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Generalized Halanay Inequalities and Relative Application to Time-Delay Dynamical Systems

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Abstract: A class of generalized Halanay inequalities is studied via the Banach fixed point method and comparison principle. The conditions to ensure the boundedness and stability of the zero solution are obtained in this study. This research provides a new approach to the study of the boundedness and stability of Halanay inequality. Numerical examples and simulation results verify the validity and superiority of the conclusions obtained in this study.

Keywords: generalized Halanay inequalities; exponential stability; boundedness; fixed point method

MSC: 37B25

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1. Introduction

Dynamical systems are applied in a wide range of fields, such as medicine physics, neural networks, biology, and mathematical finance. In the theory of dynamical systems, boundedness and stability are the most extensively studied concepts. In the research of natural science, social science, and engineering technology, the future state of systems depends not only on the current state but also on the past state. Dynamical systems with various delays are considered. Therefore, studies on the boundedness and stability of delayed dynamical systems are extensive. Recently, as a generalization of dynamical systems, many authors studied the stability of Halanay inequality systems. To analyze the boundedness and stability of the following dynamical systems with delay τ,

$$\frac{dx(t)}{dt} = [-ax(t) + bx(t-\tau)], \quad t \geq t_0,$$

Halanay proposed the Halanay inequalities (1) in [1].

$$D^+x(t) \leq -\lambda x(t) + \delta \sup_{t-\tau(t)\leq s\leq t} x(s) \tag{1}$$

Here, $D^+x(t)$ is the upper-right Dini derivative and is defined as

$$D^+x(t) = \limsup_{\sigma\to 0^+} \frac{x(t+\sigma)-x(t)}{\sigma}. \tag{2}$$

Subsequently, Halanay obtained the following Lemma 1.

Lemma 1 (Halanay's inequality)**.** *Let $\lambda > \delta$. If $x(t)$ satisfies functional differential inequalities (1), then there exist $\gamma > 0$ and $k > 0$ such that $x(t) \le ke^{-\gamma(t-t_0)}$ for $t \ge t_0$.*

The authors in [2–6] generalized the Halanay inequality as follows:

$$D^+u(t) \le \gamma(t) - \alpha(t)u(t) + \beta(t) \sup_{t-\tau(t)\le s\le t} u(s), \quad t \ge t_0 \tag{3}$$

By means of (3), many studies have been conducted. In 2004, Tian [2] researched the boundedness and exponential stability (ES) of dynamic systems with constant delays. In 2008, L. Wen [3] obtained the dissipativity results of VFDEs by applying the generalization of Halanay's inequality. In 2011, based on [3], B. Liu [4] considered the boundedness and ES of neural networks with unbounded delays. In 2015, L.V. Hien [5] considered the boundedness and global generalized ES of nonlinear nonautonomous systems with time delays. In 2019, D. Ruan [6] studied the boundedness and ES of (3) by integral inequalities. Furthermore, the authors in [7,8] used the inequality (3) to study stochastic differential systems.

When studying the stability of dynamical systems, most studies (such as [9–16]) use Lyapunov's direct method. However, there are many problems that make this method inappropriate. For example, Lyapunov's direct method usually requires the boundedness of delays. Recently, Burton and authors ([17–21]) studied the stability of various dynamical systems using the fixed point method. The results show that the fixed point method can overcome many problems in the study of the stability of dynamical systems.

Lemma 2 (Banach fixed point theorem)**.** *Let (X, d) be a nonempty complete metric space. Let $T : X \to X$ be a compressed map on X. That is, there exists a non-negative real number $q < 1$, such that for all $x, y \in X$, there are $d(T(x), T(y)) \le q \cdot d(x, y)$ Then, the mapping T has and has only one immobile point x within X.*

Based on the existing discussion, we also study the inequality (3) and derive new generalized ES and boundedness conditions using the fixed point method. The obtained results improve and generalize the conclusions of existing papers (see the examples in Section 4).

The remaining part of this paper is organized as follows. In Section 2, we introduce the generalized Halanay's inequality system and provide the results of some of the existing studies. In Section 3, the main theoretical results are proposed and proved. Examples with numerical simulations are illustrated in Section 4. The conclusions are given in Section 5.

2. Preliminaries

Let $R = (-\infty, +\infty)$, $R^+ = (0, +\infty)$, $C(A, \Omega)$ be a continuous function from A to Ω. Consider the generalized Halanay's inequality with external perturbation

$$\begin{cases} D^+x(t) \le \theta(t) - \lambda(t)x(t) + \delta(t)\sup_{t-\tau(t)\le s\le t} x(s), & t \ge t_0, \\ x(t) = |\phi(t)|, & t \le t_0. \end{cases} \tag{4}$$

Here, $\lambda(t) \ge 0$, $\delta(t) \ge 0$ and $\theta(t) \ge 0$. $\tau(t)$ is a time delay function .

Many experts have studied the boundedness and exponential stability of the system (4). See details in Lemmas 3–6.

Lemma 3 (L. Wen [3])**.** *If $x(t) \ge 0$ satisfies functional differential inequality (4), and when $t \ge t_0$, the continuous functions $\theta(t) \ge 0, \delta(t) \ge 0, \lambda(t) \le 0$ and $\tau(t) \ge 0$ exist. If there is $\sigma > 0$ such that*

$$-\lambda(t) + \delta(t) \le -\sigma < 0 \text{ for } t \ge t_0,$$

then we have

$$x(t) \leq \frac{\theta^*}{\sigma} + Ge^{-\varphi^*(t-t_0)}, \quad t \geq t_0,$$

where $G = \sup_{\xi \leq t_0} |\phi(\xi)|$ *and* $\theta^* = \sup_{t \geq t_0} |\theta(t)|$. $\varphi^* \geq 0$ *is defined as*

$$\varphi^* = \inf_{t \geq t_0} \{\varphi(t) : \varphi(t) - \lambda(t) + \delta(t)e^{\varphi(t)\tau(t)} = 0\}.$$

Furthermore, when $t - \tau(t) \to \infty$ *as* $t \to \infty$, *we have*

$$x(t) \leq \frac{\theta^*}{\sigma} + G, \quad t \geq t_0.$$

Based on [3], B. Liu ([4]) further studied the boundedness and stability of the system (4) and obtained the following conclusion.

Lemma 4 (B. Liu [4])**.** *If* $x(t) \geq 0, t \in (-\infty, +\infty)$ *satisfies the functional differential inequality (4), and all conditions of Lemma 4 are satisfied, then we have*

$$x(t) \leq \frac{\theta^*}{\sigma} + \{\sup_{s \leq t_0} e^{\varphi^*(s-t_0)} x(s) - \frac{\theta^*}{\sigma}\} \times e^{-\varphi^*(t-t_0)}, \quad t \geq t_0.$$

Furthermore, when $t - \tau(t) \to \infty$ *as* $t \to \infty$, *we have*

$$x(t) \leq max\{\frac{\theta^*}{\sigma}, G\}, \quad t \geq t_0.$$

L.V. Hien [5] considered the boundedness and global generalized ES of Halanay-type nonautonomous functional differential inequalities and obtained the following conclusion.

Lemma 5 (L.V. Hien [5])**.** *Let* $T_* = \inf\{T \geq \tau_{ev} : \sup_{t \geq T} \frac{\delta(t)}{\lambda(t)} < 1\}$, *where* $\tau_{ev} := \inf\{\tau \geq t_0 : t - \tau(t) \geq t_0, \forall t \geq \tau\}$; *define* $\varrho = \sup_{t \geq T_*} \frac{\delta(t)}{\lambda(t)}$, $I(\lambda) = \max\{\sup_{t \geq T_*} \int_{t-\tau(t)}^{t} \lambda(s)ds\}$ *and* β_* *is the unique positive solution of the scalar equation* $H(\beta) = \beta + \varrho e^{\beta I(\lambda)} - 1 = 0$; *the factor* N *was given by* $N = \exp\left(\beta_* \int_{t_0}^{T_*} \lambda(s)ds\right)$. *Suppose the continuous function* $x(t) \geq 0, t \in (-\infty, +\infty)$ *satisfies the inequality (4). If*

$(A.1)$

$$\lim_{t \to +\infty} (t - \tau(t)) = +\infty;$$

$(A.2)$

$$\lim_{t \to +\infty} \int_{t_0}^{t} \lambda(s)ds = +\infty;;$$

$(A.3)$

$$\sup_{t \geq t_0} \int_{t-\tau(t)}^{t} \lambda(s)ds < +\infty;$$

$(A.4)$

$$\sup_{t \geq t_0} \frac{\delta(t)}{\lambda(t)} < 1;$$

then, the following conclusion is derived.

$$x(t) \leq N\left(\|\phi\|_\infty - \frac{\theta_\lambda}{1 - \delta_\infty^0}\right)^+ \exp\left(-\beta_* \int_{t_0}^{t} \lambda(s)ds\right) + \frac{\theta_\lambda}{1 - \delta_\infty^0}, \quad t \geq t_0,$$

where $\theta_\lambda = \sup_{t \geq t_0} \frac{\theta(t)}{\lambda(t)}$, $\delta_\infty^0 = \sup_{t \geq t_0} \frac{\delta(t)}{\lambda(t)}$.

In addition, D Ruan [6] researched the boundedness and ES of inequality (4) by integral inequalities and obtained the following conclusion.

Lemma 6 (D. Ruan [6])**.** *Let $x(t) \geq 0, t \in (-\infty, +\infty)$ be a continuous functional satisfying inequality (4). If the assumptions A.1–A.4 hold,*

$(A.1)$

$$\lim_{t\to+\infty}\int_{t_0}^{t}\lambda(s)ds = +\infty;;$$

$(A.2)$

$$\lim_{t\to+\infty}(t-\tau(t)) = +\infty;$$

$(A.3)$

$$\sup_{t\geq t_0}\frac{\delta(t)}{\lambda(t)} := \delta < 1;$$

$(A.4)$

$$\sup_{t\geq t_0}\int_{t-\tau(t)}^{t}\lambda(s)ds := N < +\infty;$$

$(A.5)$ *There exists a number $\iota \geq 0$ such that*

$$\int_{t_0}^{t} e^{-\int_s^t \lambda(u)du}\theta(s)ds \leq \iota;$$

then, there exists a constant $\theta \in (0,1]$ such that

$$x(t) \leq \|\phi\|_\infty \exp\left\{-\theta\int_{t_0}^{t}\lambda(s)ds\right\} + \frac{\iota}{1-\delta}, \quad t \in (-\infty, +\infty).$$

Our aim here is to generalize the above Lemma and show that some of the conditions for time delays and coefficients are unnecessary.

3. Main Results

We use the Banach fixed point method to study the boundedness of inequality (4) in this study. Through the analysis, it can be observed that the conclusions of the Halanay inequality in this study will improve the results of many related studies.

Theorem 1. *Let the continuous function $x(t)$ satisfy the inequality (4). There exists a continuous function $h(t) : [0, +\infty) \to R^+$. If the following assumptions hold,*

$(H.1)$

$$\lim_{t\to+\infty}\int_{t_0}^{t}h(s)ds = +\infty;$$

$(H.2)$ *there exists a positive number β such that*

$$|h(s)-\lambda(s)| + e^{\beta\int_{s-\tau(s)}^{s}h(\mu)d\mu}|\delta(s)| \leq (1-\beta)h(s);$$

$(H.3)$ *there exists $0 < \alpha < 1$ such that*

$$\sup_{t\geq t_0}\int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}[|h(s)-\lambda(s)| + |\delta(s)|]ds \leq \alpha < 1;$$

$(H.4)$ *there exists $\rho > 0$ such that*

$$\int_{t_0}^{t} e^{-\int_s^t h(u)du}\theta(s)ds \leq \rho;$$

then,

$$x(t) \leq \|\phi\|_\infty \exp\left\{-\beta \int_{t_0}^{t} h(s)ds\right\} + \frac{\rho}{1-\alpha}, \quad t \geq t_0,$$

where $\|\phi\|_\infty = \sup_{s \leq t_0} |\phi(s)|$.

Proof. Define the following delay differential equations:

When $t \leq t_0$,$x(t) = |\phi(t)|$. Moreover, when $t \geq t_0$

$$dx(t) = (\theta(t) - \lambda(t)x(t) + \delta(t) \sup_{t-\tau(t) \leq s \leq t} x(s))dt. \tag{5}$$

Considering the derivative of $\int_{t_0}^{s} x(s)$, we obtain

$$x(t) = \phi(t_0)e^{-\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} e^{-\int_s^t h(u)du}\delta(s) \sup_{s-\tau(s) \leq u \leq s} x(u)ds + \int_{t_0}^{t} e^{-\int_s^t h(u)du}\theta(s)ds, \qquad t \geq t_0.$$

Denote by S a complete metric space $\|C(R,R)\|_\infty = \{x(t) \in C(R,R) | \|x\| = \sup_{t \in R}$ $|x(t)| < +\infty\}$.

Moreover, $\psi(s) = |\phi(s)|$ for $s \in (-\infty, t_0]$. Additionally, when $t \geq t_0$, we have

$$\psi(t) \leq \|\phi\|_\infty \exp\left\{-\beta \int_{t_0}^{t} h(s)ds\right\} + \frac{\rho}{1-\alpha}. \tag{6}$$

where ρ and α were introduced previously.

Define an operator $\Psi : \mathcal{S} \to \mathcal{S}$ by $(\pi x)(t) = \phi(s)$ for $t \in (-\infty, t_0]$ and for $t \geq t_0$,

$$\begin{aligned}(\Psi x)(t) = {} & |\phi(t_0)|e^{-\int_{t_0}^{t} h(\mu)d\mu} + \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}\theta(s)ds \\ & + \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}[h(s) - \lambda(s)]x(s)ds + \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}\delta(s) \sup_{s-\tau(s) \leq \nu \leq s} x(\nu)ds.\end{aligned} \tag{7}$$

Ψ is continuous on $(-\infty, +\infty)$. Furthermore, we show that $\Psi(\mathcal{S}) \subset \mathcal{S}$.

For any $x(t) \in \mathcal{S}$ and $t \geq t_0$, from (H.1) to (H.4) and (6) , we have

$$\begin{aligned}(\pi x)(t) = {} & |\phi(t_0)|e^{-\int_{t_0}^{t} h(\mu)d\mu} + \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}\theta(s)ds \\ & + \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}[h(s) - \lambda(s)]x(s)ds + \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}\delta(s) \sup_{s-\tau(s) \leq \nu \leq s} x(\nu)ds \\ \leq {} & \rho + \|\phi\|_\infty e^{-\int_{t_0}^{t} h(\mu)d\mu} + \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}|h(s) - \lambda(s)|(\|\phi\|_\infty e^{-\beta\int_{t_0}^{s} h(\mu)d\mu} + \frac{\rho}{1-\alpha})ds \\ & + \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}|\delta(s)|(\|\phi\|_\infty e^{-\beta\int_{t_0}^{s-\tau(s)} h(\mu)d\mu} + \frac{\rho}{1-\alpha})ds \\ \leq {} & \rho + \|\phi\|_\infty e^{-\int_{t_0}^{t} h(\mu)d\mu} + \frac{\alpha\rho}{1-\alpha} \\ & + \|\phi\|_\infty \int_{t_0}^{t} e^{-\int_{t_0}^{t} h(\mu)d\mu} e^{\int_{t_0}^{s} h(\mu)d\mu}(|h(s) - \lambda(s)|e^{-\beta\int_{t_0}^{s} h(\mu)d\mu} + |\delta(s)|e^{-\beta\int_{t_0}^{s-\tau(s)} h(\mu)d\mu})ds \\ \leq {} & \rho + \|\phi\|_\infty e^{-\int_{t_0}^{t} h(\mu)d\mu} + \frac{\alpha\rho}{1-\alpha} + \|\phi\|_\infty e^{-\int_{t_0}^{t} h(\mu)d\mu} \int_{t_0}^{t} e^{(1-\beta)\int_{t_0}^{s} h(\mu)d\mu}(1-\beta)h(s)ds \\ = {} & \rho + \|\phi\|_\infty e^{-\int_{t_0}^{t} h(\mu)d\mu} + \frac{\alpha\rho}{1-\alpha} + \|\phi\|_\infty e^{-\int_{t_0}^{t} h(\mu)d\mu}(e^{(1-\beta)\int_{t_0}^{t} h(\mu)d\mu} - 1) \\ \leq {} & \|\phi\|_\infty \exp\left\{-\beta \int_{t_0}^{t} h(s)ds\right\} + \frac{\rho}{1-\alpha}.\end{aligned}$$

Therefore, from the above analysis, we arrive at the conclusion that $\Psi(\mathcal{S}) \subset \mathcal{S}$.

In addition, we prove that the mapping Ψ is contractive. For $\xi, \eta \in \mathcal{S}$, we can obtain

$$\begin{aligned}\sup_{t\geq t_0} |(\Psi\xi)(t) - (\Psi\eta)(t)| &\leq \sup_{t\geq t_0} |((\xi)(t) - (\eta)(t)) \int_{t_0}^{t} e^{-\int_s^t h(\mu)d\mu}(|h(s) - \lambda(s)| + |\delta(s)|)ds \\ &\leq \alpha \sup_{t\geq t_0} |((\xi)(t) - (\eta)(t)).\end{aligned}$$

which implies

$$\sup_{t\geq t_0} |(\Psi\xi)(t) - (\Psi\eta)(t)| \leq \alpha \sup_{t\geq t_0} |((\xi)(t) - (\eta)(t)). \tag{8}$$

As $\alpha \in (0,1)$, we know that the mapping Ψ is a contractive by (8). As a result, based on the contractive mapping principle, there exists a unique fixed point $x(t)$ for Ψ, which is a solution of inequality (4) with $x(s) = |\phi(s)|$ on $s \in (-\infty, t_0]$ and $x(t) \leq \|\phi\|_\infty \exp\left\{-\beta \int_{t_0}^t h(s)ds\right\} + \frac{\rho}{1-\alpha}$ on $t \in [t_0, +\infty)$. This completes the proof. □

Remark 1. *As can be seen from the proof of Theorem 1, the conclusion of this paper does not require the time lag to be bounded, so it overcomes the difficulties encountered in Lyapunov's direct method.*

If we order $h(s) \equiv \lambda(s)$ in Theorem 1, we obtain Theorem 2.

Theorem 2. *Let $x(t) \geq 0$ be a continuous function satisfying (4). If the assumptions H.1–H.4 hold.*
$(H.1)$

$$\lim_{t\to+\infty} \int_{t_0}^t \lambda(s)ds = +\infty;$$

$(H.2)$ *there exists a positive number β such that*

$$e^{\beta \int_{s-\tau(s)}^s \lambda(\mu)d\mu} |\delta(s)| \leq (1-\beta)\lambda(s);$$

$(H.3)$ *there exists $0 < \alpha < 1$ such that*

$$\sup_{t\geq t_0} \int_{t_0}^t e^{-\int_s^t \lambda(\mu)d\mu} |\delta(s)|ds \leq \alpha < 1;$$

$(H.4)$ *there exists $\rho > 0$ such that*

$$\int_{t_0}^t e^{-\int_s^t \lambda(u)du} \theta(s)ds \leq \rho;$$

then,

$$x(t) \leq \|\phi\|_\infty \exp\left\{-\beta \int_{t_0}^t \lambda(s)ds\right\} + \frac{\rho}{1-\alpha}, \quad t \geq t_0,$$

where $\|\phi\|_\infty = \sup_{s\leq t_0} |\phi(s)|$.

Remark 2. *If we let $\theta(t) = 0$, we obtain exponential stability.*

Remark 3. *We do not require the boundedness of the time delay $\tau(t)$. In addition, we also do not require $t - \tau(t) \to \infty$, as $t \to \infty$, which improves the result of many previous studies. For example, Refs. [3–6].*

Remark 4. *In Theorem 1, we do not require $\lambda(t) > \delta(t)$. This considerably improves the conclusions of the Refs. [1,3,4,6]. We do not require the external perturbation $\theta(t)$ to be bounded, which* *improves the conclusion of previously published Refs. [3,4]. Moreover, we do not require* $\frac{\delta(t)}{\lambda(t)}$ *and* $\frac{\theta(t)}{\lambda(t)}$ *to have an upper bound, which improves the results of the Refs. [5,6].*

4. Examples

In this section, some examples and simulations are given to illustrate our main results.

Example 1. *Consider a delay differential system*

$$dx(t) = [t + 1 - 2tx(t) + 0.8e^{-1.2}tx(t - \frac{1}{2+t})]dt, \quad t \geq 0. \tag{9}$$

When $t \in [-2, 0]$, $x(t) = 10$. *Let* $h(t) \equiv \lambda(t) = 2t$, $\delta(t) = 0.8e^{-1.2}t$, *and* $\theta(t) = t + 1$. *Let* $\beta = 0.6$; *then,*

$$e^{0.6\int_{s-\frac{1}{2+s}}^{s} 2\mu d\mu}(0.8e^{-1.2}s) = e^{0.6[\frac{2s}{2+s} - \frac{1}{(2+s)^2}]}(0.8e^{-1.2}s) \leq (1 - 0.6)\lambda(s) = 0.8s.$$

Because

$$\sup_{t\geq 0} \int_0^t e^{-\int_s^t 2\mu d\mu}|0.8e^{-1.2}s|ds \leq 0.4e^{-1.2} < 1.$$

So, $\alpha = 0.4e^{-1.2}$.

In addition,

$$\int_0^t e^{-\int_s^t h(u)du}\theta(s)ds = \int_0^t e^{-t^2+s^2}(s+1)ds = e^{-t^2}(\frac{e^{t^2}}{2} + \frac{\sqrt{\pi}erfi(t)}{2} - \frac{1}{2}), t \geq 0.$$

where the function ***erfi(t)*** *is a imaginary error function. Figure 1 is the graph of function* $f(x) = e^{-x^2}(\frac{e^{x^2}}{2} + \frac{\sqrt{\pi}\mathbf{erfi}(x)}{2} - \frac{1}{2})$.

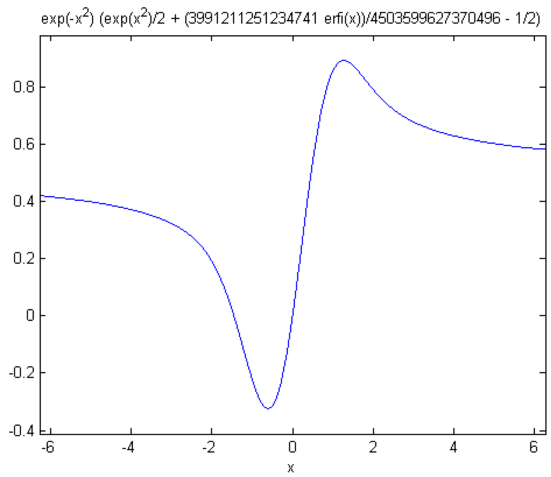

Figure 1. The graph of function $f(x) = e^{-x^2}(\frac{e^{x^2}}{2} + \frac{\sqrt{\pi}\mathbf{erfi}(x)}{2} - \frac{1}{2})$.

Additionally, $f(t) = e^{-t^2}(\frac{e^{t^2}}{2} + \frac{\sqrt{\pi}erfi(t)}{2} - \frac{1}{2}) \leq 0.9$. For

$$\int_0^t e^{-\int_s^t h(u)du}\theta(s)ds = e^{-t^2}(\frac{e^{t^2}}{2} + \frac{\sqrt{\pi}erfi(t)}{2} - \frac{1}{2}) \leq 0.9, t \geq 0.$$

So, $\rho = 0.9$. By Theorem 1, we have

$$x(t) \leq 10e^{-0.6t^2} + \frac{0.9}{1 - 0.4e^{-1.2}}$$

The simulation result presented in Figure 2 shows the validity of our theoretical results. Figure 2 is the graph of function $x(t)$ and $y(t) = 10e^{-0.6t^2} + \frac{0.9}{1-0.4e^{-1.2}}$.

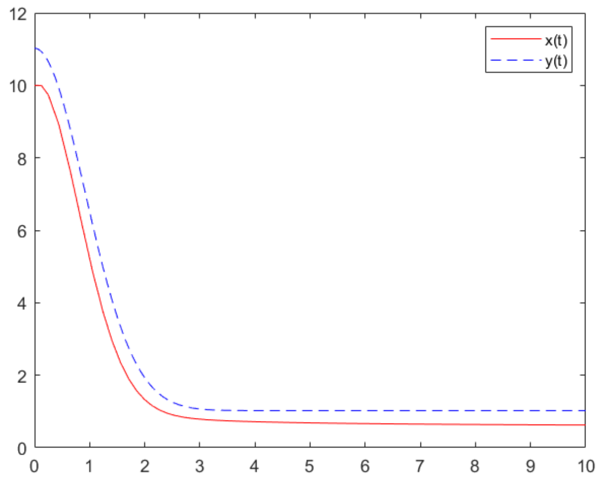

Figure 2. The graph of function $x(t)$ and $y(t)$ (Example 1).

Remark 5. *The Refs. [3,4] required that* $-\lambda(t) + \delta(t) \leq -\vartheta < 0$ *(*$t \geq 0$*,* ϑ *is a positive constant) and* $\theta(t) > 0$ *is bounded. The Ref. [5] asked for* $\sup_{t\geq 0} \frac{\theta(t)}{\lambda(t)} < +\infty$*. Obviously, in Example 1,* $\lambda(0) + \delta(0) = 0$, $\lim_{t\to+\infty} \theta(t) = +\infty$ *and* $\sup_{t\geq 0} \frac{\theta(t)}{\lambda(t)} = +\infty$*. Thus, the Refs. [3–5] are invalid for Example 1.*

Remark 6. *Let* $h(t) = 3t$*; then,* $h(t) - \lambda(t) = t$*. Let* $\beta = 0.4$*; then,*

$$e^{0.4\int_{s-\frac{1}{2+s}}^{s} 3\mu d\mu}(0.8e^{-1.2}s) + s \leq (1-0.4)h(s) = 1.8s,$$

and $\sup_{t\geq 0}\{\int_0^t e^{-\int_s^t 3\mu d\mu}|0.8e^{-1.2}s|ds\} \leq \frac{4}{15}e^{-1.2} = \alpha$*. In addition, when t≈ 0.994085,*

$$\sup_{t\geq 0}\left\{\int_0^t e^{-\int_s^t h(u)du}\theta(s)ds\right\} = \max\{0.333333 + 0.723601e^{-1.5t^2} erfi(1.22474t) - 0.333333e^{-1.5t^2}\} = 0.668644 = \rho.$$

So, $\rho = 0.668644$*. By Theorem 1, we have.*

$$x(t) \leq 10e^{-0.4t^2} + \frac{0.668644}{1 - \frac{4}{15}e^{-1.2}}$$

The simulation result presented in Figure 3 shows the validity of our theoretical result. Figure 3 is the graph of function $x(t)$, $y(t) = 10e^{-0.6t^2} + \frac{0.9}{1-0.4e^{-1.2}}$ and $z(t) = 10e^{-0.4t^2} + \frac{0.668644}{1-\frac{4}{15}e^{-1.2}}$.

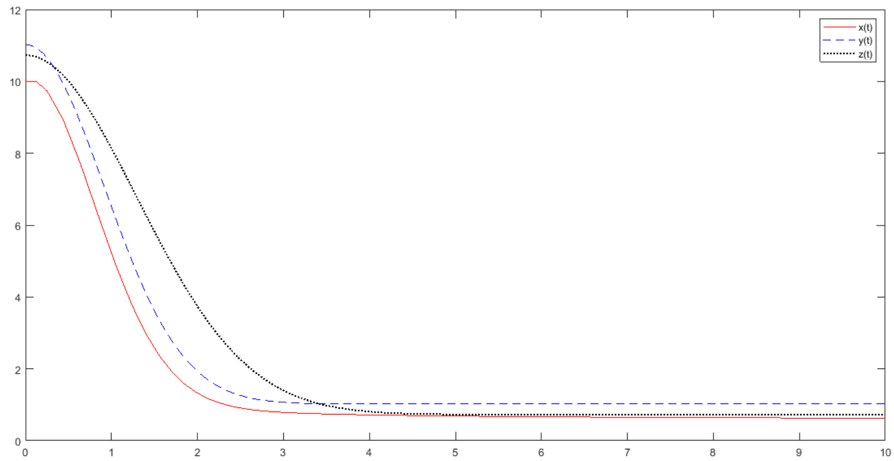

Figure 3. The graph of function $x(t)$, $y(t)$ and $z(t)$ (Example 1).

From Figure 3, it can be observed that the result of Theorem 1 is better than that of the Ref. [6], owing to the choice of an appropriate $h(t)$ function. In fact, in Theorem 1, the flexibility to choose the $h(t)$ function makes the study easier and the results better.

Example 2. *Consider a delay differential system*

$$dx(t) = (\frac{1}{t} - \frac{1}{t}x(t) + 0.8e^{-0.2\ln 2}\frac{1}{t}x(t - \frac{t}{2}))dt, t \geq 1. \tag{10}$$

When $t \in [0,1], x(t) = 10$, *let* $h(t) \equiv \lambda(t) = \frac{1}{t}$, $\delta(t) = 0.8e^{-0.2\ln 2}\frac{1}{t}$ *and* $\theta(t) = \frac{1}{t}$. *Obviously,* $\beta = 0.2$.

For $\sup_{t\geq 2}\int_{t_0}^{t} e^{-\int_s^t \frac{1}{\mu}d\mu}(0.8e^{-0.2\ln 2})ds \leq 0.8e^{-0.2\ln 2} < 1$, *so* $\alpha = 0.8e^{-0.2\ln 2}$. *In addition, when* $t \geq 1$,

$$\int_1^t e^{-\int_s^t \lambda(u)du}\theta(s)ds = \int_1^t e^{-\int_s^t \frac{1}{u}du}\frac{1}{s}ds = 1 - \frac{1}{t} < 1 = \rho.$$

Hence, by Theorem 1, we have

$$x(t) \leq 10e^{-0.2\ln t} + \frac{1}{1 - 0.8e^{-0.2\ln 2}}.$$

Figure 4 is the graph of function $x(t)$ and $y(t) = 10e^{-0.2\ln t} + \frac{1}{1-0.8e^{-0.2\ln 2}}$.

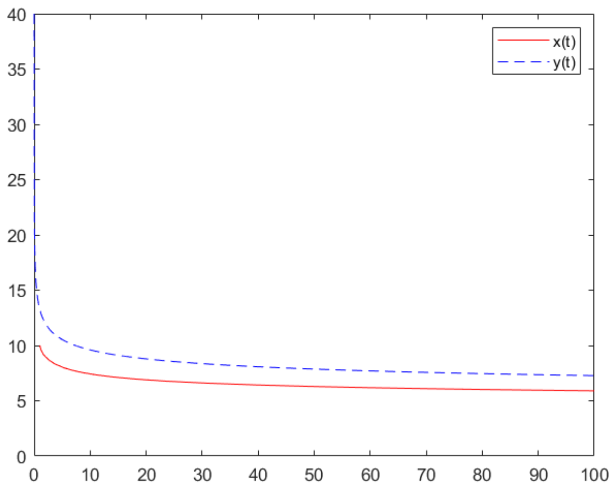

Figure 4. The graph of function $x(t)$ and $y(t)$ (Example 2).

Remark 7. *The Ref. [2] is invalid because $\tau(t) = \frac{t}{2}$ is unbounded.*

5. Conclusions

We used the fixed point method to study new types of generalized Halanay inequalities and obtained some sufficient conditions. The contributions of this study are as follows:

1. It proposes a novel approach to study the boundedness and stability of the Halanay inequality by using the fixed point method, as well as to verify the main conclusions of a paper using a numerical simulation. Simultaneously, the research of this paper extends the methods and ideas of the Halanay inequality.

2. This study relaxes the requirements of time delays and coefficients. For example, we do not require the boundedness of the time delay $\tau(t)$. In addition, it is not necessary that $t - \tau(t) \to \infty$, as $t \to \infty$. Moreover, $\lambda(t) > \delta(t)$ is not required.

3. The fixed point method is used to improve and extend the results of many previous studies; for example, [1–6] (See Remarks 2, 3 and 5–7 for more details).

4. Unlike most of the previously published papers, this paper verifies the reliability of the conclusion and the superiority of related studies through examples and numerical simulation.

5. Because it is not always easy to find the $h(s)$ that satisfies the condition of Theorem 1, there is room for more optimization of the conclusions of this paper. In addition, this study can be extended to the study of stochastic dynamical systems, which is also the direction of the group's future research.

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