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On the Extended Version of Krasnoselśkiĭ's Theorem for Kannan-Type Equicontractive Mappings

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Abstract: The purpose of the paper is to establish a sufficient condition for the existence of a solution to the equation $\mathcal{T}(u,\mathcal{C}(u)) = u$ using Kannan-type equicontractive mappings, $\mathcal{T} : \mathcal{A} \times \overline{\mathcal{C}(\mathcal{A})} \to \mathbb{Y}$, where $\mathcal{C}$ is a compact mapping, $\mathcal{A}$ is a bounded, closed and convex subset of a Banach space $\mathbb{Y}$. To achieve this objective, the authors have presented Sadovskii's theorem, which utilizes the measure of noncompactness. The relevance of the obtained results has been illustrated through the consideration of various initial value problems.

Keywords: Hausdorff measure of noncompactness; compact mapping; Kannan-type equicontraction; initial value problem

MSC: 47H10; 54A20

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1. Introduction

The objective of the present article is to extend the Krasnoselśkiĭ's fixed point theorem to an implicit form by considering a special class of contractive mappings. More specifically, a sufficient condition for the existence of a solution to the equation $\mathcal{T}(u,\mathcal{C}(u)) = u$ has been obtained using Kannan-type equicontractive mappings, $\mathcal{T} : \mathcal{A} \times \overline{\mathcal{C}(\mathcal{A})} \to \mathbb{Y}$, where $\mathcal{C}$ is a compact mapping, $\mathcal{A}$ is a bounded, closed and convex subset of a Banach space $\mathbb{Y}$. One can find many contributions, where the authors have shown interest on the idea of extension of the well-known Krasnoselśkiĭ's fixed point theorem. Burton substituted the Banach contractive condition with the more general contraction in [1]. Then, in [2], Burton and Kirk assimilated the contraction mapping theorem and Schaefer's theorem [3] to provide an extension version of Krasnoselśkiĭ's fixed point theorem. Karakostas gave a sufficient condition for having a fixed point to the operator of the form $\mathcal{E}(x) := \mathcal{T}(x,\mathcal{C}(x))$ by assuming an equicontractive family [4]. Recently, Wardowski provided an extensive version in [5] by using F-contraction, which was introduced by himself [6,7]. Contrary to some recent papers [4,5] that only dealt with the contractive mappings which are continuous with regard to the first co-ordinate, the class of contractive mappings, as considered by us, also include some discontinuous mappings. In our results, the domain of definition of generic member (of our class), say $\mathcal{T}$, is considered over a product space, where one of the factor spaces is heavily dependent on the concerned compact mapping (also continuous). The reader may refer to [8,9] and the references therein, for a detailed study on the contractive mappings over product spaces.

In the next section, we consider two different classes, corresponding to the domain of the Kannan-type equicontractive mapping $\mathcal{T}$, and derive some basic properties concerning the individual classes. We provide some examples to conclude that the class of the Kannan-type equicontractive mappings is independent of the class of the general equicontractive mappings and equicontractive singular mappings, which were discussed in the literature [4,5,10]. Finally, we discuss the utility of our results in the setting of Banach spaces to ensure the existence of a solution to two particular classes of initial value problems. More applications in this area can be found in [11–14].

Throughout this paper, unless otherwise specified, we always assume $\mathbb{R}$ as the set of all real numbers, $\mathbb{N}$ as the set of all positive integers.

2. Preliminaries

At this point, we state the Krasnoselśkiǐ's fixed point theorem [1,15–17], which is the integral theme of the present work:

Theorem 1 ([15])**.** *Let $\mathcal{H}$ be a nonempty bounded, closed and convex subset of a Banach space $\mathbb{Y}$. Suppose that $\mathcal{T}, \mathcal{C} : \mathcal{H} \to \mathbb{Y}$ are two mappings such that $\mathcal{T}$ is a contraction and $\mathcal{C}$ is a compact mapping. If for all $u, v \in \mathcal{H}, \mathcal{T}(u) + \mathcal{C}(v) \in \mathcal{H}$, then there is a $v_0 \in \mathcal{H}$ such that $\mathcal{T}(v_0) + \mathcal{C}(v_0) = v_0$.*

Among the various existing notions of measure of noncompactness, we use the setting of Hausdorff measure of noncompactness in our results. For more information about the properties of measure of noncompactness and its relevant directions, the readers can see [4,5,18–21] and the references therein.

Definition 1 ([18])**.** *Let (ζ, ρ) be a metric space and Ω be a nonempty bounded subset of ζ. The Hausdorff measure of noncompactness and Kuratowski measure of noncompactness are denoted by $\beta(\Omega)$, $\alpha(\Omega)$, respectively, and defined as*

$$\beta(\Omega) = \inf\left\{\varepsilon > 0 : \; \Omega \subset \bigcup_{i=1}^{p} B(u_i, r_i), \; u_i \in \zeta, \; r_i < \varepsilon \; (i = 1, 2, \ldots, p), \; p \in \mathbb{N}\right\}$$

and

$$\alpha(\Omega) = \inf\left\{\varepsilon > 0 : \; \Omega \subset \bigcup_{j=1}^{q} B_j, \; B_j \subset \zeta, \; \mathrm{diam}(B_j) < \varepsilon \; (j = 1, 2, \ldots, q), \; q \in \mathbb{N}\right\},$$

respectively, where $B(u_i, r_i) = \{x : \rho(u_i, x) < r_i, x \in \zeta\}$.

We now state some basic properties of β which will be required in our subsequent section.

Proposition 1 ([19])**.** *Let (ζ, ρ) be a metric space, Ω, Ω_1 and Ω_2 be nonempty bounded subsets of ζ. Then, the Hausdorff measure of noncompactness $\beta(\Omega)$ has the following properties:*

(a) $\beta(\Omega) = 0 \iff \Omega$ *is totally bounded;*
(b) $\Omega_1 \subseteq \Omega_2$ *implies* $\beta(\Omega_1) \leq \beta(\Omega_2)$*;*
(c) $\beta(\Omega) = \beta(\overline{\Omega})$*;*
(d) $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$*;*
(e) $\beta(p\Omega) = |p|\beta(\Omega), \; p \in \mathbb{R}$*;*
(f) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$.

Darbo [22] originally proved a remarkable result which assimilates the measure of noncompactness with the fixed point theory. Later, Sadovskii [23] provided a sufficient condition for a continuous mapping having a fixed point. Here, we quote those theorems as follows:

Theorem 2 ([22] Darbo's theorem)**.** *If T is a continuous self-mapping of a nonempty, bounded, closed and convex subset C of a Banach space X such that*

$$\alpha(T(Q)) \leq k\alpha(Q) \quad \text{for all} \quad Q \subset C,$$

where $k \in (0,1)$ is a constant, then T has at least one fixed point in the set C.

Theorem 3 ([23] Sadovskii's theorem)**.** *Let $\mathbb{Y}$ be a Banach space and $\mathcal{H}$ be a nonempty bounded, closed and convex subset of $\mathbb{Y}$. Suppose that $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is a continuous mapping. If for any nonempty subset $\mathcal{A}$ of $\mathcal{H}$ with $\beta(\mathcal{A}) > 0$ it satisfies*

$$\beta(\mathcal{T}(\mathcal{A})) < \beta(\mathcal{A}),$$

then $\mathcal{T}$ has at least one fixed point in $\mathcal{H}$.

3. Main Results

Let us begin this section with the definition of Kannan-type equicontractive mapping:

Definition 2. *Let $\mathbb{Y}$ be a Banach space and $\mathbb{S}$ be a closed subset of a complete metric space $(\mathbb{Y}, \rho)$. The mapping $\mathcal{T} : \mathbb{Y} \times \mathbb{S} \to \mathbb{Y}$ is called a Kannan-type equicontractive mapping if for all $u_1, u_2 \in \mathbb{Y}$ and $v \in \mathbb{S}$, it satisfies*

$$\|\mathcal{T}(u_1, v) - \mathcal{T}(u_2, v)\| \leq k(\|\mathcal{T}(u_1, v) - u_1\| + \|\mathcal{T}(u_2, v) - u_2\|)$$

for some $k \in [0, \frac{1}{2})$.

It is clear that for a Kannan-type equicontractive mapping $\mathcal{T} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is a Banach space, the corresponding family $\mathcal{S}_u = \{u \mapsto \mathcal{T}(u, v) : v \in \mathcal{A}\}$ contains the Kannan-type contraction.

Kannan-type equicontractive condition is independent from the equicontractive singular condition (Definition 1.1 in [5]) and general equicontractive condition (Definition 2.1 in [4]). The situation is illustrated in the following example.

Example 1. *Let $\mathcal{A} = [0, 1] \subseteq \mathbb{R}$ and let $\mathcal{C} : \mathcal{A} \to \mathcal{A}$ be a compact mapping such that $\mathcal{C}(u) = u$. Put $I := (\mathcal{A} \times \mathcal{A}) \setminus \{(1, 1)\}$. Let $\mathcal{T} : \mathcal{A} \times \overline{\mathcal{C}(\mathcal{A})} \to \mathcal{A}$ be a mapping such that*

$$\mathcal{T}(u, v) = \begin{cases} \dfrac{u+v}{4}, & (u, v) \in I, \\ \dfrac{1}{5}, & (u, v) = (1, 1). \end{cases}$$

It is easy to see that $\mathcal{T}$ satisfies Kannan-type equicontractive condition. Indeed, let $u_1, u_2 \in \mathcal{A}$ and $v \in \overline{\mathcal{C}(\mathcal{A})}$ be fixed. On simplification, we obtain

$$\left|\mathcal{T}(u_1, v) - u_1\right| + \left|\mathcal{T}(u_2, v) - u_2\right| \geq \left|\frac{3u_1 - 3u_2}{4}\right|,$$

and

$$\left|\mathcal{T}(u_1, v) - \mathcal{T}(u_2, v)\right| = \left|\frac{u_1 - u_2}{4}\right|.$$

Consequently, it follows that

$$\left|\mathcal{T}(u_1, v) - \mathcal{T}(u_2, v)\right| \leq \frac{2}{5}\Big(\left|\mathcal{T}(u_1, v) - u_1\right| + \left|\mathcal{T}(u_2, v) - u_2\right|\Big).$$

Moreover, one has

$$\sup_{(u_1,v)\in I}\left|\mathcal{T}(u_1,v)-\frac{1}{5}\right| = \sup_{(u_1,v)\in I}\left|\frac{5u_1+5v-4}{20}\right| = 0.3,$$

and,

$$\inf_{(u_1,v)\in I}\left(\left|\mathcal{T}(u_1,v)-u_1\right|+\frac{4}{5}\right) = \inf_{(u_1,v)\in I}\left(\left|\frac{3u_1-v}{4}\right|+\frac{4}{5}\right) = 0.8.$$

Choose $k=\frac{2}{5}$, then for any $u_1,u_2\in\mathcal{A}$, it follows that

$$\left|\mathcal{T}(u_1,v)-\mathcal{T}(u_2,v)\right| \le k\Big(\left|\mathcal{T}(u_1,v)-u_1\right|+\left|\mathcal{T}(u_2,v)-u_2\right|\Big),$$

as desired. Since $\mathcal{T}$ is not continuous at $(1,1)$, $\mathcal{T}$ does not satisfy the equicontractive singular condition as well as the general equicontractive condition.

In the next theorem, we will use the classical Schauder's theorem to check whether the equation $\mathcal{T}(u,\mathcal{C}(u))=u$ exists a solution or not by considering the compact mapping in the sense of Krasnoselśkiĭ.

Theorem 4. *Let $\mathcal{A}$ be a bounded, closed and convex subset of a Banach space $\mathbb{Y}$ and $\mathcal{C}:\mathcal{A}\to\zeta$ be a compact mapping, where (ζ,ρ) is a complete metric space. If the mapping $\mathcal{T}:\mathcal{A}\times\overline{\mathcal{C}(\mathcal{A})}\to\mathcal{A}$ satisfies Kannan-type equicontractive condition, then $\mathcal{T}(u,\mathcal{C}(u))=u$ has a solution in $\mathcal{A}$.*

Proof. For each $v\in\overline{\mathcal{C}(\mathcal{A})}$, the mapping $\mathcal{T}(v):\mathcal{A}\to\mathcal{A}$ is a Kannan-type contraction. As $\mathcal{A}$ is complete, then $\mathcal{T}(v)$ has a unique fixed point in $\mathcal{A}$. Thus, for each $v\in\overline{\mathcal{C}(\mathcal{A})}$, $\mathcal{T}(u,v)=u$ has a unique solution in $\mathcal{A}$.

Let $u=f(v)$ be the fixed point of $\mathcal{T}(v)$ in $\mathcal{A}$. Define $f:\overline{\mathcal{C}(\mathcal{A})}\to\mathcal{A}$ such that $f(v)=\mathcal{T}(f(v),v)=\mathcal{T}(v)(f(v))$. Now, the mapping f is continuous at any point of $\overline{\mathcal{C}(\mathcal{A})}$ which is the same as saying that the function $\mathcal{T}$ is continuous at the point $(f(v),v)$. The continuity of the mapping $\mathcal{T}(v)$ at its corresponding fixed point $f(v)$ ensures us the conclusion. Let $\{f(v_n)\}$ be a convergent sequence in $\mathcal{A}$ such that $f(v_n)\to f(v)$ as $n\to\infty$. Now, we arrive at

$$\begin{aligned}\left\|\mathcal{T}(v)(f(v_n))-\mathcal{T}(v)(f(v))\right\| &\le k\Big(\left\|\mathcal{T}(v)(f(v_n))-f(v_n)\right\|+\left\|\mathcal{T}(v)(f(v))-f(v)\right\|\Big)\\ &\le k\Big(\left\|\mathcal{T}(v)(f(v_n))-\mathcal{T}(v)(f(v))\right\|+\left\|f(v_n)-f(v)\right\|\Big),\end{aligned}$$

which implies that

$$\left\|\mathcal{T}(v)(f(v_n))-\mathcal{T}(v)(f(v))\right\| \le \frac{k}{1-k}\left\|f(v_n)-f(v)\right\| \to 0 \ \ (n\to\infty).$$

Therefore, the mapping $\mathcal{T}(v)$ is continuous at its fixed point $f(v)$. As v is chosen arbitrarily, then $\mathcal{T}(v)$ is continuous at its fixed point. Consequently, $\mathcal{T}$ is continuous at the point $(f(v),v)$. As a result, $f:\overline{\mathcal{C}(\mathcal{A})}\to\mathcal{A}$ is continuous. Accordingly, $f\circ\mathcal{C}$ is also continuous on a bounded, closed and convex set $\mathcal{A}$. Now, let $\mathcal{S}=\overline{Co(f(\mathcal{C}(\mathcal{A})))}\subseteq\mathcal{A}$. Apply the Schauder fixed point theorem to $f\circ\mathcal{C}$ on the compact set $\mathcal{S}$, there exists $u_0\in\mathcal{A}$ such that $f(\mathcal{C}(u_0))=u_0$. Therefore,

$$\mathcal{T}(u_0,\mathcal{C}(u_0))=\mathcal{T}(f(\mathcal{C}(u_0)),\mathcal{C}(u_0))=f(\mathcal{C}(u_0))=u_0.$$

The proof is completed. □

The following corollary is an immediate outcome of the above theorem:

Corollary 1. *Let $\mathcal{A}$ be a bounded, closed and convex subset of a Banach space $\mathbb{Y}$ and $\mathcal{C} : \mathcal{A} \to \zeta$ be a compact mapping, where (ζ, ρ) is a complete metric space. If the mapping $\mathcal{T} : \mathcal{A} \times \overline{\mathcal{C}(\mathcal{A})} \to \mathcal{A}$ is continuous and satisfies Kannan-type equicontractive condition, then $\mathcal{T}(u, \mathcal{C}(u)) = u$ has a solution in $\mathcal{A}$.*

Example 2. *Let l_p be written as the space of all p-power summable sequences of real numbers, with the p-norm, e_1 be written as the unit vector of l_p, π_1 be written as the projection to the first component. Consider $\mathbb{U} := \operatorname{span}\{e_1\}$, which is an one dimensional subspace of the Banach space l_p. Define*

$$B_{\mathbb{U}} := \{te_1 : t \in [0,1]\}.$$

Note that $B_{\mathbb{U}}$ is a bounded, closed and convex subset of l_p. Let $\mathcal{C} : B_{\mathbb{U}} \to l_p$ be the inclusion mapping. Let us define a mapping $\mathcal{T} : B_{\mathbb{U}} \times \mathcal{C}(B_{\mathbb{U}}) \to l_p$ such that

$$\mathcal{T}(x,y) = \begin{cases} \frac{1}{16}\pi_1(x+y)e_1, & (x,y) \in (B_{\mathbb{U}} \times B_{\mathbb{U}}) \setminus \{(e_1,e_1)\}, \\ \frac{1}{18}e_1, & (x,y) = (e_1,e_1). \end{cases}$$

For $x_n = y_n = \left(1 + \frac{1}{n}\right)e_1$ and letting n tend to ∞ we obtain

$$(x_n, y_n) \to (e_1, e_1), \quad \mathcal{T}(x_n, y_n) = \frac{1}{16}\pi_1(x_n + y_n)e_1 \to \frac{1}{8}e_1 \neq \mathcal{T}(e_1, e_1).$$

So, $\mathcal{T}$ is not continuous at (e_1, e_1). Let us take two elements $u = t_u e_1$ and $y = t_y e_1$ in $B_{\mathbb{U}} \setminus \{e_1\}$. Then, we acquire

$$\begin{aligned} \|\mathcal{T}(u,y) - u\| &= \left\| \frac{1}{16}\pi_1(u+y)e_1 - u \right\| \\ &= \left\| \frac{1}{16}\pi_1(u+y)e_1 - ue_1 \right\| \\ &= \left| \frac{t_u + t_y}{16} - t_u \right| \\ &= \left| \frac{t_y - 15t_u}{16} \right|. \end{aligned}$$

Similarly, for $v = t_v e_1 \in B_{\mathbb{U}} \setminus \{e_1\}$, we have

$$\|\mathcal{T}(v,y) - v\| = \left| \frac{t_y - 15t_v}{16} \right|.$$

On the other hand, we obtain

$$\|\mathcal{T}(u,y) - u\| + \|\mathcal{T}(v,y) - v\| \geq \left| \frac{15t_u - 15t_v}{16} \right|$$

and

$$\|\mathcal{T}(u,y) - \mathcal{T}(v,y)\| = \left| \frac{t_u - t_v}{16} \right|.$$

Therefore, we obtain

$$\|\mathcal{T}(u,y) - \mathcal{T}(v,y)\| \leq \frac{1}{14}\Big(\|\mathcal{T}(u,y) - u\| + \|\mathcal{T}(v,y) - v\| \Big).$$

Now, if $(u,y) \in (B_{\mathbb{U}} \times B_{\mathbb{U}}) \setminus \{(e_1,e_1)\}$ *and* $(v,y) = (e_1,e_1)$*, then*

$$\|\mathcal{T}(u,y) - \mathcal{T}(v,y)\| = \left\|\frac{1}{16}\pi_1(u+y)e_1 - \frac{1}{18}e_1\right\| = \left|\frac{t_u + t_y}{16} - \frac{1}{18}\right|$$

and

$$\|\mathcal{T}(u,y) - u\| = \left\|\frac{1}{16}\pi_1(u+y)e_1 - u\right\| = \left|\frac{t_y - 15t_u}{16}\right|, \quad \|\mathcal{T}(v,y) - v\| = \frac{17}{18}.$$

It is clear that $t_u, t_v \in [-1,1]$*, then*

$$\begin{aligned}
\|\mathcal{T}(u,y) - \mathcal{T}(v,y)\| &\leq \sup \|\mathcal{T}(u,y) - \mathcal{T}(v,y)\| \\
&= \sup\left|\frac{t_u + t_y}{16} - \frac{1}{18}\right| \\
&\leq \frac{1}{7}\inf\left(\left|\frac{t_y - 15t_u}{16}\right| + \frac{17}{18}\right) \\
&= \frac{1}{7}\inf\Big(\|\mathcal{T}(u,y) - u\| + \|\mathcal{T}(v,y) - v\|\Big) \\
&\leq \frac{1}{7}\Big(\|\mathcal{T}(u,y) - u\| + \|\mathcal{T}(v,y) - v\|\Big).
\end{aligned}$$

So, for any $u, v \in B_{\mathbb{U}}$*, it is easy to see that*

$$\|\mathcal{T}(u,y) - \mathcal{T}(v,y)\| \leq \frac{1}{7}\Big(\|\mathcal{T}(u,y) - u\| + \|\mathcal{T}(v,y) - v\|\Big).$$

As $\mathcal{T}$ *is not continuous at* (e_1,e_1)*, it does not satisfy the equicontractive singular condition as well as the general equicontractive condition. However,* $\mathcal{T}$ *satisfies Kannan-type equicontractive condition and hence, we conclude from Theorem 4 that* $\mathcal{T}(u,\mathcal{C}(u)) = u$ *has a solution in* $B_{\mathbb{U}}$*.*

In the following, we introduce the definition of m-th invariant mapping which is important in the due course of our events.

Definition 3. *Let* (ζ,ρ) *be a metric space and* $\emptyset \neq \mathcal{A} \subseteq \zeta$*. A mapping* $\mathcal{T} : \mathcal{A} \to \zeta$ *is said to be a m-th invariant mapping if for some* $q \in [0,1)$*,* $\mathcal{T}^m(B_i) \subseteq qB_i$*, for any bounded set* $B_i \subseteq \mathcal{A}$*.*

We now present an extended version of Krasnoselśkiǐ fixed point theorem for Kannan-type equicontraction of 1-st invariant and compact mapping by utilizing Sadovskii's theorem in the measure of noncompactness.

Theorem 5. *Let* $\mathcal{A}$ *be a bounded, closed and convex subset of an infinite dimensional Banach space* $\mathbb{Y}$ *and* $\mathcal{C} : \mathcal{A} \to \zeta$ *be a compact mapping, where* (ζ,ρ) *is a complete metric space. If the Kannan-type equicontractive mapping* $\mathcal{T} : \mathcal{A} \times \mathcal{C}(\mathcal{A}) \to \mathbb{Y}$ *satisfies the following conditions:*

(a) The mapping $\eta : \mathcal{A} \to \mathcal{A}$ *satisfying* $\eta(u) = \mathcal{T}(u,\mathcal{C}(u))$ *is continuous and* 1*-st invariant;*

(b) The family, say $\mathcal{S}_v = \{v \mapsto \mathcal{T}(u,v) : u \in \mathcal{A}\}$*, is equicontinuous uniformly,*

then the equation $\mathcal{T}(u,\mathcal{C}(u)) = u$ *has a solution in* $\mathcal{A}$*.*

Proof. Let us assume that G is a bounded subset of $\mathcal{A}$ with $\beta(G) > 0$. As $\mathcal{C}(G)$ is relatively compact, we can find $v_1, v_2, \ldots, v_p$ such that

$$\mathcal{C}(G) \subseteq \bigcup_{j=1}^{p} B(v_j,\delta). \tag{1}$$

We consider a finite collection $\{B(u_i, \mathcal{R})\}$ $(i = 1, 2, \ldots, n)$ such that

$$G \subseteq \bigcup_{j=1}^{n} B(u_i, \mathcal{R})$$

with $\mathcal{R} > 0$ and $\mathcal{R} \leq \beta(G) + \varepsilon$. Now, assume that η is a 1-st invariant mapping, then, for some $q \in [0, 1)$, we have

$$\eta(B(u_i, \mathcal{R})) \subseteq qB(u_i, \mathcal{R}),$$

that is,

$$\mathcal{T}\big(B(u_i, \mathcal{R}) \times \mathcal{C}(B(u_i, \mathcal{R}))\big) \subseteq qB(u_i, \mathcal{R}),$$

for any $B(u_i, \mathcal{R}) \subseteq \mathcal{A}$. As the family, $\mathcal{S}_v$ is equicontinuous uniformly, then corresponding to arbitrarily chosen $\varepsilon > 0$, there exists $\delta > 0$ such that for all $u \in \mathcal{A}$ and $v_1, v_2 \in \mathcal{C}(\mathcal{A})$, we have

$$\big\|\mathcal{T}(u, v_1) - \mathcal{T}(u, v_2)\big\| \leq \varepsilon, \tag{2}$$

so long as $\rho(v_1, v_2) < \delta$ for any member of $\mathcal{S}_v$.

Apparently,

$$G \subseteq \bigcup_{j=1}^{p} \bigcup_{i=1}^{n} \Big(B(u_i, \mathcal{R}) \cap \mathcal{C}^{-1}(B(v_j, \delta))\Big). \tag{3}$$

Now, we will prove $\beta(\eta(G)) < \beta(G)$. Let us assume that $h \in \eta(G)$, so $h = \eta(g) = \mathcal{T}(g, \mathcal{C}(g))$ for some $g \in G$. By (3), we can assume two natural numbers k $(1 \leq k \leq n)$ and l $(1 \leq l \leq p)$ such that

$$g \in B(u_k, \mathcal{R}) \;\cap\; \mathcal{C}^{-1}(B(v_l, \delta)).$$

Here, $g \in B(u_k, \mathcal{R})$ implies that $\|g - u_k\| < \mathcal{R}$. Further, $\rho(\mathcal{C}(g), v_l) < \delta$. Then, for some $\lambda \in [0, \frac{1}{2})$,

$$\begin{aligned}
\big\|h - \mathcal{T}(u_k, v_l)\big\| &= \big\|\mathcal{T}(g, \mathcal{C}(g)) - \mathcal{T}(u_k, v_l)\big\| \\
&\leq \big\|\mathcal{T}(g, \mathcal{C}(g)) - \mathcal{T}(u_k, \mathcal{C}(g))\big\| + \big\|\mathcal{T}(u_k, \mathcal{C}(g)) - \mathcal{T}(u_k, v_l)\big\| \\
&\leq \lambda\Big(\big\|\mathcal{T}(g, \mathcal{C}(g)) - g\big\| + \big\|\mathcal{T}(u_k, \mathcal{C}(g)) - u_k\big\|\Big) + \varepsilon.
\end{aligned}$$

Now, from the relationship $\mathcal{T}\big(B(u_k, \mathcal{R}) \times \mathcal{C}(B(u_k, \mathcal{R}))\big) \subseteq qB(u_k, \mathcal{R})$ we obtain

$$\big\|h - \mathcal{T}(u_k, v_l)\big\| < 2q\lambda\mathcal{R} + \varepsilon \leq 2q\lambda(\beta(G) + \varepsilon) + \varepsilon = 2q\lambda\beta(G) + (2q\lambda + 1)\varepsilon.$$

Since ε is chosen arbitrarily, then

$$\big\|h - \mathcal{T}(u_k, v_l)\big\| < 2q\lambda\beta(G) < \beta(G).$$

Let $\mathcal{R}_1 = 2q\lambda\beta(G)$, $h \in B(\mathcal{T}(u_k, v_l), \mathcal{R}_1)$. Therefore, we claim that

$$\eta(G) \subseteq \bigcup_{i=1}^{n} \left(\bigcup_{j=1}^{p} B(\mathcal{T}(u_i, v_j), \mathcal{R}_1)\right).$$

Now, the definition of β in Definition 1 implies $\beta(\eta(G)) < \beta(G)$. □

Remark 1. *It is clear that for a finite dimensional Banach space we can conclude the above theorem using Darbo's theorem in the measure of non-compactness by ignoring the* 1st *invariant condition of the mapping η along with the last condition (b).*

Indeed, for a finite dimensional Banach space $\mathbb{Y}$, the bounded, closed and convex subset $\mathcal{A}$ of $\mathbb{Y}$ is compact and hence totally bounded. So, for any bounded subset G of $\mathcal{A}$, it is also totally bounded. Hence, for the Kuratowski measure of noncompactness of G, one has $\alpha(G) = 0$. Now, we claim that

$$\begin{aligned}\alpha(G) = 0 &\implies \overline{G} \text{ is compact}\\ &\implies \eta(\overline{G}) \text{ is compact (as, } \eta \text{ is continuous in the above theorem)}\\ &\implies \alpha(\eta(\overline{G})) = 0\\ &\implies \alpha(\eta(G)) = 0 \text{ (as, } \eta(G) \subseteq \eta(\overline{G})).\end{aligned}$$

Consequently, there exists some $k > 0$ such that $\alpha(\eta(G)) \leq k\alpha(G)$. Therefore, Darbo's theorem implies that the equation $\mathcal{T}(u, \mathcal{C}(u)) = u$ has a solution in $\mathcal{A}$.

Remark 2. *To apply Sadovskii's theorem in the measure of noncompactness, it is necessary to consider the continuous condition of the mapping η in Theorem 5.*

Remark 3. *The continuity of the mapping η, as the mentioned above, does not ensure the continuity of the mapping $\mathcal{T}$. Indeed, if we consider the mapping $\mathcal{T}$ as in Example 1 and the compact mapping as $\mathcal{C}(u) = \frac{u}{4}$, then the mapping η is continuous. However, for the inclusion map $\mathcal{C}$, $\mathcal{T}$ as well as η fails to be continuous in its domain of definition. In particular, we have dealt with the continuity of $\mathcal{T}$ only on the set $\{(u, \mathcal{C}(u)) : u \in \mathcal{A}\}$.*

Remark 4. *Theorem 5 can also be concluded by using Darbo's theorem.*

4. Applications

In this section, we furnish two classes of initial value problems that can be efficiently solved by applying the results obtained in the preceding section.

4.1. Application I

Let us consider an initial value problem as follows:

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} + \omega^2 u = f(t, u(t)) \displaystyle\int_0^t G(t,s)u(s)\mathrm{d}s, \\ u(0) = a, \quad u'(0) = b, \end{cases} \tag{4}$$

where $\omega \neq 0$, $t \in I = [0, 1]$, also,

$$G(t,s) = \frac{1}{\omega}\sin(\omega(t-s))H(t-s),\ \omega \neq 0$$

and $H(t-s)$ is a heaviside step function. Let $\mathbb{Y}$ be the vector space of all real valued continuous functions over I equipping with the norm:

$$\|u(t)\|_{\mathbb{Y}} = \sup_{t\in I}|u(t)|\mathrm{e}^{-\gamma t}, \quad \gamma \in \mathbb{R}.$$

Clearly, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ forms a Banach space.

The following theorem illustrates that the above mentioned initial value problem has a solution in the closed unit ball $\mathcal{B}(0,1) := \{u \in \mathbb{Y} : \|u\| \leq 1\}$.

Theorem 6. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a nonzero function. There exists $M > 0$ such that $|f(t,u)| \leq M$, for all $t \in I$ and $u \in \mathbb{R}$. Then, the above initial value problem (4) has a solution in the closed unit ball $\mathcal{B}(0,1)$ if the following conditions are satisfied:*

(i) $|f(s, u(s)) - f(s, v(s))| \leq D(t, u(s), v(s))$, *where*

$$D(t, u(s), v(s)) = \left|u_1(s) - a\cos(\omega t) - b\sin(\omega t) - \int_0^t G(t,s)f(s, u_1(s))v(s)\mathrm{d}s\right| +$$

$$\left|u_2(s) - a\cos(\omega t) - b\sin(\omega t) - \int_0^t G(t,s)f(s,u_2(s))v(s)\mathrm{d}s\right|;$$

(ii) $M \leq |\omega|^2(1-|a|-|b|)$;
(iii) $|\gamma + 2\omega| < \dfrac{|\omega(\gamma^2+\omega^2)|}{2}$.

Proof. Consider the operator $\mathcal{C} : \mathbb{Y} \to \mathbb{Y}$ such that

$$\mathcal{C}(u(t)) = \int_0^t G(t,s)u(s)\mathrm{d}s, \quad u \in \mathbb{Y},$$

where

$$G(t,s) = \frac{1}{\omega}\sin(\omega(t-s))H(t-s) = \begin{cases} 0, & 0 \leq t < s, \\ \dfrac{1}{\omega}\sin(\omega(t-s)), & s < t < \infty, \end{cases} \quad \omega \neq 0.$$

Now, for each $u(t) \in \mathcal{B}(0,1)$, one has

$$\left|\mathcal{C}(u(t))\right| = \left|\int_0^t G(t,s)u(s)\mathrm{d}s\right| \leq \frac{1}{|\omega|}.$$

Accordingly, $\mathcal{C}$ maps $\mathcal{B}(0,1)$ to $\mathcal{B}\left(0, \frac{1}{|\omega|}\right)$. Thus, we can easily check that $\mathcal{C}$ is a compact mapping. Consider the mapping $\mathcal{T} : \mathcal{B}(0,1) \times \mathcal{C}(\mathcal{B}(0,1)) \to \mathcal{B}(0,1)$ defined by

$$\mathcal{T}(u(t),v(t)) = a\cos(\omega t) + b\sin(\omega t) + \int_0^t G(t,s)f(s,u(s))v(s)\mathrm{d}s.$$

Now, we have

$$\begin{aligned} D(t,u(s),v(s))\mathrm{e}^{-\gamma t} &\leq \sup_{s\in I}\left\{\left|u_1(s) - \mathcal{T}(u_1(t),v(t))\right| + \left|u_2(s) - \mathcal{T}(u_2(t),v(t))\right|\right\}\mathrm{e}^{-\gamma t} \\ &= \left\{\left\|u_1(t) - \mathcal{T}(u_1(t),v(t))\right\| + \left\|u_2(t) - \mathcal{T}(u_2(t),v(t))\right\|\right\}\mathrm{e}^{-\gamma(t-s)}, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \left\|\mathcal{T}(u_1(t),v(t)) - \mathcal{T}(u_2(t),v(t))\right\| &= \sup_{t\in I}\left|\mathcal{T}(u_1(t),v(t)) - \mathcal{T}(u_2(t),v(t))\right|\mathrm{e}^{-\gamma t} \\ &= \sup_{t\in I}\left|\int_0^t G(t,s)[f(s,u_1(s)) - f(s,u_2(s))]v(s)\mathrm{d}s\right|\mathrm{e}^{-\gamma t} \\ &\leq \sup_{t\in I}\left|\int_0^t G(t,s)D(t,u(s),v(s))\mathrm{d}s\right|\mathrm{e}^{-\gamma t}. \end{aligned}$$

Then from the given condition (i) and (5), we have

$$\begin{aligned} &\left\|\mathcal{T}(u_1(t),v(t)) - \mathcal{T}(u_2(t),v(t))\right\| \\ &\leq \left\{\left\|u_1(t) - \mathcal{T}(u_1(t),v(t))\right\| + \left\|u_2(t) - \mathcal{T}(u_2(t),v(t))\right\|\right\}\sup_{t\in I}\left|\int_0^t G(t,s)\mathrm{e}^{-\gamma(t-s)}\mathrm{d}s\right| \\ &= \left\{\left\|u_1(t) - \mathcal{T}(u_1(t),v(t))\right\| + \left\|u_2(t) - \mathcal{T}(u_2(t),v(t))\right\|\right\}\sup_{t\in I}\left|\int_0^t \frac{1}{\omega}\sin(\omega(t-s))\mathrm{e}^{-\gamma(t-s)}\mathrm{d}s\right| \\ &= \left\{\left\|u_1(t) - \mathcal{T}(u_1(t),v(t))\right\| + \left\|u_2(t) - \mathcal{T}(u_2(t),v(t))\right\|\right\}\sup_{t\in I}\left|\int_0^t \frac{1}{\omega}\sin(\omega r)\mathrm{e}^{-\gamma r}\mathrm{d}r\right|, \end{aligned} \tag{6}$$

where $r = t - s$. Let

$$\begin{aligned} L &= \int_0^t \frac{1}{\omega} \sin(\omega r) \mathrm{e}^{-\gamma r} \mathrm{d}r \\ &= \frac{1}{\omega} \int_0^t \sin(\omega r) e^{-\gamma r} \mathrm{d}r \\ &= \frac{1}{\omega} \left[\frac{\mathrm{e}^{-\gamma t}(\omega \mathrm{e}^{\gamma t} - \gamma \sin(\omega t) - \omega \cos(\omega t))}{\gamma^2 + \omega^2} \right], \end{aligned}$$

which means that

$$k := \sup_{t \in I} |L| = \frac{|\gamma + 2\omega|}{|\omega|(\gamma^2 + \omega^2)}.$$

On simplification, the inequality (6) produces

$$\|\mathcal{T}(u_1(t), v(t)) - \mathcal{T}(u_2(t), v(t))\| \leq k\Big\{\|u_1(t) - \mathcal{T}(u_1(t), v(t))\| + \|u_2(t) - \mathcal{T}(u_2(t), v(t))\|\Big\}.$$

Thus, $\mathcal{T}$ satisfies Kannan-type equicontractive condition. Let us take $u(t) \in \mathcal{B}(0,1)$, then by the condition (ii), we have

$$\|\mathcal{T}(u(t), \mathcal{C}(u(t)))\| \leq |a| + |b| + \frac{M}{|\omega|^2} \leq 1.$$

It now follows immediately from Theorem 4 that $\mathcal{T}(u(t), \mathcal{C}(u(t))) = u(t)$ has a solution in $\mathcal{B}(0,1)$. Consequently, the above initial value problem has a solution in $\mathcal{B}(0,1)$. □

4.2. Application II

Let us now consider another initial value problem as follows:

$$\begin{cases} \dfrac{\mathrm{d}^2 u}{\mathrm{d}t^2} = u(t) + f(t, u(t)) - \omega \cdot \displaystyle\int_0^t G(t,s)(u(s) + f(s, u(s)))\mathrm{d}s, \quad \omega \neq 0, \\ u(0) = 0, \quad u'(0) = 0, \end{cases}$$

where each function, variable and constant satisfy the same properties as mentioned in *Application I*. As an application of Theorem 5, the existence of solution(s) of the immediate above problem is assured from the following theorem.

Theorem 7. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a nonzero function which is linear in the second co-ordinate. There exists $M > 0$ such that $|f(t,u)| \leq M$, for all $t \in I$ and $u \in \mathbb{R}$. Then, the above initial value problem has a solution in the closed unit ball $\mathcal{B}(0,1)$ if the following conditions are satisfied:*

(i) $|f(s, u(s)) - f(s, v(s))| \leq D(t, u(s), v(s))$, *where*

$$\begin{aligned} D(t, u(s), v(s)) = &\left| u_1(s) - v(s) - \int_0^t G(t,s) f(s, u_1(s)) \mathrm{d}s \right| + \\ &\left| u_2(s) - v(s) - \int_0^t G(t,s) f(s, u_2(s)) \mathrm{d}s \right|; \end{aligned}$$

(ii) $M \leq |\omega| - 1$;

(iii) $|\gamma + 2\omega| < \dfrac{|\omega|(\gamma^2 + \omega^2)}{2}$.

Proof. Take the compact operator $\mathcal{C} : \mathbb{Y} \to \mathbb{Y}$ as follows:

$$\mathcal{C}(u(t)) = \int_0^t G(t,s) u(s) \mathrm{d}s, \quad u \in \mathbb{Y},$$

where $G(t,s)$ has already been described. Consider the mapping $\mathcal{T}:\mathcal{B}(0,1)\times\mathcal{B}(0,1/|\omega|)\to\mathbb{Y}$ such that

$$\mathcal{T}(u(t),v(t))=v(t)+\int_0^t G(t,s)f(s,u(s))\mathrm{d}s,\quad \textit{where } v(t)\in\mathcal{C}(\mathcal{B}(0,1)).$$

Take the family $\mathcal{S}_{v(t)}=\{u(t)\in\mathcal{B}(0,1):v(t)\mapsto\mathcal{T}(u(t),v(t))\}$ which is uniformly equicontinuous. We consider $u(t)\in\mathcal{B}(0,1)$ and $v_1(t),v_2(t)\in\mathcal{C}(\mathcal{B}(0,1))$. Corresponding to arbitrarily chosen $\varepsilon>0$, there exists $\delta>0$ such that $\|v_1(t)-v_2(t)\|<\delta$. Then we obtain

$$\|\mathcal{T}(u(t),v_1(t))-\mathcal{T}(u(t),v_2(t))\|=\|v_1(t)-v_2(t)\|<\delta=\varepsilon.$$

Now, by similar calculations as the previous theorem we conclude that $\mathcal{T}$ satisfies Kannan-type equicontractive condition. Consider $u(t)$ in $\mathcal{B}(0,1)$, then by the condition (ii), it is easy to see that

$$\|\mathcal{T}(u(t),\mathcal{C}(u(t)))\|\leq\frac{1}{|\omega|}+\frac{M}{|\omega|}\leq 1.$$

Here, the mapping $\eta:\mathcal{B}(0,1)\to\mathcal{B}(0,1)$ satisfying $\eta(u)=\mathcal{T}(u,\mathcal{C}(u))$ is clearly continuous and also is 1st invariant, moreover, $\eta(u)$ is a linear mapping. It now follows from Theorem 5 that $\mathcal{T}(u(t),\mathcal{C}(u(t)))=u(t)$ has a solution in $\mathcal{B}(0,1)$. This assures the existence of solution(s) of the initial value problem in $\mathcal{B}(0,1)$. □

5. Conclusions

The well-known Krasnoselśkiĭ's fixed point theorem has been extended to an implicit form by using a new type of contractive mappings. The classes of these types of mappings are special because it includes discontinuous mappings along with continuous mappings satisfying the contractive condition. Theorem 4 guarantees the existence of solution to the equation $\mathcal{T}(u,\mathcal{C}(u))=u$ and we construct an application of this result in Section 4.1. Further, by using the theory of measure of non-compactness (especially, Sadovskii's theorem), Theorem 5 assures the existence of the said equation. Finally, another application has been incorporated to authenticate our obtained result in Section 4.2.

In view of the obtained results, it is perhaps appropriate to end the present article with the following question:

Question: By considering Kannan-type equicontraction and the idea of the measure of noncompactness, could you prove Theorem 5 by eliminating the continuous condition of η?

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