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Integral Inequalities Involving Strictly Monotone Functions

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Abstract: Functional inequalities involving special functions are very useful in mathematical analysis, and several interesting results have been obtained in this topic. Several methods have been used by many authors in order to derive upper or lower bounds of certain special functions. In this paper, we establish some general integral inequalities involving strictly monotone functions. Next, some special cases are discussed. In particular, several estimates of trigonometric and hyperbolic functions are deduced. For instance, we show that Mitrinović-Adamović inequality, Lazarevic inequality, and Cusa-Huygens inequality are special cases of our obtained results. Moreover, an application to integral equations is provided.

Keywords: integral inequalities; strictly monotone functions; functional inequalities

MSC: 26D15; 26D05; 33B10

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1. Introduction

The use of integral inequalities is very frequent in various branches of mathematics such as differential and partial differential equations, numerical analysis, stability analysis and measure theory. Due to this fact, the study of integral inequalities is of particular importance.

Several results related to the development of integral inequalities involving monotone functions have been published. One of the most useful inequalities in analysis is due to Bellman [1]: Let $\iota, \tau, \kappa \in C([\alpha, \beta])$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $\iota > 0$ and $\tau, \kappa \geq 0$. If ι is monotonic nondecerasing, and

$$\tau(x) \leq \iota(x) + \int_\alpha^x \kappa(s)\tau(s)\, ds$$

for all $x \in [\alpha, \beta]$, then

$$\tau(x) \leq \iota(x) \exp\left(\int_\alpha^x \kappa(s)\, ds\right)$$

for all $x \in [\alpha, \beta]$. Another important inequality is due to Chebyshev (see e.g., [2]). This inequality can be stated as follows. Let $\omega_i \in L^1([\alpha, \beta])$, $i = 1, 2$, ω_i is decreasing for all i, or ω_i is increasing for all i. Let $\vartheta \in L^1([\alpha, \beta])$ and $\vartheta > 0$. Then

$$\prod_{i=1}^2 \left(\int_\alpha^\beta \omega_i(x)\vartheta(x)\, dx\right) \leq \left(\int_\alpha^\beta \vartheta(x)\, dx\right)\left(\int_\alpha^\beta \omega_1(x)\omega_2(x)\vartheta(x)\, dx\right). \tag{1}$$

An extension of the above inequality to higher dimensions have been derived in [3]. In [4–7], reversed inequalities of Hölder, Hardy and Poincaré type have been proved. Some results related to integral inequalities for operator monotonic functions can be found in [8]. Other integral inequalities involving monotone functions can be found in [9–13].

In [14], using inequality (1), Qi, Cui and Xu established several inequalities involving trigonometric functions and other inequalities involving the integral of $\frac{\sin x}{x}$. Motivated by the above mentioned contribution and also by the importance of trigonometric inequalities in real analysis, we establish in this paper new integral inequalities for strictly monotone functions, which can be useful for obtaining several functional inequalities involving trigonometric and hyperbolic functions.

Before stating our main results, let us fix some notations:

- $\mathbb{N}$: The set of positive integers.
- $a, b \in \mathbb{R}$, $a < b$.
- $f \in \mathcal{V}([a,b])$ means that $f : [a,b] \to \mathbb{R}$ is C^1, $f([a,b]) \subset [0, +\infty[$ and

$$f'(]a,b[) \subset]0, +\infty[\text{ or } f'(]a,b[) \subset]-\infty, 0[.$$

We present below our results.

Theorem 1. *Let $\sigma \in \mathbb{R} \backslash \{0, -2\}$, $f \in \mathcal{V}([a,b])$, $w \in C([a,b[)$ and $w(]a,b[) \subset]0, +\infty[$. Then, for every $n \in \mathbb{N}$ and $x \in]a,b[$, it holds that*

$$\int_a^x (x-t)^{n-1} f^\sigma(t) \left(\frac{f^2(a)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t)\, dt < \frac{2f^{\sigma+2}(a)}{\sigma(\sigma+2)} \int_a^x (x-t)^{n-1} w(t)\, dt. \quad (2)$$

Theorem 2. *Let $\sigma \in \mathbb{R} \backslash \{0, -2\}$, $f \in \mathcal{V}([a,b])$, $w \in C(]a,b])$ and $w(]a,b[) \subset]0, +\infty[$. Then, for every $n \in \mathbb{N}$ and $x \in]a,b[$, it holds that*

$$\int_x^b (t-x)^{n-1} f^\sigma(t) \left(\frac{f^2(b)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t)\, dt < \frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)} \int_x^b (t-x)^{n-1} w(t)\, dt. \quad (3)$$

Theorem 3. *Let $f \in C^1([a,b])$. Assume that $f'(]a,b[) \subset]-\infty, 0[$. Then, for every $n \in \mathbb{N}$, $n \geq 2$, and $x \in]a,b[$, it holds that*

$$\int_a^x (x-t)^{n-1} f(t)\, dt > (n-1) \int_a^x (x-t)^{n-2} (t-a) f(t)\, dt. \quad (4)$$

In the case when $f'(]a,b[) \subset]0, +\infty[$, we have the following result.

Theorem 4. *Let $f \in C^1([a,b])$. Assume that $f'(]a,b[) \subset]0, +\infty[$. Then, for every $n \in \mathbb{N}$, $n \geq 2$, and $x \in]a,b[$, it holds that*

$$\int_a^x (x-t)^{n-1} f(t)\, dt < (n-1) \int_a^x (x-t)^{n-2} (t-a) f(t)\, dt.$$

Theorem 5. *Let $f \in C^1([a,b])$. Assume that $f'(]a,b[) \subset]-\infty, 0[$. Then, for every $n \in \mathbb{N}$, $n \geq 2$, and $x \in]a,b[$, it holds that*

$$\int_x^b (t-x)^{n-1} f(t)\, dt < (n-1) \int_x^b (t-x)^{n-2} (b-t) f(t)\, dt.$$

Theorem 6. *Let $f \in C^1([a,b])$. Assume that $f'(]a,b[) \subset]0, +\infty[$. Then, for every $n \in \mathbb{N}$, $n \geq 2$, and $x \in]a,b[$, it holds that*

$$\int_x^b (t-x)^{n-1} f(t)\, dt > (n-1) \int_x^b (t-x)^{n-2} (b-t) f(t)\, dt$$

for all integer $n \geq 2$ and $a < x < b$.

The proofs of The above theorems are given in Section 2. Next, some special cases are discussed in Section 3. Finally, in Section 4, an application to integral equations is provided.

2. The Proofs

Proof of Theorem 1. Let

$$F(t) = -f'(t)f^{\sigma-1}(t)\Big(f^2(t) - f^2(a)\Big)$$

for all $t \in]a,b[$. Due to the assumptions on f and f', we have two possible cases:

$$f'(t) < 0,\ 0 \le f(b) < f(t) < f(a), \quad a < t < b$$

or

$$f'(t) > 0,\ 0 \le f(a) < f(t) < f(b), \quad a < t < b.$$

Observe that in both cases, we have

$$f(]a,b[) \subset]0,+\infty[,\ F(]a,b[) \subset]-\infty,0[.$$

Then, for all $s \in]a,b[$, it holds that

$$\int_a^s F(t)\,dt < 0,$$

which is equivalent to

$$\int_a^s \Big(-f'(t)f^{\sigma+1}(t) + f^2(a)f'(t)f^{\sigma-1}(t)\Big)\,dt < 0. \tag{5}$$

On the other hand, we have

$$\begin{aligned}
&\int_a^s \Big(-f'(t)f^{\sigma+1}(t) + f^2(a)f'(t)f^{\sigma-1}(t)\Big)\,dt \\
&= \left[-\frac{f^{\sigma+2}(t)}{\sigma+2} + \frac{f^2(a)f^{\sigma}(t)}{\sigma}\right]_{t=a}^{s} \\
&= -\frac{f^{\sigma+2}(s)}{\sigma+2} + \frac{f^2(a)f^{\sigma}(s)}{\sigma} + \frac{f^{\sigma+2}(a)}{\sigma+2} - \frac{f^{\sigma+2}(a)}{\sigma} \\
&= f^{\sigma}(s)\left(\frac{f^2(a)}{\sigma} - \frac{f^2(s)}{\sigma+2}\right) - \frac{2}{\sigma(\sigma+2)}f^{\sigma+2}(a),
\end{aligned}$$

which implies by (5) that

$$f^{\sigma}(s)\left(\frac{f^2(a)}{\sigma} - \frac{f^2(s)}{\sigma+2}\right) < \frac{2}{\sigma(\sigma+2)}f^{\sigma+2}(a).$$

Multiplying by w and integrating over $]a,x[$, where $x]a,b[$, we obtainr

$$\int_a^x f^{\sigma}(s)\left(\frac{f^2(a)}{\sigma} - \frac{f^2(s)}{\sigma+2}\right)w(s)\,ds < \frac{2}{\sigma(\sigma+2)}f^{\sigma+2}(a)\int_a^x w(s)\,ds,$$

which shows that (2) holds for $n = 1$.

Let us now assume that (2) holds for some $p \in \mathbb{N}$, that is,

$$\int_a^y (y-t)^{p-1}f^{\sigma}(t)\left(\frac{f^2(a)}{\sigma} - \frac{f^2(t)}{\sigma+2}\right)w(t)\,dt < \frac{2f^{\sigma+2}(a)}{\sigma(\sigma+2)}\int_a^y (y-t)^{p-1}w(t)\,dt$$

for all $y \in]a,b[$. Integrating over $]a,x[$, where $x \in]a,b[$, we obtain

$$\begin{aligned}&\int_a^x \left(\int_a^y (y-t)^{p-1} f^\sigma(t) \left(\frac{f^2(a)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t)\,dt \right) dy \\ &< \frac{2f^{\sigma+2}(a)}{\sigma(\sigma+2)} \int_a^x \left(\int_a^y (y-t)^{p-1} w(t)\,dt \right) dy.\end{aligned} \tag{6}$$

On the other hand, by Fubini's theorem, we have

$$\begin{aligned}&\int_a^x \left(\int_a^y (y-t)^{p-1} f^\sigma(t) \left(\frac{f^2(a)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t)\,dt \right) dy \\ &= \int_a^x f^\sigma(t) \left(\frac{f^2(a)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t) \left(\int_t^x (y-t)^{p-1}\,dy \right) dt \\ &= \frac{1}{p} \int_a^x (x-t)^p f^\sigma(t) \left(\frac{f^2(a)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t)\,dt.\end{aligned} \tag{7}$$

Similarly, we have

$$\begin{aligned}&\int_a^x \left(\int_a^y (y-t)^{p-1} w(t)\,dt \right) dy \\ &= \int_a^x w(t) \left(\int_t^x (y-t)^{p-1}\,dy \right) dt \\ &= \frac{1}{p} \int_a^x (x-t)^p w(t)\,dt.\end{aligned} \tag{8}$$

Thus, it follows from (6)–(8) that

$$\int_a^x (x-t)^p f^\sigma(t) \left(\frac{f^2(a)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t)\,dt < \frac{2f^{\sigma+2}(a)}{\sigma(\sigma+2)} \int_a^x (x-t)^p w(t)\,dt,$$

which shows that (2) holds for $p+1$. Thus, by induction, (2) holds for every $n \in \mathbb{N}$. □

Proof of Theorem 2. Let

$$G(t) = -f'(t) f^{\sigma-1}(t) (f^2(b) - f^2(t))$$

for all $t \in]a,b[$. Due to the assumptions on f and f', we have

$$f(]a,b[) \subset]0,+\infty[, \; G(]a,b[) \subset]-\infty,0[.$$

Then, for every $s \in]a,b[$, there holds

$$\int_s^b G(t)\,dt < 0,$$

which is equivalent to

$$\int_s^b \left(-f'(t) f^{\sigma-1}(t) f^2(b) + f'(t) f^{\sigma+1}(t) \right) dt < 0. \tag{9}$$

On the other hand, we have

$$\int_s^b \left(-f'(t) f^{\sigma-1}(t) f^2(b) + f'(t) f^{\sigma+1}(t) \right) dt = f^\sigma(s) \left(\frac{f^2(b)}{\sigma} - \frac{f^2(s)}{\sigma+2} \right) - \frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)},$$

which implies by (9) that

$$f^\sigma(s) \left(\frac{f^2(b)}{\sigma} - \frac{f^2(s)}{\sigma+2} \right) < \frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)}, \quad a < s < b.$$

Multiplying the above inequality by $w(s)$, we get

$$f^{\sigma}(s)\left(\frac{f^2(b)}{\sigma}-\frac{f^2(s)}{\sigma+2}\right)w(s)<\frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)}w(s),\quad a<s<b.$$

Integrating the above inequality over $]x,b[$, where $a<x<b$, we obtain

$$\int_x^b f^{\sigma}(s)\left(\frac{f^2(b)}{\sigma}-\frac{f^2(s)}{\sigma+2}\right)w(s)\,ds<\frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)}\int_x^b w(s)\,ds,$$

which shows that (3) holds for $n=1$.

The rest of the proof is similar to that of the previous theorem. □

Proof of Theorem 3. We provide two different proofs of Theorem 3. The second proof was suggested by one of the referees of the paper.

Proof 1. Let

$$H(t)=-(t-a)f'(t)$$

for all $t\in]a,b[$. Due to the assumption on f', we have

$$H(]a,b[)\subset]0,+\infty[,$$

which implies that

$$\int_a^s H(t)\,dt>0 \tag{10}$$

for every $s\in]a,b[$. Integrating by parts, we get

$$\begin{aligned}\int_a^s H(t)\,dt &= -\int_a^s (t-a)f'(t)\,dt\\ &= -\left([(t-a)f(t)]_{t=a}^{s}-\int_a^s f(t)\,dt\right)\\ &= -\left((s-a)f(s)-\int_a^s f(t)\,dt\right)\\ &= -(s-a)f(s)+\int_a^s f(t)\,dt,\end{aligned}$$

which implies by (10) that

$$\int_a^s f(t)\,dt>(s-a)f(s).$$

Integrating over $]a,x[$, where $x\in]a,b[$, we obtain

$$\int_a^x\left(\int_a^s f(t)\,dt\right)ds>\int_a^x (s-a)f(s)\,ds. \tag{11}$$

Furthermore, an integration by parts yields

$$\begin{aligned}\int_a^x\left(\int_a^s f(t)\,dt\right)ds &= \left[s\int_a^s f(t)\,dt\right]_{s=a}^{x}-\int_a^x sf(s)\,ds\\ &= x\int_a^x f(t)\,dt-\int_a^x sf(s)\,ds,\end{aligned}$$

that is,

$$\int_a^x\left(\int_a^s f(t)\,dt\right)ds=\int_a^x (x-t)f(t)\,dt,$$

which implies by (11) that

$$\int_a^x (x-t)f(t)\,dt > \int_a^x (t-a)f(t)\,dt, \quad a < x < b.$$

This shows that (4) holds for $n = 2$.

Let us now assume that (4) is satisfied for some $p \in \mathbb{N}$, $p \geq 2$, that is,

$$\int_a^y (y-t)^{p-1} f(t)\,dt > (p-1)\int_a^y (y-t)^{p-2}(t-a)f(t)\,dt$$

for all $y \in]a,b[$. Integrating over $]a,x[$, where $x \in]a,b[$, we obtain

$$\int_a^x \left(\int_a^y (y-t)^{p-1} f(t)\,dt \right) dy > (p-1)\int_a^x \left(\int_a^y (y-t)^{p-2}(t-a)f(t)\,dt \right) dy. \tag{12}$$

On the other hand, by Fubini's theorem, we have

$$\begin{aligned} &\int_a^x \left(\int_a^y (y-t)^{p-1} f(t)\,dt \right) dy \\ &= \int_a^x f(t) \left(\int_t^x (y-t)^{p-1}\,dy \right) dt \\ &= \frac{1}{p}\int_a^x (x-t)^p f(t)\,dt \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\int_a^x \left(\int_a^y (y-t)^{p-2}(t-a)f(t)\,dt \right) dy \\ &= \int_a^x (t-a)f(t) \left(\int_t^x (y-t)^{p-2}\,dy \right) dt \\ &= \frac{1}{p-1}\int_a^x (x-t)^{p-1}(t-a)f(t)\,dt. \end{aligned} \tag{14}$$

Thus, it follows form (12)–(14) that

$$\int_a^x (x-t)^p f(t)\,dt > p\int_a^x (x-t)^{p-1}(t-a)f(t)\,dt,$$

which shows that (4) holds for $p+1$. Hence, by induction, (4) holds for all $n \in \mathbb{N}$, $n \geq 2$.

Proof 2. Observe first that (4) is equivalent to

$$\int_a^x (x-t)^{n-2}(x-nt+(n-1)a)f(t)\,dt > 0. \tag{15}$$

On the other hand, we have

$$\begin{aligned} &\int_a^x (x-t)^{n-2}(x-nt+(n-1)a)f(t)\,dt \\ &= \int_a^{\frac{x-a}{n}+a} (x-t)^{n-2}(x-nt+(n-1)a)f(t)\,dt \\ &\quad + \int_{\frac{x-a}{n}+a}^x (x-t)^{n-2}(x-nt+(n-1)a)f(t)\,dt. \end{aligned} \tag{16}$$

Observe that

$$x-nt+(n-1)a > 0, \quad a < t < \frac{x-a}{n}+a$$

and

$$x - nt + (n-1)a < 0, \quad \frac{x-a}{n} + a < t < x.$$

Then, since $f'(t) < 0$ for all $a < t < b$, we have

$$\begin{aligned} &\int_a^{\frac{x-a}{n}+a} (x-t)^{n-2}(x-nt+(n-1)a) f(t)\, dt \\ &> f\left(\frac{x-a}{n}+a\right) \int_a^{\frac{x-a}{n}+a} (x-t)^{n-2}(x-nt+(n-1)a)\, dt \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\int_{\frac{x-a}{n}+a}^{x} (x-t)^{n-2}(x-nt+(n-1)a) f(t)\, dt \\ &> f\left(\frac{x-a}{n}+a\right) \int_{\frac{x-a}{n}+a}^{x} (x-t)^{n-2}(x-nt+(n-1)a)\, dt. \end{aligned} \tag{18}$$

Thus, (16)–(18) yield

$$\begin{aligned} &\int_a^{x} (x-t)^{n-2}(x-nt+(n-1)a) f(t)\, dt \\ &> f\left(\frac{x-a}{n}+a\right) \int_a^{x} (x-t)^{n-2}(x-nt+(n-1)a)\, dt. \end{aligned} \tag{19}$$

On the other hand, an integration by parts yields

$$\begin{aligned} &\int_a^{x} (x-t)^{n-2}(x-nt+(n-1)a)\, dt \\ &= -\frac{1}{n-1}\Big[(x-nt+(n-1)a)(x-t)^{n-1}\Big]_{t=a}^{x} - \frac{n}{n-1}\int_a^x (x-t)^{n-1}\, dt \\ &= \frac{(x-a)^n}{n-1} - \frac{(x-a)^n}{n-1} = 0. \end{aligned}$$

Hence, by (19), we obtain (15). □

Proof of Theorem 4. Applying inequality (4) with $-f$ instead of f, we obtain the result. □

Proof of Theorem 5. Introducing the function

$$I(t) = -(b-t)f'(t)$$

for all $t \in]a,b[$, and proceeding as in the proof of Theorem 3, the desired inequality follows. □

Proof of Theorem 6. Applying Theorem 5 with $-f$ instead of f, we obtain the desired inequality. □

3. Some Special Cases

Functional inequalities involving special functions are very useful in mathematical analysis, and several interesting results have been obtained in this topic. See e.g., [2,15–25].

Here, some estimates involving trigonometric and hyperbolic functions are deduced from our main results.

Corollary 1. *Let $\sigma \in \mathbb{R}\backslash\{0,-2\}$, $w \in C([0,\frac{\pi}{2}))$ and $w(t) > 0$ for every $t \in \,]0,\frac{\pi}{2}[$. Then, for evry $n \in \mathbb{N}$ and $x \in \,]0,\frac{\pi}{2}[$, it holds that*

$$\int_0^x (x-t)^{n-1} \cos^{\sigma}(t)\left(\frac{1}{\sigma} - \frac{\cos^2(t)}{\sigma+2}\right) w(t)\, dt < \frac{2}{\sigma(\sigma+2)} \int_0^x (x-t)^{n-1} w(t)\, dt. \tag{20}$$

Proof. Let

$$f(t) = \cos t$$

for all $t \in \left[0, \frac{\pi}{2}\right]$. It can be easily seen that $f \in \mathcal{V}([a,b])$ with $a = 0$ and $b = \frac{\pi}{2}$. Then, the functions f and w verify the assumptions of Theorem 1, and (2) holds for all $n \in \mathbb{N}$, $\sigma \in \mathbb{R}\backslash\{0,-2\}$ and $0 < x < \frac{\pi}{2}$. Namely, we have

$$\int_0^x (x-t)^{n-1} \cos^\sigma(t) \left(\frac{\cos^2(0)}{\sigma} - \frac{\cos^2(t)}{\sigma+2} \right) w(t)\, dt < \frac{2\cos^{\sigma+2}(0)}{\sigma(\sigma+2)} \int_0^x (x-t)^{n-1} w(t)\, dt,$$

which yields (20). □

Taking $w = 1$ in the above result, we obtain the following

Corollary 2. *Let $\sigma \in \mathbb{R}\backslash\{0,-2\}$. Then, for all $n \in \mathbb{N}$ and $0 < x < \frac{\pi}{2}$, we have*

$$\frac{1}{x^n} \int_0^x (x-t)^{n-1} \cos^\sigma(t) \left(\frac{1}{\sigma} - \frac{\cos^2(t)}{\sigma+2} \right) dt < \frac{2}{n\sigma(\sigma+2)}. \tag{21}$$

The following inequality derived by Mitrinović and Adamović [15] is a special case of Corollary 2.

Corollary 3. *For all $0 < x < \frac{\pi}{2}$, we have*

$$\left(\frac{\sin x}{x} \right)^3 > \cos x. \tag{22}$$

Proof. Taking $n = 1$ and $\sigma = -\frac{4}{3}$ in (21), we obtain

$$\frac{1}{x} \int_0^x \cos^{-\frac{4}{3}}(t) \left(-\frac{3}{4} - \frac{3\cos^2(t)}{2} \right) dt < -\frac{9}{4},$$

that is,

$$\int_0^x \cos^{-\frac{4}{3}}(t) \left(\frac{3}{4} + \frac{3\cos^2(t)}{2} \right) dt > \frac{9x}{4}. \tag{23}$$

On the other hand, for all $0 < t < x$, we have

$$\begin{aligned} \cos^{-\frac{4}{3}}(t) \left(\frac{3}{4} + \frac{3\cos^2(t)}{2} \right) &= \cos^{-\frac{4}{3}}(t) \left(\frac{3}{4}\cos^2 t + \frac{3}{4}\sin^2 t + \frac{3\cos^2(t)}{2} \right) \\ &= \frac{9}{4}\cos^{\frac{2}{3}}(t) + \frac{3}{4}\cos^{-\frac{4}{3}}(t)\sin^2 t \\ &= \frac{9}{4}\left(\cos^{\frac{2}{3}}(t) + \frac{1}{3}\cos^{-\frac{4}{3}}(t)\sin^2 t \right) \\ &= \frac{d}{dt}\left(\frac{9}{4}\sin t \cos^{\frac{-1}{3}}(t) \right), \end{aligned}$$

which yields

$$\int_0^x \cos^{-\frac{4}{3}}(t) \left(\frac{3}{4} + \frac{3\cos^2(t)}{2} \right) dt = \frac{9}{4}\sin x \cos^{\frac{-1}{3}}(x). \tag{24}$$

Finally, (22) follows from (23) and (24). □

Corollary 4. *Let $\sigma \in \mathbb{R}\backslash\{0,-2\}$ and $w \in C(\mathbb{R})$ be such that*

$$w(t) > 0$$

for every $t > 0$. Then, for all $n \in \mathbb{N}$ and $x > 0$, it holds that

$$\int_0^x (x-t)^{n-1} \cosh^\sigma(t) \left(\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2} \right) w(t)\, dt < \frac{2}{\sigma(\sigma+2)} \int_0^x (x-t)^{n-1} w(t)\, dt. \quad (25)$$

Proof. Let $b > 0$ and

$$f(t) = \cosh t$$

for every $t \in [0, b]$. It can be easily seen that $f \in \mathcal{V}([a, b])$, where $a = 0$. Then, the functions f and w verify the assumptions of Theorem 1, and (2) is satusfied for every $n \in \mathbb{N}$, $\sigma \in \mathbb{R}\backslash\{0, -2\}$ and $x > 0$ (since $b > 0$ is arbitrary chosen). Namely, we obtain

$$\int_0^x (x-t)^{n-1} \cosh^\sigma(t) \left(\frac{\cosh^2(0)}{\sigma} - \frac{\cosh^2(t)}{\sigma+2} \right) w(t)\, dt < \frac{2\cosh^{\sigma+2}(0)}{\sigma(\sigma+2)} \int_0^x (x-t)^{n-1} w(t)\, dt,$$

which yields (25). □

Taking $w = 1$ in the above result, we deduce the following inequality.

Corollary 5. *Let $\sigma \in \mathbb{R}\backslash\{0, -2\}$. Then, for all $n \in \mathbb{N}$ and $x > 0$, we have*

$$\frac{1}{x^n} \int_0^x (x-t)^{n-1} \cosh^\sigma(t) \left(\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2} \right) dt < \frac{2}{n\sigma(\sigma+2)}. \quad (26)$$

The following result due to Lazarevic [16] is a special case of Corollary 5.

Corollary 6. *We have*

$$\left(\frac{\sinh x}{x} \right)^3 > \cosh x \quad (27)$$

for every $x \neq 0$.

Proof. Without restriction of the generality, we may suppose that $x > 0$. Taking $n = 1$ and $\sigma = -\frac{4}{3}$ in (26), we obtain

$$\frac{1}{x} \int_0^x \cosh^{-\frac{4}{3}}(t) \left(-\frac{3}{4} - \frac{3\cosh^2(t)}{2} \right) dt < -\frac{9}{4},$$

that is,

$$\int_0^x \cosh^{-\frac{4}{3}}(t) \left(\frac{3}{4} + \frac{3\cosh^2(t)}{2} \right) dt > \frac{9x}{4}. \quad (28)$$

On the other hand, for all $0 < t < x$, we have

$$\begin{aligned} \cosh^{-\frac{4}{3}}(t) \left(\frac{3}{4} + \frac{3\cosh^2(t)}{2} \right) &= \cosh^{-\frac{4}{3}}(t) \left(\frac{3}{4}\cosh^2 t - \frac{3}{4}\sinh^2 t + \frac{3\cosh^2(t)}{2} \right) \\ &= \frac{9}{4}\cosh^{\frac{2}{3}}(t) - \frac{3}{4}\cosh^{-\frac{4}{3}}(t)\sinh^2 t \\ &= \frac{9}{4}\left(\cosh^{\frac{2}{3}}(t) - \frac{1}{3}\cosh^{-\frac{4}{3}}(t)\sinh^2 t \right) \\ &= \frac{d}{dt}\left(\frac{9}{4}\sinh t \cosh^{\frac{-1}{3}}(t) \right), \end{aligned}$$

which yields

$$\int_0^x \cosh^{-\frac{4}{3}}(t)\left(\frac{3}{4}+\frac{3\cosh^2(t)}{2}\right)dt = \frac{9}{4}\sinh x \cosh^{\frac{-1}{3}}(x). \tag{29}$$

Finally, (27) follows from (28) and (29). □

From Theorem 3, we deduce the following inequality.

Corollary 7. *For all $n \in \mathbb{N}$, $n \geq 2$ and $0 < x < \frac{\pi}{2}$, we have*

$$\int_0^x (x-nt)(x-t)^{n-2}\cos t\,dt > 0. \tag{30}$$

Proof. Let

$$f(t) = \cos t, \quad t \in \mathbb{R}.$$

Let $a = 0$ and $b = \frac{\pi}{2}$. One has

$$f'(t) = -\sin t < 0, \quad a < t < b.$$

Then, the function f satisfies the assumptions of Theorem 3. Hence, using (4), we obtain (30). □

From Corollary 7, we deduce the following Cusa-Huygens inequality (see [2]).

Corollary 8. *For all $0 < x < \frac{\pi}{2}$, we have*

$$\frac{\sin x}{x} < \frac{2+\cos x}{3}. \tag{31}$$

Proof. Taking $n = 3$ in (30), we obtain that

$$\int_0^x (x-t)(x-3t)\cos t\,dt > 0 \tag{32}$$

for all $0 < x < \frac{\pi}{2}$. A double integration by parts shows that

$$\int_0^x (x-t)(x-3t)\cos t\,dt = \frac{2+\cos x}{3} - \frac{\sin x}{x}. \tag{33}$$

Hence, (31) follows from (32) and (33). □

Similarly, taking $f(t) = \cosh(t)$, $t > 0$, in Theorem 4, we obtain the following result.

Corollary 9. *For all $n \in \mathbb{N}$, $n \geq 2$ and $x > 0$, we have*

$$\int_0^x (x-nt)(x-t)^{n-2}\cosh(t)\,dt < 0. \tag{34}$$

Taking $n = 3$ in (34), we obtain the following hyperbolic version of inequality (31) (see [16]).

Corollary 10. *We have*

$$\frac{\sinh x}{x} < \frac{2+\cosh x}{3}, \quad x \neq 0.$$

From Theorem 1, we deduce the following inequality.

Corollary 11. *We have*

$$\frac{\ln(\tan x + \sec x)}{x} > \left(\frac{20}{3} - \frac{2}{3}\sec^3 x - \sec x\right)\frac{\tan x}{x} - 4, \quad 0 < x < \frac{\pi}{2}. \tag{35}$$

Proof. Using Theorem 1 with $f(t) = \cos t$, $w(t) = cos^{-5}t$, $a = 0$, $b = \frac{\pi}{2}$, $\sigma = 3$ and $n = 1$, we obtain

$$\int_0^x \cos^{-2} t\left(\frac{1}{3} - \frac{\cos^2 t}{5}\right) dt < \frac{2}{15}\int_0^x \cos^{-5} t \, dt \tag{36}$$

for all $0 < x < \frac{\pi}{2}$. Moreover, we have

$$\int_0^x \cos^{-2} t\left(\frac{1}{3} - \frac{\cos^2 t}{5}\right) dt = \frac{\tan x}{3} - \frac{x}{5} \tag{37}$$

and

$$\int_0^x \cos^{-5} t \, dt = \frac{\sec^4 x \sin x + 3\left(\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln(\tan x + \sec x)\right)}{4}. \tag{38}$$

Using, (36)–(38), we obtain (35). □

4. An Application

Our aim is to investigate the the existence and uniqueness of solutions to

$$u(x) = \int_0^x (x-t)^{n-1}\cosh^\sigma(t)\left(\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2}\right)F(t, u(t)) \, dt, \quad 0 \le x \le h, \tag{39}$$

where $h > 0$, $\sigma > 0$, $n \in \mathbb{N}$ and $F : [0, h] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Namely, using Corollary 5, we shall establish the following result.

Theorem 7. *Assume that there exists $\alpha > 0$ such that*

$$|F(t, y) - F(t, z)| \le \alpha |y - z| \tag{40}$$

for all $0 < t < h$ and $y, z \in \mathbb{R}$. If

$$0 < h < \min\left\{\left(\frac{n\sigma(\sigma+2)}{2\alpha}\right)^{\frac{1}{n}}, \cosh^{-1}\left(\sqrt{1+\frac{2}{\sigma}}\right)\right\}, \tag{41}$$

then (39) admits a unique solution $u^* \in C([0, h])$. Moreover, for any $u_0 \in C([0, h])$, the Picard sequence $\{u_p\} \subset C([0, h])$ defined by

$$u_{p+1}(x) = \int_0^x (x-t)^{n-1}\cosh^\sigma(t)\left(\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2}\right)F(t, u_p(t)) \, dt, \quad 0 \le x \le h$$

converges uniformly to u^*.

Proof. Let us equip $C([0, h])$ with the norm

$$\|u\| = \max_{0 \le x \le h} |u(x)|, \quad u \in C([0, h]).$$

It is well-known that $(C([0, h]), \|\cdot\|)$ is a Banach space. We introduce the mapping

$$T : C([0, h]) \to C([0, h])$$

defined by

$$(Tu)(x) = \int_0^x (x-t)^{n-1} \cosh^\sigma(t) \left(\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2}\right) F(t,u(t))\,dt, \quad 0 \le x \le h,\ u \in C([0,h]).$$

Observe that $u \in C([0,h])$ is a solution to (39) if and only if u is a fixed point of the mapping T (i.e., $Tu = u$). On the other hand, for all $u, v \in C([0,h])$ and $0 \le x \le h$, we have

$$\begin{aligned} & |(Tu)(x) - (Tv)(x)| \\ & \le \int_0^x (x-t)^{n-1} \cosh^\sigma(t) \left|\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2}\right| |F(t,u(t)) - F(t,v(t))|\,dt. \end{aligned}$$

On the other hand, by (41), we have

$$0 < h < \cosh^{-1}\left(\sqrt{1+\frac{2}{\sigma}}\right),$$

which implies that

$$\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2} \ge 0, \quad 0 \le t \le h.$$

Hence, it holds that

$$\begin{aligned} & |(Tu)(x) - (Tv)(x)| \\ & \le \int_0^x (x-t)^{n-1} \cosh^\sigma(t) \left(\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2}\right) |F(t,u(t)) - F(t,v(t))|\,dt. \end{aligned}$$

Making use of (40), we obtain

$$\begin{aligned} & |(Tu)(x) - (Tv)(x)| \\ & \le \alpha \int_0^x (x-t)^{n-1} \cosh^\sigma(t) \left(\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2}\right) |u(t) - v(t)|\,dt \\ & \le \alpha \|u-v\| \int_0^x (x-t)^{n-1} \cosh^\sigma(t) \left(\frac{1}{\sigma} - \frac{\cosh^2(t)}{\sigma+2}\right) dt. \end{aligned}$$

Furthermore, using Corollary 5, we get

$$\begin{aligned} |(Tu)(x) - (Tv)(x)| & \le \frac{2\alpha x^n}{n\sigma(\sigma+2)} x^n \|u-v\| \\ & \le \frac{2\alpha h^n}{n\sigma(\sigma+2)} \|u-v\|. \end{aligned}$$

Consequently, we deduce that

$$\|Tu - Tv\| \le k\|u-v\|, \quad u, v \in C([0,h]),$$

where

$$k = \frac{2\alpha h^n}{n\sigma(\sigma+2)}.$$

On the other hand, due to (41), one has

$$0 < h < \left(\frac{n\sigma(\sigma+2)}{2\alpha}\right)^{\frac{1}{n}},$$

which yields

$$0 < k < 1.$$

Thus, from Banach contraction principle (see e.g., [26]), we deduce that T admits a unique fixed point $u^* \in C([0,h])$, and the Picard sequence $\{u_p\}$ defined by $u_{p+1} = Tu_p$ converges to u^* with respect to the norm $\|\cdot\|$. This completes the proof of Theorem 7. □

5. Conclusions

Some integral inequalities involving strictly monotone functions are provided. We shown that the obtained inequalities can be useful for deriving several functional inequalities involving trigonometric and hyperbolic functions. For instance, Theorem 1 unifies and generalizes Mitrinović-Adamović [15] and Lazarevic [16] inequalities, and Theorem 3 generalizes Cusa-Huygens inequality [2]. By applying Theorem 1, we also obtained a new inequality (see Corollary 11) that provides a lower bound of the function $\frac{\ln(\tan x + \sec x)}{x}$. Further inequalities can also be obtained by considering other functions f in Theorems 1–6. We also shown that our obtained results are useful for studying the existence and uniqueness of solutions to integral equations.

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