element in R satisfying (6) is pq. □

If a is $\langle p,q,n\rangle$-core invertible, then we have $a^{\circledcirc}_{p,q,n} = pa^n a^{\circledcirc}_{p,q,n}$ and $a^n a^{\circledcirc}_{p,q,n} = q$ and
$$q = a^n a^{\circledcirc}_{p,q,n} = a^n(pa^n a^{\circledcirc}_{p,q,n}) = a^n pa^n a^{\circledcirc}_{p,q,n}$$
Thus, we obtain
$$q = a^n pa^n a^{\circledcirc}_{p,q,n} = a^n pa^n pq = (a^n p)^2 q.$$

By Theorem 8, we have $q = a^n x = a^n pq$; here, x is the $\langle p,q,n\rangle$-core inverse of a (see next Theorem 11).

Lemma 4. *Let $a, p, q \in R$ and $n \in \mathbb{N}$. If a is $\langle p,q,n\rangle$-core invertible, then $q = a^n pa^n a^{\circledcirc}_{p,q,n} = (a^n p)^2 q$.*

Remark 5. *If $a \in R^D$ and $a^i, a^j \in R^\dagger$, then the $\langle p,q,n\rangle$-core inverse is a generalization of the $\langle \mathrm{i}, \mathrm{m}\rangle$-core inverse and the (j,m)-core inverse [11]. More precisely, we have the following statements:*
(1) *If $p = (a^D)^n$ and $q = a^i(a^i)^\dagger$, then the $\langle p,q,n\rangle$-core inverse coincides with the $\langle \mathrm{i}, \mathrm{m}\rangle$-core inverse;*
(2) *If $p = (a^D)^n$ and $q = a^m(a^j)^\dagger$, then the $\langle p,q,n\rangle$-core inverse coincides with the (j,m)-core inverse.*

Theorem 9. *Let $a, p, q \in R$ and $n \in \mathbb{N}$. Then the following are equivalent:*
(1) *a is $\langle p, q, n\rangle$-core invertible with $a^{\circledcirc}_{p,q,n} = x$;*
(2) *$x = pa^n x$ and $q = a^n pa^n x$;*
(3) *$x = pa^n x$, $a^n pq = a^n x$ and $q = (a^n p)^2 q$.*

Proof. $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ are trivial by Lemma 4 and the definition of the $\langle p, q, n\rangle$-core inverse.

$(2) \Rightarrow (1)$. From $q = a^n pa^n x = a^n x$ we have that x is the $\langle p, q, n\rangle$-core inverse of a.

$(3) \Rightarrow (2)$. It is sufficient to prove $q = a^n pa^n x$. We have $q = (a^n p)^2 q = a^n pa^n pq = a^n pa^n x$. □

Under certain conditions, the product of a $\langle p, q, n\rangle$-core invertible element and a $\langle r, s, n\rangle$-core invertible element is $\langle pr, sq, n\rangle$-core invertible.

Theorem 10. *Let $a, b, p, q, r, s \in R$ and $n \in \mathbb{N}$ such that a is $\langle p, q, n\rangle$-core invertible, b is $\langle r, s, n\rangle$-core invertible, $ab = ba$, $a^n r = ra^n$, $a^n s = sa^n$ and $prs = rsp$. Then ab is $\langle pr, sq, n\rangle$-core invertible and $(ab)^{\circledcirc}_{pr,sq,n} = b^{\circledcirc}_{r,s,n} a^{\circledcirc}_{p,q,n}$.*

Proof. Notice that

$$(ab)^n b^{\circledcirc}_{r,s,n} a^{\circledcirc}_{p,q,n} = a^n (b^n b^{\circledcirc}_{r,s,n}) a^{\circledcirc}_{p,q,n} = (a^n s) a^{\circledcirc}_{p,q,n} = s(a^n a^{\circledcirc}_{p,q,n}) = sq$$

and

$$pr(ab)^n b^{\circledcirc}_{r,s,n} a^{\circledcirc}_{p,q,n} = (prs)q = (rs)(pq) = b^{\circledcirc}_{r,s,n} a^{\circledcirc}_{p,q,n}$$

imply ab is $\langle pr, sq, n\rangle$-core invertible and $(ab)^{\circledcirc}_{pr,sq,n} = b^{\circledcirc}_{r,s,n} a^{\circledcirc}_{p,q,n}$. □

We also study when the sum of a $\langle p, q, n\rangle$-core invertible element and a $\langle r, s, n\rangle$-core invertible element is $\langle p + r, q + s, n\rangle$-core invertible.

Theorem 11. *Let $a, b, p, q, r, s \in R$ and $n \in \mathbb{N}$ such that a is $\langle p, q, n\rangle$-core invertible, b is $\langle r, s, n\rangle$-core invertible, $ab = ba = 0$, $a^n rs = 0 = b^n pq$ and $ps + rq = 0$. Then $a + b$ is $\langle p + r, q + s, n\rangle$-core invertible and $(a + b)^{\circledcirc}_{p+r,q+s,n} = a^{\circledcirc}_{p,q,n} + b^{\circledcirc}_{r,s,n}$.*

Proof. Let x be the $\langle p, q, n\rangle$-core inverse of a and y be the $\langle r, s, n\rangle$-core inverse of b, then by Theorem 8, we have $(p + r)(a + b)^n (x + y) = (p + r)(q + s) = pq + ps + rq + rs = pq + rs = x + y$. □

It is easy to check the following propositions by Definition 3 and Theorem 8.

Proposition 2. *Let $a, p, q \in R$ and $n \in \mathbb{N}$ such that a is $\langle p, q, n\rangle$-core invertible. Then $aa^{\circledcirc}_{p,q,n} = a^{\circledcirc}_{p,q,n} a$ if and only if $apq = pqa$.*

Proposition 3. *Let $a, p, q \in R$ and $n \in \mathbb{N}$ such that a is $\langle p, q, n\rangle$-core invertible. Then*
(1) *If $qa^n = a^n$, then $a^{\circledcirc}_{p,q,n}$ is an inner inverse of a^n and q is idempotent;*
(2) *If $q = q^2$, then $a^{\circledcirc}_{p,q,n} a^n a^{\circledcirc}_{p,q,n} = a^{\circledcirc}_{p,q,n}$;*
(3) *If $q = q^*$, then $a^n a^{\circledcirc}_{p,q,n} = (a^n a^{\circledcirc}_{p,q,n})^*$;*
(4) *If $apq = pqa$ and $q = q^*$, then $a^{\circledcirc}_{p,q,n} a^n = (a^{\circledcirc}_{p,q,n} a^n)^*$.*

5. How to Compute the (P, Q, m)-Core Inverse and $\langle P, Q, n\rangle$-Core Inverse in $\mathbb{C}^{n\times n}$

5.1. How to Compute the (p, q, m)-Core Inverse in $\mathbb{C}^{n\times n}$

Let $A, P, Q \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{N}$. We will assume in this subsection that A is (P, Q, m)-core invertible.If $A \in \mathbb{C}^{n\times n}$ is (P, Q, m)-core invertible, then we have $PA = AP$, PA

is idempotent, $X = PAX$ and $A^m X = Q$. Assume that A has the form (1). If we let $P = U\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}U^*$, where $P_1 \in \mathbb{C}^{r\times r}$, then

$$PA = U\begin{bmatrix} P_1MC & P_1MS \\ P_3MC & P_3MS \end{bmatrix}U^*; \tag{7}$$

$$AP = U\begin{bmatrix} MCP_1 + MSP_3 & MCP_2 + MSP_4 \\ 0 & 0 \end{bmatrix}U^*. \tag{8}$$

From (7) and (8) and $PA = AP$ we obtain $P_3MC = 0$ and $P_3MS = 0$. Then we have $P_3MC^2 = 0$ and $P_3MSS^* = 0$, thus $P_3MC^2 + P_3MSS^* = P_3M(C^2 + SS^*) = P_3M$ by $C^2 + SS^* = I_r$. The nonsingularity of M implies that P_3 is zero matrix, which gives

$$PA = U\begin{bmatrix} P_1MC & P_1MS \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix}U^* \tag{9}$$

Since PA is idempotent, $AP = PA$ and $(PA)^2 = U\begin{bmatrix} (P_1MC)^2 & P_1MCP_1MS \\ 0 & 0 \end{bmatrix}U^*$, hence

$$P_1MC = MCP_1 = (P_1MC)^2 \tag{10}$$

By Lemma 1, we have $Q = PAQ$. If we let $Q = U\begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}U^*$, then by (9) we have

$$\begin{aligned} PAQ =& \ U\begin{bmatrix} P_1MC & P_1MS \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}U^* \\ =& \ U\begin{bmatrix} P_1MCQ_1 + P_1MSQ_3 & P_1MCQ_2 + P_1MSQ_4 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned} \tag{11}$$

From $Q = PAQ$ we have that Q_3 and Q_4 are zero matrices and

$$\begin{cases} Q_1 = P_1MCQ_1 \\ Q_2 = P_1MCQ_2 \end{cases} \tag{12}$$

By Theorem 3, we have $A^{\circledast}_{P,Q,m} = P^mQ$. Since $P_3 = 0$, $Q_3 = 0$ and $Q_4 = 0$, thus we have $P = U\begin{bmatrix} P_1 & P_2 \\ 0 & P_4 \end{bmatrix}U^*$ and $Q = U\begin{bmatrix} Q_1 & Q_2 \\ 0 & 0 \end{bmatrix}U^*$, thus $P^m = U\begin{bmatrix} P_1^m & \bigstar \\ 0 & P_4^m \end{bmatrix}U^*$; the entries that we are not interested in are marked with $\bigstar$. Therefore

$$\begin{aligned} A^{\circledast}_{P,Q,m} =& \ P^mQ = U\begin{bmatrix} P_1^m & \bigstar \\ 0 & P_4^m \end{bmatrix}\begin{bmatrix} Q_1 & Q_2 \\ 0 & 0 \end{bmatrix}U^* \\ =& \ U\begin{bmatrix} P_1^mQ_1 & P_1^mQ_2 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned} \tag{13}$$

By $A^m = U\begin{bmatrix} (MC)^m & (MC)^{m-1}MS \\ 0 & 0 \end{bmatrix}U^*$ and $A^m A^{\circledast}_{P,Q,m} = Q$, we have

$$\begin{aligned} A^m A^{\circledast}_{P,Q,m} =& \ U\begin{bmatrix} (MC)^m & (MC)^{m-1}MS \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_1^mQ_1 & P_1^mQ_2 \\ 0 & 0 \end{bmatrix}U^* \\ =& \ U\begin{bmatrix} (MC)^mP_1^mQ_1 & (MC)^mP_1^mQ_2 \\ 0 & 0 \end{bmatrix}U^* \\ =& \ U\begin{bmatrix} Q_1 & Q_2 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned} \tag{14}$$

Thus

$$\begin{cases} Q_1 = (MC)^m P_1^m Q_1 \\ Q_2 = (MC)^m P_1^m Q_2 \end{cases} \tag{15}$$

Therefore, by (10), (12), (15) and the definition of the (P, Q, m)-core inverse, we have

$$\begin{cases} (MC)^{\circledast}_{P_1,Q_1,m} = P_1^m Q_1 \\ (MC)^{\circledast}_{P_1,Q_2,m} = P_1^m Q_2 \end{cases} \tag{16}$$

From (13) and (16) we have

$$A^{\circledast}_{P,Q,m} = U \begin{bmatrix} (MC)^{\circledast}_{P_1,Q_1,m} & (MC)^{\circledast}_{P_1,Q_2,m} \\ 0 & 0 \end{bmatrix} U^*.$$

5.2. How to Compute the $\langle p, q, n\rangle$-Core Inverse in $\mathbb{C}^{n\times n}$

Let $A, P, Q \in \mathbb{C}^{n\times n}$ and $n \in \mathbb{N}$. We will assume in this subsection that A is $\langle P, Q, n\rangle$-core invertible. Here we suppose that $AP = PA$, thus we have $P = U \begin{bmatrix} P_1 & P_2 \\ 0 & P_4 \end{bmatrix} U^*$, where $P_1 \in \mathbb{C}^{r\times r}$. Moreover, we have

$$P_1 MC = MCP_1 \tag{17}$$

and

$$PA^n = U \begin{bmatrix} P_1(MC)^n & P_1(MC)^{n-1}MS \\ 0 & 0 \end{bmatrix} U^*. \tag{18}$$

By Lemma 4, we have $Q = (A^n P)^2 Q$. If we let $Q = U \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} U^*$, then by (18) we have

$$\begin{aligned} (A^n P)^2 Q = {} & U \begin{bmatrix} P_1(MC)^n & P_1(MC)^{n-1}MS \\ 0 & 0 \end{bmatrix}^2 \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} U^* \\ = {} & U \begin{bmatrix} \bigstar & \bigstar \\ 0 & 0 \end{bmatrix} U^*. \end{aligned} \tag{19}$$

where we marked with $\bigstar$ the entries that we are not interested in. Thus, from $Q = (A^n P)^2 Q$ we have Q_3 and Q_4 which are zero matrices. Therefore, we have $A^{\circledcirc}_{P,Q,n} = PQ = U \begin{bmatrix} P_1 & P_2 \\ 0 & P_4 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} P_1 Q_1 & P_1 Q_2 \\ 0 & 0 \end{bmatrix} U^*$. It is not difficult to see that we have

$$A^{\circledcirc}_{P,Q,n} = U \begin{bmatrix} (MC)^{\circledcirc}_{P_1,Q_1,n} & (MC)^{\circledcirc}_{P_1,Q_2,n} \\ 0 & 0 \end{bmatrix} U^*.$$

6. Conclusions with Some Applications

Two new generalized core inverse are introduced, namely, the (p, q, m)-core inverse and the $\langle p, q, n\rangle$-core inverse. These inverses extend the inverses of the $\langle \mathrm{i}, \mathrm{m}\rangle$-core inverse, the (j, m)-core inverse, the core inverse, the core-EP inverse and the DMP-inverse. The (p, q, m)-core inverse and the $\langle p, q, n\rangle$-core inverse can used in some areas such as statistics and matrix generalized inverses. There are a lot of research articles about matrix ordering and element partial ordering; by using the reverse order of the (p, q, m)-core inverse and the $\langle p, q, n\rangle$-core inverse, one can get some suitable applications in statistics, electrical networks, etc. We can obtain several partial ordering by using different generalized inverses, such as the minus ordering by using the inner inverse, the sharp ordering by using the group inverse and the core ordering by using the core inverse. The main results in this paper as follows:

If a is (p,q,m)-core invertible, then the (p,q,m)-core inverse of a is $p^m q$. Let $a,p,q \in R$ and $m \in \mathbb{N}$. Then a is (p,q,m)-core invertible if and only if there exists $e \in R$ such that $e = e^2$,

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{e\times e}, \quad p = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}_{e\times e} \quad and \quad q = \begin{bmatrix} q_1 & q_2 \\ 0 & 0 \end{bmatrix}_{e\times e},$$

where $p_1a_1 = a_1p_1 = (p_1a_1)^2$, $p_2a_2 = a_2p_2 = 0$, a_1 is (p_1,q_1,m)-core invertible and (p_1,q_2,m)-core invertible. The (p,q,m)-core inverse of a is given by

$$a^{\circledast}_{p,q,m} = \begin{bmatrix} (a_1)^{\circledast}_{p_1,q_1,m} & (a_1)^{\circledast}_{p_1,q_2,m} \\ 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} p_1^m q_1 & p_1^m q_2 \\ 0 & 0 \end{bmatrix}_{e\times e}.$$

If $A \in \mathbb{C}^{n\times n}$ is (P,Q,m)-core invertible, then we have $PA = AP$, PA is idempotent, $X = PAX$, $A^m X = Q$ and

$$A^{\circledast}_{P,Q,m} = U \begin{bmatrix} (MC)^{\circledast}_{P_1,Q_1,m} & (MC)^{\circledast}_{P_1,Q_2,m} \\ 0 & 0 \end{bmatrix} U^*.$$

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