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Some Properties of the χ-Curvature under Conformal Transformation

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Abstract: In this paper, we identify the sufficient and necessary conditions for conformally related Randers metrics to have the same χ-curvature. Further, if $s_0 = 0$ holds, we conclude that the conformal transformation must be a homothety. Using the divergence theorem, we prove that, on the compact manifold, there is no nontrivial conformal transformation preserving the χ-curvature of the Randers metrics invariant.

Keywords: χ-curvature; conformal transformations; randers metrics

MSC: 53C30; 53C60

1. Introduction

In Finsler geometry, the Weyl theorem states that the projective and conformal properties of a Finsler space determine the metric properties uniquely [1,2]. Therefore, the conformal property of Finsler geometry deserves in-depth study. Let F and $\bar{F}$ be two Finsler metrics on a manifold M. The conformal transformation between F and $\bar{F}$ is defined by L: $F \to \bar{F}$, $\bar{F} = e^{\rho}F$, where the conformal factor $\rho := \rho(x)$ is a scalar function on M. We call these two metrics F and $\bar{F}$, and they are conformally related. A natural problem is determining all Finsler metrics that are conformally related to the given one, given a Finsler metric on a manifold M .

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B*á*cs*ó*-Cheng [3] characterized the conformal transformation that preserves the Riemann curvature, the Ricci curvature, and the (mean) Landsberg curvature or the **S**-curvature, respectively. Chen-Cheng-Zou [4] proved that if both conformally related (α, β)-metrics are Douglas metrics or of isotropic **S**-curvature, then the conformal transformation between them is a homothety. Later, Chen-Liu [5] characterized the conformal transformation between two almost regular (α, β)-metrics that preserves the mean Landsberg curvature. Shen [6] proved that the conformal transformation between non-Riemannian Finsler manifolds, which preserves the **S**-curvature, must be a homothety. Recently, Zhang-Feng [7] completely determined all Landsberg metrics which are conformally related to the warped product metrics of the Landsberg type and obtained a class of nonregular unicorn Finsler metrics.

In Finsler geometry, many non-Riemannian quantities are more colorful than those in Riemannian geometry. When used together with Riemannian quantities, non-Riemannian ones might lead to some global results. There are several important non-Riemannian quantities, such as the Cartan torsion, the **S**-curvature, and the **H**-curvature. We have another important quantity that is expressed in terms of the vertical derivatives of the Riemann curvature. It is the so-called χ-curvature defined by

$$\chi_i := -\frac{1}{6}\left\{2R^k{}_{i\cdot k} + R^k{}_{k\cdot i}\right\},$$

where $R^i{}_k := 2G^i_{x^k} - G^i_{x^j y^k} y^j + 2G^j G^i_{y^j y^k} - G^i_{y^j} G^j_{y^k}$, and "$\cdot$" denotes the vertical covariant derivative. It can be expressed in several forms. The χ-curvature has a great relationship with the Ricci curvature. Recently, Shen [8] studied the χ-curvature and showed some relationships among the flag curvature, the χ-curvature, and the **S**-curvature. Furthermore, if a Finsler metric is of scalar flag curvature, then the χ-curvature almost vanishes if and only if the flag curvature is almost isotropic. Mo [9] proved that for a spherically symmetric Finsler metric, the **H**-curvature vanishes if and only if the χ-curvature vanishes. Additionally, if F is R-quadratic, then it has a vanishing χ-curvature. Chen-Liu [10] showed that a Kropina metric is of almost vanishing χ-curvature or of almost vanishing **H**-curvature if and only if it is of isotropic **S**-curvature. Further, they proved that a Kropina metric is a Douglas metric if and only if the conformally related metric is also a Douglas metric. Cheng-Yuan [11] proved that, for a confomally flat polynomial (α, β)-metric, if F is of almost vanishing χ-curvature, then it must be Minkowskian. Recent research shows that the χ-curvature plays an important role in studies on spray geometry. Li-Shen [12] introduced the new notion of the Ricci curvature tensor and discussed its relationships with the Ricci curvature, the **H**-curvature, and the χ-curvature. They had a better understanding of the χ-curvature. Further, they [13] studied sprays with isotropic curvatures and showed that they are of isotropic curvature if and only if the χ-curvature vanishes. Shen [14] showed that the sprays obtained by a projective deformation using the **S**-curvature, always have a vanishing χ-curvature. Then, he established the Beltrami Theorem for sprays with $\chi = 0$.

The Randers metric was introduced by the physicist Randers in 1941 [15] in the context of general relativity. Later, these metrics were used in Ingarden's theory of electron microscopy in 1957, and he first named them Randers metrics. Randers metrics represent an important and ubiquitous class of Finsler metrics with a strong presence in the theory and application of Finsler geometry, and the study of Randers metrics is an important step in understanding the general Finsler metrics. Randers metrics are expressed in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form with $\|\beta_x\|_\alpha := \sqrt{a^{ij} b_i(x) b_j(x)} < 1$.

In recent years, the χ-curvature has become increasingly important in Finsler geometry. Thus, we studied the conformal transformation that preserves the χ-curvature of Randers metrics. If the conformal transformation is a homothety, it must preserve the χ-curvature invariant. Hence, the homothetic transformation is trivial and is omitted here.

In this paper, we determine the necessary and sufficient conditions for the conformal transformation that preserves the χ-curvature of Randers metrics and obtain the following results.

Theorem 1. *Let F be a non-Riemannian Randers metric on a compact manifold M with dimensions of $n (\geq 3)$. Then, there is no nonhomothetic conformal transformation that preserves the χ-curvature.*

2. Preliminaries

Let M be an n-dimensional smooth manifold and TM be the tangent bundle. If the function $F = F(x, y) : TM \to [0, \infty)$ satisfies the following properties, (i) F is a C^∞ function on $TM \backslash \{0\}$; (ii) $F(x, \lambda y) = \lambda F(x, y)$ for any $\lambda > 0$; and (iii) The Hessian matrix $(g_{ij}) := \frac{1}{2}\left(F^2_{y^i y^j}\right)$ is positive definite on $TM \backslash \{0\}$, then $F = F(x, y)$ is called a Finsler metric on M, and the tensor $\boldsymbol{g} = g_{ij}(x, y) dx^i \otimes dx^j$ is called the fundamental tensor of Finsler metric F. If the Hessian (g_{ij}) is independent of y, F is called a Riemannian metric.

The Cartan tensor is defined by $\mathbf{C} := C_{ijk} dx^i \otimes dx^j \otimes dx^k$, where

$$C_{ijk} := \frac{1}{2}(g_{ij})_{y^k}.$$

The mean Cartan torsion $\mathbf{I}_y := I_i dx^i : T_x M \to \mathbf{R}$ is defined by

$$I_i := g^{jk} C_{ijk}.$$

The geodesics of F are locally characterized by a system of 2nd ODEs:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i := \frac{1}{4} g^{ij} \left\{ \left[F^2\right]_{x^k y^j} y^k - \left[F^2\right]_{x^j} \right\}.$$

G^i are known as the geodesic coefficients of F.

The volume form of F is expressed as

$$dV_F = \sigma(x) dx^1 \cdots dx^n.$$

For a nonzero vector $y \in T_pM$, the **S**-curvature $\mathbf{S}(y)$ is defined by

$$\mathbf{S}(y) := G^k_{y^k} - \frac{1}{\sigma} y^k \sigma_{x^k}.$$

We say that F is of isotropic $\boldsymbol{S}$-curvature if

$$\mathbf{S} = (n+1)cF,$$

where $c = c(x)$ is a scalar function on M. If c is constant, F is said to have a constant $\boldsymbol{S}$-curvature.

The non-Riemannian quantity χ-curvature $\chi := \chi_i dx^i$ on the tangent bundle TM is defined by

$$\chi_i := -\frac{1}{6} \left\{ 2R^k{}_{i \cdot k} + R^k{}_{k \cdot i} \right\}.$$

We say that F is of almost vanishing χ-curvature if

$$\chi_i = -(n+1)F^2 \left(\frac{\theta}{F}\right)_{y^i},$$

where $\theta = \theta_i(x) y^i$ is a 1-form on M.

For a Randers metric $F = \alpha + \beta$, we have

$$g^{ij} = \frac{\alpha}{F} a^{ij} - \frac{\alpha}{F^2}\left(b^i y^j + b^j y^i\right) + \frac{b^2\alpha + \beta}{F^3} y^i y^j, \tag{1}$$

where $b^i := a^{ij} b_j$ and $b := \|\beta\|_\alpha$ denotes the norm of β with respect to α. Additionally, the mean Cartan tensor $\mathbf{I} = I_i dx^i$ of $F = \alpha + \beta$ is given by

$$I_i = \frac{n+1}{2F}\left(b_i - \frac{\beta y_i}{\alpha^2}\right), \tag{2}$$

where $y_i := a_{ij} y^j$.

Let

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right),\ s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right),\ r_{00} = r_{ij} y^i y^j,\ s^i{}_0 = a^{ij} s_{jk} y^k,$$
$$r_i = b^j r_{ij},\ s_i = b^j s_{ji},\ r_0 = r_i y^i,\ s_0 = s_i y^i,\ r^i{}_0 = a^{ij} r_{j0},\ s^i = a^{ij} s_j,\ r = b^i r_i,$$

where "$|$" denotes the covariant derivative with respect to the Levi–Civita connection of α.

Lemma 1 ([16])**.** *For a Randers metric, $F=\alpha+\beta$, the relationship between the spray coefficients G^i of F and G^i_α of α is given by*

$$G^i = G^i_\alpha + Py^i + Q^i,$$

where $P := \frac{e_{00}}{2F} - s_0$, $Q^i := \alpha s^i{}_0$, $e_{ij} := r_{ij} + b_i s_j + b_j s_i$ and $e_{00} := e_{ij}y^i y^j$.

Lemma 2 ([16])**.** *For a Randers metric $F=\alpha+\beta$, the $\mathbf{S}$-curvature is given by*

$$\mathbf{S} = (n+1)\left(\frac{e_{00}}{2F} + \frac{r_0 + b^2 s_0}{1-b^2}\right).$$

For more details about Randers metrics, one can see [16].

3. The Conformal Transformation Preserving the χ-Curvature of Randers Metrics

In [11], Cheng-Yuan obtained the formula of the conformal transformation that preserves the χ-curvature of Finsler metrics. In this section, we derive the formula of the conformal transformation that preserves the χ-curvature of Randers metrics.

Lemma 3 ([11])**.** *Let $\bar{F}$ and F be two Finsler metrics on a manifold M. If $\bar{F} = e^\rho F$, then the $\overline{\chi}$-curvature and the χ-curvature satisfy*

$$\bar{\chi}_i = \chi_i + B_{\cdot i\|j}y^j - B_{\|i} + 2H^j(\mathbf{S}_{\cdot i\cdot j} + B_{\cdot i\cdot j}),$$

where $\rho_i := \rho_{x^i}$, $\rho^i := g^{ij}\rho_j$, $B := F^2\rho^i I_i$, $H^i := \frac{1}{2}F^2\rho^i$, and "$||$" denotes the horizontal covariant derivative with respect to the Chern connection of F.

For Randers metrics, by Lemma 3, we have the following result:

Proposition 1. *Let $\bar{F}$ and F be two Randers metrics on a manifold M. If $\bar{F} = e^\rho F$, then the $\overline{\chi}$-curvature and the χ-curvature satisfy*

$$\begin{aligned}
\bar{\chi}_i - \chi_i =& \{[(\beta-\alpha)\rho_k r^k{}_0 F + (\alpha^2 + 2b^2\alpha\beta + \beta^2)\rho_k s^k{}_0 - 2\rho_k s^k \alpha\beta F]\frac{1}{\alpha F^2} + (1-b^2)\frac{\beta\rho_{0|0}}{\alpha F^2} \\
&+ 2b^k\rho_k[r_{00} + (\beta-\alpha)s_0]\frac{1}{F^2} + \frac{b^k\rho_{k|0}}{\alpha} + \{[(4-3b^2)\alpha + b^2\beta]r_{00} + 2(\alpha-\beta)r_0 F \\
&+ 2[-(1-2b^2)\alpha + (3-2b^2)\beta]\alpha s_0\}\frac{\rho_0}{\alpha F^3} + b^k\rho_k[-b^2\alpha^2 + 2(-2+b^2)\alpha\beta - \beta^2]\frac{\rho_0}{\alpha F^2} \\
&+ |\nabla\rho|^2_\alpha(1-b^2)\frac{\beta}{F} + (1-b^2)[-b^2\alpha^2 - (3-b^2)\alpha\beta - \beta^2]\frac{{\rho_0}^2}{\alpha F^3}\}y_i \\
&- \alpha b^k\rho_{k|i} + \frac{b^2\alpha + \beta}{F}(\rho_{0|i} - \rho_{i|0}) + \{\rho_k[-2\alpha r^k{}_0 F + 2(1-b^2)\alpha^2 s^k{}_0 + 2s^k\alpha^2 F]\frac{1}{F^2} \\
&- (1-b^2)\frac{\alpha\rho_{0|0}}{F^2} + 2b^k\rho_k(r_{00} - 2\alpha s_0)\frac{\alpha}{F^2} + \{[(5-4b^2)\alpha + \beta]r_{00} + 4\alpha r_0 F \\
&+ 2[-2(1-2b^2)\alpha^2 + 3\alpha\beta + \beta^2]s_0\}\frac{\rho_0}{F^3} + [(2-3b^2)\alpha^2 - 2\alpha\beta - \beta^2]\frac{b^k\rho_k\rho_0}{F^2} \\
&- (1-b^2)\frac{|\nabla\rho|^2_\alpha\alpha^2}{F} + (1-b^2)[(1-2b^2)\alpha^2 - 2\alpha\beta - \beta^2]\frac{\rho_0^2}{F^3}\}b_i \\
&+ [-2b^k\rho_k\alpha F - 2(1-b^2)\alpha]\frac{r_{i0}}{F^2} - 2[(1-b^2)\rho_0 + b^k\rho_k F]\frac{s_{i0}}{F} + \alpha\rho_k(r^k{}_i - s^k{}_i) \\
&+ 2[b^k\rho_k\alpha^2 F + (1-b^2)\alpha^2\rho_0]\frac{s_i}{F^2} \\
&+ [-[(2-b^2)\alpha + \beta]r_{00} - 2\alpha r_0 F - 2(b^2\alpha^2 + 2\alpha\beta + \beta^2)s_0 \\
&+ (1-b^2)(b^2\alpha^2 + 2\alpha\beta + \beta^2)\rho_0 + b^k\rho_k(b^2\alpha^2 + 2\alpha\beta + \beta^2)F]\frac{\rho_i}{F^2},
\end{aligned} \tag{3}$$

where $|\nabla\rho|^2_\alpha := a^{ij}\rho_i\rho_j$.

Proof. By (1) and (2), we have

$$\rho^i = g^{ik}\rho_k = \frac{\alpha}{F}a^{ik}\rho_k - \frac{\alpha}{F^2}\rho_0 b^i + \left(\frac{b^2\alpha}{F^3}\rho_0 + \frac{\beta}{F^3}\rho_0 - \frac{\alpha}{F^2}b^k\rho_k\right)y^i, \tag{4}$$

$$B = F^2\rho^i I_i = \frac{n+1}{2}\left[b^k\rho_k\alpha - \frac{(b^2-1)\alpha + F}{F}\rho_0\right]. \tag{5}$$

According to Lemma 3, we obtain

$$\begin{aligned}
\bar{\chi}_i &= \chi_i + B_{\cdot i\|j}y^j - B_{\|i} + 2H^j(\mathbf{S}_{\cdot i\cdot j} + B_{\cdot i\cdot j}) \\
&= \chi_i + \frac{\partial B_{\cdot i}}{\partial x^j}y^j - 2G^k B_{\cdot i\cdot k} - \frac{\partial B}{\partial x^i} + 2H^j(\mathbf{S}_{\cdot i\cdot j} + B_{\cdot i\cdot j}) \\
&= \chi_i + \frac{\partial B_{\cdot i}}{\partial x^j}y^j - 2(G^k_\alpha + Py^k + Q^k)B_{\cdot i\cdot k} - \frac{\partial B}{\partial x^i} + 2H^j(\mathbf{S}_{\cdot i\cdot j} + B_{\cdot i\cdot j}) \\
&= \chi_i + \left(\frac{\partial B}{\partial x^j}y^j - 2G^k_\alpha B_{\cdot k}\right)_{\cdot i} - 2\frac{\partial B}{\partial x^i} - 2Q^k B_{\cdot i\cdot k} + 2H^j(\mathbf{S}_{\cdot i\cdot j} + B_{\cdot i\cdot j}) \\
&= \chi_i + (B_{|j}y^j)_{\cdot i} - 2B_{|i} - 2Q^k B_{\cdot i\cdot k} + 2H^j(\mathbf{S}_{\cdot i\cdot j} + B_{\cdot i\cdot j}).
\end{aligned}$$

Plugging (4) and (5) into the above equation yields (3). □

Using Proposition 1, we can obtain the necessary and sufficient conditions for a conformal transformation to preserve the χ-curvature of Randers metrics.

Proposition 2. *Let F and $\bar{F}$ be two non-Riemannian Randers metrics on a manifold M. If $\bar{F} = e^\rho F$, then $\bar{\chi} = \chi$ if and only if the following equations hold*

$$\Pi_5\alpha^4 + \Pi_3\alpha^2 + \Pi_1 = 0, \tag{6}$$

$$\Pi_4\alpha^4 + \Pi_2\alpha^2 + \Pi_0 = 0, \tag{7}$$

where $\Pi_5, \Pi_4, \Pi_3, \Pi_2, \Pi_1$ and Π_0 are polynomials in y, listed in (A1)–(A6) in Appendix A.

Proof. Since $\bar{\chi} = \chi$, we have

$$\alpha(\alpha+\beta)^3(\bar{\chi}_i - \chi_i) = 0.$$

Plugging (3) into the above equation yields

$$\Pi_5\alpha^5 + \Pi_4\alpha^4 + \Pi_3\alpha^3 + \Pi_2\alpha^2 + \Pi_1\alpha + \Pi_0 = 0,$$

where $\Pi_5, \Pi_4, \Pi_3, \Pi_2, \Pi_1$, and Π_0 are polynomials in y. We obtain

$$\Pi_5\alpha^4 + \Pi_3\alpha^2 + \Pi_1 = 0,$$
$$\Pi_4\alpha^4 + \Pi_2\alpha^2 + \Pi_0 = 0.$$

□

Using Proposition 2, for a Randers metric $F = \alpha + \beta$, we can further optimize the necessary and sufficient conditions for the conformal transformation to preserve the χ-curvature.

Proposition 3. *Let F and $\bar{F}$ be two non-Riemannian Randers metrics on a manifold M. If $\bar{F} = e^{\rho}F$, then $\bar{\chi} = \chi$ if and only if the following equations hold*

$$
\begin{aligned}
&\Gamma_2^2\alpha^2 + \Gamma_2^0 = 0, && (8)\\
&(\alpha^2-\beta^2)\Gamma_i + (1-b^2)[\rho_{0|0} + 2\rho_k s^k{}_0\beta + (2s_0 + c\beta + b^k\rho_k\beta)\rho_0 && (9)\\
&\quad - (1-\frac{1}{2}b^2)\rho_0^2]\beta^2(y_i - \beta b_i) = 0, &&\\
&2r_{00} + 4s_0\beta - (1-b^2)\beta\rho_0 = c(\alpha^2-\beta^2), && (10)
\end{aligned}
$$

where $c = c(x)$ is a scalar function on M, and Γ_2^2, Γ_2^0 and Γ_i are polynomials in y, as listed in (A7)–(A9) in Appendix A.

Proof. "Necessity". If $\bar{\chi} = \chi$, according to Proposition 2, we know that (6) and (7) hold.

According to (6) $\times\ \beta -$ (7), we have

$$
(\alpha^2-\beta^2)A_i + 2(1-b^2)[2r_{00} + 4s_0\beta - (1-b^2)\beta\rho_0]\beta\rho_0(y_i - \beta b_i) = 0, \tag{11}
$$

where

$$
\begin{aligned}
A_i =& \{2\beta b^k\rho_{k|i} + b^2(\rho_{i|0} - \rho_{0|i}) + [2\rho_k r^k{}_0 - 2(1-b^2)\rho_k s^k{}_0 + 4b^k\rho_k s_0\\
&- 2\rho_k s^k\beta + (1-b^2)|\nabla\rho|_\alpha^2\beta - (2-3b^2)b^k\rho_k\rho_0]b_i + 2b^k\rho_k r_{i0} + 2b^k\rho_k s_{i0}\\
&- 2\beta\rho_k r^k{}_i + 2\beta\rho_k s^k{}_i + [-2b^k\rho_k\beta - 2(1-b^2)\rho_0]s_i + [2r_0 + 2b^2 s_0\\
&- (2+b^2)b^k\rho_k\beta - b^2(1-b^2)\rho_0]\rho_i\}\alpha^2\\
&+ \beta^2(\rho_{i|0} - \rho_{0|i}) + [-2b^k\rho_{k|0}\beta - 2b^k\rho_k r_{00} + 2\rho_k s^k\beta^2 - 2b^2\rho_k s^k_0\beta - 2r_0\rho_0\\
&- 2b^k\rho_k s_0\beta + 2(1-2b^2)s_0\rho_0 - (1-b^2)|\nabla\rho|_\alpha^2\beta^2 + 2(2-b^2)b^k\rho_k\beta\rho_0\\
&+ b^2(1-b^2)\rho_0^2]y_i\\
&+ [-(5-4b^2)r_{00}\rho_0 - 2(5-4b^2)s_0\beta\rho_0 + b^k\rho_k\beta^2\rho_0 + (1-b^2)(3-2b^2)\beta\rho_0^2]b_i\\
&+ [2b^k\rho_k\beta + 2(1-b^2)\rho_0]\beta s_{i0} + [r_{00} + 2s_0\beta - b^k\rho_k\beta^2 - (1-b^2)\beta\rho_0]\beta\rho_i.
\end{aligned}
$$

Since $\alpha^2 - \beta^2$ is irreducible and $\rho_0 \neq 0$, it is easy to see from (11) that $2r_{00} + 4s_0\beta - (1-b^2)\beta\rho_0$ is divisible by $\alpha^2 - \beta^2$. Thus, a scalar function $c = c(x)$ exists on M such that

$$
2r_{00} + 4s_0\beta - (1-b^2)\beta\rho_0 = c(\alpha^2-\beta^2). \tag{12}
$$

According to (12), we obtain

$$
\begin{aligned}
r_{00} =& \frac{1}{2}c(\alpha^2-\beta^2) - 2s_0\beta + \frac{1}{2}(1-b^2)\beta\rho_0, && (13)\\
r_0 =& -b^2 s_0 + \frac{1}{2}c(1-b^2)\beta + \frac{1}{4}(1-b^2)b^k\rho_k\beta + \frac{1}{4}b^2(1-b^2)\rho_0, &&\\
r^k{}_0 =& \frac{1}{2}cy^k - \beta s^k + [-s_0 - \frac{1}{2}c\beta + \frac{1}{4}(1-b^2)\rho_0]b^k + \frac{1}{4}(1-b^2)\beta a^{ik}\rho_i. && (14)
\end{aligned}
$$

By substituting the above equations into (11) and (6), we obtain

$$
\begin{aligned}
&\Gamma_1^4\alpha^4 + \Gamma_1^2\alpha^2 + \Gamma_1^0 = 0, && (15)\\
&\Gamma_2^2\alpha^2 + \Gamma_2^0 = 0, && (16)
\end{aligned}
$$

where Γ_1^4, Γ_1^2, Γ_1^0, Γ_2^2 and Γ_2^0 are polynomials in y, listed in (A7)–(A8) and (A10)–(A12) in Appendix A.

Then, (16) $\times(\alpha^2+\beta^2)$ + (15) $\times\beta$ yields

$$(\alpha^2-\beta^2)\Gamma_i+(1-b^2)[\,rho_{0|0}+2\rho_k s^k{}_0\beta+2s_0\rho_0+c\beta\rho_0+b^k\rho_k\beta\rho_0 - (1-\frac{1}{2}b^2)\rho_0^2]\beta^2(y_i-b_i\beta)=0.$$

"Sufficiency". If we assume that (8)–(10) hold, we can easily have (6) and (7). Based on Proposition 2, we know that the conformal transformation preserves the χ-curvature of Randers metrics. This completes the proof. □

4. Proof of the Main Theorem

Now we can prove our characterization theorem for the conformal transformation preserving the χ-curvature of Randers metrics.

Theorem 2. *Let F and $\bar{F}$ be two non-Riemannian Randers metrics on a manifold M. If $\bar{F}=e^{\rho}F$, then $\bar{\chi}=\chi$ if and only if one of the following cases holds:*

(i) $(b^k\rho_k=0)$ *$\rho(x)$ satisfies*

$$\rho_0=-\frac{4s_0}{b^2}, \tag{17}$$

and $\beta=b_i(x)y^i$ satisfies

$$b^2b_{i|j}=-2s_ib_j, \tag{18}$$

$$2s_ks^ka_{ij}-b^2s_{i|j}-4s_is_j=0; \tag{19}$$

(ii) $(b^k\rho_k\neq 0)$ *$\rho(x)$ satisfies*

$$\rho_0=-\frac{4s_0}{b^2}+\frac{b^k\rho_k\beta}{b^2}, \tag{20}$$

and $\beta=b_i(x)y^i$ satisfies

$$\begin{aligned}2b^4s_0b_{i|j}=&b^2\Big(2\psi_0-b^2b^k\rho_ks_0\Big)a_{ij}+\Big(b^2b^k\rho_ks_0-2\psi_0\Big)b_ib_j+2\beta(2\psi_ib_j-b_i\psi_j)\\&+2b^2s_0(b_is_j-2s_ib_j)-2b^2\Big(\beta s_i+b^2s_{i0}\Big)s_j-2b^2(2\psi_iy_j-y_i\psi_j),\end{aligned} \tag{21}$$

$$4b^2s_ks^ka_{ij}=2b_i\psi_j+2\psi_ib_j+8b^2s_is_j-2b^2b^k\rho_ks_ib_j-b^4b^k\rho_ks_{ij}+2b^4s_{i|j}, \tag{22}$$

where $\psi_0=b^2s_ks^k{}_0+s_ks^k\beta$ and $\psi_i=b^2s_ks^k{}_i+s_ks^kb_i$.

Proof. "Necessity". If $\bar{\chi}=\chi$, based on Proposition 3, we have (9). Because $\alpha^2-\beta^2$ are relatively prime polynomials in y, there is a scalar function $d=d(x)$ on M such that

$$\rho_{0|0}+2\rho_ks^k{}_0\beta+2s_0\rho_0+c\beta\rho_0+b^k\rho_k\beta\rho_0-(1-\frac{1}{2}b^2)\rho_0^2=d(\alpha^2-\beta^2).$$

Based on the above equation, we have

$$\begin{aligned}\rho_{0|0}=&d\alpha^2-2\rho_ks^k{}_0\beta-2s_0\rho_0-d\beta^2-c\beta\rho_0-b^k\rho_k\beta\rho_0+(1-\frac{1}{2}b^2)\rho_0^2,\\ \rho_{0|i}=&2dy_i-\rho_{i|0}-[2\rho_ks^k{}_0+2d\beta+c\rho_0+b^k\rho_k\rho_0]b_i-2\beta\rho_ks^k{}_i-2\rho_0s_i\\&+[-2s_0-c\beta-b^k\rho_k\beta+2(1-\frac{1}{2}b^2)\rho_0]\rho_i.\end{aligned}$$

By substituting the above equations into (8), (9) and (15), by a direct computation, we obtain

$$\Gamma_4^4\alpha^4+\Gamma_4^2\alpha^2+\Gamma_4^0=0, \tag{23}$$
$$\Gamma_3^2\alpha^2+\Gamma_3^0=0, \tag{24}$$
$$\Gamma_5^2\alpha^2+\Gamma_5^0=0, \tag{25}$$

where Γ_4^4, Γ_4^2, Γ_4^0, Γ_3^2, Γ_3^0, Γ_5^2 and Γ_5^0 are polynomials in y, listed in (A13)–(A19) in Appendix A.

By calculating (24) $\times$ $(\alpha^2+2\beta^2)$ + (23) $\times$ 2β + (25) $\times$ $2\beta^2$, we obtain

$$\Gamma_6^4\alpha^2+\Gamma_6^2=0, \tag{26}$$

where Γ_6^4 and Γ_6^2 are polynomials in y, listed in (A20)–(A21) in Appendix A.

Contracting (26) with b^i yields

$$\Gamma_7^2\alpha^2+\Gamma_7^0\beta^2=0, \tag{27}$$

where Γ_7^2 and Γ_7^0 are polynomials in y, listed in (A22)–(A23) in Appendix A.

Since α^2 is irreducible, (27) is equivalent to $\Gamma_7^2=0$ and $\Gamma_7^0=0$. That is,

$$\begin{aligned}&2b^2b^k\rho_{k|0}-2b^2(1-2b^2)\rho_k s^k{}_0+2(1+b^2)b^k\rho_k s_0-c(\frac{1}{2}-b^2)b^k\rho_k\beta\\&-4db^2(1-b^2)\beta-2b^2\rho_k s^k\beta-(1-b^2)(b^k\rho_k)^2\beta\\&-\frac{1}{2}b^2(1-b^2)|\nabla\rho|_\alpha^2\beta-\frac{3}{2}b^2c(1-2b^2)\rho_0-\frac{1}{2}b^2(7-9b^2)b^k\rho_k\rho_0=0,\end{aligned} \tag{28}$$

$$\begin{aligned}&-2b^k\rho_{k|0}+2(1-2b^2)\rho_k s^k{}_0-4b^k\rho_k s_0-\frac{1}{2}cb^k\rho_k\beta\\&+4d(1-b^2)\beta+2\rho_k s^k\beta+\frac{1}{2}(1-b^2)|\nabla\rho|_\alpha^2\beta+\frac{1}{2}c(4-7b^2)\rho_0\\&+\frac{1}{2}(9-11b^2)b^k\rho_k\rho_0=0.\end{aligned} \tag{29}$$

By calculating (28) + (29) and (28) + (29)$\times b^2$, we obtain

$$\begin{aligned}&-2b^k\rho_{k|0}+2(1-2b^2)\rho_k s^k{}_0-2b^k\rho_k s_0-cb^k\rho_k\beta+4d(1-b^2)\beta\\&-(b^k\rho_k)^2\beta+2\rho_k s^k\beta+\frac{1}{2}(1-b^2)|\nabla\rho|_\alpha^2\beta+c(2-3b^2)\rho_0\\&+\frac{9}{2}(1-b^2)b^k\rho_k\rho_0=0,\end{aligned} \tag{30}$$

$$2b^k\rho_k s_0-\frac{1}{2}cb^k\rho_k\beta-(b^k\rho_k)^2\beta+\frac{1}{2}cb^2\rho_0+b^2b^k\rho_k\rho_0=0. \tag{31}$$

From (30), we obtain

$$\begin{aligned}b^k\rho_{k|0}=&(1-2b^2)\rho_k s^k{}_0-b^k\rho_k s_0-\frac{1}{2}cb^k\rho_k\beta+2d(1-b^2)\beta\\&-\frac{1}{2}(b^k\rho_k)^2\beta+\rho_k s^k\beta+\frac{1}{4}(1-b^2)|\nabla\rho|_\alpha^2\beta+c(1-\frac{3}{2}b^2)\rho_0\\&+\frac{9}{4}(1-b^2)b^k\rho_k\rho_0,\end{aligned} \tag{32}$$

$$\begin{aligned}b^k\rho_{k|i}=&[-\frac{1}{2}cb^k\rho_k+2d(1-b^2)-\frac{1}{2}(b^k\rho_k)^2+\rho_k s^k+\frac{1}{4}(1-b^2)|\nabla\rho|_\alpha^2]b_i\\&+(1-2b^2)\rho_k s^k{}_i-b^k\rho_k s_i+c(1-\frac{3}{2}b^2)\rho_i+\frac{9}{4}(1-b^2)b^k\rho_k\rho_i.\end{aligned} \tag{33}$$

Based on (31), we divide the problem into two cases: (i) $b^k\rho_k=0$; (ii) $b^k\rho_k\neq 0$.

Case (i): $b^k\rho_k = 0$. Based on (31), we obtain $\frac{1}{2}cb^2\rho_0 = 0$. Thus, $c = 0$.

Furthermore, based on (14), $0 = (b^k\rho_k)_{|i} = b^k{}_{|i}\rho_k + b^k\rho_{k|i} = \rho_k(r^k{}_i + s^k{}_i) + b^k\rho_{k|i} = [\frac{1}{4}(1-b^2)|\nabla\rho|^2_\alpha - \rho_k s^k]b_i + \rho_k s^k{}_i + b^k\rho_{k|i}$. Thus

$$b^i b^k\rho_{k|i} = -\frac{1}{4}b^2(1-b^2)|\nabla\rho|^2_\alpha + \rho_k s^k(1+b^2). \tag{34}$$

On the other hand, contracting (33) with b^i yields

$$b^i b^k\rho_{k|i} = 2db^2(1-b^2) - (1-3b^2)\rho_k s^k + \frac{1}{4}b^2(1-b^2)|\nabla\rho|^2_\alpha. \tag{35}$$

Based on (34) and (35), we have

$$d = \frac{1}{b^2}\rho_k s^k - \frac{1}{4}|\nabla\rho|^2_\alpha. \tag{36}$$

By contracting (24) with b^i and plugging $b^k\rho_k = 0$, $c = 0$, (32), (33) and (36) into it, we can conclude that

$$(b^2\alpha^2 - \beta^2)(\frac{1}{2}b^2|\nabla\rho|^2_\alpha + 2\rho_k s^k) + (4s_0 + b^2\rho_0)\rho_0 = 0.$$

Since $b^2\alpha^2 - \beta^2$ is irreducible, $\rho_0 = -\frac{4s_0}{b^2}$.

For $\rho_0 = -\frac{4s_0}{b^2}$, we have

$$\begin{aligned} &\rho_k s^k = -\frac{4s_k s^k}{b^2},\ |\nabla\rho|^2_\alpha = \frac{16s_k s^k}{b^4},\ d = -\frac{8s_k s^k}{b^4}, \\ &\rho_{i|0} = -\frac{4s_{i|0}}{b^2},\ b^k\rho_{k|i} = \frac{4s_k s^k{}_i}{b^2} - \frac{4s_k s^k b_i}{b^4}. \end{aligned} \tag{37}$$

By plugging the above equations into (25), we get

$$\begin{aligned} &\{2b^2 s_k s^k y_i - b^4 s_{i|0} + b^2(1-2b^2)s_k s^k{}_0 b_i + (1-3b^2)s_k s^k\beta b_i \\ &- b^4\beta s_k s^k{}_i - 4b^2 s_0 s_i\}\alpha^2 - (1-b^2)(b^2 s_k s^k{}_0 + s_k s^k\beta)\beta y_i = 0. \end{aligned} \tag{38}$$

Contracting (38) with b^i yields

$$\begin{aligned} &\{(3-2b^2)b^2 s_k s^k\beta - b^4 b^k s_{k|0} + b^4(1-2b^2)s_k s^k{}_0\}\alpha^2 - (1-b^2)(b^2 s_k s^k{}_0 \\ &+ s_k s^k\beta)\beta^2 = 0. \end{aligned}$$

Since α^2 is irreducible, then we can easily have

$$b^2 s_k s^k{}_0 + s_k s^k\beta = 0. \tag{39}$$

Plugging (39) into (38) yields

$$b^2 s_{i|0} = -4s_0 s_i + 2s_k s^k y_i, \tag{40}$$

which is (19).

Substituting (37), (39) and (40) into (24) and (12) yields

$$\begin{aligned} &b^2 s_{i0} = s_0 b_i - \beta s_i, \\ &b^2 r_{00} = -2s_0\beta. \end{aligned}$$

Hence

$$b^2 b_{i|j} = -2s_i b_j,$$

which is (18).

Case (ii): $b^k \rho_k \neq 0$. Based on (31), we obtain

$$s_0 = \frac{1}{4}(c + 2b^k \rho_k)\beta - \frac{1}{4}(\frac{1}{b^k \rho_k} + 2)b^2 \rho_0. \tag{41}$$

By contracting (24) with b^i and plugging in (32), (33), and (41) into it, we can conclude that

$$\begin{aligned} &\{6db^2(1-b^2)b^k\rho_k + (1-b^2)(4b^2 b^k \rho_k + cb^2)|\nabla\rho|^2_\alpha \\ &+ (2-5b^2)(c+b^k\rho_k)(b^k\rho_k)^2\}\alpha^2 - 2d(1-b^2)b^k\rho_k\beta^2 \\ &- (1-b^2)|\nabla\rho|^2_\alpha b^k\rho_k\beta^2 + 3(c+b^k\rho_k)(b^k\rho_k)^2\beta^2 \\ &+ 2(1-b^2)(c+b^k\rho_k)b^k\rho_k\beta\rho_0 - b^2(1-b^2)(c+b^k\rho_k)\rho_0^2 = 0. \end{aligned} \tag{42}$$

Differentiating (42) with respect to y^i and contracting it with b^i yields

$$d = -\frac{|\nabla\rho|^2_\alpha}{4b^k\rho_k}(c + 3b^k\rho_k) - \frac{3b^k\rho_k}{4b^2}(c + b^k\rho_k). \tag{43}$$

By plugging (43) into (42), we have

$$\begin{aligned} &(c+b^k\rho_k)\{-b^2[b^2(1-b^2)|\nabla\rho|^2_\alpha + (5+b^2)(b^k\rho_k)^2]\alpha^2 \\ &+ b^2(1-b^2)|\nabla\rho|^2_\alpha\beta^2 + 3(1+b^2)(b^k\rho_k)^2\beta^2 + 4b^2(1-b^2)b^k\rho_k\beta\rho_0 \\ &- 2b^4(1-b^2)\rho_0^2\} = 0. \end{aligned} \tag{44}$$

Based on (44), we divide the problem into two cases:

$$\begin{aligned} &\text{(ii-i)}\ \ c + b^k\rho_k = 0; \\ &\text{(ii-ii)} - b^2[b^2(1-b^2)|\nabla\rho|^2_\alpha + (5+b^2)(b^k\rho_k)^2]\alpha^2 + b^2(1-b^2)|\nabla\rho|^2_\alpha\beta^2 \\ &+ 3(1+b^2)(b^k\rho_k)^2\beta^2 + 4b^2(1-b^2)b^k\rho_k\beta\rho_0 - 2b^4(1-b^2)\rho_0^2 = 0. \end{aligned} \tag{45}$$

Case (ii-i): By plugging $c = -b^k\rho_k$ into (31) and (43), we obtain

$$\rho_0 = -\frac{4s_0}{b^2} + \frac{b^k\rho_k\beta}{b^2}, \tag{46}$$

$$d = -\frac{1}{2}|\nabla\rho|^2_\alpha. \tag{47}$$

Since $\rho_0 = -\frac{4s_0}{b^2} + \frac{b^k\rho_k\beta}{b^2}$, we obtain

$$\begin{aligned} &\rho_i = -\frac{4s_i}{b^2} + \frac{b^k\rho_k b_i}{b^2},\ \rho_k s^k = -\frac{4}{b^2}s_k s^k, \\ &\rho_k s^k{}_i = -\frac{4}{b^2}s_k s^k{}_i + \frac{1}{b^2}b^k\rho_k s_i,\ |\nabla\rho|^2_\alpha = \frac{16}{b^4}s_k s^k + \frac{1}{b^2}(b^k\rho_k)^2. \end{aligned} \tag{48}$$

Plugging $c = -b^k\rho_k$, (32), (33) and (47) into (25) yields

$$\rho_{i|0} = \frac{1}{b^4}\{-[8s_ks^k + \frac{b^2}{2}(b^k\rho_k)^2]y_i + [-4(1-2b^2)s_ks^k{}_0 - 3b^k\rho_k s_0 + 8s_ks^k\beta + (b^k\rho_k)^2\beta]b_i - b^2b^k\rho_k s_{i0} + 4\beta s_ks^k{}_i + (16s_0 - 5b^k\rho_k\beta)s_i\}. \tag{49}$$

By substituting $c = -b^k\rho_k$, (32), (33), (41), (46), (47), (48), and (49) into (24) and (12), we obtain

$$-(s_ks^kb_i + b^2s_ks^k{}_i)\alpha^2 + (b^2s_ks^k{}_0 + s_ks^k\beta)y_i + (s_0^2 - s_ks^k{}_0\beta)b_i - b^2s_0s_{i0} + \beta^2s_ks^k{}_i - s_0\beta s_i = 0, \tag{50}$$

$$2b^2r_{00} + 4s_0\beta = b^k\rho_k(\beta^2 - b^2\alpha^2). \tag{51}$$

Based on $\rho_i = -\frac{4s_i}{b^2} + \frac{b^k\rho_kb_i}{b^2}$, (13), (14) and (48), we obtain

$$\begin{aligned}\rho_{i|0} &= [(-4s_i + b^k\rho_kb_i)_{|j}b^{-2} + (-4s_i + b^k\rho_kb_i)(b^{-2})_{|j}]y^j \\ &= -\frac{1}{2b^2}(b^k\rho_k)^2y_i - \frac{4}{b^2}s_{i|0} + [-\frac{8}{b^4}(1-b^2)s_ks^k{}_0 - \frac{8}{b^6}(1-b^2)s_ks^k\beta \\ &\quad - \frac{3}{b^4}b^k\rho_ks_0 + \frac{1}{b^4}(b^k\rho_k)^2\beta]b_i + \frac{1}{b^2}b^k\rho_ks_{i0} - \frac{1}{b^4}b^k\rho_k\beta s_i.\end{aligned} \tag{52}$$

Clearly, based on (49) and (52), we have

$$-4b^2s_ks^ky_i + 2b^4s_{i|0} + 2\psi_0b_i + (8b^2s_0 - 2b^2b^k\rho_k\beta)s_i - b^4b^k\rho_ks_{i0} + 2\beta\psi_i = 0, \tag{53}$$

where $\psi_0 = b^2s_ks^k{}_0 + s_ks^k\beta$ and $\psi_i = b^2s_ks^k{}_i + s_ks^kb_i$. Clearly, (53) is (22).

We claim that $s_0 \neq 0$. If $s_0 = 0$, we have $s_{i0} = 0$ based on (53). Based on (46), we obtain $\rho_0 = \frac{b^k\rho_k\beta}{b^2}$. Furthermore, we can obtain

$$\rho_kr^k{}_0 = 0,\ |\nabla\rho|^2_\alpha = \frac{(b^k\rho_k)^2}{b^2},\ b^k\rho_{k|0} = 0,\ s_{i0} = 0,\ s_0 = 0,$$

$$r_{00} = \frac{b^k\rho_k}{2b^2}(-b^2\alpha^2 + \beta^2),\ r_{i0} = \frac{b^k\rho_k}{2b^2}(-b^2y_i + \beta b_i),\ r_0 = 0.$$

Plugging the above equations into (7) yields

$$\{[3-(1-b^2)^2]b^2\alpha^2b_i + [(1-b^2)^2+1]\beta^2b_i - [3-(1-b^2)^2]b^2\beta y_i\}\alpha^2 - [(1-b^2)^2+1]\beta^3y_i = 0.$$

Contracting this with b^i yields

$$\{[3-(1-b^2)^2]\alpha^2 - 2(2-b^2)\beta^2\}b^4\alpha^2 - [(1-b^2)^2+1]\beta^4 = 0,$$

which is a contradiction.

For $s_0 \neq 0$, by (50) and (51), we can easily have

$$\begin{aligned}2b^4s_0b_{i|j} &= (b^2b^k\rho_ks_0b_i - 2\psi_0b_i + 4\beta\psi_i - 4b^2s_0s_i)b_j + b^2(2\psi_0 - b^2b^k\rho_ks_0)a_{ij} \\ &\quad + 2b^2(s_0b_i - \beta s_i - b^2s_{i0})s_j - 4b^2\psi_iy_j + 2(b^2y_i - \beta b_i)\psi_j,\end{aligned}$$

which is (21).

Case (ii-ii): Differentiating (45) with respect to y^i and y^j and contracting this with ρ^i and ρ^j yields

$$[(b^k\rho_k)^2 - b^2|\nabla\rho|_\alpha^2][b^2(1-b^2)|\nabla\rho|_\alpha^2 + (1+b^2)(b^k\rho_k)^2] = 0. \tag{54}$$

We claim that (54) cannot hold. If $(b^k\rho_k)^2 - b^2|\nabla\rho|_\alpha^2 = 0$, plugging it into (45) yields

$$(b^k\rho_k)^2[-3b^2\alpha^2 + 2(2+b^2)\beta^2] + b^2(1-b^2)(2b^k\rho_k\beta - b^2\rho_0)\rho_0 = 0.$$

Since $-3b^2\alpha^2 + 2(2+b^2)\beta^2$ is irreducible, we obtain $b^k\rho_k = 0$,, which is a contradiction. Similarly, if $b^2(1-b^2)|\nabla\rho|_\alpha^2 + (1+b^2)(b^k\rho_k)^2 = 0$, then plugging it into (45) yields

$$(b^k\rho_k)^2[-2b^2\alpha^2 + (1+b^2)\beta^2] + b^2(1-b^2)(2b^k\rho_k\beta - b^2\rho_0)\rho_0 = 0.$$

Since $-2b^2\alpha^2 + (1+b^2)\beta^2$ is irreducible, we have $b^k\rho_k = 0$, which is also a contradiction.

"Sufficiency". By substituting (17)–(19) or (20)–(22) into (6) and (7), respectively, we can easily show that (6) and (7) hold. Based on Proposition 2, we know that the conformal transformation preserves the χ-curvature of Randers metrics. This completes the proof of Theorem 2. □

Lemma 4 ([17]). *$r_0 + s_0 = 0$ if and only if b^2 is a constant.*

Lemma 5. *Let F and $\bar{F}$ be two non-Riemannian Randers metrics on a manifold M. If $\bar{F} = e^\rho F$ and $\bar{\chi} = \chi$, then b^2 is a constant.*

Proof. If the conformal transformation preserves the χ-curvature of Randers metrics, based on Theorem 2, we known that (18) or (21) hold.

If (18) holds, then we have $b^2 r_{00} + 2s_0\beta = 0$. Differentiating this with respect to y^i and contracting with b^i yields

$$r_0 + s_0 = 0.$$

Similarly, if (21) holds, then we have $2b^2 r_{00} + 4s_0\beta = b^k\rho_k(\beta^2 - b^2\alpha^2)$. Differentiating this with respect to y^i and contracting with b^i yields

$$r_0 + s_0 = 0.$$

Above all, based on Lemma 4, we know that b^2 is a constant. □

5. Proofs of Other Results

Now, we are in the position to prove the other results. Firstly, assume that $s_0 = 0$. Based on Theorem 2, we have the following result:

Theorem 3. *Let F be a non-Riemannian Randers metric on a manifold M. Suppose that $s_0 = 0$. Then, there is no nonhomothetic conformal transformation, which preserves the χ-curvature.*

Proof. Based on Theorem 2, we divide the problem into two cases:

(i) $b^k\rho_k = 0$. Since $s_0 = 0$, based on (17), we obtain $\rho_0 = 0$. Thus, the conformal transformation is a homethety.

(ii) $b^k\rho_k \neq 0$. Plugging $s_0 = 0$ into (22) yields $s_{i0} = 0$. Based on (20), we obtain $\rho_0 = \frac{b^k\rho_k\beta}{b^2}$. Furthermore, we have

$$\rho_k r^k{}_0 = 0,\ |\nabla\rho|^2_\alpha = \frac{(b^k\rho_k)^2}{b^2},\ b^k\rho_{k|0} = 0,\ s_{i0} = 0,\ s_0 = 0,$$
$$r_{00} = \frac{b^k\rho_k}{2b^2}(-b^2\alpha^2 + \beta^2),\ r_{i0} = \frac{b^k\rho_k}{2b^2}(-b^2 y_i + \beta b_i),\ r_0 = 0.$$

Plugging the above equations into (7) yields

$$\{[3-(1-b^2)^2]b^2\alpha^2 b_i + [(1-b^2)^2+1]\beta^2 b_i - [3-(1-b^2)^2]b^2\beta y_i\}\alpha^2$$
$$-[(1-b^2)^2+1]\beta^3 y_i = 0.$$

Contracting this with b^i yields

$$\{[3-(1-b^2)^2]\alpha^2 - 2(2-b^2)\beta^2\}b^4\alpha^2 - [(1-b^2)^2+1]\beta^4 = 0,$$

which is a contradiction. □

If the dimensions of the manifold are $n \geq 4$, then Theorem 2 can be simplified as follows:

Corollary 1. *Let F and $\bar{F}$ be two non-Riemannian Randers metrics on a manifold M of dimensions $n(\geq 4)$. If $\bar{F} = e^\rho F$, then $\bar{\chi} = \chi$ if and only if one of the following equations holds:*

$$\rho_0 = -\frac{4s_0}{b^2} + \frac{b^k\rho_k\beta}{b^2}, \tag{55}$$

and $\beta = b_i(x)y^i$ satisfies

$$2b^2 b_{i|j} = -b^2 b^k\rho_k a_{ij} + b^k\rho_k b_i b_j - 4s_i b_j, \tag{56}$$
$$4s_k s^k a_{ij} = -b^k\rho_k b_i s_j + 8s_i s_j - b^k\rho_k s_i b_j + 2b^2 s_{i|j}. \tag{57}$$

Proof. "Necessity". Based on Theorem 2, we divide the problem into two cases:

(i) $b^k\rho_k = 0$. Based on case (i) of Theorem 2, it is easy to check that (55)–(57) hold.

(ii) $b^k\rho_k \neq 0$. Based on case (ii) of Theorem 2, we have $\rho_0 = -\frac{4s_0}{b^2} + \frac{b^k\rho_k\beta}{b^2}$. Meanwhile, based on the proof of Theorem 2, (50) holds. Differentiating (50) with respect to y^j and contracting it with a^{ij} yields

$$(n-3)(b^2 s_k s^k{}_0 + s_k s^k\beta) = 0.$$

Thus

$$b^2 s_k s^k{}_i + s_k s^k b_i = 0.$$

By plugging it into (22) and (50), we obtain

$$s_0(b^2 s_{i0} - s_0 b_i + \beta s_i) = 0, \tag{58}$$
$$-4s_k s^k y_i + 2b^2 s_{i|0} + 8s_0 s_i - 2b^k\rho_k\beta s_i - b^2 b^k\rho_k s_{i0} = 0. \tag{59}$$

If $s_0 = 0$, based on Theorem 3, we know the conformal transformation is a homethety. Thus, based on (58), we have

$$b^2 s_{i0} - s_0 b_i + \beta s_i = 0, \tag{60}$$

i.e., $s_{i0} = \dfrac{1}{b^2}(s_0 b_i - \beta s_i)$. By plugging it into (59), we obtain

$$-4s_k s^k y_i + 2b^2 s_{i|0} - b^k \rho_k s_0 b_i + 8s_0 s_i - b^k \rho_k \beta s_i = 0, \tag{61}$$

which is (57).

Based on $2b^2 r_{00} + 4s_0\beta = b^k\rho_k(\beta^2 - b^2\alpha^2)$ and (60), we can easily obtain

$$2b^2 b_{i|j} = -4s_i b_j - b^2 b^k \rho_k a_{ij} + b^k \rho_k b_i b_j, \tag{62}$$

which is (56).

"Sufficiency". Since $\rho_0 = -\dfrac{4s_0}{b^2} + \dfrac{b^k\rho_k\beta}{b^2}$, (61), and (62) hold, we obtain (6) and (7). Based on Proposition 2, we know that the conformal transformation preserves the χ-curvature of Randers metrics. This completes the proof of Corollary 1. □

Corollary 2. *Let F be a non-Riemannian Randers metric on a manifold M. Then, there is no non-homothetic conformal transformation that preserves the vanishing χ-curvature ($\chi = \tilde{\chi} = 0$).*

To prove Corollary 2, we require the following lemmas.

Lemma 6 ([16])**.** *For a Randers metric $F = \alpha + \beta$, $\mathbf{S} = (n+1)c(x)F$ if and only if $r_{00} + 2s_0\beta = 2c'(x)(\alpha^2 - \beta^2)$, where $c = c(x)$ and $c' = c'(x)$ are scalar functions on M.*

Lemma 7 ([8])**.** *Let $F = \alpha + \beta$ be a Randers metric. It is of isotropic* **S***-curvature if and only if its χ-curvature almost vanishes. In particular, it is of constant* **S***-curvature if and only if $\chi = 0$.*

Proof. For a Randers metric $F = \alpha + \beta$, based on Lemmas 6 and 7, its χ-curvature vanishes if and only if it is of constant **S**-curvature. This means that

$$r_{00} + 2s_0\beta = 2c'\left(\alpha^2 - \beta^2\right), \tag{63}$$

where c' is a constant.

Meanwhile, when the conformal transformation preserves the χ-curvature, based on Proposition 3, we have

$$2r_{00} + 4s_0\beta - (1 - b^2)\beta\rho_0 = c(\alpha^2 - \beta^2),$$

where $c = c(x)$ is a scalar function on M.

Plugging it into (63) yields

$$(1 - b^2)\beta\rho_0 = (4c' - c)(\alpha^2 - \beta^2).$$

Because $\alpha^2 - \beta^2$ is irreducible, we obtain $\rho_0 = 0$. Thus, the conformal transformation is a homothety. □

6. Proof of Theorem 1

Now we assume that the manifold is a compact space. Because the conformal transformation preserves the χ-curvature of Randers metrics, we have a better rigidity result.

Proof. If the conformal transformation preserves the χ-curvature of Randers metrics, based on Theorem 2, (19) or (22) hold.

When (19) or (22) holds, differentiating (19) or (22) with respect to y^j and contracting them with a^{ij} yields

$$s^k{}_{|k} = \frac{2(n-2)}{b^2}|s_k|^2_\alpha,$$

where $|s_k|^2_\alpha = s_k s^k$. Based on the Divergence theorem, on the n-dimensional manifold M, we have $\int_M s^k{}_{|k} dx^1 \cdots dx^n = 0$. Thus, based on the above equation, we obtain that

$$\int_M s^k{}_{|k} dx^1 \cdots dx^n = 2(n-2)\int_M \frac{|s_k|^2_\alpha}{b^2} dx^1 \cdots dx^n = 0,$$

which means that $s_k = 0$. By Theorem 3, we know that the conformal transformation is a homothety. □

7. Conclusions

The research presented in this paper is driven by two motivations. The first motivation is that research on the χ-curvature has become more and more important in recent years. The second motivation comes from the following question: is there a nonhomothetic conformal transformation in Finsler geometry that preserves the invariance of certain curvature properties? Based on Theorem 1, we know that on a compact manifold M of dimensions $n(\geq 3)$, there is no nonhomothetic conformal transformation that preserves the χ-curvature on the Randers metric. From Corollary 1, we obtain three characterization equations for the conformal transformation preserving the χ-curvature of Randers metrics on a manifold M of dimensions $n(\geq 4)$.

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Appendix A

In this appendix, we give some coefficients appearing in Sections 3 and 4.

$$\Pi_5 = -b^k\rho_{k|i} + [-(1-b^2)|\nabla\rho|^2_\alpha + 2\rho_k s^k]b_i + \rho_k r^k{}_i - \rho_k s^k{}_i + 2b^k\rho_k s_i + b^2 b^k \rho_k \rho_i, \tag{A1}$$

$$\begin{aligned}\Pi_4 = &-3\beta b^k\rho_{k|i} + b^2(\rho_{0|i} - \rho_{i|0}) + \{-4b^k\rho_k s_0 - 2\rho_k r^k{}_0 + 2(1-b^2)\rho_k s^k{}_0 + 4\rho_k s^k\beta \\ &- 2(1-b^2)|\nabla\rho|^2_\alpha\beta + (2-3b^2)b^k\rho_k\rho_0\}b_i \\ &- 2b^k\rho_k r_{i0} - 2b^k\rho_k s_{i0} + 3\beta\rho_k r^k{}_i - 3\beta\rho_k s^k{}_i + 2[2b^k\rho_k\beta + (1-b^2)\rho_0]s_i \\ &+ [-2r_0 - 2b^2 s_0 + 2(1+b^2)b^k\rho_k\beta + b^2(1-b^2)\rho_0]\rho_i,\end{aligned} \tag{A2}$$

$$\begin{aligned}\Pi_3 = &\{b^k\rho_{k|0} - \rho_k r^k{}_0 + \rho_k s^k{}_0 - 2b^k\rho_k s_0 - 2\rho_k s^k\beta + (1-b^2)|\nabla\rho|^2_\alpha\beta - b^2 b^k\rho_k\rho_0\}y_i \\ &+ (1+2b^2)\beta(\rho_{0|i} - \rho_{i|0}) - 3\beta^2 b^k\rho_{k|i} \\ &+ \{-(1-b^2)\rho_{0|0} + 2b^k\rho_k r_{00} - 4\rho_k r^k{}_0\beta + 2(1-b^2)\rho_k s^k{}_0\beta + 4r_0\rho_0 - 4b^k\rho_k s_0\beta \\ &- 4(1-2b^2)s_0\rho_0 + 2\rho_k s^k\beta^2 - (1-b^2)|\nabla\rho|^2_\alpha\beta^2 - 3b^2 b^k\rho_k\beta\rho_0 + (1-b^2)(1-2b^2)\rho_0^2\}b_i \\ &- 2[2b^k\rho_k\beta + (1-b^2)\rho_0]r_{i0} - 2[3b^k\rho_k\beta + (1-b^2)\rho_0]s_{i0} \\ &+ 3\beta^2\rho_k r^k{}_i - 3\beta^2\rho_k s^k{}_i + 2[b^k\rho_k\beta + (1-b^2)\rho_0]\beta s_i \\ &+ [-(2-b^2)r_{00} - 4r_0\beta - 2(2+b^2)s_0\beta + (5+b^2)b^k\rho_k\beta^2 \\ &+ (1-b^2)(2+b^2)\beta\rho_0]\rho_i,\end{aligned} \tag{A3}$$

$$\begin{aligned}\Pi_2 =&\{3b^k\rho_{k|0}\beta + 2b^k\rho_k r_{00} - \rho_k r^k{}_0\beta + (1+2b^2)\rho_k s^k{}_0\beta + 2r_0\rho_0 - 2(1-2b^2)s_0\rho_0\\ &- 4\rho_k s^k\beta^2 + 2(1-b^2)|\nabla\rho|^2_\alpha\beta^2 - (4-b^2)b^k\rho_k\beta\rho_0 - b^2(1-b^2)\rho_0^2\}y_i\\ &+ (2+b^2)\beta^2(\rho_{0|i} - \rho_{i|0}) - \beta^3 b^k\rho_{k|i}\\ &+ \{-(1-b^2)\rho_{0|0}\beta + 2b^k\rho_k r_{00}\beta + (5-4b^2)r_{00}\rho_0 - 2\rho_k r^k{}_0\beta^2 + 4r_0\beta\rho_0 + 6s_0\beta\rho_0\\ &- 3b^k\rho_k\beta^2\rho_0 - 2(1-b^2)\beta\rho_0^2\}b_i\\ &- 2[b^k\rho_k\beta + (1-b^2)\rho_0]\beta r_{i0} - 2[3b^k\rho_k\beta + 2(1-b^2)\rho_0]\beta s_{i0} + \beta^3\rho_k r^k{}_i - \beta^3\rho_k s^k{}_i\\ &+ [-(3-b^2)r_{00} - 2r_0\beta - 6s_0\beta + 4b^k\rho_k\beta^2 + 3(1-b^2)\beta\rho_0]\beta\rho_i,\end{aligned} \tag{A4}$$

$$\begin{aligned}\Pi_1 =&\{3b^k\rho_{k|0}\beta^2 + (1-b^2)\rho_{0|0}\beta + 2b^k\rho_k r_{00}\beta + (4-3b^2)r_{00}\rho_0 + \rho_k r^k{}_0\beta^2\\ &+ (1+2b^2)\rho_k s^k{}_0\beta^2 + 2b^k\rho_k s_0\beta^2 + 2(3-2b^2)s_0\beta\rho_0 - 2\rho_k s^k\beta^3 + (1-b^2)|\nabla\rho|^2_\alpha\beta^3\\ &- (5-2b^2)b^k\rho_k\beta^2\rho_0 - (3-b^2)(1-b^2)\beta\rho_0^2\}y_i\\ &+ \beta^3(\rho_{0|i} - \rho_{i|0}) + [r_{00}\rho_0 + 2s_0\rho_0\beta - b^k\rho_k\beta^2\rho_0 - (1-b^2)\beta\rho_0^2]\beta b_i\\ &- 2[b^k\rho_k\beta + (1-b^2)\rho_0]\beta^2 s_{i0} + [-r_{00} - 2s_0\beta + b^k\rho_k\beta^2 + (1-b^2)\beta\rho_0]\beta^2\rho_i,\end{aligned} \tag{A5}$$

$$\begin{aligned}\Pi_0 =&\{b^k\rho_{k|0}\beta^2 + (1-b^2)\rho_{0|0}\beta + b^2 r_{00}\rho_0 + \rho_k r^k{}_0\beta^2 + \rho_k s^k{}_0\beta^2 - 2r_0\beta\rho_0 - b^k\rho_k\beta^2\rho_0\\ &- (1-b^2)\beta\rho_0^2\}\beta y_i,\end{aligned} \tag{A6}$$

$$\begin{aligned}\Gamma_2^2 =&2\beta b^k\rho_{k|i} + b^2(\rho_{i|0} - \rho_{0|i}) + [-2(1-b^2)\rho_k s^k{}_0 - cb^k\rho_k\beta - 2\rho_k s^k\beta + (1-b^2)|\nabla\rho|^2_\alpha\beta\\ &- c(\frac{3}{2} - 2b^2)\rho_0 - (1-2b^2)b^k\rho_k\rho_0]b_i + 2b^k\rho_k s_{i0} + 2\rho_k s^k{}_i\beta - 2[b^k\rho_k\beta + (1-b^2)\rho_0]s_i\\ &+ \frac{1}{2}[c(11-2b^2)\beta - 3(1+b^2)b^k\rho_k\beta - b^2(1-b^2)\rho_0]\rho_i,\end{aligned} \tag{A7}$$

$$\begin{aligned}\Gamma_2^0 =&\beta^2(\rho_{i|0} - \rho_{0|i}) + [-2b^k\rho_{k|0}\beta - 2b^2\rho_k s^k{}_0\beta + 2b^k\rho_k s_0\beta + 2(1-b^2)s_0\rho_0 + cb^k\rho_k\beta^2\\ &+ 2\rho_k s^k\beta^2 - (1-b^2)|\nabla\rho|^2_\alpha\beta^2 + c(1-b^2)\beta\rho_0 + \frac{1}{2}(5-b^2)b^k\rho_k\beta\rho_0 + \frac{1}{2}(1-b^2)\rho_0^2]y_i\\ &+ \frac{1}{2}[c\beta\rho_0 + 2b^k\rho_k\beta\rho_0 + (1-b^2)\rho_0^2]\beta b_i + [2b^k\rho_k\beta + (1-b^2)\rho_0]\beta s_{i0}\\ &- \frac{1}{2}[c\beta + 2b^k\rho_k\beta + (1-b^2)\rho_0]\beta^2\rho_i,\end{aligned} \tag{A8}$$

$$\begin{aligned}\Gamma_i =&\{\beta b^k\rho_{k|i} + b^2(\rho_{i|0} - \rho_{0|i}) + [-2(1-b^2)\rho_k s^k{}_0 - \frac{1}{2}cb^k\rho_k\beta - \rho_k s^k\beta\\ &+ \frac{1}{4}(1-b^2)|\nabla\rho|^2_\alpha\beta - c(\frac{3}{2} - 2b^2)\rho_0 - (1-2b^2)b^k\rho_k\rho_0]b_i\\ &+ 2b^k\rho_k s_{i0} + \beta\rho_k s^k{}_i - [b^k\rho_k\beta + 2(1-b^2)\rho_0]s_i\\ &+ [-\frac{1}{2}cb^2\beta - \frac{1}{4}(5+3b^2)b^k\rho_k\beta - \frac{1}{2}b^2(1-b^2)\rho_0]\rho_i\}\alpha^2\\ &+ [-b^k\rho_{k|0}\beta + (1-2b^2)\rho_k s^k{}_0\beta + b^k\rho_k s_0\beta + 2(1-b^2)s_0\rho_0 + \frac{1}{2}cb^k\rho_k\beta^2 + \rho_k s^k\beta^2\\ &- \frac{1}{4}(1-b^2)|\nabla\rho|^2_\alpha\beta^2 + \frac{3}{2}c(1-b^2)\beta\rho_0 + \frac{1}{4}(9-5b^2)b^k\rho_k\beta\rho_0 + \frac{1}{2}b^2(1-b^2)\rho_0^2]y_i\\ &+ [-(1-b^2)\rho_{0|0} - 2(1-b^2)\rho_k s^k{}_0\beta - 2(1-b^2)s_0\rho_0 - c(1-b^2)\beta\rho_0\\ &- (1-b^2)b^k\rho_k\beta\rho_0 + (1-\frac{1}{2}b^2)(1-b^2)\rho_0^2]\beta b_i,\end{aligned} \tag{A9}$$

$$\begin{aligned}\Gamma_1^4 =& - b^k\rho_{k|i} + [\frac{1}{2}cb^k\rho_k + \rho_k s^k - \frac{3}{4}(1-b^2)|\nabla\rho|^2_\alpha]b_i - \rho_k s^k{}_i + b^k\rho_k s_i\\ &+ \frac{1}{4}[-2c(1-b^2) + (1+3b^2)b^k\rho_k]\rho_i,\end{aligned} \tag{A10}$$

$$\begin{aligned}
\Gamma_1^2 =&[b^k\rho_{k|0} + \rho_k s^k{}_0 - b^k\rho_k s_0 - \frac{1}{2}cb^k\rho_k\beta - \rho_k s^k\beta + \frac{3}{4}(1-b^2)|\nabla\rho|_\alpha^2\beta \\
&+ \frac{1}{2}c(1-b^2)\rho_0 - \frac{1}{4}(1+3b^2)b^k\rho_k\rho_0]y_i \\
&- 3\beta^2 b^k\rho_{k|i} + (1+2b^2)\beta(\rho_{0|i} - \rho_{i|0}) + [-(1-b^2)\rho_{0|0} + 2(1-b^2)\rho_k s^k{}_0\beta \\
&- 2(1-b^2)s_0\rho_0 + \frac{3}{2}cb^k\rho_k\beta^2 + 3\rho_k s^k\beta^2 - \frac{5}{4}(1-b^2)|\nabla\rho|_\alpha^2\beta^2 + 3c(1-b^2)\beta\rho_0 \\
&- 3b^2b^k\rho_k\beta\rho_0 + \frac{1}{2}(1-b^2)^2\rho_0^2]b_i - 2[3b^k\rho_k\beta + (1-b^2)\rho_0]s_{i0} - 3\beta^2\rho_k s^k{}_i \\
&+ [3b^k\rho_k\beta + 4(1-b^2)\rho_0]\beta s_i \\
&+ \frac{1}{4}[6cb^2\beta + 3(5+3b^2)b^k\rho_k\beta + 2(1-b^2)(1+2b^2)\rho_0]\beta\rho_i, \qquad \text{(A11)}
\end{aligned}$$

$$\begin{aligned}
\Gamma_1^0 =&\frac{1}{4}[12b^k\rho_{k|0}\beta + 4(1-b^2)\rho_{0|0} + 4(1+2b^2)\rho_k s^k{}_0\beta - 6cb^k\rho_k\beta^2 - 12b^k\rho_k s_0\beta \\
&- 8(1-b^2)s_0\rho_0 - 12\rho_k s^k\beta^2 + 5(1-b^2)|\nabla\rho|_\alpha^2\beta^2 - 3(5-b^2)b^k\rho_k\beta\rho_0 \\
&- 6(1-b^2)\beta\rho_0 - 2(1-b^2)(2+b^2)\rho_0^2]\beta y_i \\
&+ \beta^3(\rho_{0|i} - \rho_{i|0}) - 2[b^k\rho_k + (1-b^2)]\beta^3 s_{i0} - \frac{1}{2}[c\beta\rho_0 + 2b^k\rho_k\rho_0\beta \\
&+ (1-b^2)\rho_0^2]\beta^2 b_i + \frac{1}{2}[c\beta + (1-b^2)\rho_0]\beta^3\rho_i, \qquad \text{(A12)}
\end{aligned}$$

$$\begin{aligned}
\Gamma_4^4 =& - b^k\rho_{k|i} + \frac{1}{4}[2cb^k\rho_k - 4d(1-b^2) + 4\rho_k s^k - 3(1-b^2)|\nabla\rho|_\alpha^2]b_i \\
&- \rho_k s^k{}_i + b^k\rho_k s_i + \frac{1}{4}[-2c(1-b^2) + (1+3b^2)b^k\rho_k]\rho_i, \qquad \text{(A13)}
\end{aligned}$$

$$\begin{aligned}
\Gamma_4^2 =&[b^k\rho_{k|0} + \rho_k s^k{}_0 - b^k\rho_k s_0 - \frac{1}{2}cb^k\rho_k\beta + 3d(1+b^2)\beta - \rho_k s^k\beta + \frac{3}{4}(1-b^2)|\nabla\rho|_\alpha^2\beta \\
&+ \frac{1}{2}c(1-b^2)\rho_0 - \frac{1}{4}(1+3b^2)b^k\rho_k\rho_0]y_i - 3\beta^2 b^k\rho_{k|i} - 2(1+2b^2)\beta\rho_{i|0} \\
&+ [2(1-4b^2)\rho_k s^k{}_0\beta + \frac{3}{2}cb^k\rho_k\beta^2 - d(1+5b^2)\beta^2 + 3\rho_k s^k\beta^2 - \frac{5}{4}(1-b^2)|\nabla\rho|_\alpha^2\beta^2 \\
&+ 3c(1-2b^2)\beta\rho_0 - 6b^2b^k\rho_k\beta\rho_0 - \frac{1}{2}(1-b^2)\rho_0^2]b_i \\
&- 2[3b^k\rho_k\beta + (1-b^2)\rho_0]s_{i0} - (5+4b^2)\beta^2\rho_k s^k{}_i + [3b^k\rho_k\beta + 2(1-4b^2)\rho_0]\beta s_i \\
&+ [-2(1+2b^2)s_0 - c(1+\frac{1}{2}b^2)\beta + \frac{1}{4}(11+b^2)b^k\rho_k\beta + \frac{1}{2}(5-3b^2)(1+2b^2)\rho_0]\beta\rho_i, \qquad \text{(A14)}
\end{aligned}$$

$$\begin{aligned}
\Gamma_4^0 =&[3b^k\rho_{k|0}\beta - (1-4b^2)\rho_k s^k{}_0\beta - 3b^k\rho_k s_0\beta - 4(1-b^2)s_0\rho_0 + \frac{5}{4}(1-b^2)|\nabla\rho|_\alpha^2\beta^2 \\
&- \frac{3}{2}cb^k\rho_k\beta^2 + d(1+b^2)\beta^2 - 2\rho_k s^k\beta^2 - \frac{5}{2}c(1-b^2)\beta\rho_0 - \frac{1}{4}(19-7b^2)b^k\rho_k\beta\rho_0 \\
&+ b^2\rho_0^2]\beta y_i \\
&- 2\beta^3\rho_{i|0} + [-2\rho_k s^k{}_0\beta - 2d\beta^2 - \frac{3}{2}\beta\rho_0 - 2b^k\rho_k\beta\rho_0 + \frac{1}{2}\rho_0^2]\beta^2 b_i \\
&- [2b^k\rho_k + 2(1-b^2)]\beta^3 s_{i0} - 2\beta^4\rho_k s^k{}_i - 2\beta^3\rho_0 s_i \\
&+ [-2s_0 - c\beta - b^k\rho_k\beta + \frac{1}{2}(5-3b^2)\rho_0]\beta^3\rho_i, \qquad \text{(A15)}
\end{aligned}$$

$$\begin{aligned}
\Gamma^2_3 =& -2db^2y_i + 2\beta b^k\rho_{k|i} + 2b^2\rho_{i|0} + [-2(1-2b^2)\rho_k s^k{}_0 - cb^k\rho_k\beta + 2db^2\beta - 2\rho_k s^k\beta \\
& + (1-b^2)|\nabla\rho|^2_\alpha\beta - \frac{3}{2}c(1-2b^2)\rho_0 - (1-3b^2)b^k\rho_k\rho_0]b_i \\
& + 2b^k\rho_k s_{i0} + 2(1+b^2)\beta\rho_k s^k{}_i - 2[b^k\rho_k\beta + (1-2b^2)\rho_0]s_i \\
& + [2b^2s_0 + \frac{11}{2}c\beta - \frac{1}{2}(3+b^2)b^k\rho_k\beta - \frac{1}{2}b^2(5-3b^2)\rho_0]\rho_i, && \text{(A16)}\\
\Gamma^0_3 =& [-2b^k\rho_{k|0}\beta - 2b^2\rho_k s^k{}_0\beta + 2b^k\rho_k s_0\beta + 2(1-b^2)s_0\rho_0 + cb^k\rho_k\beta^2 - 2d\beta^2 + 2\rho_k s^k\beta^2 \\
& - (1-b^2)|\nabla\rho|^2_\alpha\beta^2 + c(1-b^2)\beta\rho_0 + \frac{1}{2}(5-b^2)b^k\rho_k\beta\rho_0 + \frac{1}{2}(1-b^2)\rho_0^2]y_i + 2\beta^2\rho_{i|0} \\
& + [2\rho_k s^k{}_0\beta + 2d\beta^2 + \frac{3}{2}c\beta\rho_0 + 2b^k\rho_k\beta\rho_0 + \frac{1}{2}(1-b^2)\rho_0^2]\beta b_i \\
& + [2b^k\rho_k\beta + 2(1-b^2)\rho_0]\beta s_{i0} + 2\beta^3\rho_k s^k{}_i + 2\beta^2\rho_0 s_i \\
& + [2s_0 + \frac{1}{2}c\beta - \frac{3}{2}(1-b^2)\rho_0]\beta^2\rho_i, && \text{(A17)}\\
\Gamma^2_5 =& -2b^2dy_i + \beta b^k\rho_{k|i} + 2b^2\rho_{i|0} + [-2(1-2b^2)\rho_k s^k{}_0 - \frac{1}{2}cb^k\rho_k\beta - (1-3b^2)d\beta \\
& - \rho_k s^k\beta + \frac{1}{4}(1-b^2)|\nabla\rho|^2_\alpha\beta - (1-3b^2)b^k\rho_k\rho_0 - \frac{3}{2}c(1-2b^2)\rho_0]b_i \\
& + 2b^k\rho_k s_{i0} + (1+2b^2)\beta\rho_k s^k{}_i - [b^k\rho_k\beta + 2(1-2b^2)\rho_0]s_i \\
& + [2b^2s_0 + \frac{1}{2}cb^2\beta - \frac{1}{4}(5-b^2)b^k\rho_k\beta - \frac{1}{2}b^2(5-3b^2)\rho_0]\rho_i, && \text{(A18)}\\
\Gamma^0_5 =& [-b^k\rho_{k|0}\beta + (1-2b^2)\rho_k s^k{}_0\beta + b^k\rho_k s_0\beta + 2(1-b^2)s_0\rho_0 + \frac{1}{2}cb^k\rho_k\beta^2 \\
& + d(1-b^2)\beta^2 + \rho_k s^k\beta^2 - \frac{1}{4}(1-b^2)|\nabla\rho|^2_\alpha\beta^2 + \frac{3}{2}c(1-b^2)\beta\rho_0 \\
& + \frac{1}{4}(9-5b^2)b^k\rho_k\beta\rho_0 + \frac{1}{2}b^2(1-b^2)\rho_0^2]y_i, && \text{(A19)}\\
\Gamma^4_6 =& -2b^2dy_i + 2b^2\rho_{i|0} + [-2d(1-2b^2)\beta - 2(1-2b^2)\rho_k s^k{}_0 - \frac{1}{2}(1-b^2)|\nabla\rho|^2_\alpha\beta \\
& - \frac{3}{2}c(1-2b^2)\rho_0 - (1-3b^2)b^k\rho_k\rho_0]b_i \\
& + 2b^k\rho_k s_{i0} + 2b^2\beta\rho_k s^k{}_i - 2(1-2b^2)\rho_0 s_i \\
& + [2b^2s_0 - c(\frac{1}{2}-b^2)\beta - (1-b^2)b^k\rho_k\beta - \frac{1}{2}b^2(5-3b^2)\rho_0]\rho_i, && \text{(A20)}\\
\Gamma^2_6 =& -2\beta^2\rho_{i|0} + [2(1-b^2)\rho_k s^k{}_0\beta + 2(1-b^2)s_0\rho_0 + 2(2-b^2)d\beta^2 \\
& + \frac{1}{2}(1-b^2)|\nabla\rho|^2_\alpha\beta^2 + 2c(1-b^2)\beta\rho_0 + 2(1-b^2)b^k\rho_k\beta\rho_0 + \frac{1}{2}b^2(1-b^2)\rho_0^2]y_i \\
& + [-2\rho_k s^k{}_0\beta - 2d\beta^2 - \frac{3}{2}c\beta\rho_0 - 2b^k\rho_k\beta\rho_0 - \frac{1}{2}(1-b^2)\rho_0^2]\beta b_i \\
& - 2[b^k\rho_k\beta + (1-b^2)\rho_0]\beta s_{i0} - 2\beta^3\rho_k s^k_i - 2\beta^2\rho_0 s_i \\
& + [-2s_0 - \frac{1}{2}c\beta + \frac{1}{2}(5-3b^2)\rho_0]\beta^2\rho_i, && \text{(A21)}\\
\Gamma^2_7 =& 2b^2b^k\rho_{k|0} - 2b^2(1-2b^2)\rho_k s^k{}_0 + 2(1+b^2)b^k\rho_k s_0 - c(\frac{1}{2}-b^2)b^k\rho_k\beta \\
& - 4db^2(1-b^2)\beta - 2b^2\rho_k s^k\beta - (1-b^2)(b^k\rho_k)^2\beta - \frac{1}{2}b^2(1-b^2)|\nabla\rho|^2_\alpha\beta \\
& - \frac{3}{2}cb^2(1-2b^2)\rho_0 - \frac{1}{2}b^2(7-9b^2)b^k\rho_k\rho_0, && \text{(A22)}
\end{aligned}$$

$$\Gamma_7^0 = -2b^k\rho_{k|0} + 2(1-2b^2)\rho_k s^k{}_0 - 4b^k\rho_k s_0 - \frac{1}{2}cb^k\rho_k\beta + 4d(1-b^2)\beta + 2\rho_k s^k\beta$$
$$+ \frac{1}{2}(1-b^2)|\nabla\rho|_\alpha^2\beta + \frac{1}{2}c(4-7b^2)\rho_0 + \frac{1}{2}(9-11b^2)b^k\rho_k\rho_0. \quad \text{(A23)}$$

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