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The Cauchy Exponential of Linear Functionals on the Linear Space of Polynomials

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Abstract: In this paper, we introduce the notion of the Cauchy exponential of a linear functional on the linear space of polynomials in one variable with real or complex coefficients using a functional equation by using the so-called moment equation. It seems that this notion hides several properties and results. Our purpose is to explore some of these properties and to compute the Cauchy exponential of some special linear functionals. Finally, a new characterization of the positive-definiteness of a linear functional is given.

Keywords: cauchy power of linear functional; cauchy exponential of linear functional; weakly-regular linear functional; regular linear functional; positive-definite linear functional; orthogonal polynomial sequence; D_u-Laguerre–Hahn operator

MSC: 33C45; 42C05; 46F10

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1. Introduction

We start with a brief overview of some basic notions and results about the linear space of polynomials in one variable $\mathbb{P}_{\mathbb{K}} := \mathbb{K}[x]$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{P}'_{\mathbb{K}}$ be the algebraic dual space of $\mathbb{P}_{\mathbb{K}}$, i.e., the set of all linear functionals from $\mathbb{P}_{\mathbb{K}}$ to $\mathbb{K}$. Here, $\langle u, p \rangle$ is the action of $u \in \mathbb{P}'_{\mathbb{K}}$ on $p \in \mathbb{P}_{\mathbb{K}}$. We denote by $(u)_n := \langle u, x^n \rangle,\ n \geq 0$, the moment of order n of the linear functional $u \in \mathbb{P}'_{\mathbb{K}}$. In the sequel, we recall some useful operations in $\mathbb{P}'_{\mathbb{K}}$ and some of their properties. For u and v in $\mathbb{P}'_{\mathbb{K}}$, $f(x) = \sum_{\nu=0}^{m} a_\nu x^\nu$ in $\mathbb{P}_{\mathbb{K}}$, a, b and c in $\mathbb{K}$, with $a \neq 0$, let $Du = u'$, fu, uv, $(x-c)^{-1}u$, $\mathfrak{h}_a(u)$, $\mathfrak{t}_b(u)$ and $\sigma(u)$ be the linear functionals defined by duality [1–4].

- *The derivative of a linear functional*

$$\langle u', p \rangle := -\langle u, p' \rangle,\ p \in \mathbb{P}_{\mathbb{K}}.$$

Its moments are $(u')_n = -n(u)_{n-1},\ n \geq 0,\ (u)_{-1} = 0.$

- *The left-multiplication of a linear functional by a polynomial* $f(x) = \sum_{k=0}^{m} a_k x^k$.

$$\langle fu, p \rangle := \langle u, fp \rangle,\ p \in \mathbb{P}_{\mathbb{K}}.$$

The corresponding moments are $(fu)_n = \sum_{\nu=0}^{m} a_\nu (u)_{n+\nu},\ n \geq 0.$

- *The Cauchy product of two linear functionals.*

$$\langle uv, p \rangle := \langle u, vp \rangle,\ p \in \mathbb{P}_{\mathbb{K}},$$

where the right-multiplication of v by p is a polynomial given by

$$(vp)(x) := \langle v_y, \frac{xp(x) - yp(y)}{x-y} \rangle,\ p \in \mathbb{P}_c.$$

Its moments are $(uv)_n = \sum_{\nu=0}^{n} (u)_\nu (v)_{n-\nu},\ n \geq 0.$

- *The Dirac delta linear functional at a point c.*

  Given $c \in \mathbb{K}$, δ_c is the Dirac linear functional at point c, defined by

$$\langle \delta_c, p \rangle := p(c),\ p \in \mathbb{P}_{\mathbb{K}}.$$

  In the sequel, we denote $\delta = \delta_0$. Notice that δ is the unit element for the Cauchy product of linear functionals.

- *The division of a linear functional by a polynomial of first degree.*

$$\langle (x-c)^{-1}u, p \rangle := \langle u, \theta_c(p) \rangle = \langle u, \frac{p(x)-p(c)}{x-c} \rangle,\ p \in \mathbb{P}_{\mathbb{K}}.$$

  Its moments are $\left((x-c)^{-1}u\right)_n = \sum_{\nu=0}^{n-1} c^\nu (u)_{n-1-\nu},\ n \geq 0.$

- *The dilation of a linear functional.*

$$\langle \mathfrak{h}_a(u), p \rangle :=\ \langle u, \mathfrak{h}_a(p) \rangle = \langle u, p(ax) \rangle,\ p \in \mathbb{P}_{\mathbb{K}}.$$

  The corresponding moments are $\left(\mathfrak{h}_a(u)\right)_n = a^n (u)_n,\ n \geq 0.$

- *The shift of a linear functional.*

$$\langle \mathfrak{t}_b(u), p \rangle :=\ \langle u, \mathfrak{t}_{-b}(p) \rangle = \langle u, p(x+b) \rangle,\ p \in \mathbb{P}_{\mathbb{K}}.$$

  Its moments are $(\mathfrak{t}_b(u))_n = \sum_{\nu=0}^{n} \left(\begin{array}{c} n \\ \nu \end{array} \right) b^\nu (u)_{n-\nu},\ n \geq 0.$

- *The σ-transformation of a linear functional.*

$$\langle \sigma(u), p \rangle := \langle u, \sigma(p) \rangle = \langle u, p(x^2) \rangle,\ p \in \mathbb{P}_{\mathbb{K}}.$$

  Its moments are $\left(\sigma(u)\right)_n = (u)_{2n},\ n \geq 0.$

As usual, $u^{(n)}$ will denote the nth derivative of $u \in \mathbb{P}'_{\mathbb{K}}$, with the convention $u^{(0)} = u$. By referring to [3], $u \in \mathbb{P}'_{\mathbb{K}}$ has an inverse for the Cauchy product, denoted by u^{-1}, *i.e.*, $uu^{-1} = u^{-1}u = \delta$, if and only if $(u)_0 \neq 0$.
Recall that $u \in \mathbb{P}'_{\mathbb{K}}$ is said to be symmetric if $(u)_{2n+1} = 0$, for all $n \geq 0$. Moreover, u is symmetric if and only if $\sigma(xu) = 0$, or, equivalently, $\mathfrak{h}_{-1}u = u$.

Definition 1 ([5]). *A linear functional $u \in \mathbb{P}'_{\mathbb{K}}$ is said to be weakly-regular if $\phi u = 0$, where $\phi \in \mathbb{P}_{\mathbb{K}}$, then $\phi \equiv 0$.*

Definition 2 ([1,3]). *A linear functional $u \in \mathbb{P}'_{\mathbb{K}}$ is said to be regular (quasi-definite, according to [6]), if there exists a sequence of monic polynomials $\{B_n(x)\}_{n\geq 0}$ in $\mathbb{P}_{\mathbb{K}}$, $\deg B_n = n$, $n \geq 0$, such that $\langle u, B_n B_m \rangle = r_n \delta_{n.m},\ n,\ m \geq 0$, where $r_n \in \mathbb{K}$, $r_n \neq 0$, $n \geq 0$, ($\delta_{n,m}$ is the Kronecker delta).*

In this case, $\{B_n(x)\}_{n\geq 0}$ is said to be a monic orthogonal polynomial sequence with respect to u (in short, MOPS). Any regular linear functional on polynomials is weakly-regular. The converse is not true; see [5].

Definition 3 ([1,6,7])**.** *A linear functional $u \in \mathbb{P}'_{\mathbb{R}}$ is said to be positive (resp. positive-definite), if $\langle u, p^2 \rangle \geq 0$, (resp. $\langle u, p^2 \rangle > 0$), for all $p \in \mathbb{P}_{\mathbb{R}}$, $p \not\equiv 0$.*

Proposition 1 ([1,6,7])**.** *Let $u \in \mathbb{P}'_{\mathbb{R}}$. The following statements are equivalent.*

(i) *u is positive-definite.*
(ii) *There exists a MOPS $\{B_n(x)\}_{n\geq 0}$ in $\mathbb{P}_{\mathbb{R}}$ such that $\langle u, B_n B_m \rangle = r_n \delta_{n,m}$, for every $n, \ m \geq 0$, where $r_n > 0$, for all $n \geq 0$.*

This contribution aims to introduce the analog of the exponential function in the framework of linear functionals and then provide some of its properties. First of all, we must specify that the Cauchy exponential of a linear functional is also a linear functional. We will denote it as e^u. On the other hand, it satisfies

$$\begin{aligned} e^{\lambda \delta} &= e^{\lambda} \delta, \ \lambda \in \mathbb{P}_{\mathbb{K}}. \\ e^{u+v} &= e^u e^v, \ u, v \in \mathbb{P}'_{\mathbb{K}}. \end{aligned}$$

Here, $e^u e^v$ is the Cauchy product of e^u and e^v. The Cauchy exponential of a linear functional on the linear space of polynomials can be defined in several equivalent ways. The easiest one, which fits best with the theory of linear functionals on the linear space of polynomials, is based on its moments. Indeed, the moments of e^u can be defined in an iterate way as follows:

$$(e^u)_0 = e^{(u)_0}, \quad n(e^u)_n = \sum_{\nu=0}^{n-1} (n-\nu)(e^u)_\nu (u)_{n-\nu}, \ n \geq 1.$$

Once defined, we highlight several formulas and properties satisfied by the Cauchy exponential map as a function from $\mathbb{P}'_{\mathbb{K}}$ to $\mathbb{P}'_{\mathbb{K}}$, and to compute the Cauchy exponential of some classical linear functionals (see [6,8,9]).

$$\begin{aligned} e^{2\delta'} &= \mathcal{B}(1/2): \text{ Bessel linear functional with parameter } \alpha = 1/2. \\ e^{-(1/8)\delta''} &= \mathcal{B}[0]: \text{ Symmetric } D\text{−semiclassical linear functional of class 1.} \\ e^{\alpha \delta_{-2}} &= \mathfrak{t}_{-1} \mathcal{J}(\alpha, -1-\alpha): \text{ Shifted Jacobi linear functional.} \end{aligned}$$

Among others, the following formulas: are deduced.

$$\begin{aligned} \mathfrak{h}_a(e^u) &= e^{\mathfrak{h}_a u}, \\ \delta_b^{-1} \mathfrak{t}_b(e^u) &= e^{\delta_b^{-1} \mathfrak{t}_b(u)}, \\ \sigma(e^u) &= e^{\frac{1}{2}\sigma(u)}, \end{aligned}$$

for every u in $\mathbb{P}'_{\mathbb{K}}$ and every a, b in $\mathbb{K}$, where $a \neq 0$.

The manuscript is structured as follows. In Section 2, we first introduce the notion of the Cauchy exponential of a linear functional on the linear space of polynomials. Second, we establish several formulas and properties satisfied by the Cauchy exponential map. In Section 3, we compute the Cauchy power of some special linear functionals by using some properties of the Cauchy exponential map. In Section 4, we give necessary and sufficient conditions on a given linear functional on the linear space of polynomials for its Cauchy exponential will be weakly-regular. In Section 5, we establish a necessary and sufficient condition on a given linear functional in the linear space of polynomials so that its Cauchy exponential will be positive-definite. This enables us to give a new characterization of the positive-definite of a linear functional on the linear space of polynomials. Finally, some open problems concerning orthogonal polynomials associated with the Cauchy exponential function of a linear functional are stated.

2. The Cauchy Exponential of a Linear Functional on the Linear Space of Polynomials

2.1. Definition and Basic Properties

For any $u \in \mathbb{P}'_{\mathbb{K}}$, let $\mathcal{M}(u)$ be the linear functional in $\mathbb{P}'_{\mathbb{K}}$ that is the solution of the following functional equation:

$$\big(\mathcal{M}(u)\big)_0 = e^{(u)_0}, \quad \big(x\mathcal{M}(u)\big)' = (xu)'\mathcal{M}(u). \tag{1}$$

Equivalently, the sequence of moments $\{(\mathcal{M}(u))_n\}_{n\geq 0}$ satisfies the following recurrence relation:

$$\big(\mathcal{M}(u)\big)_0 = e^{(u)_0}, \quad n\big(\mathcal{M}(u)\big)_n = \sum_{\nu=0}^{n-1}(n-\nu)\big(\mathcal{M}(u)\big)_\nu (u)_{n-\nu}, \; n \geq 1. \tag{2}$$

To list some properties of $\mathcal{M}$, we need the following formulas.

Lemma 1 ([2,3])**.** *For any u, v in $\mathbb{P}'_{\mathbb{K}}$, any $f \in \mathbb{P}_{\mathbb{K}}$, and any a, c in $\mathbb{K}$ with $a \neq 0$, we have*

$$(x-c)\big((x-c)^{-1}u\big) = u, \tag{3}$$

$$(x-c)^{-1}\big((x-c)u\big) = u - (u)_0\delta_c, \tag{4}$$

$$uv = vu, \; \delta u = u, \tag{5}$$

$$(uv)' = u'v + uv' + x^{-1}(uv), \tag{6}$$

$$(fu)' = fu' + f'u, \tag{7}$$

$$x^{-1}(uv) = (x^{-1}u)v = u(x^{-1}v). \tag{8}$$

Following (3), where $c = 0$, (1) is equivalent to

$$\big(\mathcal{M}(u)\big)_0 = e^{(u)_0}, \quad \mathcal{M}(u)' = -x^{-1}\mathcal{M}(u) + x^{-1}(xu)'\mathcal{M}(u). \tag{9}$$

Proposition 2. *For any u, v in $\mathbb{P}'_{\mathbb{K}}$, any $\tau \in \mathbb{K}$, and any non-negative integer n, we have the following properties*

(i) $\mathcal{M}(\tau\delta) = e^{\tau}\delta$.

(ii) $\mathcal{M}(u+v) = \mathcal{M}(u)\mathcal{M}(v)$.

(iii) $\big(\mathcal{M}(u)\big)^n = \mathcal{M}(nu)$.

Proof. From (1) taken with $u = \tau\delta$, where $\tau \in \mathbb{K}$, we get $\big(\mathcal{M}(\tau\delta)\big)_0 = e^{\tau}$ and $\big(x\mathcal{M}(\tau\delta)\big)' = 0$. Thus, $x\mathcal{M}(\tau\delta) = 0$. Then, $\mathcal{M}(\tau\delta) = \big(\mathcal{M}(\tau\delta)\big)_0\delta = e^{\tau}\delta$, according to (4) when $c = 0$. Hence, (i) holds.

Let u, v in $\mathbb{P}'_{\mathbb{K}}$. Putting $v_1 = \mathcal{M}(u)$, $v_2 = \mathcal{M}(v)$, $w_1 = \mathcal{M}(u+v)$ and $w_2 = v_1v_2$. From (9), we have

$$(v_1)_0 = e^{(u)_0}, \;\; v_1' = -x^{-1}v_1 + x^{-1}(xu)'v_1, \tag{10}$$

$$(v_2)_0 = e^{(v)_0}, \;\; v_2' = -x^{-1}v_2 + x^{-1}(xv)'v_2, \tag{11}$$

$$(w_1)_0 = e^{(u+v)_0}, \;\; w_1' = -x^{-1}w_1 + x^{-1}\big(x(u+v)\big)'w_1. \tag{12}$$

Clearly, $(w_2)_0 = (v_1v_2)_0 = (v_1)_0(v_2)_0 = e^{(u)_0}e^{(v)_0} = e^{(u+v)_0}$.

From (6), (8), (10) and (11), we obtain

$$\begin{aligned} w_2' &= (v_1v_2)' = v_1'v_2 + v_1v_2' + x^{-1}(v_1v_2) \\ &= \big(-x^{-1}v_1 + x^{-1}(xu)'v_1\big)v_2 + \big(-x^{-1}v_2 + x^{-1}(xv)'v_2\big)v_1 + x^{-1}(v_1v_2) \\ &= -x^{-1}v_1v_2 + x^{-1}\Big(\big(x(u+v)\big)'v_1v_2\Big). \end{aligned}$$

Therefore,

$$(w_2)_0 = e^{(u+v)_0}, \;\; w_2' = -x^{-1}v_1v_2 + x^{-1}\Big(\big(x(u+v)\big)'v_1v_2\Big). \tag{13}$$

From (12), (13), and by the definition of the operator $\mathcal{M}$, we infer that $w_1 = w_2$, *i.e.*, $\mathcal{M}(u+v) = \mathcal{M}(u)\mathcal{M}(v)$. Hence, (ii) holds.

The property (iii) is a straightforward consequence of (i) and (ii). □

In a natural way, it is convenient to use the following notation

$$e^u := \mathcal{M}(u), \quad \text{for every } u \in \mathbb{P}'_{\mathbb{K}}. \tag{14}$$

Definition 4. *For any $u \in \mathbb{P}'_{\mathbb{K}}$, the Cauchy exponential of u, that we denote by e^u, is the unique linear functional in $\mathbb{P}'_{\mathbb{K}}$ that satisfies*

$$\left(e^u\right)_0 = e^{(u)_0}, \quad \left(xe^u\right)' = (xu)'e^u.$$

By an iteration process, we deduce

$$\begin{aligned}
\left(e^u\right)_1 &= e^{(u)_0}(u)_1,\\
\left(e^u\right)_2 &= e^{(u)_0}\big(\frac{1}{2}(u)_1^2 + (u)_2\big),\\
\left(e^u\right)_3 &= e^{(u)_0}\big(\frac{1}{6}(u)_1^3 + (u)_1(u)_2 + (u)_3\big).
\end{aligned}$$

From Proposition 2 and Definition 4, the following formulas hold.

$$e^{\tau\delta} = e^{\tau}\delta, \tag{15}$$

$$e^{u+v} = e^u e^v, \tag{16}$$

$$(e^u)^n = e^{nu}, \tag{17}$$

for any u, v in $\mathbb{P}'_{\mathbb{K}}$, any $\tau \in \mathbb{K}$ and any non-negative integer n.

2.2. Some Properties of the Cauchy Exponential Map

The linear functional Cauchy exponential induces a map in the algebraic dual space $\mathbb{P}'_{\mathbb{K}}$ as follows

$$\begin{aligned}
Exp_{\mathbb{P}'_{\mathbb{K}}} : \mathbb{P}'_{\mathbb{K}} &\longrightarrow \mathbb{P}'_{\mathbb{K}}\\
u &\longmapsto Exp_{\mathbb{P}'_{\mathbb{K}}}(u) = e^u.
\end{aligned}$$

Proposition 3. *For any u, v in $\mathbb{P}'_{\mathbb{K}}$, the following properties hold.*

(i) *When $\mathbb{K} = \mathbb{C}$, then $e^u = e^v$ if and only if there exists an integer k such that $u = v + (2k\pi i)\delta$, where $i^2 = -1$.*

(ii) *When $\mathbb{K} = \mathbb{R}$, then $e^u = e^v$ if and only $u = v$.*

(iii) *$Exp_{\mathbb{P}'_{\mathbb{R}}}$ is an isomorphism of Abelian groups from $(\mathbb{P}'_{\mathbb{R}}, +)$ to $({\mathbb{P}'_{\mathbb{R}}}^{+}, .)$, where ${\mathbb{P}'_{\mathbb{R}}}^{+} = \{v \in \mathbb{P}'_{\mathbb{R}} | (v)_0 > 0\}$.*

Proof. Assume that u, v in $\mathbb{P}'_{\mathbb{C}}$ are such that $e^u = e^v$. Then,

$$\begin{aligned}
(e^u)_0 = e^{(u)_0}, \quad (e^u)' = -x^{-1}e^u + x^{-1}(xu)'e^u,\\
(e^v)_0 = e^{(v)_0}, \quad (e^v)' = -x^{-1}e^v + x^{-1}(xv)'e^v.
\end{aligned}$$

Since $e^{(u)_0} = e^{(v)_0}$ in $\mathbb{C}$, then there exists an integer k such that $(u)_0 = (v)_0 + 2k\pi i$, $i^2 = -1$. Moreover, we can see that $x^{-1}\Big(\big(x(u-v)\big)'e^u\Big) = 0$. Thus, $\big(x(u-v)\big)'e^u = 0$, according to (3) for $c = 0$. However, since e^u is invertible, $(e^u)_0 \neq 0$, then $\big(x(u-v)\big)' = 0$. This requires that, $x(u-v) = 0$. Thus, $u - v = \big((u)_0 - (v)_0\big)\delta = (2k\pi i)\delta$, on account of (4) taken with $c = 0$.

Conversely, assume that u and v are in $\mathbb{P}'_{\mathbb{C}}$ such that $u = v + (2k\pi i)\delta$. From (15) and (16), we get $e^u = e^{v+(2k\pi i)\delta} = e^v e^{(2k\pi i)\delta} = e^v(e^{2k\pi i}\delta) = e^v$.
Hence, (i) holds.

The property (ii) is a straightforward consequence of (i).

For any $v \in {\mathbb{P}'_{\mathbb{R}}}^{+}$, let u be the unique linear functional defined by

$$(u)_0 = \ln\big((v)_0\big), \quad n(u)_n(v)_0 = n(v)_n - \sum_{\nu=1}^{n-1}(n-\nu)(u)_{n-\nu}(v)_\nu, \ n \geq 1. \tag{18}$$

Equivalently,

$$(v)_0 = e^{(u)_0}, \quad (xv)' = (xu)'v. \tag{19}$$

By Definition 4, we infer that $v = e^u$. This concludes the proof of (iii). □

Furthermore, we need the following formulas.

Lemma 2 ([2,3])**.** *For any u, v in $\mathbb{P}'_{\mathbb{K}}$, any $f \in \mathbb{P}_{\mathbb{K}}$, and any a, c in $\mathbb{K}$ with $a \neq 0$, we have the following formulas.*

$$\mathfrak{h}_a(u') = a\big(\mathfrak{h}_a(u)\big)', \tag{20}$$
$$x^{-1}\mathfrak{h}_a(u) = a^{-1}\mathfrak{h}_a(x^{-1}u), \tag{21}$$
$$\mathfrak{h}_a(fu) = f(a^{-1}x)\mathfrak{h}_a u, \tag{22}$$
$$\mathfrak{h}_a(uv) = \mathfrak{h}_a(u)\mathfrak{h}_a(v), \tag{23}$$
$$\mathfrak{t}_b(u') = \big(\mathfrak{t}_b(u)\big)', \tag{24}$$
$$\mathfrak{t}_b(fu) = \mathfrak{t}_b(f)\mathfrak{t}_b(u), \tag{25}$$
$$\mathfrak{t}_b(uv) = \mathfrak{t}_b(u)\mathfrak{t}_b(v)\delta_b^{-1}, \tag{26}$$
$$f(uv) = (fu)v + x(u\theta_0 f)(x)v, \tag{27}$$
$$\sigma\big(f(x^2)u\big) = f(x)\sigma(u), \tag{28}$$
$$\sigma(u') = 2\big(\sigma(xu)\big)', \tag{29}$$
$$2\big(\sigma(u)\big)' = \sigma\big((xu)'\big), \tag{30}$$
$$\sigma(uv) = \sigma(u)\sigma(v), \quad \text{if either} \quad u \quad \text{or} \quad v \quad \text{is symmetric.} \tag{31}$$

Proposition 4. *For any a, b in $\mathbb{K}$, where $a \neq 0$, we have*

(i) $Exp_{\mathbb{P}'_{\mathbb{K}}} \circ \mathfrak{h}_a = \mathfrak{h}_a \circ Exp_{\mathbb{P}'_{\mathbb{K}}}$.
(ii) $\mathfrak{t}_b(e^u)\delta_b^{-1} = e^{\delta_b^{-1}\mathfrak{t}_b(u)}$, *for all* $u \in \mathbb{P}'_{\mathbb{K}}$.
(iii) $\sigma\big(e^u\big) = e^{\sigma(u)}$, *for all symmetric* $u \in \mathbb{P}'_{\mathbb{K}}$.
(iv) e^u *is symmetric if and only if u is symmetric.*

Proof. Let $a \in \mathbb{K}$, with $a \neq 0$, and u in $\mathbb{P}'_{\mathbb{K}}$. Putting $w_1 = e^{\mathfrak{h}_a(u)}$, then

$$(w_1)_0 = e^{(\mathfrak{h}_a(u))_0} = e^{(u)_0}, \quad w_1' = -x^{-1}w_1 + x^{-1}\big(x\mathfrak{h}_a(u)\big)'w_1. \tag{32}$$

Using (20)–(23) and (27), we can derive

$$\begin{aligned}
\mathfrak{h}_{a^{-1}}w_1' &= a^{-1}(\mathfrak{h}_{a^{-1}}w_1)',\\
\mathfrak{h}_{a^{-1}}(x^{-1}w_1) &= a^{-1}x^{-1}\mathfrak{h}_{a^{-1}}w_1,\\
\mathfrak{h}_{a^{-1}}(x\mathfrak{h}_a u)' &= a^{-1}\big(\mathfrak{h}_{a^{-1}}(x\mathfrak{h}_a u)\big)' = (xu)',\\
\mathfrak{h}_{a^{-1}}\big((x\mathfrak{h}_a u)'w_1\big) &= (xu)'\mathfrak{h}_{a^{-1}}w_1,\\
\mathfrak{h}_{a^{-1}}\Big(x^{-1}\big(x\mathfrak{h}_a(u)\big)'w_1\Big) &= a^{-1}x^{-1}\mathfrak{h}_{a^{-1}}\Big(\big(x\mathfrak{h}_a(u)\big)'w_1\Big) = a^{-1}x^{-1}(xu)'\mathfrak{h}_{a^{-1}}w_1.
\end{aligned}$$

Applying the operator $\mathfrak{h}_{a^{-1}}$ in both sides of (32), it follows that

$$(\mathfrak{h}_{a^{-1}}w_1)_0 = e^{(u)_0}, \quad (\mathfrak{h}_{a^{-1}}w_1)' = -x^{-1}\mathfrak{h}_{a^{-1}}w_1 + x^{-1}(xu)'\mathfrak{h}_{a^{-1}}w_1.$$

From the uniqueness of the solution of the last equation, we can say that $\mathfrak{h}_{a^{-1}}w_1 = e^u$ and, then, $w_1 = \mathfrak{h}_a(e^u)$. Hence, (i) holds.

Assume that $b \in \mathbb{K}$ and u in $\mathbb{P}'_{\mathbb{K}}$. Let first establish the following formula

$$\mathfrak{t}_b(v\delta_{-b}) = \mathfrak{t}_b(v)\delta_b^{-1},\ v \in \mathbb{P}'_{\mathbb{K}}. \tag{33}$$

Indeed, by (26), $\mathfrak{t}_b(v\delta_{-b}) = \mathfrak{t}_b(v)\mathfrak{t}_b(\delta_{-b})\delta_b^{-1}$. Since $\mathfrak{t}_b(\delta_{-b}) = \delta$, then we have $\mathfrak{t}_b(v\delta_{-b}) = \mathfrak{t}_b(v)\delta_b^{-1}$. Setting $w = \mathfrak{t}_b(e^u)\delta_b^{-1}$. Clearly, $\big(\mathfrak{t}_b(u)\delta_b^{-1}\big)_0 = (u)_0$ and $(w)_0 = (e^u)_0 = e^{(u)_0}$. On the other hand, by (25), (33) and (27),

$$\begin{aligned}
xw &= x\big(\mathfrak{t}_b(e^u)\delta_b^{-1}\big)\\
&= x\mathfrak{t}_b(e^u\delta_{-b})\\
&= \mathfrak{t}_b\Big((x+b)\big(e^u\delta_{-b}\big)\Big)\\
&= \mathfrak{t}_b\Big(\big((x+b)\delta_{-b}\big)e^u + x\big(\delta_{-b}\theta_0(x+b)\big)(x)e^u\Big).
\end{aligned}$$

However, from $(x+b)\delta_{-b} = 0$ and $\big(\delta_{-b}\theta_0(x+b)\big)(x) = 1$, we get $xw = \mathfrak{t}_b(xe^u)$. From Definition 4, and while using (24), (26) and (33), we obtain

$$\begin{aligned}
(xw)' &= \mathfrak{t}_b\Big(\big((xe^u)'\Big)\\
&= \mathfrak{t}_b\big((xu)'e^u\big)\\
&= \mathfrak{t}_b\big((xu)'\big)\mathfrak{t}_b\big(e^u\big)\delta_b^{-1} = \mathfrak{t}_b\big((xu)'\big)w\\
&= \big(\mathfrak{t}_b(xu)\big)'w.
\end{aligned}$$

From (27), we have $xu = \big((x+b)\delta_{-b}\big)u + x\big(\delta_{-b}\theta_0(x+b)\big)u = (x+b)(u\delta_{-b})$. By (25) and (26), we deduce

$$\begin{aligned}
\mathfrak{t}_b(xu) &= \mathfrak{t}_b\big((x+b)(\delta_{-b}u)\big)\\
&= x\mathfrak{t}_b(\delta_{-b}u)\\
&= x\mathfrak{t}_b(\delta_{-b})\mathfrak{t}_b(u)\delta_b^{-1}\\
&= x\mathfrak{t}_b(u)\delta_b^{-1}.
\end{aligned}$$

Accordingly, we have $(w)_0 = e^{(\mathfrak{t}_b(u)\delta_b^{-1})_0}$ and $(xw)' = \big(x\big(\mathfrak{t}_b(u)\delta_b^{-1}\big)\big)'w$. From the uniqueness of the solution of the last equation, we get $w = e^{\mathfrak{t}_b(u)\delta_b^{-1}}$ and, as a consequence, $\mathfrak{t}_b\big(e^u\big)\delta_b^{-1} = e^{\mathfrak{t}_b(u)\delta_b^{-1}}$. Hence, (ii) holds.

Next, assume that u is a symmetric linear functional, i.e., $\sigma(xu) = 0$. If $w_2 = e^u$, then

$$(w_2)_0 = e^{(u)_0}, \quad (xw_2)' = -(xu)'w_2. \tag{34}$$

Since u is symmetric, then $(xu)'$ is also symmetric. By (31), (29), and (28), it follows that

$$\begin{aligned}\sigma\big((xu)'w_2\big) &= \sigma\big((xu)'\big)\ \sigma\, w_2 \\ &= 2\big(\sigma(x^2u)\big)'\ \sigma\, w_2 \\ &= 2\big(x\sigma(u)\big)'\ \sigma w_2.\end{aligned}$$

Therefore, if we apply the operator σ in both hand sides of (34), then

$$(\sigma w_2)_0 = e^{(u)_0}, \quad \big(x\sigma(w_2)\big)' = -\big(x\sigma(u)\big)'\sigma(w_2).$$

The uniqueness of the solution of the last equation yields $\sigma w_2 = e^{\sigma u}$.
Hence, (iii) holds.
Assume that u is symmetric, i.e., $h_{-1}u = u$. By (i), taken with $a = -1$, we obtain $h_{-1}(e^u) = e^{h_{-1}(u)} = e^u$. Thus, e^u is also symmetric.
Conversely, assume that e^u is symmetric, i.e., $\mathfrak{h}_{-1}(e^u) = e^u$. Again by (i), when $a = -1$, we deduce $e^{\mathfrak{h}_{-1}(u)} = e^u$. Notice that

$$\begin{aligned}(e^u)_0 = e^{(u)_0}, &\quad (xe^u)' = (xu)'e^u. \\ (e^{\mathfrak{h}_{-1}(u)})_0 = e^{(u)_0}, &\quad (xe^u)' = (x\mathfrak{h}_{-1}(u))'e^u.\end{aligned}$$

This implies $(xu)'e^u = (x\mathfrak{h}_{-1}(u))'e^u$. If we multiply both hand sides of the last equation by e^{-u}, then $(xu)' = (x\mathfrak{h}_{-1}(u))'$, and so that $xu = x\mathfrak{h}_{-1}(u)$. Since $(\mathfrak{h}_{-1}u)_0 = (u)_0$, then $\mathfrak{h}_{-1}u = u$, by (4) taken with $c = 0$. Hence, u is symmetric. Thus, the statement (iv) is proved. □

3. Cauchy Power of a Linear Functional

We start recalling the following formulas.

Lemma 3 ([2,3,10]). *For any u, v in $\mathbb{P}'_{\mathbb{K}}$, any $f \in \mathbb{P}_{\mathbb{K}}$ and any a, c in $\mathbb{K}$ where $a \neq 0$, we have*

$$(u^{-1})' = -u^{-2}u' - 2x^{-1}u^{-1}. \tag{35}$$

For any u in $\mathbb{P}'_{\mathbb{K}}$ and any arbitrary non-negative integer number n, we can define the Cauchy power of order n of u, denoted by u^n, as follows

$$u^n = \underbrace{u...u}_{n-times}\,,\ u^0 = \delta.$$

When $(u)_0 \neq 0$, recall that u is invertible. In such a case, we can extend the definition of u^n to negative integer numbers n as follows $u^n = \underbrace{u^{-1}...u^{-1}}_{(-n)-times}$.

In [11], we have deduced that $(u^2)' = 2uu' + x^{-1}u^2$. More generally, we have

Proposition 5. *For any $u \in \mathbb{P}'_{\mathbb{K}}$, the following properties hold.*

(i) *For every positive integer number n we have*

$$(u^n)' = nu^{n-1}u' + (n-1)x^{-1}u^n.$$

(ii) *If* $(u)_0 \neq 0$*, then for every integer number* n*,*

$$(u^n)' = nu^{n-1}u' + (n-1)x^{-1}u^n.$$

Proof. We proceed by induction. If $n = 1$, then $u' = \delta u'$. Therefore, the statement is true. We assume that the statement is true for $n = k$, i.e., $(u^k)' = ku^{k-1}u' + (k-1)x^{-1}u^k$. From the previous Lemma, we get

$$\begin{aligned}(u^{k+1})' &= (u^k u)' \\ &= (u^k)'u + u^k u' + x^{-1}u^{k+1} \\ &= \big(ku^{k-1}u' + (k-1)x^{-1}u^k\big)u + u^k u' + x^{-1}u^{k+1} \\ &= (k+1)u^k u' + (k)x^{-1}u^{k+1}.\end{aligned}$$

Thus, if the statement is true for $n = k$, then it also holds for $n = k+1$. Hence, (i) holds.

Assume that $(u)_0 \neq 0$. Then u is invertible and $uu^{-1} = u^{-1}u = \delta$. Clearly, the statement (ii) is true, for $n = 0$, it comes back to $\delta' = -x^{-1}\delta$. Let n be a negative integer number n. By (i) and Lemma 3, we have

$$\begin{aligned}(u^n)' &= \big((u^{-1})^{-n}\big)' \\ &= -n(u^{-1})^{-n-1}(u^{-1})' - (n+1)x^{-1}(u^{-1})^{-n} \\ &= -nu^{n+1}(u^{-1})' - (n+1)x^{-1}u^n \\ &= -nu^{n+1}(-u^{-2}u' - 2x^{-1}u^{-1}) - (n+1)x^{-1}u^n \\ &= nu^{n-1}u' + (n-1)x^{-1}u^n.\end{aligned}$$

Hence, (ii) holds. □

First application. Recall that the moments of the classical Bessel linear functional $\mathcal{B}(1/2)$, with parameter $\alpha = \frac{1}{2}$, are $(\mathcal{B}(1/2))_n = \frac{(-2)^n}{n!}$, $n \geq 0$. Equivalently, see [7–9],

$$\big(\mathcal{B}(1/2)\big)_0 = 1, \quad \big(\mathcal{B}(1/2)\big)' - (x+2)\mathcal{B}(1/2) = 0.$$

Proposition 6. *For any integer number* m *and* $\lambda \in \mathbb{K}$*,* $\lambda \neq 0$*, we have*

(i) $\mathfrak{h}_{-\frac{\lambda}{2}} e^{-2\,\delta'} = e^{\lambda\delta'}$.

(ii) $\big(\mathcal{B}(1/2)\big)^m = \mathfrak{h}_m\big(\mathcal{B}(1/2)\big)$.

Proof. We start by showing that $e^{-2\,\delta'} = \mathcal{B}(\frac{1}{2})$. Indeed, observe that $(xe^{-2\,\delta'})' + \delta' e^{-2\,\delta'} = 0$. If we compute the first moments of $e^{-2\,\delta'}$ and multiply the last equation by x, after using (27) and an easy computation, we find $\big(e^{-2\,\delta'}\big)_0 = 1$, $\big(x^2 e^{-2\,\delta'}\big)' - (x+2)e^{-2\,\delta'} = 0$. By the uniqueness of the solution of the last equation, $e^{-2\,\delta'} = \mathcal{B}(\frac{1}{2})$. By Proposition 4, (i), we get $\mathfrak{h}_{-\frac{\lambda}{2}} e^{-2\,\delta'} = e^{\mathfrak{h}_{-\frac{\lambda}{2}}(-2\delta')}$. Since $\langle \mathfrak{h}_{-\frac{\lambda}{2}}(-2\delta'), p\rangle = \langle -2\delta', p(-\frac{\lambda}{2}x)\rangle = -\lambda p'(0)$, $p \in \mathbb{P}_{\mathbb{K}}$, then $\mathfrak{h}_{-\frac{\lambda}{2}}(-2\delta') = \lambda\delta'$. Thus, $\mathfrak{h}_{-\frac{\lambda}{2}}(e^{-2\,\delta'}) = e^{\lambda\delta'}$. Hence, (i) holds.

Let m be a non-zero integer. By (17) and the last property (i), we get $\big(\mathcal{B}(\frac{1}{2})\big)^m = (e^{-2\,\delta'})^m = e^{-2m\,\delta'} = \mathfrak{h}_m\big(\mathcal{B}(\frac{1}{2})\big)$. Hence, (ii) holds. □

Second application. Let first recall that the moments of the generalized Bessel linear functional $\mathcal{B}[0]$ with parameter $\nu = 0$, a symmetric D—semi-classical linear functional of class one, see [8,9], are

$$(\mathcal{B}[0])_{2n+1} = 0, \quad (\mathcal{B}[0])_{2n} = \frac{(-1)^n}{2^{2n} n!},\ n \geq 0.$$

Equivalently, $\mathcal{B}[0]$ satisfies the Pearson equation:

$$\big(x^3 \mathcal{B}[0]\big)' - (2x^2 + \frac{1}{2})\mathcal{B}[0] = 0,\ \text{where}\ (\mathcal{B}[0])_0 = 1\ \text{and}\ (\mathcal{B}[0])_1 = 0.$$

Proposition 7. *For any integer number m and $\lambda \in \mathbb{K}$, $\lambda \neq 0$, we have*

(i) $\mathfrak{h}_{2i\sqrt{2\lambda}} e^{\frac{1}{4}\delta''} = e^{\lambda \delta''}$.

(ii) $\big(\mathcal{B}[0]\big)^m = \mathfrak{h}_{\sqrt{m}}\big(\mathcal{B}[0]\big)$.

Proof. First, let us show that $e^{-\frac{1}{8}\delta''} = \mathcal{B}[0]$. Indeed, we have $(xe^{-\frac{1}{8}\delta''})' - \frac{1}{4}\delta'' e^{-\frac{1}{8}\delta''} = 0$. If we compute the first moments of $e^{-\frac{1}{8}\delta''}$ and then multiply the last equation by x^2, we get after using (27) and an easy computation, $\big(x^3 e^{-\frac{1}{8}\delta''}\big)' - (2x^2 + \frac{1}{2})e^{-\frac{1}{8}\delta''} = 0$, with $(e^{\frac{1}{4}\delta''})_0 = 1$, and $(e^{-\frac{1}{8}\delta''})_1 = 0$. By the uniqueness of the solution of this equation, we get $e^{-\frac{1}{8}\delta''} = \mathcal{B}[0]$. By Proposition 4, (i), we get $\mathfrak{h}_{2i\sqrt{2\lambda}}(e^{-\frac{1}{8}\delta''}) = e^{-\frac{1}{8}\mathfrak{h}_{2i\sqrt{2\lambda}}(\delta'')}$. However, since $\mathfrak{h}_{2i\sqrt{2\lambda}}(\delta'') = -8\lambda\delta''$, it follows that $\mathfrak{h}_{2i\sqrt{2\lambda}} e^{-\frac{1}{8}\delta''} = e^{\lambda\delta''}$. Hence, (i) holds.

Let m be a non-zero integer number. By (17) and the last property (i), we get $\big(\mathcal{B}[0]\big)^m = \big(e^{-\frac{1}{8}\delta''}\big)^m = e^{-\frac{m}{2}\delta''} = \mathfrak{h}_{\sqrt{m}}\big(\mathcal{B}[0]\big)$. Hence, (ii) holds. ☐

Third application. Recall that the moments with respect to the sequence $\{(x-1)^n\}_{n\geq 0}$ of the classical Jacobi linear functional $\mathcal{J}(\alpha, -1-\alpha)$ with parameter α, a non-integer number, are

$$\big(\mathcal{J}(\alpha, -1-\alpha)\big)_{n,1} = \langle \mathcal{J}(\alpha, -1-\alpha), (x-1)^n \rangle = (-2)^n \frac{\Gamma(n-\alpha)\Gamma(\alpha)}{\Gamma(-\alpha) n!},\ n \geq 0.$$

Equivalently, (see [1,7,8])

$$\big(\mathcal{J}(\alpha, -1-\alpha)\big)_0 = 1,\ \big((x^2-1)\mathcal{J}(\alpha, -1-\alpha)\big)' + (-x + 2\alpha + 1)\mathcal{J}(\alpha, -1-\alpha) = 0.$$

Notice that the shifted linear functional $w = \mathfrak{t}_{-1}\mathcal{J}(\alpha, -1-\alpha)$ satisfies

$$(w)_0 = 1,\ \big(x(x+2)w\big)' + (-x + 2\alpha)w = 0.$$

Proposition 8. *For any non-zero complex number c and any positive integer number n, we have*

(i) *For any non-integer complex number α such $n\alpha$ is a non-integer number, $\Big(\mathfrak{t}_{-1}\mathcal{J}(\alpha, -1-\alpha)\Big)^n = \mathfrak{t}_{-1}\mathcal{J}(n\alpha, -1-n\alpha)$. Equivalently,*

$$\Big(\mathcal{J}(\alpha, -1-\alpha)\Big)^n = \mathcal{J}(n\alpha, -1-n\alpha)\ \delta_1^{n-1}.$$

(ii) *For any pair of non-integer complex numbers (α, γ) such that $\alpha + \gamma$ is a non-integer number,*

$$\Big(\mathfrak{t}_{-1}\mathcal{J}(\alpha, -1-\alpha)\Big)\Big(\mathfrak{t}_{-1}\mathcal{J}(\gamma, -1-\gamma)\Big) = \mathfrak{t}_{-1}\mathcal{J}(\alpha+\gamma, -1-\alpha-\gamma).$$

Equivalently, $\mathcal{J}(\alpha, -1-\alpha)\mathcal{J}(\gamma, -1-\gamma) = \mathcal{J}(\alpha+\gamma, -1-\alpha-\gamma)\ \delta_1$.

Proof. Let α be a fixed non-integer complex number. First, let's show that $e^{\alpha\delta_{-2}} = \mathfrak{t}_{-1}\mathcal{J}(\alpha, -1-\alpha)$. Indeed, if we put $w = e^{\alpha\delta_{-2}}$, then $(w)_0 = 1$, $w' - x^{-1}(\delta_{-2}w) = x^{-1}w$. Since, $\delta_{-2}w = (w)_0\delta - 2(x+2)^{-1}w = \delta - 2(x+2)^{-1}w$, then $(w)_0 = 1$, $w' - x^{-1}\big(w - 2(x+2)^{-1}w\big) = x^{-1}w$. If we multiply both hand sides of the last equation by $x(x+2)$, we get $x(x+2)w' + \big(x + 2(\alpha+1)\big)w = 0$, i.e., $\big(x(x+2)w\big)' + (-x+2\alpha)w = 0$. This implies that $w = \mathfrak{t}_{-1}\mathcal{J}(\alpha, -1-\alpha)$. By Proposition 4, (i), $\mathfrak{h}_{-\frac{c}{2}}(e^{\alpha\,\delta_{-2}}) = e^{\alpha\,\mathfrak{h}_{-\frac{c}{2}}\delta_{-2}}$. Since, $\mathfrak{h}_{-\frac{c}{2}}(\delta_{-2}) = \delta_c$, then $\mathfrak{h}_{-\frac{c}{2}}(e^{\alpha\,\delta_{-2}}) = e^{\alpha\,\delta_c}$. Hence, the first statement in (i) holds.

Let n be a non-zero integer number and α be a non-integer complex number such that $n\alpha$ is a non-integer number. From (17) and the previous property (i), we get $\big(\mathfrak{t}_{-1}\mathcal{J}(\alpha, -1-\alpha)\big)^n = e^{n\alpha\,\delta_{-2}} = \mathfrak{t}_{-1}\mathcal{J}(n\alpha, -1-n\alpha)$. Therefore, $\big(\mathfrak{t}_{-1}\mathcal{J}(\alpha, -1-\alpha)\big)^n = \mathfrak{t}_{-1}\mathcal{J}(n\alpha, -1-n\alpha)$. By applying the operator $\mathfrak{t}_1$ and using (26), we get $\big(\mathcal{J}(\alpha, -1-\alpha)\big)^n\,\delta_1^{-n+1} = \mathcal{J}(n\alpha, -1-n\alpha)$. This yields $\big(\mathcal{J}(\alpha, -1-\alpha)\big)^n = \mathcal{J}(n\alpha, -1-n\alpha)\,\delta_1^{n-1}$.
Hence, the second statement in (i) holds.

Let (α, γ) be a pair of non-integer complex numbers such that $\alpha+\gamma$ is a non-integer number. We can write

$$\begin{aligned}\big(\mathfrak{t}_{-1}\mathcal{J}(\alpha, -1-\alpha)\big)\big(\mathfrak{t}_{-1}\mathcal{J}(\gamma, -1-\gamma)\big) &= e^{\alpha\,\delta_{-2}}e^{\gamma\,\delta_{-2}}\\ &= e^{(\alpha+\gamma)\,\delta_{-2}}\\ &= \mathfrak{t}_{-1}\mathcal{J}(\alpha+\gamma, -1-\alpha-\gamma).\end{aligned}$$

Finally, if we apply the operator $\mathfrak{t}_1$ and we use (26), we find

$$\mathcal{J}(\alpha, -1-\alpha)\mathcal{J}(\gamma, -1-\gamma) = \mathcal{J}(\alpha+\gamma, -1-\alpha-\gamma)\,\delta_1.$$

Hence, (ii) holds. □

4. Weak-Regularity Property

We start with the following Lemma.

Lemma 4. *For any $u \in \mathbb{P}'_{\mathbb{K}}$, if $(xu)'$ is weakly-regular, then e^u is also weakly-regular.*

Proof. Assume that $u \in \mathbb{P}'_{\mathbb{K}}$ is such that $(xu)'$ is weakly-regular. Suppose that there exists $\phi \in \mathbb{P}_{\mathbb{K}}$, $\phi \not\equiv 0$ such that $\phi e^u = 0$. Necessarily, $\deg(\phi) \geq 1$. Indeed, if we suppose that $\deg(\phi) = 0$, then $0 = (\phi e^u)_0 = \phi e^{(u)_0}$. This is a contradiction, because $\phi \neq 0$ and $e^{(u)_0} \neq 0$. From (7), (27) and the definition of Cauchy exponential of a linear functional, we obtain

$$\begin{aligned}0 &= (\phi x e^u)'\\ &= \phi'(xe^u) + \phi(xe^u)'\\ &= \phi'(xe^u) + \phi\big((xu)'e^u\big)\\ &= \phi'(xe^u) + (\phi e^u)(xu)' + x\big(e^u\theta_0\phi\big)(x)(xu)'\\ &= \phi'(xe^u) + x\big(e^u\theta_0\phi\big)(x)(xu)'.\end{aligned}$$

Multiplying both hand sides of the last equation by ϕ and assuming $\phi e^u = 0$, we get $x\phi(e^u\theta_0\phi)(x)(xu)' = 0$. This is a contradiction, taking into account $(xu)'$ is weakly-regular and the fact that $\deg(\phi) \geq 1$, $(e^u)_0 \neq 0$ and so that $\deg(e^u\theta_0\phi) \geq 0$. □

Proposition 9. *For any u in $\mathbb{P}'_{\mathbb{K}}$, the following statements are equivalent.*

(i) e^u *is weakly-regular.*
(ii) $(xu)'$ *is weakly-regular. Otherwise, we must have*

$$\min\{\deg(A) \mid A \in \mathbb{P}_{\mathbb{K}},\ A \not\equiv 0 \text{ and } A(xu)' = 0\} \geq 2.$$

Proof. $(i) \Rightarrow (ii)$. Assume that e^u is weakly-regular. Suppose that $(xu)'$ is not weakly-regular. Then there exists $A \in \mathbb{P}_{\mathbb{K}}$, $A \not\equiv 0$, with minimum degree, such that $A(xu)' = 0$ and $\deg A \geq 2$. We have to treat two cases.

First case: $\deg(A) = 0$. In such a situation $(xu)' = 0$, and then $u = (u)_0\delta$. In this case, $e^u = e^{(u)_0}e^{\delta} = e^{(u)_0}\delta$ and then $xe^u = 0$. This contradicts the assumption e^u is weakly-regular.

Second case: $\deg(A) = 1$. Therefore, there exists $c \in \mathbb{K}$ such that $(x-c)(xu)' = 0$. Thus, $(xu)' = ((xu)')_0\delta = 0$ and so that $u = (u)_0\delta$. This is a contradiction.

Hence, $\min\{\deg(A) \mid A \in \mathbb{P}_{\mathbb{K}},\ A \not\equiv 0 \text{ and } A(xu)' = 0\} \geq 2$.

$(ii) \Rightarrow (i)$. By Lemma 4, if $(xu)'$ is weakly-regular, e^u is also weakly-regular. Assume that $\min\{\deg(A) \mid A \in \mathbb{P}_{\mathbb{K}},\ A \not\equiv 0 \text{ and } A(xu)' = 0\} \geq 2$. Then, there exists $A \in \mathbb{P}_{\mathbb{K}}$, $\deg(A) \geq 2$, with minimum degree that satisfies $A(xu)' = 0$. We have

$$\begin{aligned} A(xe^u)' &= A\big((xu)'e^u\big) \\ &= A(xu)'e^u + x\big((xu)'\theta_0 A\big)e^u \\ &= x\big((xu)'\theta_0 A\big)e^u. \end{aligned}$$

Equivalently,

$$(Axe^u)' - \Big(A'(x) + \big((xu)'\theta_0 A\big)(x)\Big)xe^u = 0.$$

The last equation can not be simplified. Otherwise, suppose that it can be simplified by $x-c$, where $A(c) = 0$. Then,

$$(x-c)\theta_c(A)\big(xe^u\big)' - \big[(xu)'\theta_0\big((x-c)\theta_c(A)\big)(x)\big]xe^u = 0.$$

Notice that

$$\begin{aligned} (xu)'\theta_0\big((x-c)\theta_c(A)\big)(x) &= \langle (yu)', \frac{(x-c)\theta_c(A)(x) - (y-c)\theta_c(A)(y)}{x-y} \rangle \\ &= \langle (yu)', (x-c)\frac{\theta_c(A)(x) - \theta_c(A)(y)}{x-y} + \theta_c(A)(y) \rangle \\ &= (x-c)\langle (yu)', \frac{\theta_c(A)(x) - \theta_c(A)(y)}{x-y} \rangle + \langle (yu)', \theta_c(A)(y) \rangle. \end{aligned}$$

Then, $(x-c)\Big(\theta_c(A)\big(xe^u\big)' - \big((xu)'\theta_0\theta_c(A)\big)xe^u\Big) - \langle (yu)', \theta_c(A)(y)\rangle xe^u = 0$. The simplification by $(x-c)$ requires the two following conditions:

$$\begin{cases} \langle \theta_c(A)(xe^u)' - \big((xu)'\theta_0\theta_c(A)\big)(x)xe^u, 1\rangle & = 0, \\ \langle (yu)', \theta_c(A)(y)\rangle & = 0. \end{cases}$$

The simplification gives $\theta_c(A)\big(xe^u\big)' - \big((xu)'\theta_0\theta_c(A)\big)\big(xe^u\big) = 0$. By the definition of the Cauchy exponential, $\theta_c(A)(xu)'e^u - (xu)'\theta_0\theta_c(A)(xe^u) = 0$. By (27), it follows that $\big(\theta_c(A)(xu)'\big)e^u = 0$. If we multiply both hand sides of the last equation by e^{-u} and we use the property $e^{-u}e^u = e^u e^{-u} = \delta$, we get $\theta_c(A)(xu)' = 0$. This contradicts the fact that A is of minimum degree such that $A(xu)' = 0$.

If $V = xe^u$, then it satisfies $(AV)' - \big(A' + \big((xu)'\theta_0 A\big)\big)V = 0$, where $\deg A \geq 2$, which can not be simplified. Moreover, $V \neq 0$. Indeed, if $V = 0$, then $e^u = e^{(u)_0}\delta$. This implies $(xu)' = 0$. This is a contradiction. For the sequel, notice that V is weakly-regular if and only if e^u is weakly-regular. Indeed, suppose that there exists a non-zero polynomial Φ with a minimal degree such that $\Phi V = 0$. Thus, we have

$$AV' = \big((xu)'\theta_0 A\big)V, \tag{36}$$

$$\Phi V' = -\Phi' V. \tag{37}$$

Since the pseudo-class (see [11]) of V is equal to $\deg(A)$, then A divides Φ. So, there exists $Q \in \mathbb{P}_{\mathbb{K}}$ such that $\Phi = AQ$. From (36) and (37), we have

$$QAV' = -(QA)'V, \tag{38}$$
$$Q\big((xu)'\theta_0 A\big)V = -(QA)'V. \tag{39}$$

So, $BV = 0$, where $B = Q\big((xu)'\theta_0 A\big) + (QA)'$. Since $\deg(A) \geq 2$, then $\deg(B) = \deg(Q) + \deg(A) - 1 \geq \deg(Q) + 1$. Moreover, $\deg(B) < \deg(\Phi)$. This contradicts the fact that Φ is of minimal degree such that $\Phi V = 0$. Thus, V is weakly-regular and then e^u is also weakly-regular. □

5. A $\mathbf{D}_u$-Laguerre–Hahn Property

In what follows, let $\mathbb{P}_{\mathbb{K}}'^{\star} = \{u \in \mathbb{P}'_{\mathbb{K}} \mid (u)_0 \neq -n, \text{ for all integer } n \geq 1\}$. For any u in $\mathbb{P}_{\mathbb{K}}'^{\star}$, the non-singular lowering operator $\mathbf{D}_u$ on the linear space of polynomials is defined by [10,11]

$$\mathbf{D}_u(p)(x) := p'(x) + u\theta_0 p(x) = p'(x) + \langle u_y, \frac{p(x) - p(y)}{x - y} \rangle, \quad p \in \mathbb{P}_{\mathbb{K}}. \tag{40}$$

Let us give some fundamental properties satisfied by the non-singular lowering operator $\mathbf{D}_u$.

Linearity: $\mathbf{D}_u(\alpha p + \beta q) = \alpha \mathbf{D}_u(p) + \beta \mathbf{D}_u(q),\ p, q \in \mathbb{P}_{\mathbb{K}},\ \alpha, \beta \in \mathbb{K}.$

Lowering of degrees:

$$\begin{aligned} \mathbf{D}_u(x^n)(x) &= \big(n + (u)_0\big)x^{n-1} + \sum_{\nu=0}^{n-2} (u)_{n-\nu-1} x^{\nu},\ n \geq 1,\ \big(\sum_{\nu=0}^{-1} = 0\big), \\ \mathbf{D}_u(1) &= 0. \end{aligned}$$

Under the condition $(u)_0 \neq -n$, for all integer $n \geq 1$, we can see that $\deg(\mathbf{D}_u(p)) = \deg(p) - 1$, for all $p \in \mathbb{P}_{\mathbb{K}}$.

Symmetry:

When u is symmetric, *i.e.*, $(u)_{2n+1} = 0$, $n \geq 0$, and the MPS $\{B_n(x)\}_{n\geq 0}$ is symmetric, then the polynomial sequence $\{Q_n(x)\}_{n\geq 0}$ defined by $Q_n(x) = \mathbf{D}_u(B_{n+1})(x)$, $n \geq 0$, is also symmetric.

The product rule:

$$\mathbf{D}_u(fg) = \mathbf{D}_u(f)g + f\mathbf{D}_u(g) + u\theta_0(fg) - (u\theta_0 f)g - (u\theta_0 g)f,\ f, g \in \mathbb{P}_{\mathbb{K}}. \tag{41}$$

In particular, we have

$$\mathbf{D}_u(xf)(x) = x\mathbf{D}_u(f)(x) + f(x) + \langle u, f\rangle,\ f \in \mathbb{P}_{\mathbb{K}}. \tag{42}$$

By transposition of the operator $\mathbf{D}_u$, we obtain

$$\begin{aligned} \langle {}^t\mathbf{D}_u(w), p\rangle &= \langle w, \mathbf{D}_u(p)\rangle \\ &= \langle w, p' + u\theta_0 p\rangle \\ &= \langle -w' + x^{-1}wu, p\rangle, \quad p \in \mathbb{P}_{\mathbb{K}},\ w \in \mathbb{P}'_{\mathbb{K}}. \end{aligned}$$

Then, ${}^t\mathbf{D}_u(w) = -w' + x^{-1}(wu),\ w \in \mathbb{P}'_{\mathbb{K}}$. If we set $\mathbf{D}_u := -{}^t\mathbf{D}_u$, we have

$$\mathbf{D}_u(w) = w' - x^{-1}(uw),\ w \in \mathbb{P}'_{\mathbb{K}}, \tag{43}$$

and we can write

$$\langle \mathbf{D}_u(w), p\rangle = -\langle w, \mathbf{D}_u(p)\rangle,\ p \in \mathbb{P}_{\mathbb{K}}. \tag{44}$$

The following product rule is a straightforward consequence of the previous definitions and formulas

$$\mathbf{D}_u(fw) = \mathbf{D}_u(f)w + f\mathbf{D}_u w + (w\theta_0 f)u - (u\theta_0 f)w,\ f \in \mathbb{P}_{\mathbb{K}},\ w \in \mathbb{P}'_{\mathbb{K}}. \tag{45}$$

For any $u \in \mathbb{P}'^{\star}_{\mathbb{K}}$, let $S = S(u)$ be the unique linear functional defined by [2]

$$\begin{cases} (S)_0 = 1, \\ \mathbf{D}_u(S) = -\big((u)_0 + 1\big)x^{-1}S. \end{cases} \tag{46}$$

Equivalently,

$$\begin{cases} (S)_0 = 1, \\ S' - x^{-1}(uS) = -\big((u)_0 + 1\big)x^{-1}S. \end{cases} \tag{47}$$

i.e.,

$$\begin{cases} (S)_0 = 1, \\ (xS)' - \big(u - (u)_0\delta\big)S = 0. \end{cases} \tag{48}$$

Let $\{e_n(x;u)\}_{n\geq 0}$ be the sequence of monic polynomials defined by

$$e_n := e_n(x;u) = S^{-1}x^n, \quad n \geq 0, \tag{49}$$

where S is given by (46). Observe that

$$\mathbf{D}_u(e_n) = (n + (u)_0)e_{n-1},\ n \geq 0. \tag{50}$$

Clearly, $\{e_n(x;u)\}_{n\geq 0}$ is an Appell sequence with respect to $\mathbf{D_u}$. In addition, the polynomial sequence $\{e_n(x)\}_{n\geq 0}$ can be characterized by

$$e_0(x) = 1, \quad e_{n+1}(x) = xe_n(x) + (S^{-1})_{n+1},\ n \geq 0. \tag{51}$$

Proposition 10. *For any $v \in \mathbb{P}'_{\mathbb{K}}$, we have*

$$\mathbf{D}_{(xv)'}(e^v) = -x^{-1}e^v. \tag{52}$$

Proof. Assume that $v \in \mathbb{P}'_{\mathbb{K}}$ and recall that e^v is defined by

$$(e^v)_0 = e^{(v)_0}, \quad (xe^v)' = (xv)'e^v. \tag{53}$$

Observe that $(xv)' \in \mathbb{P}'^{\star}_{\mathbb{K}}$, because $\big((xv)'\big)_0 = 0 \neq -n$, $n \geq 1$. From (48) taken with $u = (xv)'$, we have

$$\big(S\big((xv)'\big)\big)_0 = 1, \quad \big(xS\big((xv)'\big)\big)' - (xv)'S\big((xv)'\big) = 0. \tag{54}$$

By the uniqueness of the solution of each of (53) and (54), we deduce

$$e^v = e^{(v)_0}S\big((xv)'\big). \tag{55}$$

This yields the desired result, according to (46), where $u = (xv)'$. □

Setting $\tilde{e}_n(x) = e_n(x;(xv)') = S\big((xv)'\big)^{-1}x^n$, $n \geq 0$. According to (49) and (50), we can say that

$$\tilde{e}_n(x) = e^{(v)_0}e^{-v}x^n,\ n \geq 0. \tag{56}$$

$$\mathbf{D}_u(\tilde{e}_n) = n\tilde{e}_{n-1},\ n \geq 0. \tag{57}$$

$$\tilde{e}_0(x) = 1, \quad \tilde{e}_{n+1}(x) = x\tilde{e}_n(x) + e^{(v)_0}(e^{-v})_{n+1},\ n \geq 0. \tag{58}$$

From (56), observe that

$$\langle e^v, \tilde{e}_n\rangle = e^{(v)_0}\delta_{n,0}, \quad n \geq 0. \tag{59}$$

Lemma 5. *For any $v \in \mathbb{P}'_{\mathbb{K}}$, the monic polynomial sequence $\{\tilde{e}_n(x)\}_{n\geq 0}$ defined by $\tilde{e}_n(x) = e^{(v)_0}e^{-v}x^n,\ n \geq 0$, satisfies*

$$x\tilde{e}'_n(x) + (xv)'\tilde{e}_n(x) = n\tilde{e}_n(x), \quad n \geq 0. \tag{60}$$

Proof. Assume that $v \in \mathbb{P}'_{\mathbb{K}}$. Notice that (57) can be rewritten as $\tilde{e}'_n(x) + (xv)'\theta_0\tilde{e}_n(x) = n\tilde{e}_{n-1}(x),\ n \geq 0$. If we multiply both hand sides of the last equation by x and we use (58), then we obtain

$$x\tilde{e}'_n(x) + x\big((xv)'\theta_0\tilde{e}_n(x)\big)(x) = n\big(\tilde{e}_n - e^{(v)_0}(e^{-v})_n\big), \quad n \geq 0. \tag{61}$$

However, from $(ye^{-v})' = -(yv)'e^{-v}$ and while taking into account (56), we get

$$\begin{aligned} x\big((xv)'\theta_0\tilde{e}_n\big)(x) &= \langle (yv)', \frac{x\tilde{e}_n(x) - y\tilde{e}_n(y)}{x-y} - \tilde{e}_n(y)\rangle \\ &= (xv)'\tilde{e}_n(x) - \langle (yv)', \tilde{e}_n(y)\rangle \\ &= (xv)'\tilde{e}_n(x) - e^{(v)_0}\langle (yv)'e^{-v}, y^n\rangle \\ &= (xv)'\tilde{e}_n(x) + e^{(v)_0}\langle (ye^{-v})', y^n\rangle \\ &= (xv)'\tilde{e}_n(x) - ne^{(v)_0}\big(e^{-v}\big)_n, \quad n \geq 0. \end{aligned}$$

Then, (61) gives $x\tilde{e}'_n + (xv)'\tilde{e}_n - ne^{(v)_0}\big(e^{-v}\big)_n = n\big(\tilde{e}_n - e^{(v)_0}(e^{-v})_n\big) = n\tilde{e}_n$, $n \geq 0$. Hence, the desired result. □

6. A New Characterization of Positive-Definiteness

We start with the two following technical Lemmas.

Lemma 6 ([5])**.** *For any $w \in \mathbb{P}_{\mathbb{R}}'$, the following statements are equivalent.*

(i) *w is positive-definite.*
(ii) *w is weakly-regular and positive.*

Lemma 7. *For any $g \in \mathbb{P}_{\mathbb{K}}$, there exists $p \in \mathbb{P}_{\mathbb{K}}$, with $\deg(p) = \deg(g)$, such that*

$$g(x) - e^{-(v)_0}\langle e^v, g\rangle = xp'(x) + \big((xv)'p\big)(x). \tag{62}$$

Proof. Assume that $g \in \mathbb{P}_{\mathbb{K}}$. We always have $g = \sum_{\nu=0}^{N} \theta_\nu \tilde{e}_\nu$, where $\theta_\nu \in \mathbb{K}$, $0 \le \nu \le N$. From (59) and (60), we have

$$\begin{aligned} g(x) - e^{-(v)_0}\langle e^v, g\rangle &= \sum_{\nu=0}^{N} \theta_\nu \big(\tilde{e}_\nu(x) - e^{-(v)_0}\langle e^v, \tilde{e}_\nu\rangle\big) \\ &= \sum_{\nu=0}^{N} \theta_\nu \big(\tilde{e}_\nu(x) - e^{-(v)_0} e^{(v)_0}\delta_{\nu,0}\big) \\ &= \sum_{\nu=1}^{N} \theta_\nu \tilde{e}_\nu(x) \\ &= \sum_{\nu=1}^{N} \frac{\theta_\nu}{\nu}\big(x\tilde{e}'_\nu(x) + (xv)'\tilde{e}_\nu(x)\big) \\ &= xp'(x) + \big((xv)'p\big)(x), \end{aligned}$$

where $p(x) = \sum_{\nu=1}^{N} \frac{\theta_\nu}{\nu}\tilde{e}_\nu(x)$. □

Theorem 1. *For any linear functional* $v \in \mathbb{P}_{\mathbb{R}}{}'$ *such that* e^v *is weakly-regular, the following statements are equivalent.*

(i) e^v *is positive-definite.*

(ii) *For any* $p \in \mathbb{P}_{\mathbb{R}}$, $\deg(p) = 2l$, $l \ge 1$, *the polynomial* $xp'(x) + \big((xv)'p\big)(x)$ *has at least one real zero.*

Proof. $(i) \Rightarrow (ii)$. Let $v \in \mathbb{P}_{\mathbb{R}}{}'$ such that e^v is positive-definite. Suppose that there exists $p \in \mathbb{P}_{\mathbb{R}}$, $\deg(p) = 2l$, $l \ge 1$, and such that $xp'(x) + \big((xv)'p\big)(x)$ has not real zeros. Clearly, $\deg\big(xp' + (xv)'p\big) = 2l$. Without loss of generality, we can suppose that the leading coefficient of p is positive. Then $xp'(x) + \big((xv)'p\big)(x)$ is a positive polynomial. Under the assumption e^v is positive-definite, then we get $\langle e^v, xp' + (xv)'p\rangle > 0$. This is a contradiction, because $\langle e^v, xp' + (xv)'p\rangle = \langle -(xe^v)' + (xv)'e^v, p\rangle = 0$, by the definition of e^v. Thus, $xp'(x) + \big((xv)'p\big)(x)$ must have at least one real zero.

$(ii) \Rightarrow (i)$. Let $g \in \mathbb{P}_{\mathbb{R}}$, $p \not\equiv 0$ and $g \ge 0$. Let $\deg(g) = 2l, l \ge 0$.

If $l = 0$, i.e., $g(x) = m > 0$, then we have $\langle e^v, g\rangle = me^{(v)_0} > 0$.

If $l \ge 1$, there exists $p \in \mathbb{P}_{\mathbb{R}}$, $\deg(p) = 2l$, such that $g(x) - e^{-(v)_0}\langle e^v, g\rangle = xp'(x) + \big((xv)'p\big)(x)$, by virtue of Lemma 7. By the assumption, there exists $c \in \mathbb{R}$, such that $g(c) - e^{-(v)_0}\langle e^v, g\rangle = 0$. Then, $\langle e^v, g\rangle = e^{(v)_0}g(c) \ge 0$. Thus, e^v is a positive linear functional. Since e^v is weakly-regular, it follows that e^v is positive-definite, according to Lemma 6. □

Corollary 1. *For any weakly-regular linear functional* $w \in \mathbb{P}'^{+}_{\mathbb{R}}$, *the following statements are equivalent.*

(i) w *is positive-definite.*

(ii) *For any* $p \in \mathbb{P}_{\mathbb{R}}$, $\deg(p) = 2l$, $l \ge 1$, *the polynomial* $w^{-1}x(wp)'(x)$ *has at least one real zero.*

Proof. Let $w \in \mathbb{P}'^{+}_{\mathbb{R}}$. By Proposition 3, (iii), there exists a unique $v \in \mathbb{P}_{\mathbb{R}}{}'$ such that $w = e^v$. By Lemma 7, Theorem 1, and under the assumption w is weakly-regular, we infer that w is positive-definite, if and only if $xp'(x) + \big((xv)'p\big)(x)$ has at least one real zero, for all $p \in \mathbb{P}_{\mathbb{R}}$, where $\deg(p) = 2l$, $l \ge 1$. Let $p \in \mathbb{P}_{\mathbb{R}}$, $\deg(p) = 2l$, $l \ge 1$. We always have $p(x) = \sum_{\nu=0}^{2l} \theta_\nu \tilde{e}_\nu(x)$, where $\tilde{e}_n(x) = e^{(v)_0}e^{-v}x^n$, $n \ge 0$. Then,

$$
\begin{aligned}
xp'(x) + \big((xv)'p\big)(x) &= \sum_{\nu=0}^{2l} \theta_\nu \big(x\tilde{e}_\nu'(x) + (xv)'\tilde{e}_\nu(x)\big) \\
&= \sum_{\nu=0}^{2l} \nu\theta_\nu \tilde{e}_\nu(x) \\
&= e^{(v)_0} e^{-v} \sum_{\nu=0}^{2l} \nu\theta_\nu x^\nu \\
&= e^{(v)_0} e^{-v} x \Big(\sum_{\nu=0}^{2l} \theta_\nu e^{-(v)_0} e^{v} x^\nu \Big)' \\
&= w^{-1} x \Big(w \sum_{\nu=0}^{2l} \theta_\nu x^\nu \Big)' \\
&= w^{-1} x (wp)'(x).
\end{aligned}
$$

This concludes the proof. □

7. Concluding Remarks

In this contribution, the Cauchy exponential of a linear functional in the linear space of polynomials with either real or complex coefficients has been introduced. Some analytic and algebraic properties are studied. The Cauchy power of a linear functional is defined. Some illustrative examples of Jacobi and Bessel's classical linear functionals are discussed. A characterization of the weak regularity of the Cauchy exponential of a linear functional is given. A characterization of the positive definiteness of the Cauchy exponential of a linear functional is presented.

As further work, we are dealing with the following problems.

(i) Given a regular linear functional u such that its Cauchy exponential e^u is also a regular linear functional there exists a connection formula between the corresponding sequences of orthogonal polynomials?

(ii) Assuming u is a D—semiclassical linear functional, see [3], is e^u a $D-$semiclassical linear functional?

(iii) Can do you define other analytic functions of linear functionals in a natural way, by using the corresponding Taylor expansions?

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