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Calibration-Based Mean Estimators under Stratified Median Ranked Set Sampling

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Abstract: Using auxiliary information, the calibration approach modifies the original design weights to enhance the mean estimates. This paper initially proposes two families of estimators based on an adaptation of the estimators presented by recent researchers, and then, it presents a new family of calibration estimators with the set of some calibration constraints under stratified median ranked set sampling (MRSS). The result has also been implemented to the situation of two-stage stratified median ranked set sampling (MRSS). To best of our knowledge, we are presenting for the first time calibration-based mean estimators under stratified MRSS, so the performance evaluation is made between adapted and proposed estimators on behalf of the simulation study with real and artificial datasets. For real-world data or applications, we use information on the body mass index (BMI) of 800 people in Turkey in 2014 as a research variable and age as an auxiliary variable.

Keywords: median ranked set sampling; two-stage median ranked set sampling; auxiliary information; calibration-type estimators

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1. Introduction

In many real-life studies, specifically in ecological and environmental research, the variable of interest, say Y, may not be effectively perceptible; the measurements might be costly, tedious, intrusive or even destructive on the subjects being measured. Despite the difficulties or complexities in data collection, ranking the sampled units may be relatively straightforward at no extra cost or with almost no expense. Consider the following example: calliphoridae flies detect and colonize on a food source, such as a decaying corpse, as a natural means of survival within minutes of death. Thus, forensic entomologists frequently use calliphoridae fly larvae to estimate a cadaver's time since death during their post-mortem investigations. As soon as the larvae reach their largest size, they cease eating. Because their anterior intestine is always empty during the course of their future development, forensic entomologists can accurately determine the post-mortem interval by observing how full their intestines are. However, it is challenging to determine changes in the intestinal contents of maggots using radiographic techniques [1].

Meanwhile, since the larvae appear to lengthen in a continuous manner during their growth, it is relatively easy to measure and rank their length. As another example, in a health-related study, suppose that the interest is in estimating the mean cholesterol level of a population. Instead of performing an invasive blood test on all subjects in the sample, subjects can be ranked with respect to their weights, even just visually, and the blood sample can be taken only on a small number of subjects.

For such circumstances, as described in examples, ranked set sampling (RSS) is a method for handling data collecting and processing. In order to estimate mean pasture yields, McIntyre originally proposed RSS in 1952. Takahasi and Wakimoto [2] later developed the RSS theory under the presumption of perfect ranking. The RSS is carried out as follows: the population is divided into a simple random sample of size ϑ, each unit is rated according to subjective criteria, the smallest unit in the sample is measured, and the remaining units are eliminated. After ranking each unit according to the same criteria, a second simple random sample of size ϑ is chosen from the population. The second smallest unit is then measured, and the remaining units are discarded. Until the ordered units are measured, this process is repeated. A cycle is defined as the ordered observations $Y_{i[1]}, Y_{i[2]}, \ldots, Y_{i[\vartheta]}$, where $i = 1, ..., m$ denotes the cycle number. A total sample size of ϑ is produced once the cycles are repeated m times.

Since its inception, RSS has attracted a great deal of attention from scholars, and it continues to be a very active research area. Beyond its initial horticultural-based origins in the foundational work by McIntyre [3], it has now begun to find its way into commercial applications. For more details regarding RSS, intrigued readers may refer to Chen et al. [4], Samawi and Muttlak [5], Bouza [6], Jeelani and Bouza [7], Eftekharian and Razmkhah [8] and Koyuncu [9]. In order to estimate the population mean, Muttlak [10] suggested median ranked set sampling (MRSS) and demonstrated that it produces estimates that are more accurate than RSS. MRSS can be thought of as a modified form of RSS, where the median of each sample in a cycle is measured rather than the k^{th} $(k = 1, 2, ..., \vartheta)$ smallest unit in each ranked sample.

The most popular estimator of the population mean in sampling theory is the classic ratio estimator when there is a high positive correlation between the study variable (Y) and the auxiliary variable (X) [11]. Al-Omari [12] took the MRSS scheme into consideration when proposing new ratio-type estimators that are based on the first and third quartiles of the auxiliary variable. The original structure of the MRSS proposed by Al-Omari [12] requires the use of ϑ independent samples of each size ϑ bivariate units from a finite population. The variable of interest Y is ranked by individual judgment, such as by a visual examination, or by means of the utilization of an accompanying variable associated with Y. Al-Omari [12] considered ranking on the auxiliary variable X as follows: When ϑ is odd in a cycle, the $\left(\frac{\vartheta+1}{2}\right)^{th}$ smallest X and the corresponding Y are chosen from each sample. When ϑ is even, the $\left(\frac{\vartheta}{2}\right)^{th}$ smallest X and the associated Y are chosen from the first $\frac{\vartheta}{2}$ set and the $\left(\frac{\vartheta+2}{2}\right)^{th}$ smallest X and associated Y are chosen from the remaining $\frac{\vartheta}{2}$ set. For more information, see Al-Omari [12]. The cycles can be repeated $m \geqslant 1$ times to obtain a total sample size of $m\vartheta$. Later, Koyuncu [13] expanded on Al-Omari [12] concept and introduced estimators of the regression, exponential, and difference types. However, all this work is completed on traditional ratio and regression-type mean estimation under MRSS. In this paper, taking motivation from these, we have made an attempt to develop calibration-type mean estimators under stratified MRSS.

The remainder of this article is structured as follows: In Section 2, we present a calibration technique and present adapted estimators under stratified MRSS. In Section 3, we propose a new family of estimators with a set of calibration constraints. Section 4 is dedicated to a two-stage MRSS scheme. In Section 5, where we compare the effectiveness of our suggested estimators with modified estimators, we conducted a thorough simulation analysis. Finally, in Section 6, we offer our concluding remarks.

2. Adapted Estimators under Stratified MRSS Design

The effectiveness of the mean estimator from a finite population can be increased at various stages when auxiliary information is supplied. There are many instances in everyday life where the research variable Y and the auxiliary variable X have a linear relationship. Think about your height and weight, as taller people tend to weigh more; think about your GPA and SAT scores, as students with higher GPAs typically perform better on the SAT test; think about the relationship between depression and suicide: severe depression increases the chance of suicide compared to those who do not have depression [14]; take body mass index (BMI) and total cholesterol as an example. It has been demonstrated that these two variables have a direct and positive association [15].

A basic method of adjusting the initial weights with the goal of minimizing a specified distance measure while taking into account auxiliary data is known as calibration estimation. By creating new calibration weights in stratified sampling, researchers have attempted to boost estimates of the population parameter in the literature. A distance metric and a set of calibration constraints are the two fundamental building blocks in the creation of new calibration weights.

The development of calibration estimation in survey sampling dates back to Deville and Sarndal [16]. In the presence of auxiliary information, they created the calibration restrictions. They claim that when the sample sum of the weighted auxiliary variable equals the known population total for that auxiliary variable, the calibrated weights may provide accurate estimations. Because there is a significant correlation between the study variable and the auxiliary variables, weights that are effective for the auxiliary variable should also be effective for the research variable. Numerous authors have investigated calibration estimates utilizing various calibration constraints in survey sampling in the wake of Deville and Sarndal [16]. The first extended calibration method for a stratified sampling design was introduced by Singh, Horn, and Yu [17]. Koyuncu and Kadilar [13] provided corrected expressions of Tracy et al. [18] calibrated weights, and also new improved calibration weights are introduced. Furthermore, Sinha et al. [19] and Garg and Pachori [20] have extended the work in the two-stage stratified sampling scheme. Taking motivation from these important studies, we are adapting Sinha et al. [19] and Garg and Pachori [20] estimators under MRSS.

2.1. Sinha et al. (2017) Estimator [19]

In a stratified sampling design, a random sample of size n_δ, is drawn without replacement from a population of size N_δ in stratum δ, $(\delta = 1, 2, ...\gamma)$. Let $(X_{i(1)}, Y_{i[1]}), (X_{i(2)}, Y_{i[2]}), \ldots, (X_{i(n_\delta)}, Y_{i[n_\delta]})$ be the order statistics of $X_{i1}, X_{i2}, \ldots, X_{in_\delta}$ and the judgment order of $Y_{i1}, Y_{i2}, \ldots, Y_{in_\delta}$, in δ^{th} stratum, for $(i = 1, 2, ...n_\delta)$. Furthermore, () and [] indicate that the ranking of X is perfect and the ranking of Y has errors. For odd and even sample sizes, the units measured using MRSS are denoted by M(O) and M(E), respectively.

As per each reviewer suggestion, let us provide small examples for sample selection in case of even and odd sample sizes so that readers catch the true spirit of this article as given below:

- In case of an even sample size in the δ^{th} stratum, the $\left(\frac{n_\delta}{2}\right)^{th}$ smallest X and the associated Y are chosen from the first $\frac{n_\delta}{2}$ set and the $\left(\frac{n_\delta+2}{2}\right)^{th}$ smallest X and associated Y are chosen from the remaining $\frac{n_\delta}{2}$ set. Let us take a small example of MRSS for the even sample size in Table 1 for (i = 1,2,...,4). Clearly, for $n_\delta = 4$, $X_{(\frac{n_\delta}{2})} = X_{(\frac{4}{2})} = X_{(2)}$ with an associated Y is selected for the first and second cycles, i.e., $(X_{1(2)}, Y_{1[2]})$ and $(X_{2(2)}, Y_{2[2]})$. Furthermore, $X_{(\frac{n_\delta+2}{2})} = X_{(\frac{4+2}{2})} = X_{(\frac{6}{2})} = X_{(3)}$ with associated Y is selected for the remaining two cycles, i.e., $(X_{3(3)}, Y_{3[3]})$ and $(X_{4(3)}, Y_{4[3]})$.
- In case of odd sample size in the δ^{th} stratum, the $\left(\frac{n_{\delta+}}{2}\right)^{th}$ smallest X and the associated Y are chosen from each set. Let us take a small example of MRSS for the odd sample size in Table 2 for (i = 1, 2 , 3). Clearly, for $n_\delta = 3$, $X_{(\frac{n_\delta+1}{2})} = X_{(\frac{3+1}{2})} = X_{(\frac{4}{2})} = X_{(2)}$ with associated Y is selected from each cycle, i.e., $(X_{1(2)}, Y_{1[2]})$, $(X_{2(2)}, Y_{2[2]})$ and $(X_{3(2)}, Y_{3[2]})$.

Table 1. MRSS for even sample size, i.e., $n_\delta = 4$.

$(X_{1(1)}, Y_{1[1]})$	$(\mathbf{X_{1(2)}}, \mathbf{Y_{1[2]}})$	$(X_{1(3)}, Y_{1[3]})$	$(X_{1(4)}, Y_{1[4]})$
$(X_{2(1)}, Y_{2[1]})$	$(\mathbf{X_{2(2)}}, \mathbf{Y_{2[2]}})$	$(X_{2(3)}, Y_{2[3]})$	$(X_{2(4)}, Y_{2[4]})$
$(X_{3(1)}, Y_{3[1]})$	$(X_{3(2)}, Y_{3[2]})$	$(\mathbf{X_{3(3)}}, \mathbf{Y_{3[3]}})$	$(X_{3(4)}, Y_{3[4]})$
$(X_{4(1)}, Y_{4[1]})$	$(X_{4(2)}, Y_{4[2]})$	$(\mathbf{X_{4(3)}}, \mathbf{Y_{4[3]}})$	$(X_{4(4)}, Y_{4[4]})$

Table 2. MRSS for odd sample size i.e., $n_\delta = 3$.

$(X_{1(1)}, Y_{1[1]})$	$(\mathbf{X_{1(2)}}, \mathbf{Y_{1[2]}})$	$(X_{1(3)}, Y_{1[3]})$
$(X_{2(1)}, Y_{2[1]})$	$(\mathbf{X_{2(2)}}, \mathbf{Y_{2[2]}})$	$(X_{2(3)}, Y_{2[3]})$
$(X_{3(1)}, Y_{3[1]})$	$(\mathbf{X_{3(2)}}, \mathbf{Y_{3[2]}})$	$(X_{3(3)}, Y_{3[3]})$

For odd sample size, let $(X_{1(\frac{n_\delta+1}{2})}, Y_{1[\frac{n_\delta+1}{2}]}), (X_{2(\frac{n_\delta+1}{2})}, Y_{2[\frac{n_\delta+1}{2}]}), \ldots, (X_{n_\delta(\frac{n_\delta+1}{2})}, Y_{n_\delta[\frac{n_\delta+1}{2}]})$ denote the observed units by M(O) in δ^{th} stratum. Let $\bar{x}_{st(M(O))} = \sum_{\delta=1}^{\gamma} W_\delta \bar{x}_{\delta(M(O))}$ and $\bar{y}_{st(M(O))} = \sum_{\delta=1}^{\gamma} W_\delta \bar{y}_{\delta(M(O))}$ be the overall sample means of δ^{th} strata for X and Y, respectively. Furthermore, $\bar{y}_{\delta(M(O))} = \frac{1}{n_\delta} \sum_{i=1}^{n_\delta} Y_{\delta i[\frac{n_\delta+1}{2}]}$ and $\bar{x}_{\delta(M(O))} = \frac{1}{n_\delta} \sum_{i=1}^{n_\delta} X_{\delta i(\frac{n_\delta+1}{2})}$, be the sample means in δ^{th} stratum. In addition, $Var(\bar{x}_{st(M(O))}) = \sum_{\delta=1}^{\gamma} \frac{W_\delta^2}{2n_\delta} \sigma^2_{x(\frac{n_\delta+1}{2})}$ and $Var(\bar{y}_{st(M(O))}) = \sum_{\delta=1}^{\gamma} \frac{W_\delta^2}{2n_\delta} \sigma^2_{x[\frac{n_\delta+1}{2}]}$, where $\sigma^2_{x(\frac{n_\delta+1}{2})} = \frac{1}{n_\delta^2} \sum_{\delta=1}^{\gamma} Var\left(X_{\delta i(\frac{n_\delta+1}{2})}\right)$ and $\sigma^2_{y[\frac{n_\delta+1}{2}]} = \frac{1}{n_\delta^2} \sum_{\delta=1}^{\gamma} Var\left(Y_{\delta i[\frac{n_\delta+1}{2}]}\right)$. Note that $Var(\bar{y}_{st(M(O))})$ and $Var(\bar{x}_{st(M(O))})$ are the overall sample variances of δ^{th} strata for Y and X, respectively. The notations $Y_{\delta i[\frac{n_\delta+1}{2}]}$ and $X_{\delta i(\frac{n_\delta+1}{2})}$ are representing the selected MRSS sample values of study and auxiliary variables for odd sample size.

For even sample size, let $(X_{1(\frac{n_\delta}{2})}, Y_{1[\frac{n_\delta}{2}]}), (X_{2(\frac{n_\delta}{2})}, Y_{2[\frac{n_\delta}{2}]}), \ldots, (X_{\frac{n}{2}(\frac{n_\delta}{2})}, Y_{\frac{n_\delta}{2}[\frac{n_\delta}{2}]}), (X_{\frac{n_\delta+2}{2}(\frac{n_\delta+2}{2})}, Y_{\frac{n_\delta+2}{2}[\frac{n_\delta+2}{2}]}), (X_{\frac{n_\delta+4}{2}(\frac{n_\delta+2}{2})}, Y_{\frac{n_\delta+4}{2}[\frac{n_\delta+2}{2}]}), \ldots, (X_{n_\delta(\frac{n_\delta}{2})}, Y_{n_\delta[\frac{n_\delta}{2}]})$ denote the observed units by M(E) in the δ^{th} stratum. Let $\bar{x}_{st(M(E))} = \sum_{\delta=1}^{\gamma} W_\delta \bar{x}_{\delta(M(E))}$ and $\bar{y}_{st(M(E))} = \sum_{\delta=1}^{\gamma} W_\delta \bar{y}_{\delta(M(E))}$ be the overall sample means of δ^{th} strata for X and Y, respectively. Here, $\bar{x}_{\delta(M(E))} = \frac{1}{n_\delta}\left(\sum_{i=1}^{\frac{n_\delta}{2}} X_{\delta i(\frac{n_\delta}{2})} + \sum_{i=\frac{n_\delta+2}{2}}^{n_\delta} X_{\delta i(\frac{n_\delta+2}{2})}\right)$ and $\bar{y}_{\delta(M(E))} = \frac{1}{n_\delta}\left(\sum_{i=1}^{\frac{n_\delta}{2}} Y_{\delta i[\frac{n_\delta}{2}]} + \sum_{i=\frac{n_\delta+2}{2}}^{n_\delta} Y_{\delta i[\frac{n_\delta+2}{2}]}\right)$ are the sample means in δ^{th} stratum. In addition, $Var(\bar{x}_{st(M(E))}) = \sum_{\delta=1}^{\gamma} \frac{W_\delta^2}{2n_\delta}\left(\sigma^2_{x(\frac{n_\delta}{2})} + \sigma^2_{x(\frac{n_\delta+2}{2})}\right)$ and $Var(\bar{y}_{st(M(E))}) = \sum_{\delta=1}^{\gamma} \frac{W_\delta^2}{2n_\delta}\left(\sigma^2_{y[\frac{n_\delta}{2}]} + \sigma^2_{y[\frac{n_\delta+2}{2}]}\right)$, where $\sigma^2_{x(\frac{n_\delta}{2})} = \frac{1}{n_\delta^2} \sum_{\delta=1}^{\gamma} Var\left(X_{\delta i(\frac{n_\delta}{2})}\right)$, $\sigma^2_{x(\frac{n_\delta+2}{2})} = \frac{1}{n_\delta^2} \sum_{\delta=1}^{\gamma} Var\left(X_{\delta i(\frac{n_\delta+2}{2})}\right)$, $\sigma^2_{y[\frac{n_\delta}{2}]} = \frac{1}{n_\delta^2} \sum_{\delta=1}^{\gamma} Var\left(Y_{\delta i[\frac{n_\delta}{2}]}\right)$ and $\sigma^2_{y[\frac{n_\delta+2}{2}]} = \frac{1}{n_\delta^2} \sum_{\delta=1}^{\gamma} Var\left(Y_{\delta i[\frac{n_\delta+2}{2}]}\right)$. Note that $Var(\bar{y}_{st(M(E))})$ and $Var(\bar{x}_{st(M(E))})$ are the overall sample variances of δ^{th} strata for Y and X, respectively. The notations $Y_{\delta i[\frac{n_\delta}{2}]}, Y_{\delta i[\frac{n_\delta+2}{2}]}, X_{\delta i(\frac{n_\delta}{2})}, X_{\delta i(\frac{n_\delta+2}{2})}$ are representing the selected MRSS sample values of study and auxiliary variables for even sample size. For more details about MRSS notations, interested readers may refer to Koyuncu [13].

Let $j = (E, O)$ denote the sample size even or odd; we are adapting Sinha et al.'s [19] calibration estimator under MRSS design as given by

$$\bar{y}_{SM} = \sum_{\delta=1}^{\gamma} \Phi_\delta \bar{y}_{\delta(M(j))}, \tag{1}$$

subject to the constraints

$$\sum_{\delta=1}^{\gamma} \Phi_\delta = \sum_{\delta=1}^{\gamma} W_\delta, \tag{2}$$

$$\sum_{\delta=1}^{\gamma} \Phi_{\delta} \bar{x}_{\delta(M(j))} = \sum_{\delta=1}^{\gamma} W_{\delta} \bar{X}_{\delta(M)} \tag{3}$$

where $\bar{X}_{\delta(M)}$ is the population mean of auxiliary variable in the δ^{th} stratum. By defining $\lambda_{1(M(j))}$ and $\lambda_{2(M(j))}$ as Lagrange multipliers, the Lagrange function is given by

$$\Delta_{(M(j))} = \sum_{\delta=1}^{\gamma} \frac{(\Phi_{\delta} - W_{\delta})^2}{Q_{\delta} W_{\delta}} - 2\lambda_{1(M(j))} \left[\sum_{\delta=1}^{\gamma} \Phi_{\delta} - \sum_{\delta=1}^{\gamma} W_{\delta} \right] - 2\lambda_{2(M(j))} \left[\sum_{\delta=1}^{\gamma} \Phi_{\delta} \bar{x}_{\delta(M(j))} - \sum_{\delta=1}^{\gamma} W_{\delta} \bar{X}_{\delta(M)} \right]. \tag{4}$$

Differentiating $\Delta_{(M(j))}$ according to calibration weight and obtaining the optimum value of Φ_{δ}

$$\Phi_{\delta} = W_{\delta} + Q_{\delta} W_{\delta} \left[\lambda_{2(M(j))} \bar{x}_{\delta(M(j))} + \lambda_{1(M(j))} \right], \tag{5}$$

putting (5) in (2) and (3), we obtain

$$\lambda_{1(M(j))} = -\frac{\sum_{\delta=1}^{\gamma} W_{\delta} \left[\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}^2_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right]^2}, \tag{6}$$

$$\lambda_{2(M(j))} = \frac{\sum_{\delta=1}^{\gamma} W_{\delta} \left[\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}^2_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right]^2}. \tag{7}$$

By substituting these weights in (5), we obtain calibration weights as

$$\Phi_{\delta} = W_{\delta} + Q_{\delta} W_{\delta} \frac{\left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right] \bar{x}_{\delta(M(j))} - \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}^2_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right]^2} \sum_{\delta=1}^{\gamma} W_{\delta} \left[\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))} \right], \tag{8}$$

Finally, by putting Φ_{δ} in $\bar{y}_{SM}$ and obtaining the calibrated mean estimator of study variable

$$\bar{y}_{SM} = \sum_{\delta=1}^{\gamma} W_{\delta} \bar{y}_{\delta(M(j))} + \frac{\left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{y}_{\delta(M(j))} \bar{x}_{\delta(M(j))} \right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{y}_{\delta(M(j))} \right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}^2_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right]^2} \sum_{\delta=1}^{\gamma} W_{\delta} \left[\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))} \right]. \tag{9}$$

This estimator can be rewritten as

$$\bar{y}_{SM} = \bar{y}_{st(M(j))} + \hat{b}_{(j)} \left[\sum_{\delta=1}^{\gamma} W_{\delta} (\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}) \right], \tag{10}$$

where

$$\hat{b}_{(j)} = \frac{\left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{y}_{\delta(M(j))} \bar{x}_{\delta(M(j))} \right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{y}_{\delta(M(j))} \right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}^2_{\delta(M(j))} \right] \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta} W_{\delta} \bar{x}_{\delta(M(j))} \right]^2}. \tag{11}$$

$$\bar{y}_{SM} = \begin{cases} \bar{y}_{st(M(O))} + \hat{b}_{(O)}\left[\sum_{\delta=1}^{\gamma} W_\delta(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(O))})\right] & \text{when n is odd} \\ \bar{y}_{st(M(E))} + \hat{b}_{(E)}\left[\sum_{\delta=1}^{\gamma} W_\delta(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(E))})\right] & \text{when n is even.} \end{cases} \tag{12}$$

2.2. *Garg and Pachori (2019) Estimator [20]*

Let $j = (E, O)$ denote the sample size even or odd; we are adapting Garg and Pachori's (2019) calibration estimator under MRSS design as given by

$$\bar{y}_{GM} = \sum_{\delta=1}^{\gamma} \Phi_\delta \bar{y}_{\delta(M(j))}, \tag{13}$$

subject to the constraints

$$\sum_{\delta=1}^{\gamma} \Phi_\delta = \sum_{\delta=1}^{\gamma} W_\delta, \tag{14}$$

$$\sum_{\delta=1}^{\gamma} \Phi_\delta \hat{C}x_{\delta(M(j))} = \sum_{\delta=1}^{\gamma} W_\delta C x_{\delta(M)} \tag{15}$$

where $(\hat{C}x_{\delta(M(j))}, Cx_{\delta(M)})$ represent the sample and population coefficient of variation (CV) of the auxiliary variable in the δ^{th} stratum. By defining $\lambda_{1(M(j))}$ and $\lambda_{2(M(j))}$ as Lagrange multipliers, the Lagrange function is given by

$$\Delta_{(M(j))} = \sum_{\delta=1}^{\gamma} \frac{(\Phi_\delta - W_\delta)^2}{Q_\delta W_\delta} - 2\lambda_{1(M(j))}\left[\sum_{\delta=1}^{\gamma} \Phi_\delta - \sum_{\delta=1}^{\gamma} W_\delta\right] - 2\lambda_{2(M(j))}\left[\sum_{\delta=1}^{\gamma} \Phi_\delta \hat{C}x_{\delta(M(j))} - \sum_{\delta=1}^{\gamma} W_\delta C x_{\delta(M)}\right]. \tag{16}$$

Differentiating $\Delta_{(M(j))}$ according to calibration weight and obtaining the optimum value of Φ_δ

$$\Phi_\delta = W_\delta + Q_\delta W_\delta\left[\lambda_{2(M(j))}\hat{C}x_{\delta(M(j))} + \lambda_{1(M(j))}\right], \tag{17}$$

putting (17) in (14) and (15), we obtain

$$\lambda_{1(M(j))} = -\frac{\sum_{\delta=1}^{\gamma} W_\delta\left[Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x^2_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right] - \left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]^2}, \tag{18}$$

$$\lambda_{2(M(j))} = \frac{\sum_{\delta=1}^{\gamma} W_\delta\left[Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right]}{\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x^2_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right] - \left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]^2}. \tag{19}$$

By substituting these weights in (17), we obtain calibration weights as

$$\Phi_\delta = W_\delta + Q_\delta W_\delta \frac{\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right]\hat{C}x_{\delta(M(j))} - \left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x^2_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right] - \left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]^2} \sum_{\delta=1}^{\gamma} W_\delta\left[Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right], \tag{20}$$

By putting Φ_δ in $\bar{y}_{GM}$ and obtaining the calibrated mean estimator of study variable

$$\bar{y}_{GM} = \sum_{\delta=1}^{\gamma} W_\delta \bar{y}_{\delta(M(j))} + \frac{\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))} \hat{C}x_{\delta(M(j))}\right] - \left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x^2_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right] - \left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]^2} \sum_{\delta=1}^{\gamma} W_\delta \left[Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right]. \tag{21}$$

This estimator can be rewritten as

$$\bar{y}_{GM} = \bar{y}_{st(M(j))} + \hat{b}_j \left[\sum_{\delta=1}^{\gamma} W_\delta (Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))})\right], \tag{22}$$

where

$$\hat{b}_j = \frac{\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))} \hat{C}x_{\delta(M(j))}\right] - \left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x^2_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right] - \left[\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right]^2}. \tag{23}$$

$$\bar{y}_{GM} = \begin{cases} \bar{y}_{st(M(O))} + \hat{b}_{(O)} \left[\sum_{\delta=1}^{\gamma} W_\delta (Cx_{\delta(M)} - \hat{C}x_{\delta(M(O))})\right] & \text{when n is odd} \\ \bar{y}_{st(M(E))} + \hat{b}_{(E)} \left[\sum_{\delta=1}^{\gamma} W_\delta (Cx_{\delta(M)} - \hat{C}x_{\delta(M(E))})\right] & \text{when n is even.} \end{cases} \tag{24}$$

3. Proposed Family of Estimators in MRSS

Taking motivation from Sinha et al. [19] and Garg and Pachori [20], we propose the following estimator under stratified MRSS

$$\bar{y}_{PM} = \sum_{\delta=1}^{\gamma} \Phi_\delta \bar{y}_{\delta(M(j))}, \tag{25}$$

subject to the constraints

$$\sum_{\delta=1}^{\gamma} \Phi_\delta \bar{x}_{\delta(M(j))} = \sum_{\delta=1}^{\gamma} W_\delta \bar{X}_{\delta(M)} \tag{26}$$

$$\sum_{\delta=1}^{\gamma} \Phi_\delta \hat{C}x_{\delta(M(j))} = \sum_{\delta=1}^{\gamma} W_\delta Cx_{\delta(M)} \tag{27}$$

$$\sum_{\delta=1}^{\gamma} \Phi_\delta = \sum_{\delta=1}^{\gamma} W_\delta, \tag{28}$$

Defining $\lambda_{1(M(j))}$, $\lambda_{2(M(j))}$ and $\lambda_{3(M(j))}$ as Lagrange multipliers, the Lagrange function is given by

$$\begin{aligned} \Delta_{(M(j))} = \sum_{\delta=1}^{\gamma} \frac{(\Phi_\delta - W_\delta)^2}{Q_\delta W_\delta} - 2\lambda_{1(M(j))} \left[\sum_{\delta=1}^{\gamma} \Phi_\delta \bar{x}_{\delta(M(j))} - \sum_{\delta=1}^{\gamma} W_\delta \bar{X}_{\delta(M)} \right] \\ - 2\lambda_{2(M(j))} \left[\sum_{\delta=1}^{\gamma} \Phi_\delta \hat{C}x_{\delta(M(j))} - \sum_{\delta=1}^{\gamma} W_\delta Cx_{\delta(M)}\right] - 2\lambda_{3(M(j))} \left[\sum_{\delta=1}^{\gamma} \Phi_\delta - \sum_{\delta=1}^{\gamma} W_\delta\right]. \end{aligned} \tag{29}$$

Differentiating $\Delta_{(M(j))}$ w.r.t Φ_δ and equating to zero, we obtain the calibration weight

$$\Phi_\delta = W_\delta + Q_\delta W_\delta \left[\lambda_{1(M(j))} \bar{x}_{\delta(M(j))} + \lambda_{2(M(j))} \hat{C}x_{\delta(M(j))} + \lambda_{3(M(j))}\right], \tag{30}$$

Substituting (30) in (26), (27), and (28), respectively, we obtain a system of equations containing three equations. The system of equations in matrix form

$$G_{(3\times3)}\lambda_{(3\times1)} = F_{(3\times1)}, \quad (31)$$

where

$$\lambda_{(3\times1)} = \begin{bmatrix} \lambda_{1(M(j))} \\ \lambda_{2(M(j))} \\ \lambda_{3(M(j))} \end{bmatrix},$$

$$F_{(3\times1)} = \begin{bmatrix} \sum_{\delta=1}^{\gamma} W_\delta\left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}\right) \\ \sum_{\delta=1}^{\gamma} W_\delta\left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right) \\ 0 \end{bmatrix},$$

$$G_{(3\times3)} = \begin{bmatrix} \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right) & \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\hat{C}x_{\delta(M(j))}\right) & \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right) \\ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right) & \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2x_{\delta(M(j))}\right) & \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right) \\ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right) & \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right) & \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right) \end{bmatrix}$$

.

Solving the system of equations for lambdas, we obtain

$$\lambda_{1(M(j))} = \frac{D_1}{H}, \quad \lambda_{2(M(j))} = \frac{D_2}{H}, \quad \lambda_{3(M(j))} = \frac{D_3}{H},$$

where

$$\begin{aligned} D_{1(M(j))} = {} & \sum_{\delta=1}^{\gamma} W_\delta\left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2x_{\delta(M(j))}\right) \\ & - \sum_{\delta=1}^{\gamma} W_\delta\left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)^2 \\ & + \sum_{\delta=1}^{\gamma} W_\delta\left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right) \\ & - \sum_{\delta=1}^{\gamma} W_\delta\left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right), \end{aligned}$$

$$\begin{aligned} D_{2(M(j))} = {} & \sum_{\delta=1}^{\gamma} W_\delta\left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right) \\ & - \sum_{\delta=1}^{\gamma} W_\delta\left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)^2 \\ & - \sum_{\delta=1}^{\gamma} W_\delta\left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right) \\ & + \sum_{\delta=1}^{\gamma} W_\delta\left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right), \end{aligned}$$

$$D_{3(M(j))} = \sum_{\delta=1}^{\gamma} W_\delta \left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)$$
$$- \sum_{\delta=1}^{\gamma} W_\delta \left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right)$$
$$+ \sum_{\delta=1}^{\gamma} W_\delta \left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\hat{C}x_{\delta(M(j))}\right)$$
$$- \sum_{\delta=1}^{\gamma} W_\delta \left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right).$$

$$H = \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right) - \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)^2\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)^2 - \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)^2\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right)$$
$$+ 2\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\hat{C}x_{\delta(M(j))}\right).$$

Substituting these values in (30) and (25), we obtain the calibrated estimator for study variable

$$\bar{y}_{PM} = \bar{y}_{st(M(j))} + \lambda_{1(M(j))}\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right) + \lambda_{2(M(j))}\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right) + \lambda_{3(M(j))}\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right), \tag{32}$$

$$= \sum_{\delta=1}^{\gamma} W_\delta \bar{y}_{\delta(M(j))} + \hat{b}_1\left[\sum_{\delta=1}^{\gamma} W_\delta \left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(j))}\right)\right] + \hat{b}_2\left[\sum_{\delta=1}^{\gamma} W_\delta \left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(j))}\right)\right], \tag{33}$$

where

$$\hat{b}_{1(j)} = \frac{D_4}{H}, \quad \hat{b}_{2(j)} = \frac{D_5}{H},$$

where

$$D_{4(M(j))} = \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)^2$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)$$
$$+ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)$$
$$+ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right),$$

$$D_{5(M(j))} = \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)$$
$$+ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\hat{C}x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)$$
$$+ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)^2.$$

$$\bar{y}_{PM} = \begin{cases} \bar{y}_{st(M(O))} + \hat{b}_{1(O)}\left[\sum_{\delta=1}^{\gamma} W_\delta\left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(O))}\right)\right] + \hat{b}_{2(O)}\left[\sum_{\delta=1}^{\gamma} W_\delta\left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(O))}\right)\right] & \text{when n is odd} \\ \bar{y}_{st(M(E))} + +\hat{b}_{1(E)}\left[\sum_{\delta=1}^{\gamma} W_\delta\left(\bar{X}_{\delta(M)} - \bar{x}_{\delta(M(E))}\right)\right] + \hat{b}_{2(E)}\left[\sum_{\delta=1}^{\gamma} W_\delta\left(Cx_{\delta(M)} - \hat{C}x_{\delta(M(E))}\right)\right] & \text{when n is even.} \end{cases} \quad (34)$$

4. Two Stage Stratified MRSS

In real life, the double sampling approach can be used to estimate population mean when the population mean of an auxiliary variable is unknown. This section makes the assumption that the auxiliary variable's mean is not available. As a result, using the two-stage MRSS approach described in Al-Omari [12] and Koyuncu [13], basic random sampling is utilized at the first stage and median ranked set sampling is utilized at the second stage. Keep in mind that the first-phase sample is $n_a = n^2$ and the second-phase sample is n. Let $\bar{x}_{a\delta(M)}$, $\hat{C}_{xa\delta(M)}$ be the first-phase sample mean and CV of the auxiliary variable. While $\bar{x}_{\delta(M(j))}$, $\bar{y}_{\delta(M(j))}$ and $\hat{C}_{x\delta(M(j))}$ are the second-phase sample characteristics of the auxiliary variable and study variable.

4.1. Adapted Estimators in Two-Stage Stratified MRSS

Sinha et al.'s [19] estimator under two-stage stratified MRSS is as follows

$$\bar{y}_{SaM} = \sum_{\delta=1}^{\gamma} \Phi a_\delta \bar{y}_{\delta(M(j))}, \quad (35)$$

subject to the constraints

$$\sum_{\delta=1}^{\gamma} \Phi a_\delta = \sum_{\delta=1}^{\gamma} W_\delta, \quad (36)$$

$$\sum_{\delta=1}^{\gamma} \Phi a_\delta \bar{x}_{\delta(M(j))} = \sum_{\delta=1}^{\gamma} W_\delta \bar{x}_{a\delta(M)} \quad (37)$$

Defining $\lambda_{1(M(j))}$ and $\lambda_{2(M(j))}$ as Lagrange multipliers, the Lagrange function is given by

$$\Delta_{(M(j))} = \sum_{\delta=1}^{\gamma} \frac{(\Phi a_\delta - W_\delta)^2}{Q_\delta W_\delta} - 2\lambda_{1(M(j))}\left[\sum_{\delta=1}^{\gamma} \Phi a_\delta - \sum_{\delta=1}^{\gamma} W_\delta\right] - 2\lambda_{2(M(j))}\left[\sum_{\delta=1}^{\gamma} \Phi a_\delta \bar{x}_{\delta(M(j))} - \sum_{\delta=1}^{\gamma} W_\delta \bar{x}_{a\delta(M)}\right]. \quad (38)$$

Differentiating $\Delta_{(M(j))}$ according to calibration weights, we obtain

$$\Phi a_{\delta} = W_{\delta} + Q_{\delta}W_{\delta}\left[\lambda_{2(M(j))}\bar{x}_{\delta(M(j))} + \lambda_{1(M(j))}\right], \tag{39}$$

putting (39) in (36) and (37), we obtain

$$\lambda_{1(M(j))} = -\frac{\sum_{\delta=1}^{\gamma} W_{\delta}\left[\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]^{2}}, \tag{40}$$

$$\lambda_{2(M(j))} = \frac{\sum_{\delta=1}^{\gamma} W_{\delta}\left[\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]^{2}}. \tag{41}$$

By substituting these weights in (39), we obtain the optimum calibration weight as

$$\Phi a_{\delta} = W_{\delta} + Q_{\delta}W_{\delta}\frac{\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right]\bar{x}_{\delta(M(j))} - \left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]^{2}}\sum_{\delta=1}^{\gamma} W_{\delta}\left[\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))}\right], \tag{42}$$

By putting Φa_{δ} in $\bar{y}_{SaM}$

$$\bar{y}_{SaM} = \sum_{\delta=1}^{\gamma} W_{\delta}\bar{y}_{\delta(M(j))} + \frac{\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{y}_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{y}_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]^{2}}\sum_{\delta=1}^{\gamma} W_{\delta}\left[\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))}\right]. \tag{43}$$

This estimator can be rewritten as

$$\bar{y}_{SaM} = \bar{y}_{st(M(j))} + \hat{b}_{j}\left[\sum_{\delta=1}^{\gamma} W_{\delta}(\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))})\right], \tag{44}$$

where

$$\hat{b}_{j} = \frac{\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{y}_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{y}_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\right] - \left[\sum_{\delta=1}^{\gamma} Q_{\delta}W_{\delta}\bar{x}_{\delta(M(j))}\right]^{2}}. \tag{45}$$

$$\bar{y}_{SaM} = \begin{cases} \bar{y}_{st(M(O))} + \hat{b}_{O}\left[\sum_{\delta=1}^{\gamma} W_{\delta}(\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))})\right] & \text{when n is odd} \\ \bar{y}_{st(M(E))} + \hat{b}_{E}\left[\sum_{\delta=1}^{\gamma} W_{\delta}(\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))})\right] & \text{when n is even.} \end{cases} \tag{46}$$

Garg and Pachori's [20] estimator under two-stage MRSS is given below

$$\bar{y}_{GaM} = \sum_{\delta=1}^{\gamma} \Phi a_{\delta}\bar{y}_{\delta(M(j))}, \tag{47}$$

subject to the constraints

$$\sum_{\delta=1}^{\gamma}\Phi a_{\delta}=\sum_{\delta=1}^{\gamma}W_{\delta}, \tag{48}$$

$$\sum_{\delta=1}^{\gamma}\Phi a_{\delta}\hat{C}_{x\delta(M(j))}=\sum_{\delta=1}^{\gamma}W_{\delta}\hat{C}_{xa\delta(M)} \tag{49}$$

Defining $\lambda_{1(M(j))}$ and $\lambda_{2(M(j))}$ as Lagrange multipliers, the Lagrange function is given by

$$\Delta_{(M(j))}=\sum_{\delta=1}^{\gamma}\frac{(\Phi a_{\delta}-W_{\delta})^{2}}{Q_{\delta}W_{\delta}}-2\lambda_{1(M(j))}\left[\sum_{\delta=1}^{\gamma}\Phi a_{\delta}-\sum_{\delta=1}^{\gamma}W_{\delta}\right]-2\lambda_{2(M(j))}\left[\sum_{\delta=1}^{\gamma}\Phi a_{\delta}\hat{C}_{x\delta(M(j))}-\sum_{\delta=1}^{\gamma}W_{\delta}\hat{C}_{xa\delta(M)}\right]. \tag{50}$$

Differentiating $\Delta_{(M(j))}$ according to calibration weights, we obtain

$$\Phi a_{\delta}=W_{\delta}+Q_{\delta}W_{\delta}\left[\lambda_{2(M(j))}\hat{C}_{x\delta(M(j))}+\lambda_{1(M(j))}\right], \tag{51}$$

putting (51) in (48) and (49), we obtain

$$\lambda_{1(M(j))}=-\frac{\sum_{\delta=1}^{\gamma}W_{\delta}\left[\hat{C}_{xa\delta(M)}-\hat{C}_{x\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]-\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]^{2}}, \tag{52}$$

$$\lambda_{2(M(j))}=\frac{\sum_{\delta=1}^{\gamma}W_{\delta}\left[\hat{C}_{xa\delta(M)}-\hat{C}_{x\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]}{\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]-\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]^{2}}. \tag{53}$$

By substituting these weights in (51), we obtain calibration weights as

$$\Phi a_{\delta}=W_{\delta}+Q_{\delta}W_{\delta}\frac{\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]\hat{C}_{x\delta(M(j))}-\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]-\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]^{2}}\sum_{\delta=1}^{\gamma}W_{\delta}\left[\hat{C}_{xa\delta(M)}-\hat{C}_{x\delta(M(j))}\right], \tag{54}$$

By putting Φa_{δ} in $\bar{y}_{GaM}$

$$\bar{y}_{GaM}=\sum_{\delta=1}^{\gamma}W_{\delta}\bar{y}_{\delta(M(j))}+\frac{\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{y}_{\delta(M(j))}\hat{C}_{x\delta(M(j))}\right]-\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{y}_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]-\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]^{2}}\sum_{\delta=1}^{\gamma}W_{\delta}\left[\hat{C}_{xa\delta(M)}-\hat{C}_{x\delta(M(j))}\right]. \tag{55}$$

This estimator can be rewritten as

$$\bar{y}_{GaM}=\bar{y}_{st(M(j))}+\hat{b}_{j}\left[\sum_{\delta=1}^{\gamma}W_{\delta}(\hat{C}_{xa\delta(M)}-\hat{C}_{x\delta(M(j))})\right], \tag{56}$$

where

$$\hat{b}_{j}=\frac{\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{y}_{\delta(M(j))}\hat{C}_{x\delta(M(j))}\right]-\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{y}_{\delta(M(j))}\right]}{\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\bar{x}^{2}_{\delta(M(j))}\right]\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\right]-\left[\sum_{\delta=1}^{\gamma}Q_{\delta}W_{\delta}\hat{C}_{x\delta(M(j))}\right]^{2}}. \tag{57}$$

$$\bar{y}_{GaM} = \begin{cases} \bar{y}_{st(M(O))} + \hat{b}_O\left[\sum_{\delta=1}^{\gamma} W_\delta(\hat{C}_{xa\delta(M)} - \hat{C}_{x\delta(M(O))})\right] & \text{when n is odd} \\ \bar{y}_{st(M(E))} + \hat{b}_E\left[\sum_{\delta=1}^{\gamma} W_\delta(\hat{C}_{xa\delta(M)} - \hat{C}_{x\delta(M(E))})\right] & \text{when n is even.} \end{cases} \quad (58)$$

4.2. Proposed Family of Estimators in Two Stage Stratified MRSS

The proposed estimator under stratified MRSS is given below

$$\bar{y}_{PaM} = \sum_{\delta=1}^{\gamma} \Phi_{a\delta}\bar{y}_{\delta(M(j))}, \quad (59)$$

subject to the constraints

$$\sum_{\delta=1}^{\gamma} \Phi_{a\delta}\bar{x}_{\delta(M(j))} = \sum_{\delta=1}^{\gamma} W_\delta \bar{x}_{a\delta(M)} \quad (60)$$

$$\sum_{\delta=1}^{\gamma} \Phi_{a\delta}\hat{C}x_{\delta(M(j))} = \sum_{\delta=1}^{\gamma} W_\delta \hat{C}_{xa\delta(M)} \quad (61)$$

$$\sum_{\delta=1}^{\gamma} \Phi_{a\delta} = \sum_{\delta=1}^{\gamma} W_\delta, \quad (62)$$

Defining $\lambda_{1(M(j))}$, $\lambda_{2(M(j))}$ and $\lambda_{3(M(j))}$ as Lagrange multipliers, the Lagrange function is given by

$$\begin{aligned}\Delta_{(M(j))} = \sum_{\delta=1}^{\gamma} \frac{(\Phi_{a\delta} - W_\delta)^2}{Q_\delta W_\delta} - 2\lambda_{1(M(j))}\left[\sum_{\delta=1}^{\gamma} \Phi_{a\delta}\bar{x}_{\delta(M(j))} - \sum_{\delta=1}^{\gamma} W_\delta \bar{x}_{a\delta(M)}\right] \\ - 2\lambda_{2(M(j))}\left[\sum_{\delta=1}^{\gamma} \Phi_{a\delta}\hat{C}x_{\delta(M(j))} - \sum_{\delta=1}^{\gamma} W_\delta \hat{C}_{xa\delta(M)}\right] - 2\lambda_{3(M(j))}\left[\sum_{\delta=1}^{\gamma} \Phi_{a\delta} - \sum_{\delta=1}^{\gamma} W_\delta\right].\end{aligned} \quad (63)$$

Differentiating $\Delta_{(M(j))}$ according to calibration weights, we obtain

$$\Phi_{a\delta} = W_\delta + Q_\delta W_\delta\left[\lambda_{1(M(j))}\bar{x}_{\delta(M(j))} + \lambda_{2(M(j))}\hat{C}x_{\delta(M(j))} + \lambda_{3(M(j))}\right], \quad (64)$$

Substituting (64) in (60), (61), and (62), respectively, we obtain a system of equations containing three equations. The system of equations in matrix form

$$G_{(3\times3)}\lambda_{(3\times1)} = F_{(3\times1)}, \quad (65)$$

where

$$\lambda_{(3\times1)} = \begin{bmatrix} \lambda_{1(M(j))} \\ \lambda_{2(M(j))} \\ \lambda_{3(M(j))} \end{bmatrix},$$

$$F_{(3\times1)} = \begin{bmatrix} \sum_{\delta=1}^{\gamma} W_\delta\left(\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))}\right) \\ \sum_{\delta=1}^{\gamma} W_\delta\left(\hat{C}_{xa\delta(M)} - \hat{C}x_{\delta(M(j))}\right) \\ 0 \end{bmatrix},$$

$$G_{(3\times 3)}=\begin{bmatrix}\left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right) & \left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta \bar{x}_{\delta(M(j))}\hat{C}x_{\delta(M(j))}\right) & \left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\\ \left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right) & \left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta \hat{C}^2x_{\delta(M(j))}\right) & \left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)\\ \left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right) & \left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right) & \left(\sum\limits_{\delta=1}^{\gamma}Q_\delta W_\delta\right)\end{bmatrix}.$$

By substituting the values of $G_{(3\times 3)}$,$\lambda_{(3\times 1)}$, $F_{(3\times 1)}$ in (65) and then solving the system of equations for lambdas, we obtain

$$\lambda_{1(M(j))}=\frac{D_{1(M(j))}}{H},\quad \lambda_{2(M(j))}=\frac{D_{2(M(j))}}{H},\quad \lambda_{3(M(j))}=\frac{D_{1(M(j))}}{H},$$

where

$$\begin{aligned}D_{1(M(j))}=&\sum_{\delta=1}^{\gamma}W_\delta\left(\bar{x}_{a\delta(M)}-\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}^2x_{\delta(M(j))}\right)\\&-\sum_{\delta=1}^{\gamma}W_\delta\left(\bar{x}_{a\delta(M)}-\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}x_{\delta(M(j))}\right)^2\\&+\sum_{\delta=1}^{\gamma}W_\delta\left(\hat{C}_{xa\delta(M)}-\hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}x_{\delta(M(j))}\right)\\&-\sum_{\delta=1}^{\gamma}W_\delta\left(\hat{C}_{xa\delta(M)}-\hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right),\end{aligned}$$

$$\begin{aligned}D_{2(M(j))}=&\sum_{\delta=1}^{\gamma}W_\delta\left(\hat{C}_{xa\delta(M)}-\hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\bar{x}^2_{\delta(M(j))}\right)\\&-\sum_{\delta=1}^{\gamma}W_\delta\left(\hat{C}_{xa\delta(M)}-\hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\bar{x}_{\delta(M(j))}\right)^2\\&-\sum_{\delta=1}^{\gamma}W_\delta\left(\bar{x}_{a\delta(M)}-\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\right)\\&+\sum_{\delta=1}^{\gamma}W_\delta\left(\bar{x}_{a\delta(M)}-\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}x_{\delta(M(j))}\right),\end{aligned}$$

$$\begin{aligned}D_{3(M(j))}=&\sum_{\delta=1}^{\gamma}W_\delta\left(\bar{x}_{a\delta(M)}-\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)\\&-\sum_{\delta=1}^{\gamma}W_\delta\left(\bar{x}_{a\delta(M)}-\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}^2x_{\delta(M(j))}\right)\\&+\sum_{\delta=1}^{\gamma}W_\delta\left(\hat{C}_{xa\delta(M)}-\hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\bar{x}_{\delta(M(j))}\hat{C}x_{\delta(M(j))}\right)\\&-\sum_{\delta=1}^{\gamma}W_\delta\left(\hat{C}_{xa\delta(M)}-\hat{C}x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\bar{x}^2_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma}Q_\delta W_\delta\hat{C}x_{\delta(M(j))}\right).\end{aligned}$$

$$H = \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right) - \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)^2\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\right)^2 - \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\right)^2\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right)$$
$$+ 2\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))} \hat{C} x_{\delta(M(j))}\right).$$

Substituting these values in (64) and (59), we have

$$\bar{y}_{PaM} = \bar{y}_{st(M(j))} + \lambda_{1(M(j))}\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right) + \lambda_{2(M(j))}\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right) + \lambda_{3(M(j))}\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right),$$

$$= \sum_{\delta=1}^{\gamma} W_\delta \bar{y}_{\delta(M(j))} + \hat{b}_1\left[\sum_{\delta=1}^{\gamma} W_\delta\left(\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(j))}\right)\right] + \hat{b}_2\left[\sum_{\delta=1}^{\gamma} W_\delta\left(\hat{C}_{xa\delta(M)} - \hat{C} x_{\delta(M(j))}\right)\right], \quad (66)$$

where

$$\hat{b}_{1(j)} = \frac{D_{4(M(j))}}{H}, \quad \hat{b}_{2(j)} = \frac{D_{5(M(j))}}{H},$$

where

$$D_{4(M(j))} = \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)$$
$$+ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\right)$$
$$+ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}^2 x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C} x_{\delta(M(j))}\right)^2,$$

$$D_{5(M(j))} = \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)$$
$$+ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\hat{C}x_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\right)$$
$$+ \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}^2_{\delta(M(j))}\right)$$
$$- \left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \hat{C}x_{\delta(M(j))}\bar{y}_{\delta(M(j))}\right)\left(\sum_{\delta=1}^{\gamma} Q_\delta W_\delta \bar{x}_{\delta(M(j))}\right)^2.$$

$$\bar{y}_{PaM} = \begin{cases} \bar{y}_{st(M(O))} + \hat{b}_{1(O)}\left[\sum_{\delta=1}^{\gamma} W_\delta\left(\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(O))}\right)\right] + \hat{b}_{2(O)}\left[\sum_{\delta=1}^{\gamma} W_\delta\left(\hat{C}_{xa\delta(M)} - \hat{C}x_{\delta(M(O))}\right)\right] & \text{when n is odd} \\ \bar{y}_{st(M(E))} + \hat{b}_{1(E)}\left[\sum_{\delta=1}^{\gamma} W_\delta\left(\bar{x}_{a\delta(M)} - \bar{x}_{\delta(M(E))}\right)\right] + \hat{b}_{2(E)}\left[\sum_{\delta=1}^{\gamma} W_\delta\left(\hat{C}_{xa\delta(M)} - \hat{C}x_{\delta(M(E))}\right)\right] & \text{when n is even.} \end{cases} \quad (67)$$

Note that all the family members of the adapted and proposed classes of estimators based on different values Q_δ are provided in Table 3. It is worth mentioning that Q_δ is a suitably chosen weight for determining various types of estimators as provided in Table 3.

Table 3. Family members of all classes.

MRSS Estimators	Q_δ	Two-Stage MRSS Estimators
$\bar{y}_{SM_I}$	1	$\bar{y}_{SaM_I}$
$\bar{y}_{SM_{II}}$	$1/\hat{C}_{x\delta(M(j))}$	$\bar{y}_{SaM_{II}}$
$\bar{y}_{SM_{III}}$	$1/\bar{x}_{\delta(M(j))}$	$\bar{y}_{SaM_{III}}$
$\bar{y}_{GM_I}$	1	$\bar{y}_{GaM_I}$
$\bar{y}_{GM_{II}}$	$1/\hat{C}_{x\delta(M(j))}$	$\bar{y}_{GaM_{II}}$
$\bar{y}_{GM_{III}}$	$1/\bar{x}_{\delta(M(j))}$	$\bar{y}_{GaM_{III}}$
$\bar{y}_{PM_I}$	1	$\bar{y}_{PaM_I}$
$\bar{y}_{PM_{II}}$	$1/\hat{C}_{x\delta(M(j))}$	$\bar{y}_{PaM_{II}}$
$\bar{y}_{PM_{III}}$	$1/\bar{x}_{\delta(M(j))}$	$\bar{y}_{PaM_{III}}$

5. Simulation Study

5.1. Simulation Design

The simulation experiments considered in this section are designed to provide insight into the efficiency of the proposed estimators $\bar{y}_{PM_I}$, $\bar{y}_{PM_{II}}$, $\bar{y}_{PM_{III}}$, $\bar{y}_{PaM_I}$, $\bar{y}_{PaM_{II}}$ and $\bar{y}_{PaM_{III}}$ compared to the estimators $\bar{y}_{SM_I}$, $\bar{y}_{SM_{II}}$, $\bar{y}_{SM_{III}}$, $\bar{y}_{SaM_I}$, $\bar{y}_{SaM_{II}}$, $\bar{y}_{SaM_{III}}$, $\bar{y}_{GM_I}$, $\bar{y}_{GM_{II}}$, $\bar{y}_{GM_{III}}$, $\bar{y}_{GaM_I}$, $\bar{y}_{GaM_{II}}$, and $\bar{y}_{GaM_{III}}$. All samples were generated from a finite stratified population having size $\Omega = 1000$ in each stratum, using four distinctive (with respect to variance–covariance matrix) bivariate Gaussian distributions for each stratum with $(\mu_x = 2, \mu_y = 4)$ and the variance–covariance matrix given, respectively, by

- Stratum 1

$$\Sigma = \begin{bmatrix} 1 & 0.90 \\ 0.90 & 1 \end{bmatrix}$$

- Stratum 2

$$\Sigma = \begin{bmatrix} 1 & 0.76 \\ 0.76 & 1 \end{bmatrix}$$

- Stratum 3

$$\Sigma = \begin{bmatrix} 1 & 0.55 \\ 0.55 & 1 \end{bmatrix}$$

- Stratum 4

$$\Sigma = \begin{bmatrix} 1 & 0.30 \\ 0.30 & 1 \end{bmatrix}.$$

Taking motivation from Koyuncu [13], we select samples from the above-mentioned stratified population. As they used stratified SRS, however, we are adapting their framework under stratified MRSS design. For the fair comparison among adapted and proposed estimators, we draw different sample sizes regarding even and odd sample sizes under MRSS. For increasing the readability of the article, we are providing the considered sample sizes in Table 4, where A_1, A_2, A_3, A_4, B_1, B_2, B_3, and B_4 represent the overall selected strata sample sizes at the first and second stage.

Table 4. Details of different sample sizes for simulation study.

	MRSS (n_1, n_2, n_3, n_4)	**Table**	**Two-Stage MRSS** (na_1, na_2, na_3, na_4) (n_1, n_2, n_3, n_4)
A_1	$(3,5,5,3)$	Table 5	$(9,25,25,9)$
B_1		Table 5	$(3,5,5,3)$
A_2	$(4,6,6,4)$	Table 6	$(16,36,36,16)$
B_2		Table 6	$(4,6,6,4)$
A_3	$(5,7,7,5)$	Table 7	$(25,49,49,25)$
B_3		Table 7	$(5,7,7,5)$
A_4	$(6,8,8,6)$	Table 8	$(36,64,64,36)$
B_4		Table 8	$(6,8,8,6)$

Table 5. PRE values for (A_1, B_1).

	PRE MRSS				**PRE Two-Stage MRSS**		
$\hat{\phi}$	$\bar{y}_{PM_I}$	$\bar{y}_{PM_{II}}$	$\bar{y}_{PM_{III}}$	$\hat{\phi}$	$\bar{y}_{PaM_I}$	$\bar{y}_{PaM_{II}}$	$\bar{y}_{PaM_{III}}$
$\bar{y}_{SM_I}$	133.0992	132.9015	133.0761	$\bar{y}_{SaM_I}$	118.0992	119.9015	116.5761
$\bar{y}_{SM_{II}}$	132.8973	132.7000	132.8742	$\bar{y}_{SaM_{II}}$	127.8973	126.7000	126.3742
$\bar{y}_{SM_{III}}$	133.1524	132.9547	133.1293	$\bar{y}_{SaM_{III}}$	129.1524	127.9547	129.6293
$\bar{y}_{GM_I}$	536.1575	535.3614	536.0645	$\bar{y}_{GaM_I}$	534.1575	532.3614	532.7645
$\bar{y}_{GM_{II}}$	530.6326	529.8446	530.5405	$\bar{y}_{GaM_{II}}$	522.6326	522.8446	520.3405
$\bar{y}_{GM_{III}}$	536.6391	535.8423	536.5460	$\bar{y}_{GaM_{III}}$	525.6391	525.8423	522.1460

Table 6. PRE values for (A_2, B_2).

	PRE MRSS				**PRE Two-Stage MRSS**		
$\hat{\phi}$	$\bar{y}_{PM_I}$	$\bar{y}_{PM_{II}}$	$\bar{y}_{PM_{III}}$	$\hat{\phi}$	$\bar{y}_{PaM_I}$	$\bar{y}_{PaM_{II}}$	$\bar{y}_{PaM_{III}}$
$\bar{y}_{SM_I}$	1017.100	1017.528	1017.366	$\bar{y}_{SaM_I}$	992.0999	994.5282	1009.366
$\bar{y}_{SM_{II}}$	1018.984	1019.413	1019.251	$\bar{y}_{SaM_{II}}$	1003.9842	1003.4134	1009.751
$\bar{y}_{SM_{III}}$	1016.884	1017.312	1017.149	$\bar{y}_{SaM_{III}}$	1002.8836	1002.3119	1010.649
$\bar{y}_{GM_I}$	1560.270	1553.899	1563.418	$\bar{y}_{GaM_I}$	1557.2701	1562.8991	1561.118
$\bar{y}_{GM_{II}}$	1647.891	1650.885	1648.266	$\bar{y}_{GaM_{II}}$	1645.8907	1649.8852	1648.766
$\bar{y}_{GM_{III}}$	1561.818	1564.453	1563.970	$\bar{y}_{GaM_{III}}$	1559.8180	1567.4527	1562.370

Table 7. PRE values for (A_3, B_3).

	PRE MRSS				PRE Two-Stage MRSS		
$\hat{\phi}$	$\bar{y}_{PM_I}$	$\bar{y}_{PM_{II}}$	$\bar{y}_{PM_{III}}$	$\hat{\phi}$	$\bar{y}_{PaM_I}$	$\bar{y}_{PaM_{II}}$	$\bar{y}_{PaM_{III}}$
$\bar{y}_{SM_I}$	122.9272	124.3531	122.8358	$\bar{y}_{SaM_I}$	101.92718	101.3531	114.8358
$\bar{y}_{SM_{II}}$	126.9125	128.3847	126.8182	$\bar{y}_{SaM_{II}}$	111.91254	112.3847	117.3182
$\bar{y}_{SM_{III}}$	123.1588	124.5874	123.0673	$\bar{y}_{SaM_{III}}$	109.15884	109.5874	116.5673
$\bar{y}_{GM_I}$	1627.4355	1704.3119	1622.5118	$\bar{y}_{GaM_I}$	1615.43551	1691.3119	1617.2118
$\bar{y}_{GM_{II}}$	1720.3714	1798.3259	1715.3786	$\bar{y}_{GaM_{II}}$	1702.37144	1791.3259	1705.8786
$\bar{y}_{GM_{III}}$	1658.3218	1735.5565	1653.3751	$\bar{y}_{GaM_{III}}$	1637.32178	1715.5565	1637.7751

Table 8. PRE values for (A_4, B_4).

	PRE MRSS				PRE Two-Stage MRSS		
$\hat{\phi}$	$\bar{y}_{PM_I}$	$\bar{y}_{PM_{II}}$	$\bar{y}_{PM_{III}}$	$\hat{\phi}$	$\bar{y}_{PaM_I}$	$\bar{y}_{PaM_{II}}$	$\bar{y}_{PaM_{III}}$
$\bar{y}_{SM_I}$	782.1886	800.9280	772.5701	$\bar{y}_{SaM_I}$	757.1886	777.9280	764.5701
$\bar{y}_{SM_{II}}$	788.7738	807.6710	779.0744	$\bar{y}_{SaM_{II}}$	773.7738	791.6710	769.5744
$\bar{y}_{SM_{III}}$	781.7996	800.5298	772.1859	$\bar{y}_{SaM_{III}}$	767.7996	785.5298	765.6859
$\bar{y}_{GM_I}$	1485.6027	1521.5577	1467.9012	$\bar{y}_{GaM_I}$	1484.6027	1520.5577	1466.6012
$\bar{y}_{GM_{II}}$	1484.5264	1520.1681	1467.9857	$\bar{y}_{GaM_{II}}$	1482.5264	1519.1681	1465.4857
$\bar{y}_{GM_{III}}$	1492.3204	1528.9696	1474.7493	$\bar{y}_{GaM_{III}}$	1490.3204	1526.9696	1472.1493

For single-stage stratified MRSS, $K_1 = 7000$ samples of sizes $n = A_1, A_2, A_3, A_4$ were chosen independently under the stratified MRSS design from the population, and for the kth sample, the estimate $(\hat{\phi}^{(k_1)}, \hat{\phi}^{(k_2)})$ of μ_y was calculated, where

$$\hat{\phi}^{(k_1)} = \bar{y}_{SM_I}, \bar{y}_{SM_{II}}, \bar{y}_{SM_{III}}, \bar{y}_{GM_I}, \bar{y}_{GM_{II}}, \bar{y}_{SM_{III}}.$$

$$\hat{\phi}^{(k_2)} = \bar{y}_{PM_I}, \bar{y}_{PM_{II}}, \bar{y}_{PM_{III}}.$$

Al-Omari [12] and Koyuncu [13] considered double MRSS design. However, we are adapting their strategy for double MRSS in δ^{th} stratum where $K_1 = 7000$ samples of sizes $n_{a\delta} = n_\delta \times n_\delta = A_1, A_2, A_3, A_4$ were chosen independently under the SRS at the first stage and then stratified MRSS samples of sizes $n_\delta = B_1, B_2, B_3, B_4$ were chosen from $n_\delta \times n_\delta$ at the second stage, and for the k^{th} sample, the estimate $(\hat{\phi}^{(k_1)}, \hat{\phi}^{(k_2)})$ of μ_y was calculated, where

$$\hat{\phi}^{(k_1)} = \bar{y}_{SaM_I}, \bar{y}_{SaM_{II}}, \bar{y}_{SaM_{III}}, \bar{y}_{GaM_I}, \bar{y}_{GaM_{II}}, \bar{y}_{SaM_{III}}.$$

$$\hat{\phi}^{(k_2)} = \bar{y}_{PaM_I}, \bar{y}_{PaM_{II}}, \bar{y}_{PaM_{III}}.$$

The bias and MSE were calculated from the formula given below

$$\text{Bias}(\hat{\phi}^{(k_1)}) = \sum_{k_1=1}^{K_1} (\hat{\phi}^{(k_1)} - \mu_y)/K_1.$$

$$\text{MSE}(\hat{\phi}^{(k_1)}) = \sum_{k_1=1}^{K_1} (\hat{\phi}^{(k_1)} - \mu_y)^2/K_1.$$

$$\text{Bias}(\hat{\phi}^{(k_2)}) = \sum_{k_1=1}^{K_1} (\hat{\phi}^{(k_2)} - \mu_y)/K_1.$$

$$\text{MSE}(\hat{\phi}^{(k_2)}) = \sum_{k_1=1}^{K_1} (\hat{\phi}^{(k_2)} - \mu_y)^2/K_1.$$

The calculated bias values are presented in Table 9.

Table 9. Bias values of estimators for simulation study.

$\hat{\phi}$	(A_1, B_1)	(A_2, B_2)	(A_3, B_3)	(A_4, B_4)
MRSS				
$\bar{y}_{SM_I}$	0.9587	0.8480	0.8210	0.7302
$\bar{y}_{SM_{II}}$	0.6095	0.4988	0.4718	0.3810
$\bar{y}_{SM_{III}}$	0.7104	0.5997	0.5727	0.4819
$\bar{y}_{GM_I}$	0.7864	0.6757	0.6487	0.5579
$\bar{y}_{GM_{II}}$	0.6381	0.5274	0.5004	0.4096
$\bar{y}_{GM_{III}}$	0.5753	0.4646	0.4376	0.3468
$\bar{y}_{PM_I}$	0.5550	0.4443	0.4173	0.3265
$\bar{y}_{PM_{II}}$	0.4471	0.3364	0.3094	0.2186
$\bar{y}_{PM_{III}}$	0.4261	0.3154	0.2884	0.1976
Two stage MRSS				
$\bar{y}_{SaM_I}$	0.6656	0.5549	0.5279	0.4371
$\bar{y}_{SaM_{II}}$	0.5114	0.4007	0.3737	0.2829
$\bar{y}_{SaM_{III}}$	0.5163	0.4056	0.3786	0.2878
$\bar{y}_{GaM_I}$	0.6020	0.4913	0.4643	0.3735
$\bar{y}_{GaM_{II}}$	0.4816	0.3709	0.3439	0.2531
$\bar{y}_{GaM_{III}}$	0.4398	0.3291	0.3021	0.2113
$\bar{y}_{PaM_I}$	0.2537	0.1430	0.1160	0.0252
$\bar{y}_{PaM_{II}}$	0.2739	0.1632	0.1362	0.0454
$\bar{y}_{PaM_{III}}$	0.2593	0.1486	0.1216	0.0308

After calculating the MSE values separately, the efficiency of the estimators was compared by using the percent relative efficiency (PRE) formula

$$\text{PRE}(\hat{\phi}^{(k_1)}, \hat{\phi}^{(k_2)}) = \frac{\text{MSE}(\hat{\phi}^{(k_1)})}{\text{MSE}(\hat{\phi}^{(k_2)})} \times 100,$$

We provide our PRE results in Tables 5–8.

5.2. Real Life Application

We also assessed the properties of the proposed estimators using a real-life example. We use the data concerning body mass index (BMI) as a study variable and the weight as auxiliary variables for 800 people in Turkey in 2014. The open-access dataset belongs to a health survey prepared by the Turkish Statistical Institute (TSI) that examines the determinants "factors which may affect obesity" of health-related behaviors in Turkey for 800 people. All the dataset information is already available in Cetin and Koyuncu [21]. The collected data consist of $N = 800$ observations with $\rho_{xy} = 0.86$, $\mu_y = 23.77$, $\mu_x = 67.55$, $C_x = 0.20$ and $C_y = 0.17$. We stratified the dataset using gender in two strata. Some major characteristics of the strata as follows

- **Stratum-I** $N_{h1} = 477$, $\rho_{yx_{h1}} = 0.90$, $\mu_{y_{h1}} = 22.36$, $\mu_{x_{h1}} = 59.99$, $C_{x_{h1}} = 0.17$, $C_{y_{h1}} = 0.17$.
- **Stratum-II** $N_{h2} = 323$, $\rho_{yx_{h2}} = 0.80$, $\mu_{y_{h2}} = 25.85$, $\mu_{x_{h2}} = 78.72$, $C_{x_{h2}} = 0.04$, $C_{y_{h1}} = 0.13$.

The calculated bias values are presented in Table 10.

Table 10. Bias values of estimators for real-life data.

$\hat{\phi}$	**MRSS**	$\hat{\phi}$	**Two Stage MRSS**
$\bar{y}_{SM_I}$	2.9587	$\bar{y}_{SaM_I}$	2.8480
$\bar{y}_{SM_{II}}$	2.7404	$\bar{y}_{SaM_{II}}$	2.6297
$\bar{y}_{SM_{III}}$	2.6413	$\bar{y}_{SaM_{III}}$	2.5306
$\bar{y}_{GM_I}$	2.6293	$\bar{y}_{GaM_I}$	2.5186
$\bar{y}_{GM_{II}}$	2.4296	$\bar{y}_{GaM_{II}}$	2.3189
$\bar{y}_{GM_{III}}$	2.3194	$\bar{y}_{GaM_{III}}$	2.2087
$\bar{y}_{PM_I}$	2.0246	$\bar{y}_{PaM_I}$	1.9139
$\bar{y}_{PM_{II}}$	2.0024	$\bar{y}_{PaM_{II}}$	1.8917
$\bar{y}_{PM_{III}}$	2.1110	$\bar{y}_{PaM_{III}}$	2.0003

The numerical comparisons based on PRE for BMT data are provided in Tables 11 and 12.

Table 11. PRE values for BMI data for odd sample size.

	PRE MRSS (3,5)				PRE Two Stage MRSS (9,25,3,5)		
$\hat{\phi}$	$\bar{y}_{PM_I}$	$\bar{y}_{PM_{II}}$	$\bar{y}_{PM_{III}}$	$\hat{\phi}$	$\bar{y}_{PaM_I}$	$\bar{y}_{PaM_{II}}$	$\bar{y}_{PaM_{III}}$
$\bar{y}_{SM_I}$	569.1733	618.0211	579.6620	$\bar{y}_{SaM_I}$	549.1333	593.1211	531.5231
$\bar{y}_{SM_{II}}$	548.0034	598.8768	567.9848	$\bar{y}_{SaM_{II}}$	530.8009	571.5667	521.7440
$\bar{y}_{SM_{III}}$	590.3432	637.1654	591.3392	$\bar{y}_{SaM_{III}}$	567.4657	614.6755	541.3022
$\bar{y}_{GM_I}$	593.0307	642.8811	603.7382	$\bar{y}_{GaM_I}$	572.5669	617.4752	554.5779
$\bar{y}_{GM_{II}}$	571.8608	623.7368	592.0610	$\bar{y}_{GaM_{II}}$	554.2345	595.9208	544.7988
$\bar{y}_{GM_{III}}$	614.2006	662.0254	615.4154	$\bar{y}_{GaM_{III}}$	590.8993	639.0296	564.3570

Table 12. PRE values for BMI data for even sample size.

	PRE MRSS (4,6)				PRE Two Stage MRSS (16,36,4,6)		
$\hat{\phi}$	$\bar{y}_{PM_I}$	$\bar{y}_{PM_{II}}$	$\bar{y}_{PM_{III}}$	$\hat{\phi}$	$\bar{y}_{PaM_I}$	$\bar{y}_{PaM_{II}}$	$\bar{y}_{PaM_{III}}$
$\bar{y}_{SM_I}$	877.1798	931.0274	881.5154	$\bar{y}_{SaM_I}$	851.1444	896.1555	866.5199
$\bar{y}_{SM_{II}}$	856.0099	911.8831	869.8382	$\bar{y}_{SaM_{II}}$	832.8120	874.6011	856.7408
$\bar{y}_{SM_{III}}$	898.3497	950.1717	893.1926	$\bar{y}_{SaM_{III}}$	869.4768	917.7099	876.2990
$\bar{y}_{GM_I}$	906.7970	961.5401	911.2057	$\bar{y}_{GaM_I}$	880.3188	926.0914	895.9566
$\bar{y}_{GM_{II}}$	885.6271	942.3958	899.5285	$\bar{y}_{GaM_{II}}$	861.9864	904.5370	886.1775
$\bar{y}_{GM_{III}}$	927.9669	980.6844	922.8829	$\bar{y}_{GaM_{III}}$	898.6512	947.6458	905.7357

We explore the following points from numerical investigation:

- Tables 9 and 10 show bias results for proposed and existing estimators based on simulation and real-life data. It is worth mentioning that the proposed estimators have less bias as compared to existing ones. Furthermore, in the simulation study bias results, i.e., Table 9, the bias is reducing by increasing the sampling size.
- Clearly, PRE > 100, which means all the proposed estimators are performing better as compared to the adapted estimators. Although we make this conclusion based on our simulation and real-life study, we are confident that this result would be valid under different settings as well.

6. Conclusions

MRSS is a well-known sampling technique. In this paper, we adapt Sinha et al. [19] and Garg and Pachori [20] estimators under stratified MRSS design. Additionally, new calibration estimators that use the mean and the coefficient of variation of an auxiliary variable as a calibration constraint are proposed in this study to estimate the population mean in the case of stratified MRSS and stratified two-stage MRSS. It has been discovered that fresh ideas are more effective than modified ones. The proposed work has been supported by a simulation study. We hope to extend the present work in light of Koyuncu [13].

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