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Biharmonic Maps on f-Kenmotsu Manifolds with the Schouten–van Kampen Connection

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Abstract: The object of the present paper was to study biharmonic maps on f-Kenmotsu manifolds and f-Kenmotsu manifolds with the Schouten–van Kampen connection. With the help of this connection, our results provided important insights related to harmonic and biharmonic maps.

Keywords: harmonic maps; biharmonic maps; f-Kenmotsu manifolds; Schouten–van Kampen connection

MSC: 53C21; 53C25; 5350; 53E20

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1. Introduction

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds. The energy density of ϕ was the smooth function on M given by:

$$e(\phi)_p = \sum_{i=1}^{m} h(d_p\phi(e_i), d_p\phi(e_i)),$$

for any $p \in M$ and any orthonormal basis $\left\{e_i\right\}_{i=1}^{m}$ of T_pM. If M was a compact Riemannian manifold, the energy functional $E(\phi)$ was the integral of its energy density.

$$E(\phi) = \int_M e(\phi) dv^g. \tag{1}$$

For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}|_{t=0}$, we had the following:

$$\frac{d}{dt}E(\phi_t)|_{t=0} = -\int_M h(\tau(\phi), V) dv^g, \tag{2}$$

where

$$\tau(\phi) = tr_g \nabla d\phi, \tag{3}$$

is the tension field of ϕ. Then, we found that $\phi : (M^m, g) \longrightarrow (N^n, h)$ was harmonic if, and only if,

$$\tau(\phi) = 0. \tag{4}$$

If $(x^i)_{1\leq i\leq m}$ and $(y^\alpha)_{1\leq \alpha\leq n}$ denoted local coordinates on M and N, respectively, then Equation (4) took the following form:

$$\tau(\phi)^\alpha = \sum_{\substack{1\leq \alpha,\beta,\gamma\leq n \\ 1\leq i,j\leq m}} \left(\Delta\phi^\alpha + g^{ij}\, \Gamma^{N\,\alpha}_{\beta\gamma} \frac{\partial\phi^\beta}{\partial x^i} \frac{\partial\phi^\gamma}{\partial x^j} \right) = 0, \tag{5}$$

where $\Delta\phi^{\alpha} = \sum_{\substack{1\le\alpha\le n\\ 1\le i,j\le m}} \Big(\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \big(\sqrt{|g|} g^{ij} \frac{\partial \phi^{\alpha}}{\partial x^j} \Big)$ is the Laplace operator on (M^m, g), and $\overset{N}{\Gamma}{}^{\alpha}_{\beta\gamma}$ are the Christoffel symbols of the Levi-Civita connections of (N^n, h). The biharmonic maps, which provide a natural generalization of harmonic maps, were defined as the critical points of the bi-energy function:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv^g. \tag{6}$$

For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}|_{t=0}$, we had the following:

$$\frac{d}{dt} E_2(\phi_t)|_{t=0} = - \int_M h(\tau_2(\phi), V) dv_g. \tag{7}$$

The Euler–Lagrange equation attached to the bi-energy was given by the vanishing of the bitension field, as follows:

$$\tau_2(\phi) = -(\triangle\tau(\phi) + tr_g R^N(\tau(\phi), d\phi)d\phi). \tag{8}$$

where $\triangle$=trace $(\nabla^{\phi}\nabla^{\phi} - \nabla^{\phi}_{\nabla})$ is the rough Laplacian on the sections of the pull-back bundle $\varphi^{-1}TN$, ∇^{ϕ} is the pull-back connection, and R^N is the curvature tensor on N. Clearly, any harmonic map was always a biharmonic map, and a proper biharmonic map would not be harmonic. The harmonic and biharmonic maps have been studied by many authors [1–4]. Currently, the theories of harmonic and biharmonic maps have become a very important field of research in differential geometry. Najma in [5] studied the harmonic maps between the Kähler and Kenmotsu manifolds. After that, Zagane and Ouakkas in [6] studied the biharmonicity on Kenmotsu manifolds, and they calculated the stress bi-energy tensor from a Kähler manifold to a Kenmotsu manifold. Moreover, Mangione in [7] studied harmonic maps and their stability on f-Kenmotsu manifolds. In [8], Ichi Inoguchi and Eun Lee investigated the biharmonic curves on f-Kenmotsu 3D-manifolds.

Motivated by the above studies, in this paper, we obtained results concerning the harmonicity and biharmonicity of (J, φ)-holomorphic maps from a Kähler manifold (N^{2n}, J, h) to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ and we provided the necessary and sufficient conditions for the biharmonicity of the identity map $I : M \longrightarrow \overline{M}$ from an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ to an f-Kenmotsu manifold with the Schouten–van Kampen connection.

The structure of this paper is as follows: After the introduction, we described some well-known basic formulas and the properties of the f-Kenmotsu manifold and the f-Kenmotsu manifold with the Schouten–van Kampen connection.

In Section 2, we initiated a study of harmonic and biharmonic maps when the domain was a Kähler manifold (N^{2n}, J, h), and the target was an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$. We proved that for $F : N \longrightarrow M$ being a (J, φ)-holomorphic map of constant energy density $e(F)$, then F would be biharmonic if, and only if:

$$-2e(F)((f\circ F)(f'\circ F) + 2(f\circ F)^3)\xi + 3(f\circ F)dF(\mathrm{grad}(f\circ F)) + \triangle(f\circ F)\xi = 0. \tag{9}$$

On the other hand, we proved if the function $f\circ F$ was constant on N and $F : N \longrightarrow M$ was a (J, φ)-holomorphic map of constant energy density, then F would be biharmonic if, and only if:

$$(f\circ F)(f'\circ F) + 2(f\circ F)^3 = 0. \tag{10}$$

Finally, we provided an example of a (J, φ)-holomorphic map from a Kähler manifold to an f-Kenmotsu manifold, which verified Theorem 3.

In Section 3, we proved that any (J, φ)-holomorphic map from a Kähler manifold (N^{2n}, J, h) to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ with the Schouten–van Kampen connection was harmonic. In the same section, we also studied the biharmonicity of the identity map $I : M \longrightarrow \overline{M}$ from an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ to an

f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1}, f, \varphi, \xi, \eta, g)$. We obtained the following results: Firstly, the identity map $I : M \longrightarrow \overline{M}$ would be biharmonic if, and only if, the function f was harmonic. Secondly, if f was a constant function, then the identity map $I : \overline{M} \longrightarrow M$ from an f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1}, f, \varphi, \xi, \eta, g)$ to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ would be biharmonic if, and only if, ξ was biharmonic vector field.

2. Preliminaries

A $(2n+1)$ dimensional real differentiable manifold M was assumed to be an almost contact metric manifold if it had an almost contact metric structure (φ, ξ, η, g), where φ is a $(1,1)$ type tensor field, ξ a global vector field, η is a 1-form, and g is a Riemannian metric compatible with (φ, ξ, η, g), satisfying the following [9–12]:

$$\begin{aligned} \varphi^2 = & -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \\ \varphi\xi = 0, & \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi), \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{11}$$

for any vector fields $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the Lie algebra of all differentiable vector fields on M^{2n+1} and I is the identity transformation.

An almost contact metric manifold was a Kenmotsu manifold if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{12}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold, we had the following relations [13–15]:

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y). \tag{13}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{14}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{15}$$

for any vector fields X, Y on M, and R denotes the Riemannian curvature tensor on M.

We assumed that M was an f-Kenmotsu manifold if the Levi-Civita ∇ of φ satisfied the following condition [16–22]:

$$(\nabla_X \varphi)Y = f\Big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\Big), \tag{16}$$

where $f \in C^\infty(M)$, such that $df \wedge \eta = 0$. If the function f was equal to a constant $\alpha > 0$, we obtained an α-Kenmotsu manifold, which were Kenmotsu manifolds for $\alpha = 1$. If $f = 0$, then the manifold would be cosymplectic [23,24]. An f-Kenmotsu manifold was assumed to be regular if $f^2 + f' \neq 0$, where $f' = \xi(f)$. For an f-Kenmotsu manifold from (11) and (16), it followed that:

$$\nabla_X \xi = f\Big(X - \eta(X)\xi\Big), \tag{17}$$

then using (17), we had

$$(\nabla_X \eta)(Y) = f\Big(g(X, Y) - \eta(X)\eta(Y)\Big). \tag{18}$$

The condition $df \wedge \eta = 0$ held if $\dim(M) \geq 5$; however, this did not hold, in general, if we had $\dim(M) = 3$ [25]. The characteristic vector field of an f-Kenmotsu manifold also satisfied:

$$R(X,Y)\xi = (f^2 + f')\Big(\eta(X)Y - \eta(Y)X\Big), \tag{19}$$

$$R(\xi,Y)Z = (f^2 + f')\Big(\eta(Z)Y - g(Y,Z)\xi\Big), \tag{20}$$

$$\eta[R(\xi,Y)Z] = (f^2 + f')\Big(g(Y,Z)\eta(Z)\eta(Y)\Big). \tag{21}$$

The Schouten–van Kampen connection $\overset{\star}{\nabla}$ associated with the Levi-Civita connection ∇ was given by [26–29]:

$$\overset{\star}{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi, \tag{22}$$

for any vector fields $X, Y \in \Gamma(TM)$. Using (13) and (14), the above equation yielded the following:

$$\overset{\star}{\nabla}_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X. \tag{23}$$

By taking $Y = \xi$ in (23) and using (14), we obtained

$$\overset{\star}{\nabla}_X \xi = 0. \tag{24}$$

Let M be an f-Kenmotsu manifold with the Schouten–van Kampen connection. Then, using (17) and (18) in (22), we obtained the following [30,31]:

$$\overset{\star}{\nabla}_X Y = \nabla_X Y + f\Big(g(X,Y)\xi - \eta(Y)X\Big). \tag{25}$$

Let R and $\overset{\star}{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten–van Kampen connection $\overset{\star}{\nabla}$, then

$$R(X,Y) = \Big(\nabla_X, \nabla_Y\Big) - \nabla_{[X,Y]}, \quad \overset{\star}{R}(X,Y) = \Big(\overset{\star}{\nabla}_X, \overset{\star}{\nabla}_Y\Big) - \overset{\star}{\nabla}_{[X,Y]}.$$

By direct calculations, we obtained the following formula connecting R and $\overset{\star}{R}$ on an f-Kenmotsu manifold M:

$$\overset{\star}{R}(X,Y)Z = R(X,Y)Z + f^2\Big(g(Y,Z)X - g(X,Z)Y\Big) \tag{26}$$

$$+ f'\Big(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \tag{27}$$

$$+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\Big). \tag{28}$$

and

$$\overset{\star}{R}(\xi,Y)Z = 0. \tag{29}$$

3. Harmonic and Biharmonic Maps on f-Kenmotsu Manifolds

Definition 1. *A smooth map $F : N \longrightarrow M$ between a Kähler manifold (N^{2n}, J, h) and an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ was assumed to be a (J,φ)-holomorphic map if it satisfied the following:*

$$dF \circ J = \varphi \circ dF.$$

Lemma 1 ([6]). *Let $F : N \longrightarrow M$ be a (J, φ)-holomorphic map from a Kähler manifold (N^{2n}, J, h) to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$. Then, we had, for any $X \in \Gamma(TN)$,*

$$(\eta \circ dF)(X) = 0.$$

We could ask now if such a map would be harmonic when the domain was a Kähler manifold.

Lemma 2. *Let $F : N \longrightarrow M$ be a (J, φ)-holomorphic map from a Kähler manifold (N^{2n}, J, h) to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$, then we had the following:*

$$g(\tau(F), \xi) = -2(f \circ F)e(F).$$

where $e(F)$ is the energy density of the map F.

Proof. Considering a local orthonormal basis $\left\{e_i\right\}_{i=1}^{2n}$ on T_pN for any $p \in N$, we obtained the following:

$$\begin{aligned}
g(\tau(F), \xi) &= \sum_{i=1}^{2n} g\Big(\nabla^F_{e_i} dF(e_i) - dF(\nabla^N_{e_i} e_i), \xi\Big) \\
&= \sum_{i=1}^{2n} g\Big(\nabla^M_{dF(e_i)} dF(e_i) - dF(\nabla^N_{e_i} e_i), \xi\Big) \\
&= \sum_{i=1}^{2n} g\Big(\nabla^M_{dF(e_i)} dF(e_i), \xi\Big) - \sum_{i=1}^{2n} g\Big(dF(\nabla^N_{e_i} e_i), \xi\Big) \\
&= \sum_{i=1}^{2n} \Big(\nabla^M_{dF(e_i)} g\Big(dF(e_i), \xi\Big) - g\Big(dF(e_i), \nabla^M_{dF(e_i)} \xi\Big)\Big) - \sum_{i=1}^{2n} g\Big(dF(\nabla^N_{e_i} e_i), \xi\Big) \\
&= \sum_{i=1}^{2n} \Big(\nabla^M_{dF(e_i)} \eta(dF(e_i)) - g\Big(dF(e_i), \nabla^M_{dF(e_i)} \xi\Big)\Big) - \sum_{i=1}^{2n} \eta(dF(\nabla^N_{e_i} e_i)).
\end{aligned}$$

As F was a (J, φ)-holomorphic map, then by using Lemma 1, we obtained $\eta(dF(e_i)) = 0$ and $\eta(dF(\nabla^N_{e_i} e_i)) = 0$. Then, we had the following:

$$g(\tau(F), \xi) = -\sum_{i=1}^{2n} g\Big(dF(e_i), \nabla^M_{dF(e_i)} \xi\Big).$$

Using the Equation (17), we obtained the following:

$$\begin{aligned}
g(\tau(F), \xi) &= -\sum_{i=1}^{2n} \Big((f \circ F)\Big(g(dF(e_i), dF(e_i)) - g(dF(e_i), \eta(dF(e_i))\xi)\Big)\Big) \\
&= -\sum_{i=1}^{2n} (f \circ F) g(dF(e_i), dF(e_i)) \\
&= -2(f \circ F)e(F).
\end{aligned}$$

□

Theorem 1. *Let $F : N \longrightarrow M$ be a (J, φ)-holomorphic map from a Kähler manifold (N^{2n}, J, h) to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$. Then, the tension field of the map F was given by:*

$$\tau(F) = -2(f \circ F)e(F)\xi, \tag{30}$$

Proof. For any (J, φ)-holomorphic map $F : N \longrightarrow M$, we have the following formula for its tension field [32]

$$\varphi(\tau(F)) = dF(divJ) - tr_h B,$$

where B is defined by $B(X,Y) = (\nabla^F_X \varphi)dFY$ for any vector fields $X, Y \in \Gamma(TN)$. Since N was a Kähler manifold, $\nabla J = 0$, then we had

$$divJ = \sum_{i=1}^{2n} (\nabla_{e_i} J)e_i = 0,$$

where $\left\{e_i\right\}_{i=1}^{2n}$ is an orthonormal local basis on TN. By using the relation (16) and doing a straightforward calculation, we obtained the following:

$$\begin{aligned} tr_h B &= \sum_{i=1}^{2n} (\nabla^F_{e_i} \varphi)dF(e_i) = \sum_{i=1}^{2n} (\nabla^M_{dF(e_i)} \varphi)dF(e_i) \\ &= \sum_{i=1}^{2n} (f \circ F)\Big(g(\varphi(dF(e_i)), dF(e_i))\xi - \eta(dF(e_i))\varphi(dF(e_i))\Big) \\ &= \sum_{i=1}^{2n} (f \circ F)\Big(- \eta(dF(e_i))\varphi(dF(e_i))\Big). \end{aligned}$$

As F was a (J, φ)-holomorphic map, then by using Lemma 1, we found the following:

$$\sum_{i=1}^{2n} (f \circ F)\Big(- \eta(dF(e_i))\varphi(dF(e_i))\Big) = 0.$$

As a result, $\varphi(\tau(F)) = 0 \Longrightarrow \varphi^2(\tau(F)) = 0$, that is,

$$\tau(F) = \eta(\tau(F))\xi = g(\tau(F), \xi)\xi = -2(f \circ F)e(F)\xi.$$

□

Theorem 2. *Let (N^{2n}, J, h) be a Kähler manifold and $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ be an f-Kenmotsu manifold. Then, any (J, φ)-holomorphic map $F : N \longrightarrow M$ would be harmonic if, and only if, it was a constant map or $f \circ F = 0$.*

Proof. According to Theorem 1, if the map F was harmonic, then $(f \circ F)e(F) = 0$. We assumed that $f \circ F \neq 0$. There existed an open subset U on M, such that $f \circ F \neq 0$ was everywhere on U. Therefore, $e(F) = 0$ was on U. From the harmonicity of F, we concluded that $e(F) = 0$ on M, that is, F was a constant map. □

Biharmonic Maps on f-Kenmotsu Manifolds

Theorem 3. *Let $F : N \longrightarrow M$ be a (J, φ)-holomorphic map from a Kähler manifold (N^{2n}, J, h) to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$. Then, the bitension field of F was given by the following:*

$$\begin{aligned} \tau_2(F) = &-2\Big(- 2(e(F))^2((f \circ F)(f' \circ F) + 2(f \circ F)^3)\xi \\ &+ 3(f \circ F)e(F)dF(grad(f \circ F)) + (f \circ F)\triangle(e(F))\xi \\ &+ e(F)\triangle(f \circ F)\xi + 2g(grad(f \circ F), grad(e(F)))\xi \\ &+ 2(f \circ F)^2 dF(grad(e(F)))\Big). \end{aligned}$$

Proof. By definition of the bitension field of the map F, we had:

$$\begin{aligned}\tau_2(F) =& tr_h(\nabla^F)^2\tau(F) + tr_h R^M(\tau(F), dF)dF \\ =& -2\Big(tr_h(\nabla^F)^2(f\circ F)e(F)\xi + tr_h R^M((f\circ F)e(F)\xi, dF)dF\Big) \\ =& -2\sum_{i=1}^{2n}\Big(\nabla^F_{e_i}\nabla^F_{e_i}(f\circ F)e(F)\xi - \nabla^F_{\nabla^N_{e_i}e_i}(f\circ F)e(F)\xi \\ & + R^M((f\circ F)e(F)\xi, dF(e_i))dF(e_i)\Big),\end{aligned} \tag{31}$$

where $\left\{e_i\right\}_{i=1}^{2n}$ is an orthonormal local basis on TN. A direct calculation provided the following:

$$\begin{aligned}\sum_{i=1}^{2n}\Big(\nabla^F_{e_i}\nabla^F_{e_i}(f\circ F)e(F)\xi\Big) &= \sum_{i=1}^{2n}\Big(\nabla^F_{e_i}((f\circ F)e(F)\nabla^F_{e_i}\xi) + \nabla^F_{e_i}(e_i((f\circ F)e(F))\xi\Big) \\ &= \sum_{i=1}^{2n}\Big((f\circ F)e(F)\nabla^F_{e_i}\nabla^F_{e_i}\xi + e_i((f\circ F)e(F))\nabla^F_{e_i}\xi \\ &\quad + e_i((f\circ F)e(F))\nabla^F_{e_i}\xi + e_i(e_i((f\circ F)e(F)))\xi\Big) \\ &= \sum_{i=1}^{2n}\Big((f\circ F)e(F)\nabla^F_{e_i}\nabla^F_{e_i}\xi + 2\nabla^F_{\mathrm{grad}((f\circ F)e(F))}\xi \\ &\quad + e_i(e_i((f\circ F)e(F)))\xi\Big),\end{aligned}$$

and

$$\sum_{i=1}^{2n}\Big(\nabla^F_{\nabla^N_{e_i}e_i}(f\circ F)e(F)\xi\Big) = \sum_{i=1}^{2n}\Big((f\circ F)e(F)\nabla^F_{\nabla_{e_i}e_i}\xi + \nabla_{e_i}e_i((f\circ F)e(F))\xi\Big).$$

Based on the following:

$$\triangle((f\circ F)e(F)) = \sum_{i=1}^{2n}\Big(e_i(e_i((f\circ F)e(F))) - \nabla_{e_i}e_i((f\circ F)e(F)\Big)\xi,$$

and

$$tr_h(\nabla^F)^2\xi = \sum_{i=1}^{2n}\Big(\nabla^F_{e_i}\nabla^F_{e_i}\xi - \nabla^F_{\nabla^N_{e_i}e_i}\xi\Big),$$

we could deduce that

$$\begin{aligned}tr_h(\nabla^F)^2(f\circ F)e(F)\xi =& (f\circ F)e(F)tr_h(\nabla^F)^2\xi + \triangle((f\circ F)e(F))\xi + 2\nabla^F_{\mathrm{grad}((f\circ F)e(F))}\xi \\ =& (f\circ F)e(F)tr_h(\nabla^F)^2\xi + (f\circ F)\triangle(e(F))\xi + e(F)\triangle(f\circ F)\xi \\ & + 2g(\mathrm{grad}(f\circ F), \mathrm{grad}(e(F))\xi + 2(f\circ F)\nabla^F_{\mathrm{grad}(e(F))}\xi \\ & + 2e(F)\nabla^F_{\mathrm{grad}(f\circ F)}\xi.\end{aligned} \tag{32}$$

After calculating the term $tr_h(\nabla^F)^2\xi$, we obtained the following:

$$tr_h(\nabla^F)^2\xi = \sum_{i=1}^{2n}\Big(\nabla^F_{e_i}\nabla^F_{e_i}\xi - \nabla^F_{\nabla^N_{e_i}e_i}\xi\Big).$$

By using Equation (17), we obtained:

$$\sum_{i=1}^{2n} \nabla^F_{e_i}\xi = \sum_{i=1}^{2n} \nabla_{dF(e_i)}\xi = \sum_{i=1}^{2n} (f\circ F)\Big(dF(e_i)) - \eta(dF(e_i))\xi\Big),$$
$$= (f\circ F)\sum_{i=1}^{2n} dF(e_i),$$

which gave us:

$$\sum_{i=1}^{2n} \nabla^F_{e_i}\nabla^F_{e_i}\xi = \sum_{i=1}^{2n} \nabla^F_{e_i}(f\circ F)dF(e_i),$$

and

$$\begin{aligned}\sum_{i=1}^{2n} \nabla^F_{\nabla^N_{e_i}e_i}\xi &= \nabla^M_{dF(\nabla^N_{e_i}e_i)}\xi \\ &= \sum_{i=1}^{2n}(f\circ F)\Big(dF(\nabla^N_{e_i}e_i) - \eta(dF(\nabla^N_{e_i}e_i))\xi\Big) \\ &= \sum_{i=1}^{2n}(f\circ F)dF(\nabla^N_{e_i}e_i),\end{aligned}$$

we conclude that

$$\begin{aligned} tr_h(\nabla^F)^2\xi &= \sum_{i=1}^{2n}\Big(\nabla^F_{e_i}(f\circ F)dF(e_i) - (f\circ F)dF(\nabla^N_{e_i}e_i)\Big) \\ &= \sum_{i=1}^{2n}\Big(e_i((f\circ F))dF(e_i) + (f\circ F)\nabla^F_{e_i}dF(e_i) - (f\circ F)dF(\nabla^N_{e_i}e_i)\Big) \\ &= dF(\mathrm{grad} f) + (f\circ F)\tau(F) \\ &= dF(\mathrm{grad} f) - 2(f\circ F)^2 e(F)\xi, \end{aligned} \tag{33}$$

Now, by simplifying the terms $\nabla^F_{\mathrm{grad}(e(F))}\xi$, and $\nabla^F_{\mathrm{grad}(f\circ F)}\xi$, we had the following:

$$\begin{aligned}\nabla^F_{\mathrm{grad}(e(F))}\xi &= \nabla^M_{dF(\mathrm{grad}(e(F))}\xi \\ &= (f\circ F)\Big(dF(\mathrm{grad}(e(F)) - \eta dF(\mathrm{grad}(e(F))\xi\Big) \\ &= (f\circ F)dF(\mathrm{grad}(e(F)),\end{aligned}$$

and

$$\begin{aligned}\nabla^F_{\mathrm{grad}(f\circ F)}\xi &= \nabla^M_{dF(\mathrm{grad}(f\circ F))}\xi \\ &= (f\circ F)\Big(dF(\mathrm{grad}(f\circ F)) - \eta dF(\mathrm{grad}(f\circ F))\xi\Big) \\ &= (f\circ F)dF(\mathrm{grad}(f\circ F)),\end{aligned}$$

which finally gave us:

$$\begin{aligned} tr_h(\nabla^F)^2(f\circ F)e(F)\xi = &-2(f\circ F)^3(e(F))^2\xi + (f\circ F)e(F)dF(\mathrm{grad}(f\circ F)) \\ &+ (f\circ F)\triangle(e(F))\xi + e(F)\triangle(f\circ F)\xi \\ &+ 2g(\mathrm{grad}(f\circ F), \mathrm{grad}(e(F)))\xi \\ &+ 2(f\circ F)^2 dF(\mathrm{grad}(e(F))) \\ &+ 2(f\circ F)e(F)dF(\mathrm{grad}(f\circ F)) \end{aligned} \tag{34}$$

By using Equation (19), we obtained the following:

$$\begin{aligned} tr_h R^M(fe(F)\xi, dF)dF =& (f\circ F)e(F)\sum_{i=1}^{2n} R(\xi, dF(e_i))dF(e_i) \\ =& (f\circ F)e(F)((f\circ F)^2+(f'\circ F))\sum_{i=1}^{2n}\Big(\eta(dF(e_i))dF(e_i)-g(dF(e_i),dF(e_i))\xi\Big) \\ =& -2(f\circ F)^3(e(F))^2\xi-2(f\circ F)(f'\circ F)(e(F))^2\xi. \end{aligned} \tag{35}$$

If we replaced (34) and (35) in (31), we arrived at the following:

$$\begin{aligned} \tau_2(F) = & -2\Big(-2(e(F))^2((f\circ F)(f'\circ F)+2(f\circ F)^3)\xi \\ & +3(f\circ F)e(F)dF(\mathrm{grad}(f\circ F))+(f\circ F)\triangle(e(F))\xi \\ & +e(F)\triangle(f\circ F)\xi+2g(\mathrm{grad}(f\circ F),\mathrm{grad}(e(F)))\xi \\ & +2(f\circ F)^2dF(\mathrm{grad}(e(F)))\Big). \end{aligned}$$

□

Corollary 1. *Let (N^{2n}, J, h) be a Kähler manifold and $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ be an f-Kenmotsu manifold. Then, any (J,φ)-holomorphic map $F : N \longrightarrow M$ would biharmonic if, and only if:*

$$\begin{aligned} & -2(e(F))^2((f\circ F)(f'\circ F)+2(f\circ F)^3)\xi \\ & +3(f\circ F)e(F)dF(grad(f\circ F))+(f\circ F)\triangle(e(F))\xi \\ & +e(F)\triangle(f\circ F)\xi+2g(grad(f\circ F),grad(e(F)))\xi \\ & +2(f\circ F)^2dF(grad(e(F)))=0. \end{aligned}$$

Corollary 2. *Let (N^{2n}, J, h) be a Kähler manifold and $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold; then, any (J,φ)-holomorphic map $F : N \longrightarrow M$ would be biharmonic if, and only if:*

$$-4e(F)^2\xi+\triangle(e(F))\xi+2dF(grad(e(F))=0.$$

Corollary 3. *Let (N^{2n}, J, h) be a Kähler manifold and $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ be an f-Kenmotsu manifold. Then, any (J,φ)-holomorphic map $F : N \longrightarrow M$ of constant energy density would biharmonic if, and only if:*

$$-2e(F)((f\circ F)(f'\circ F)+2(f\circ F)^3)\xi+3(f\circ F)dF(grad(f\circ F))+\triangle(f\circ F)\xi=0.$$

Corollary 4. *Let $F : N \longrightarrow M$ be a (J,φ)-holomorphic map of constant energy density from a Kähler manifold (N, J, h) to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$. If the function $f\circ F$ was constant on N, then F would biharmonic if, and only if, $ff'+2f^3=0$ was on $F(N)$.*

Example 1. *Let the five-dimensional manifold $M=\mathbb{R}^4\times(0,\infty)$ be equipped with the Riemannian metric $g=t^{-2\alpha}(dy_1^2+dy_2^2+dy_3^2+dy_4^2)+dt^2$, for some constant $\alpha\in\mathbb{R}$. We considered the following orthonormal basis:*

$$e_1=t^\alpha\frac{\partial}{\partial y_1},\quad e_2=t^\alpha\frac{\partial}{\partial y_2},\quad e_3=t^\alpha\frac{\partial}{\partial y_3},\quad e_4=t^\alpha\frac{\partial}{\partial y_4},\quad e_5=\frac{\partial}{\partial t}.$$

We considered a 1-form η defined by:

$$\eta(X)=g(X,e_5),\quad \forall X\in\Gamma(TM).$$

That is, we chose $e_5 = \xi$. We defined the tensor field φ by the following:

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = -e_4, \quad \varphi(e_4) = e_3, \quad \varphi(e_5) = 0.$$

By the linearity properties of g and φ, we obtained the following:

$$\eta(e_5) = 1, \quad \varphi^2 X = -X + \eta(X)e_5,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M. Therefore, $(M, \varphi, \xi, \eta, g)$ formed an almost contact metric manifold.

Otherwise, we had $[e_i, e_5] = -\alpha t^{-1} e_i$ for $i = 1, 2, 3, 4$ and $[e_i, e_j] = 0$.Let ∇ be the Levi-Civita connection of (M, g). By using Koszul's formula, we obtained the following for $i, j = 1, 2, 3, 4$ with $i \neq j$:

$$\nabla_{e_i} e_i = \frac{\alpha}{t} e_5, \quad \nabla_{e_i} e_5 = -\frac{\alpha}{t} e_i, \quad \nabla_{e_i} e_j = \nabla_{e_5} e_i = \nabla_{e_5} e_5 = 0.$$

The above relations indicated that $\nabla_X \xi = f\{X - \eta(X)\xi\}$ for $\xi = e_5$ and $f = -\frac{\alpha}{t}$. Therefore, we could say that $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ was an f-Kenmotsu manifold. Moreover, $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ was a regular f-Kenmotsu manifold if, and only if, $\alpha \neq 0, -1$, because $f^2 + f' = \frac{\alpha(\alpha+1)}{t^2}$.

Let F be a (J, φ)-holomorphic map, defined by the following:

$$\begin{aligned} F : (\mathbb{R}^2, J, h) &\longrightarrow (M^{2n+1}, f, \varphi, \xi, \eta, g), \\ (x_1, x_2) &\longmapsto (F_1(x_1, x_2), F_2(x_1, x_2), F_3(x_1, x_2), F_4(x_1, x_2), F_5(x_1, x_2)) \end{aligned}$$

where $h = dx_1^2 + dx_2^2$, $J(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_2}$, $J(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_1}$ and $F_i(x_1, x_2)$ are defined by

$$\begin{aligned} F_1(x_1, x_2) &= a_1 x_1 + a_2 x_2 + c_1 \\ F_2(x_1, x_2) &= a_2 x_1 - a_1 x_2 + c_2 \\ F_3(x_1, x_2) &= a_3 x_1 + a_4 x_2 + c_3 \\ F_4(x_1, x_2) &= a_4 x_1 - a_3 x_2 + c_4 \\ F_5(x_1, x_2) &= c_5 \end{aligned}$$

where $a_j, c_i \in \mathbb{R}$ are for all $j = 1, 2, 3, 4$ and $i = 1, 2, 3, 4, 5$. Note that the density energy of F was a constant given by $e(F) = (a_1^2 + a_2^2 + a_3^2 + a_4^2) c_5^{-\alpha}$. According to Theorem 3, the tension field of F was given by the following:

$$\tau(F) = \frac{\alpha(a_1^2 + a_2^2 + a_3^2 + a_4^2)}{c_5^{\alpha+1}} e_5.$$

As $f \circ F$ was constant on N, and the density energy of F was constant, from Corollary 4, the map F would be biharmonic if, and only if:

$$(f f' + 2f^3) \circ F = -\frac{\alpha^2(1 + 2\alpha)}{c_5^3} = 0.$$

Therefore, F was biharmonic non-harmonic if, and only if, $\alpha = -\frac{1}{2}$.

4. Biharmonic Maps on f-Kenmotsu with the Schouten–van Kampen Connection

Theorem 4. *Let (N^{2n}, J, h) be a Kähler manifold and $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ be an f-Kenmotsu manifold with the Schouten–van Kampen connection. Then, any (J, φ)-holomorphic map*

$F : N \longrightarrow M$ would be harmonic.

Proof. Based on the (J, φ)-holomorphic map, we had the following:

$$\varphi(\tau(F)) = dF(divJ) - tr_h B,$$

where B is defined by $B(X,Y) = (\nabla^F_X \varphi)dFY$ for any vector fields $X, Y \in \Gamma(TN)$. Considering a local orthonormal basis $\left\{e_i\right\}_{i=1}^{2n}$ on T_pN for any $p \in N$, we obtained the following:

$$divJ = \sum_{i=1}^{2n} (\nabla_{e_i} J)e_i = 0,$$

and by using relation (16), we found the following, as well:

$$\begin{aligned} tr_h B &= \sum_{i=1}^{2n} (\nabla^F_{e_i} \varphi)dF(e_i) = \sum_{i=1}^{2n} (\nabla^M_{dF(e_i)} \varphi)dF(e_i) \\ &= \sum_{i=1}^{2n} (f \circ F)\Big(g(\varphi(dF(e_i)), dF(e_i))\xi - \eta(dF(e_i))\varphi(dF(e_i))\Big) \\ &= \sum_{i=1}^{2n} (f \circ F)\Big(- \eta(dF(e_i))\varphi(dF(e_i))\Big). \\ &= 0. \end{aligned}$$

From the above relation, we could obtain the following: $\varphi(\tau(F)) = 0 \Rightarrow \tau(F) = g(\tau(F), \xi)\xi$. However,

$$\begin{aligned} g(\tau(F), \xi) &= \sum_{i=1}^{2n} g\Big(\nabla^F_{e_i} dF(e_i) - dF(\nabla^N_{e_i} e_i), \xi\Big) \\ &= \sum_{i=1}^{2n} g\Big(\nabla^M_{dF(e_i)} dF(e_i) - dF(\nabla^N_{e_i} e_i), \xi\Big) \\ &= \sum_{i=1}^{2n} g\Big(\nabla^M_{dF(e_i)} dF(e_i), \xi\Big) - \sum_{i=1}^{2n} g\Big(dF(\nabla^N_{e_i} e_i), \xi\Big) \\ &= \sum_{i=1}^{2n} \Big(\nabla^M_{dF(e_i)} g\Big(dF(e_i), \xi\Big) - g\Big(dF(e_i), \nabla^M_{dF(e_i)} \xi\Big)\Big) - \sum_{i=1}^{2n} g\Big(dF(\nabla^N_{e_i} e_i), \xi\Big) \\ &= \sum_{i=1}^{2n} \Big(\nabla^M_{dF(e_i)} \eta(dF(e_i)) - g\Big(dF(e_i), \nabla^M_{dF(e_i)} \xi\Big)\Big) - \sum_{i=1}^{2n} \eta(dF(\nabla^N_{e_i} e_i)). \end{aligned}$$

As F was a (J, φ)-holomorphic map, then by using Lemma 1, we found $\eta(dF(e_i)) = 0$ and $\eta(dF(\nabla^N_{e_i} e_i)) = 0$. Then, we had

$$g(\tau(F), \xi) = \sum_{i=1}^{2n} \Big(- g\Big(dF(e_i), \nabla^M_{dF(e_i)} \xi\Big)\Big).$$

In addition, from relation (24), we had $\nabla^M_{dF(e_i)} \xi = 0$, and then, $g(\tau(F), \xi) = 0$. □

Biharmonic Identity Map with the Schouten–van Kampen Connection

Theorem 5. *Let $I : M \longrightarrow \overline{M}$ be the identity map from an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ to an f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1}, f, \varphi, \xi, \eta, g)$. Then the tension field of map I was given by the following:*

$$\tau(I) = 2nf\xi \tag{36}$$

Proof. Let $\left\{e_1, ..., e_n, \varphi(e_1), ..., \varphi(e_n), \xi\right\}$ be an orthonormal local basis on TM. Then, by definition of the tension field of map I, we found the following:

$$\begin{aligned}\tau(I) =& tr_g \nabla dI \\ =& \sum_{i=1}^{2n+1} \Big(\nabla^{\overline{M}}_{dI(e_i)} dI(e_i) - dI(\nabla^{M}_{e_i} e_i) \Big).\end{aligned}$$

Using relation (25), we had the following:

$$\begin{aligned}\tau(I) =& \sum_{i=1}^{2n+1} \Big(f(g(e_i, e_i)\xi - \eta(e_i)e_i) \Big) \\ =& 2nf\xi.\end{aligned}$$

□

Theorem 6. *Let $I : M \longrightarrow \overline{M}$ be the identity map from an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ to an f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1}, f, \varphi, \xi, \eta, g)$. Then, map I would be harmonic if, and only if, M was a cosymplectic manifold.*

Theorem 7. *Let $I : M \longrightarrow \overline{M}$ be the identity map from an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$ to an f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1}, f, \varphi, \xi, \eta, g)$. Then, map I would be biharmonic if, and only if, f was a harmonic function.*

Proof. Let $\Big\{ e_1, ..., e_n, \varphi(e_1), ..., \varphi(e_n), \xi \Big\}$ be an orthonormal local basis on TM; then, by definition of the tension field of map I, we had the following:

$$\begin{aligned}\tau_2(I) &= tr_g (\nabla^I)^2 \tau(I) + tr_g R^{\overline{M}}(\tau(I), dI) dI \\ &= 2n\Big(tr_g (\nabla^I)^2 f\xi + tr_g R^{\overline{M}}(f\xi, dI) dI \Big) \\ &= \sum_{i=1}^{2n+1} \Big(\nabla^I_{e_i} \nabla^I_{e_i} 2nf\xi - \nabla^I_{\nabla^M_{e_i} e_i} 2nf\xi + R^{\overline{M}}(2nf\xi, dI(e_i)) dI(e_i) \Big).\end{aligned}$$

The combination of Equation (24) and direct calculations provided the following:

$$\sum_{i=1}^{2n+1} \Big(\nabla^I_{e_i} \nabla^I_{e_i} 2nf\xi \Big) = 2n \sum_{i=1}^{2n+1} \Big(e_i(e_i(f))\xi \Big),$$

and

$$\sum_{i=1}^{2n+1} \Big(\nabla^I_{\nabla_{e_i} e_i} 2nf\xi \Big) = 2n \sum_{i=1}^{2n+1} \Big(\nabla_{e_i} e_i(f)\xi \Big).$$

Based on the following:

$$\triangle(f) = \sum_{i=1}^{2n+1} \Big(e_i(e_i(f)) - \nabla_{e_i} e_i(f) \Big),$$

we could conclude

$$\sum_{i=1}^{2n+1} \Big(\nabla^I_{e_i} \nabla^I_{e_i} 2nf\xi \Big) = 2n\triangle(f)\xi.$$

Based on Equation (29), we found the following:

$$
\begin{aligned}
tr_g R^{\overline{M}}(2nf\xi, dI)dI = 2nf tr_g\Big(R^{\overline{M}}(\xi, dI)dI\Big) \\
= 2nf \sum_{i=1}^{2n+1} \Big(R^{\overline{M}}(\xi, e_i)e_i\Big) \\
=0.
\end{aligned}
$$

Finally, we obtained

$$\tau_2(I) = 2n\triangle(f)\xi. \tag{37}$$

□

Remark 1. *If f was a constant or harmonic function, then I would be a proper biharmonic map.*

Theorem 8. *Let $I : \overline{M} \longrightarrow M$ be the identity map from an f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1}, f, \varphi, \xi, \eta, g)$ to an f-Kenmotsu manifold $(M^{2n+1}, f, \varphi, \xi, \eta, g)$. Then, the bitension field of map I was given by the following:*

$$\tau_2(I) = -2n\Big(\overline{\Delta}(f)\xi + f\tau_2(\xi) + 2\nabla^I_{grad f}\xi\Big), \tag{38}$$

where $\overline{\Delta}$ is the Laplacian on $(\overline{M}^{2n+1}, f, \varphi, \xi, \eta, g)$.

Proof. Let $\Big\{e_1, ..., e_n, \varphi(e_1), ..., \varphi(e_n), \xi\Big\}$ be an orthonormal local basis on TM; then, by definition of the tension field of map I, we had the following:

$$
\begin{aligned}
\tau(I) =& tr_g \nabla dI \\
=& \sum_{i=1}^{2n+1} \Big(\nabla^M_{dI(e_i)} dI(e_i) - dI(\nabla^{\overline{M}}_{e_i} e_i)\Big) \\
=& -\sum_{i=1}^{2n+1} \Big(f(g(e_i, e_i)\xi - \eta(e_i)e_i)\Big) \\
=& -2nf\xi.
\end{aligned}
$$

However, we had the following:

$$
\begin{aligned}
\tau_2(I) &= tr_g(\nabla^I)^2\tau(I) + tr_g R^M(\tau(I), dI)dI \\
&= -2n\Big(tr_g(\nabla^I)^2 f\xi + tr_g R^{\overline{M}}(f\xi, dI)dI\Big) \\
&= -\sum_{i=1}^{2n+1} \Big(\nabla^I_{e_i}\nabla^I_{e_i} 2nf\xi - \nabla^I_{\nabla^{\overline{M}}_{e_i} e_i} 2nf\xi + R^M(2nf\xi, dI(e_i))dI(e_i)\Big).
\end{aligned} \tag{39}
$$

A direct calculation of

$$
\begin{aligned}
\sum_{i=1}^{2n+1} \Big(\nabla^I_{e_i}\nabla^I_{e_i} 2nf\xi\Big) =& 2n \sum_{i=1}^{2n+1} \nabla^I_{e_i}\Big(e_i(f)\xi + f\nabla^I_{e_i}\xi\Big) \\
=& 2n \sum_{i=1}^{2n+1} \Big(e_i(e_i(f))\xi + e_i(f)\nabla^I_{e_i}\xi + e_i(f)\nabla^I_{e_i}\xi + f\nabla^I_{e_i}\nabla^I_{e_i}\xi\Big) \\
=& 2n \sum_{i=1}^{2n+1} \Big(e_i(e_i(f))\xi + 2\nabla^I_{\mathrm{grad} f}\xi + f\nabla^I_{e_i}\nabla^I_{e_i}\xi\Big),
\end{aligned}
$$

and

$$\sum_{i=1}^{2n+1}\left(\nabla^{I}_{\nabla^{\overline{M}}_{e_i}e_i}2nf\xi\right)=2n\sum_{i=1}^{2n+1}\left((\nabla^{\overline{M}}_{e_i}e_i)(f)\xi+f\nabla^{I}_{\nabla^{\overline{M}}_{e_i}e_i}\xi\right)$$

finally yielded the following:

$$tr_g(\nabla^I)^2nf\xi=2n\left(\overline{\Delta}(f)\xi+ftr_g(\nabla^I)^2\xi+2\nabla^I_{\mathrm{grad}f}\xi\right). \quad (40)$$

If we replaced (40) in (39), we arrived at:

$$\tau_2(I)=-2n\left(\overline{\Delta}(f)\xi+f\tau_2(\xi)+2\nabla^I_{\mathrm{grad}f}\xi\right).$$

□

Corollary 5. *Let $I:\overline{M}\longrightarrow M$ be the identity map from an f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1},f,\varphi,\xi,\eta,g)$ to an f-Kenmotsu manifold $(M^{2n+1},f,\varphi,\xi,\eta,g)$. Then, map I would be biharmonic if, and only if:*

$$\overline{\Delta}(f)\xi+f\tau_2(\xi)+2\nabla^I_{gradf}\xi=0,$$

Corollary 6. *Let $I:\overline{M}\longrightarrow M$ be the identity map from an f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1},f,\varphi,\xi,\eta,g)$ to an f-Kenmotsu manifold $(M^{2n+1},f,\varphi,\xi,\eta,g)$. If f was a constant function, then map I would be biharmonic if, and only if, ξ was a biharmonic vector field.*

Corollary 7. *Let $I:\overline{M}\longrightarrow M$ be the identity map from an f-Kenmotsu manifold with the Schouten–van Kampen connection $(\overline{M}^{2n+1},f,\varphi,\xi,\eta,g)$ to an f-Kenmotsu manifold $(M^{2n+1},f,\varphi,\xi,\eta,g)$. If ξ was a parallel vector field (i.e., $\nabla\xi=0$), then map I would be biharmonic if, and only if, f was a harmonic function.*

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