*mathematics*

*Review*

# A Comprehensive Review of the Hermite–Hadamard Inequality Pertaining to Fractional Integral Operators

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**Abstract:** In the frame of fractional calculus, the term convexity is primarily utilized to address several challenges in both pure and applied research. The main focus and objective of this review paper is to present Hermite–Hadamard (H-H)-type inequalities involving a variety of classes of convexities pertaining to fractional integral operators. Included in the various classes of convexities are classical convex functions, $m$-convex functions, $r$-convex functions, $(\alpha, m)$-convex functions, $(\alpha, m)$-geometrically convex functions, harmonically convex functions, harmonically symmetric functions, harmonically $(\theta, m)$-convex functions, $m$-harmonic harmonically convex functions, $(s, r)$-convex functions, arithmetic–geometric convex functions, logarithmically convex functions, $(\alpha, m)$-logarithmically convex functions, geometric–arithmetically $s$-convex functions, $s$-convex functions, Godunova–Levin-convex functions, differentiable $\phi$-convex functions, $MT$-convex functions, $(s, m)$-convex functions, $p$-convex functions, $h$-convex functions, $\sigma$-convex functions, exponential-convex functions, exponential-type convex functions, refined exponential-type convex functions, $n$-polynomial convex functions, $\sigma, s$-convex functions, modified $(p, h)$-convex functions, co-ordinated-convex functions, relative-convex functions, quasi-convex functions, $(\alpha, h-m)-p$-convex functions, and preinvex functions. Included in the fractional integral operators are Riemann–Liouville (R-L) fractional integral, Katugampola fractional integral, $k$-R-L fractional integral, $(k, s)$-R-L fractional integral, Caputo-Fabrizio (C-F) fractional integral, R-L fractional integrals of a function with respect to another function, Hadamard fractional integral, and Raina fractional integral operator.

**Keywords:** Hermite–Hadamard inequality; convex function; Riemann–Liouville fractional integral; Katugampola fractional integral; Hadamard fractional integral

**MSC:** 26A33; 26A51; 26D07; 26D10; 26D15

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## 1. Introduction

The term convex functions has been intensively propagated in the renowned book *Inequalities*, which was written by Hardy, Littlewood, and Polya [1]. During the last century, the notion of convexity and its generalizations have emerged as an interesting field of pure and applied mathematics. Several mathematicians have offered their insight into this field, presenting new versions of different types of inequalities with convex sets and convex functions. The term convexity, with help from the concept of optimization, has a magnificent impact on many fields of applied sciences including finance [2], estimation and signal processing [3], control systems [4], engineering [5], data analysis and computer science [6], statistics [7], and mathematical optimization for modeling [8,9]. The combined study of convex analysis and integral inequalities presents a captivating and engrossing field of research within the area of mathematical interpretation. Because of their widespread perspectives and applications, the tactics and literature of convex analysis and integral inequalities have recently become the subject of intensive research in both contemporary and historical times.

Among all the inequalities, the most extensively used are the Hermite–Hadamard-type and Fejér-type inequalities. These inequalities involving convex functions play a consequential and fundamental role in applied mathematics. Thus, convex analysis and inequalities have been referred to as an absorbing field for mathematicians due to their wide applications in numerous branches of science. Integral inequalities have effective applications in information technology, probability, optimization theory, stochastic processes, statistics, integral operator theory, and numerical integration.

In the current decade, many mathematicians have been merging new ideas with fractional analysis to bring new dimensions with different features to the field of mathematical analysis. Fractional analysis has numerous applications in modeling [10], epidemiology [11], fluid flow [12], nanotechnology [13], mathematical biology [14], control system [15], and transform theory [16]. Due to these widespread views and their applications, fractional analysis has become an attractive field for scholars. Thus, the concept of fractional integral inequalities has many applications in applied sciences.

Mathematical inequalities play important roles in the study of mathematics as well as in other areas of mathematics because of their wide applications in mathematics and physics. One of the most significant functions used to study many interesting inequalities is the convex function, which is defined as follows:

Let $\mathbb{I} \subset \mathbb{R}$ be a non-empty interval. The function $\Pi : \mathbb{I} \to \mathbb{R}$ is called convex if

$$\Pi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\Pi(x_1) + (1-\lambda)\Pi(x_2)$$

holds for every $x_1, x_2 \in \mathbb{I}$ and $\lambda \in [0,1]$.

Currently, many researchers have been fascinated by the field of convex functions and, particularly, one of the well-known inequalities for convex functions known as the H-H inequality, which is defined as follows:

$$\Pi\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\Pi(x)dx \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}. \tag{1}$$

Inequality (1) was introduced by C. Hermite [17] and investigated by J. Hadamard [18] in 1893.

The following result is connected with the right-hand part of the inequality (1).

**Theorem 1** ([19]). *Assume that $x_1, x_2 \in \mathbb{I}^\circ$, the interior of an interval $\mathbb{I}$, with $x_1 < x_2$ and $\Pi : \mathbb{I}^\circ \subset \mathbb{R}$ is a differentiable function on $\mathbb{I}^\circ$. If $|\Pi'|$ is a convex function on $\mathbb{I}^\circ$, then*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\Pi(x)dx\right| \leq \frac{(x_2-x_1)}{8}(|\Pi'(x_1)| + |\Pi'(x_2)|). \tag{2}$$

In view of the increasing interest in fractional integral operators and the applications of H-H-type inequalities, in recent years many papers have been devoted to the generalizations of H-H inequality involving fractional integrals. A thorough literature review dealing with fractional H-H inequalities can be found in the book [20] and the paper [21]. This book [20] is devoted to H-H-type inequalities involving R-L and fractional integrals of Hadamard type by using different concepts of convexities. In [21] are presented H-H-type inequalities of $\alpha$-type real-valued convex functions, along with various classes of convexity through differentiable mapping of fractional integrals.

Our objective in this paper is to present a comprehensive and up-to-date review on H-H-type inequalities for different kinds of convexities and different kinds of fractional integral operators. In each section and subsection, we first introduce the basic definitions of different kinds of convexities and fractional integral operators and then include the results on H-H-type inequalities. We believe that the collection of almost all existing in the literature H-H-type inequalities in one file will help new researchers in the field learn about the available work on the topic before developing new results. We present the results without proof but instead provide a complete reference for the details of each result elaborated in this survey for the convenience of the reader.

The construction of this review paper is as follows. In Section 2, we introduce the reader to the basic concepts of R-L fractional integrals. In Sections 2.2–2.33, we summarize fractional H-H-type inequalities for various classes of convexities, including classical convex functions, $m$-convex functions, $r$-convex functions, $(\alpha, m)$-convex functions, $(\alpha, m)$-geometrically convex unctions, harmonically convex functions, harmonically symmetric functions, harmonically $(\theta, m)$-convex functions, $m$-harmonic harmonically convex functions, $(s, r)$-convex functions, arithmetic–geometric convex functions, logarithmically convex functions, $(\alpha, m)$-logarithmically convex functions, geometric–arithmetically $s$-convex functions, $s$-convex functions, Godunova–Levin-convex functions, differentiable $\phi$-convex functions, $MT$-convex functions, $(s, m)$-convex functions, $p$-convex functions, $h$-convex functions, $\sigma$-convex functions, exponential-convex functions, exponential-type convex functions, refined exponential-type convex functions, $n$-polynomial convex functions, $\sigma, s$-convex functions, modified $(p, h)$-convex functions, co-ordinated-convex functions, relative-convex functions, quasi-convex functions, $(\alpha, h - m) - p$-convex functions and preinvex functions. In Section 3, H-H-type inequalities via Katugampola fractional integral are discussed, while, in Section 4, H-H-type inequalities via $k$-R-L fractional integral are included. In Section 5, H-H-type inequalities via $(k, s)$-R-L fractional integral are given, in Section 6, H-H-type inequalities via C-F fractional integral are included, while in Section 7, H-H–Mercer-type inequalities via R-L fractional integrals are presented. H-H-type inequalities via R-L fractional integrals of a function with respect to another function are given in Section 8, fractional H-H inequalities via weighted symmetric function in Section 9, H-H-type inequalities via Hadamard fractional integral in Section 10, and, finally, H-H-type inequalities via proportional fractional integrals and H-H-type inequalities via Raina integral operator are in Sections 11 and 12, respectively.

Note that the main motivation of this review paper is to clarify the state of knowledge, explain apparent contradictions, identify needed research, and even create a consensus where none previously existed. The purpose of this review paper is to succinctly review recent progress in a particular topic, namely convex analysis in the frame of fractional calculus. Overall, this review paper summarizes the current state of knowledge on the topic of convexity. It creates an understanding of the topic for the reader by discussing the findings presented in recent research papers. Our goal here is a more complete and comprehensive review and, as such, the choice is made to include as many results as possible to illustrate the progress on the matter. Any proofs (which are rather long) are omitted; for this matter, the reader is accordingly referred to the relevant article.

## 2. H-H-Type Inequalities via R-L Fractional Integrals

We start by giving the definitions of left and right R-L fractional integrals.

**Definition 1** ([22])**.** *Assume that $\Pi \in L_1[x_1, x_2]$. Then, the left and right R-L integrals $J^{\alpha}_{x_1+}\Pi$ and $J^{\alpha}_{x_2-}\Pi$, $\alpha > 0$, $x_1 \geq 0$ are stated by*

$$J^{\alpha}_{x_1+}\Pi(x) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} (x - t)^{\alpha - 1}\Pi(t)dt, \ \ x > x_1,$$

*and*

$$J^{\alpha}_{x_2-}\Pi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{x_2} (t - x)^{\alpha - 1}\Pi(t)dt, \ \ x < x_2,$$

*respectively. Here $\Gamma(\alpha)$ represent the Euler Gamma function and $J^{0}_{x_1+}\Pi(x) = J^{0}_{x_2-}\Pi(x) = \Pi(x)$.*

### *2.1. Fractional H-H-Type Inequalities for Convex Functions*

H-H inequalities were represented in a fractional integral in [23].

**Theorem 2** ([23]). *Assume that $0 \leq x_1 < x_2$, $\Pi : \mathbb{I} = [x_1, x_2] \to \mathbb{R}$ is a positive function and $\Pi \in L_1[x_1, x_2]$. If $\Pi$ is a convex function on $\mathbb{I}$, then*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}, \quad \alpha > 0.$$

**Theorem 3** ([23]). *Assume that $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $\Pi : \mathbb{I} = [x_1, x_2] \to \mathbb{R}$ is a differentiable mapping on $(x_1, x_2)$, such that $\Pi' \in L_1[x_1, x_2]$. If $|\Pi'|$ is a convex function on $\mathbb{I}$, then*

$$\begin{aligned} &\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^2}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\Big| \\ \leq\; & \frac{x_2-x_1}{2(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)[|\Pi'(x_1)| + |\Pi'(x_2)|]. \end{aligned}$$

New fractional H-H integral inequalities related of right-hand inequalities for some convex functions are given in the next theorems.

**Theorem 4** ([24]). *Assume that $\Pi$ is as in Theorem 3. If $|\Pi'|^q$ is a convex function on $\mathbb{I}$, and $p > 1$, then*

$$\begin{aligned} &\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\Big| \\ \leq\; & \frac{x_2-x_1}{2(\alpha p+1)^{\frac{1}{p}}}\Big(\frac{|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{2}\Big)^{\frac{1}{q}}, \end{aligned}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 5** ([24]). *Assume that $\Pi$ is as in Theorem 3. If $|\Pi'|^q$ is a convex function on $\mathbb{I}$, and $q > 1$, then*

$$\begin{aligned} &\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\Big| \\ \leq\; & \frac{x_2-x_1}{\alpha+1}\Big(1-\frac{1}{2^\alpha}\Big)\Big(\frac{|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{2}\Big)^{\frac{1}{q}}. \end{aligned}$$

The H-H inequalities for left and right R-L fractional integrals are given in the next theorems.

**Theorem 6** ([25]). *Assume that $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $\Pi : [x_1, x_2] \to \mathbb{R}$ is a convex function. If $\Pi \in L_1[x_1, x_2]$, then R-L fractional integral inequalities are given as*

$$\Pi\Big(\frac{\alpha x_1+x_2}{\alpha+1}\Big) \leq \frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}J^\alpha_{x_1+}\Pi(x_2) \leq \frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1}, \quad \alpha > 0.$$

**Theorem 7** ([26]). *Assume that $\Pi$ is as in Theorem 6. Then R-L fractional integral inequalities are given as*

$$\Pi\Big(\frac{x_1+\alpha x_2}{\alpha+1}\Big) \leq \frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}J^\alpha_{x_2-}\Pi(x_1) \leq \frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1}, \quad \alpha > 0.$$

New refinement of fractional H-H inequalities are included in the next theorems.

**Theorem 8** ([27]). *Assume that $x_1 < x_2$ and $\Pi : \mathbb{I} = [x_1, x_2] \to \mathbb{R}$ is a positive function such that $\Pi \in L_1[x_1, x_2]$. If $\Pi$ is a convex function on $\mathbb{I}$, then $WH$ is a convex function and monotonically increasing on $[0,1]$ and*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big) = WH(0) \le WH(t) \le WH(1) = \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{x_1+}\Pi(x_2) + J^{\alpha}_{x_2-}\Pi(x_1)\Big], \ \alpha > 0,$$

*where*

$$WH(t) = \frac{\alpha}{2(x_2-x_1)^{\alpha}}\int_{x_1}^{x_2} \Pi\Big(tx + (1-t)\frac{x_1+x_2}{2}\Big)\big((x_2-x)^{\alpha-1} + (x-x_1)^{\alpha-1}\big)dx.$$

**Theorem 9** ([27]). *Assume that $\Pi$ is as in Theorem 8. If $\Pi$ is a convex function on $\mathbb{I}$, then $WP$ is monotonically non-decreasing and convex on $[0,1]$ and*

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{x_1+}\Pi(x_2) + J^{\alpha}_{x_2-}\Pi(x_1)\Big] = WP(0) &\le WP(t) \\ &\le WP(1) = \frac{\Pi(x_1)+\Pi(x_2)}{2}, \ \alpha > 0,\end{aligned}$$

*where*

$$\begin{aligned} WP(t) = {} & \frac{\alpha}{4(x_2-x_1)^{\alpha}}\int_{x_1}^{x_2}\Big[\Pi\Big(\Big(\frac{1+t}{2}\Big)x_1 + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{2x_2-x_1-x}{2}\Big)^{\alpha-1} \\ & + \Big(\frac{x-x_1}{2}\Big)^{\alpha-1}\Big) + \Pi\Big(\Big(\frac{1+t}{2}\Big)x_2 + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{x+x_2-2x_1}{2}\Big)^{\alpha-1} \\ & + \Big(\frac{x_2-x}{2}\Big)^{\alpha-1}\Big)\Big]dx. \end{aligned}$$

New weighted versions of H-H inequality for R-L fractional integrals are established in the following theorems.

**Theorem 10** ([28]). *Assume that $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $\Pi : [x_1, x_2] \to \mathbb{R}$ is a convex function such that $\Pi \in L_1[x_1, x_2]$. In addition, we suppose that $w : [x_1, x_2] \to \mathbb{R}$ is continuous and $\frac{1}{2}[w(s) + w(x_1 + x_2 - s)] = w\Big(\frac{x_1+x_2}{2}\Big)$, i.e., symmetric about the point $\Big(\frac{x_1+x_2}{2}, w\Big(\frac{x_1+x_2}{2}\Big)\Big)$. Then*

$$\Pi\Big(w\Big(\frac{x_1+x_2}{2}\Big)\Big) \le \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{x_1+}\Pi(w(x_2)) + J^{\alpha}_{x_2-}\Pi(w(x_1))\Big],$$

*and*

$$\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{x_1+}\Pi(w(x_2)) + J^{\alpha}_{x_2-}\Pi(w(x_1))\Big] \le \frac{\Pi(w(x_1)) + \Pi(w(x_2))}{2}, \ \alpha > 0,$$

*if $w$ is monotonic on $[x_1, x_2]$.*

**Theorem 11** ([28]). *Assume that $\Pi$ is as in Theorem 10. Then $WH_{\omega}$ is a convex function and monotonically increasing on $[0,1]$ and*

$$\begin{aligned}\Pi\Big(w\Big(\frac{x_1+x_2}{2}\Big)\Big) &= WH_{\omega}(0) \le WH_{\omega}(t) \le WH_{\omega}(1) \\ &\le \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{x_1+}\Pi(w(x_2)) + J^{\alpha}_{x_2-}\Pi(w(x_1))\Big], \ \alpha > 0,\end{aligned}$$

*where*

$$WH_{\omega}(t) = \frac{\alpha}{2(x_2 - x_1)^{\alpha}} \int_{x_1}^{x_2} \Pi\Big(tw(t) + (1-t)w\Big(\frac{x_1+x_2}{2}\Big)\Big)\Big((x_2 - x)^{\alpha-1} + (x - x_1)^{\alpha-1}\Big)dx.$$

**Theorem 12** ([29]). *Assume that $x_1, x_2 \in \mathbb{R}$ with $0 \leq x_1 < x_2$ and $\Pi : \mathbb{I} = [x_1, x_2] \to \mathbb{R}$ is a positive function such that $\Pi \in L_1[x_1, x_2]$. If $\Pi$ is a convex function on $\mathbb{I}$, then*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2) + J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\Big] \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}, \quad \alpha > 0.$$

**Theorem 13** ([29]). *Assume that $\Pi$ is as in Theorem 12. If $|\Pi'|^q$ is convex on $\mathbb{I}$ for $q \geq 1$, then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2) + J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\Big] - \Pi\Big(\frac{x_1+x_2}{2}\Big)\right| \\
\leq\ & \frac{x_2-x_1}{4(\alpha+1)}\Big(\frac{1}{2(\alpha+2)}\Big)^{\frac{1}{q}}\Big\{\Big((\alpha+1)|\Pi'(x_1)|^q + (\alpha+3)|\Pi'(x_2)|^q\Big)^{\frac{1}{q}} \\
&+\Big((\alpha+3)|\Pi'(x_1)|^q + (\alpha+1)|\Pi'(x_2)|^q\Big)^{\frac{1}{q}}\Big\}.
\end{aligned}$$

**Theorem 14** ([30]). *Assume $|\Pi'|$ is a convex function on $[x_1, x_2]$. Then*

$$\begin{aligned}
&\frac{[(x-x_1)^{\alpha} + (x_2-x)^{\alpha}]\Pi(x)}{x_2-x_1} + \frac{(x-x_1)^{\alpha}\Pi(x_1) + (x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1} \\
&-\frac{2^{\alpha}\Gamma(\alpha+1)}{x_2-x_1}\Big[J^{\alpha}_{x-}\Pi\Big(\frac{x+x_1}{2}\Big) + J^{\alpha}_{x_1+}\Pi\Big(\frac{x+x_1}{2}\Big) + J^{\alpha}_{x_2-}\Pi\Big(\frac{x+x_2}{2}\Big) + J^{\alpha}_{x+}\Pi\Big(\frac{x+x_2}{2}\Big) \\
\leq\ & \frac{(x-x_1)^{\alpha+1}}{x_2-x_1}\frac{|\Pi'(x)| + |\Pi'(x_1)|}{2(\alpha+1)} + \frac{(x_2-x)^{\alpha+1}}{x_2-x_1}\frac{|\Pi'(x)| + |\Pi'(x_2)|}{2(\alpha+1)}.
\end{aligned}$$

**Theorem 15** ([30]). *Assume $|\Pi'|^q$ is a convex function on $[x_1, x_2]$ for some fixed $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}
&\frac{[(x-x_1)^{\alpha} + (x_2-x)^{\alpha}]\Pi(x)}{x_2-x_1} + \frac{(x-x_1)^{\alpha}\Pi(x_1) + (x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1} \\
&-\frac{2^{\alpha}\Gamma(\alpha+1)}{x_2-x_1}\Big[J^{\alpha}_{x-}\Pi\Big(\frac{x+x_1}{2}\Big) + J^{\alpha}_{x_1+}\Pi\Big(\frac{x+x_1}{2}\Big) + J^{\alpha}_{x_2-}\Pi\Big(\frac{x+x_2}{2}\Big) + J^{\alpha}_{x+}\Pi\Big(\frac{x+x_2}{2}\Big)\Big] \\
\leq\ & \Big(\frac{1}{2}\Big)^{1+\frac{2}{q}}\Big(\frac{1}{\alpha p+1}\Big)^{\frac{1}{p}}\Big\{\frac{(x-x_1)^{\alpha+1}}{x_2-x_1}\Big[\big(3|\Pi'(x)|^q + |\Pi'(x_1)|^q\big)^{\frac{1}{q}} + \big(|\Pi'(x)|^q + 3|\Pi'(x_1)|^q\big)^{\frac{1}{q}}\Big] \\
&+\frac{(x_2-x)^{\alpha+1}}{x_2-x_1}\Big[\big(3|\Pi'(x)|^q + |\Pi'(x_1)|^q\big)^{\frac{1}{q}} + \big(|\Pi'(x)|^q + 3|\Pi'(x_1)|^q\big)^{\frac{1}{q}}\Big]\Big\}.
\end{aligned}$$

H-H-type inequalities for fractional integrals depending on a parameter are now given.

**Theorem 16** ([31]). *Assume that $0 \leq x_1 < x_2$ and $\Pi : \mathbb{I} = [x_1, x_2] \to \mathbb{R}$ is a function differentiable on $(x_1, x_2)$. If $|\Pi'|^q$, $q \geq 1$ is a convex function on $\mathbb{I}$, then we have the fractional integral inequality:*

$$\left|\frac{\Pi(\lambda x_1 + (1-\lambda)x_2) + \Pi(\lambda x_2 + (1-\lambda)x_1)}{(1-2\lambda)(x_2-x_1)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(x_2-x_1)^{\alpha+1}}\right.$$

$$\times\Big[J^{\alpha}_{(\lambda x_2+(1-\lambda)x_1)+}\Pi(\lambda x_1+(1-\lambda)x_2)+J^{\alpha}_{(\lambda x_1+(1-\lambda)x_2)+}\Pi(\lambda x_2+(1-\lambda)x_1)\Big]\Bigg|$$
$$\leq \frac{2}{\alpha+1}\Big(1-\frac{1}{2^{\alpha}}\Big)\Big[\frac{|\Pi'(\lambda x_1+(1-\lambda)x_2)|^q+|\Pi'(\lambda x_2+(1-\lambda)x_1)|^q}{2}\Big]^{\frac{1}{q}},$$

*where* $\lambda\in[0,1]\setminus\{1/2\}$ *and* $\alpha>0$.

**Theorem 17** ([31]). *Assume that* $\Pi$ *is as in Theorem 16. If* $|\Pi'|^q$, $q>1$ *is a convex function on* $\mathbb{I}$, *then fractional integral inequality is given as:*

$$\Bigg|\frac{\Pi(\lambda x_1+(1-\lambda)x_2)+\Pi(\lambda x_2+(1-\lambda)x_1)}{(1-2\lambda)(x_2-x_1)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(x_2-x_1)^{\alpha+1}}$$
$$\times\Big[J^{\alpha}_{(\lambda x_2+(1-\lambda)x_1)+}\Pi(\lambda x_1+(1-\lambda)x_2)+J^{\alpha}_{(\lambda x_1+(1-\lambda)x_2)+}\Pi(\lambda x_2+(1-\lambda)x_1)\Big]\Bigg|$$
$$\leq \Big[\frac{2}{p\alpha+1}\Big(1-\frac{1}{2^{p\alpha}}\Big)\Big]^{\frac{1}{p}}\Big[\frac{|\Pi'(\lambda x_1+(1-\lambda)x_2)|^q+|\Pi'(\lambda x_2+(1-\lambda)x_1)|^q}{2}\Big]^{\frac{1}{q}},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$, $\lambda\in[0,1]\setminus\{1/2\}$ *and* $\alpha>0$.

New fractional H-H-type inequalities are obtained in the following theorems.

**Theorem 18** ([32]). *Assume that* $x_1,x_2\in\mathbb{R}$ *with* $x_1<x_2$ *and* $\Pi:\mathbb{I}=[x_1,x_2]\to\mathbb{R}$ *is a differentiable function on* $(x_1,x_2)$. *If* $|\Pi'|^q$ *is a convex function on* $\mathbb{I}$ *for some fixed* $q\geq 1$, *then fractional integral inequality is given as:*

$$\Bigg|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\Big]\Bigg|$$
$$\leq \frac{x_2-x_1}{2^{\alpha+1+\frac{1}{q}}}\Big(\frac{2^{\alpha}-1}{\alpha+1}\Big)^{1-\frac{1}{q}}\Big\{\Big[\frac{2^{\alpha+1}-(\alpha+2)}{(\alpha+1)(\alpha+2)}|\Pi'(x_1)|^q+\frac{2^{\alpha+1}(\alpha+1)-(\alpha+2)}{(\alpha+1)(\alpha+2)}|\Pi'(x_2)|^q\Big]^{\frac{1}{q}}$$
$$+\Big[\frac{2^{\alpha+1}(\alpha+1)-(\alpha+2)}{(\alpha+1)(\alpha+2)}|\Pi'(x_1)|^q+\frac{2^{\alpha+1}-(\alpha+2)}{(\alpha+1)(\alpha+2)}|\Pi'(x_1)|^q\Big]^{\frac{1}{q}}\Big\}.$$

**Theorem 19** ([32]). *Assume that* $\Pi$ *is as in Theorem 18. If* $|\Pi'|^q$, $q>1$ *is a convex function on* $\mathbb{I}$, *then fractional integral inequality is given as:*

$$\Bigg|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\Big]\Bigg|$$
$$\leq \frac{x_2-x_1}{2^{\alpha+2-\frac{1}{q}}}\Big(\frac{2^{\alpha p}-1}{\alpha p+1}\Big)^{\frac{1}{p}}\Big\{\Big[\frac{3|\Pi'(x_1)|^q+|\Pi'(x_2)|^q}{4}\Big]^{\frac{1}{q}}+\Big[\frac{|\Pi'(x_1)|^q+3|\Pi'(x_2)|^q}{4}\Big]^{\frac{1}{q}}\Big\},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**Theorem 20** ([33]). *Assume that* $\Pi$ *is as in Theorem 18. If* $|\Pi'|^q$, $q\geq 1$ *is a convex function on* $\mathbb{I}$, $\frac{1}{p}+\frac{1}{q}=1$, *then fractional integral inequality for* $0<\alpha\leq 1$ *is given as:*

$$\Bigg|\Pi\Big(\frac{x_1+x_2}{2}\Big)-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\Big]\Bigg|$$

$$\leq \frac{x_2 - x_1}{2^{\alpha+1}(\alpha p+1)^{\frac{1}{p}}}\left[\left(\frac{3|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\Pi'(x_1)|^q + 3|\Pi'(x_2)|^q}{4}\right)^{\frac{1}{q}}\right].$$

**Theorem 21** ([33]). *Assume that $\Pi$ is as in Theorem 18. If $|\Pi'|$ is a convex function on $\mathbb{I}$, then for $0 < \alpha \leq 1$ we have*

$$\begin{aligned} &\Pi\left(\frac{x_1+x_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right] \\ \leq\ & \frac{x_2-x_1}{2^{\alpha+1}(\alpha+1)}\Big(|\Pi'(x_1)| + |\Pi'(x_2)|\Big). \end{aligned}$$

**Theorem 22** ([33]). *Assume that $\Pi$ is as in Theorem 18. If $|\Pi'|^q$, $q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, is a convex function on $\mathbb{I}$, then fractional integral inequality for $\alpha > 0$ is given as:*

$$\begin{aligned} &\left|\Pi\left(\frac{x_1+x_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right]\right| \\ \leq\ & \frac{x_2-x_1}{2^{\alpha+1}(\alpha+1)}\left[\left(\frac{(\alpha+1)|\Pi'(x_2)|^q + (\alpha+3)|\Pi'(x_1)|^q}{2(\alpha+2)}\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\frac{(\alpha+1)|\Pi'(x_1)|^q + (\alpha+3)|\Pi'(x_2)|^q}{2(\alpha+2)}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

**Theorem 23** ([33]). *Assume that $\Pi$ is as in Theorem 18. If $|\Pi'|^q$, $q \geq 1$ is a convex function on $\mathbb{I}$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then fractional integral inequality for $\alpha > 0$ is given as:*

$$\begin{aligned} &\left|\Pi\left(\frac{x_1+x_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right]\right| \\ \leq\ & \frac{x_2-x_1}{2^{\alpha+1}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|\Pi'\left(\frac{x_1+3x_2}{4}\right)\right| + \left|\Pi'\left(\frac{3x_1+x_2}{4}\right)\right|\right]. \end{aligned}$$

**Theorem 24** ([34]). *Assume that $\Pi$ is as in Theorem 18. If $|\Pi'|^q$, $q > 1$ is a convex function on $\mathbb{I}$, then for all $x \in [x_1, x_2]$ fractional integral inequality is given as:*

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{(x_2-x_1)}\left[(x-x_1)^{1-\alpha}J^\alpha_{x_2-}\Pi(x_1+x_2-x) + (x_2-x)^{1-\alpha}J^\alpha_{x_1+}\Pi(x_1+x_2-x)\right]\right. \\ &\left. -\Pi(x_1+x_2-x)\right| \\ \leq\ & \frac{1}{x_2-x_1}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-x_1)^2\left(\frac{|\Pi'(x_2)|^q + |\Pi'(x_1+x_2-x)|^q}{2}\right)^{\frac{1}{q}}\right. \\ & + \left[(x_2-x)^2\left(\frac{|\Pi'(x_1)|^q + |\Pi'(x_1+x_2-x)|^q}{2}\right)^{\frac{1}{q}}\right], \end{aligned}$$

*where* $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

In the next we present some weighted fractional inequalities for differentiable mappings whose derivatives in absolute value are convex.

**Theorem 25** ([35]). *Assume that $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $\Pi : \mathbb{I} \to \mathbb{R}$ is a differentiable function on $\mathbb{I}^\circ$ such that $\Pi' \in L_1[x_1, x_2]$ and $g : [x_1, x_2] \to \mathbb{R}$ is continuous. If $|\Pi'|$ is a convex function on $\mathbb{I}$, then fractional integral inequality is given as:*

$$\begin{aligned} &\left|\Pi\left(\frac{x_1+x_2}{2}\right)\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}g(x_1)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}g(x_2)\right]\right.\\ &\left.-\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}(\Pi g)(x_1)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}(\Pi g)(x_2)\right]\right|\\ \leq\ &\frac{(x_2-x_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)}[|\Pi'(x_1)|+|\Pi'(x_2)|], \end{aligned}$$

*where $\|g\|_\infty = \sup\{|g(t)|, t \in [x_1, x_2]\}$.*

**Theorem 26** ([35]). *Assume that $\Pi$ is as in Theorem 25 and $g : [x_1, x_2] \to \mathbb{R}$ is continuous. If $|\Pi'|^q$, $q > 1$ is a convex function on $\mathbb{I}$, then fractional integral inequality is given as:*

$$\begin{aligned} &\left|\Pi\left(\frac{x_1+x_2}{2}\right)\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}g(x_1)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}g(x_2)\right]\right.\\ &\left.-\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}(\Pi g)(x_1)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}(\Pi g)(x_2)\right]\right|\\ \leq\ &\frac{(x_2-x_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1+\frac{2}{q}}(\alpha p+1)^{1/p}\Gamma(\alpha+1)}\\ &\times\left[\left(3|\Pi'(x_1)|^q+|\Pi'(x_2)|^q\right)^{\frac{1}{q}}+\left(|\Pi'(x_1)|^q+3|\Pi'(x_2)|^q\right)^{\frac{1}{q}}\right], \end{aligned}$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

In the next theorem, we establish above and below bounds for the left- and right-hand sides of fractional H-H inequalities.

**Theorem 27** ([36]). *Assume that $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $\Pi : [x_1, x_2] \to \mathbb{R}$ is positive, twice differentiable mapping, and $\Pi \in L_1[x_1, x_2]$. If $\Pi''$ is bounded, i.e., $m \leq \Pi''(t) \leq M, t \in [x_1, x_2]$, $m, M \in \mathbb{R}$, then fractional integral inequalities are given as:*

$$\begin{aligned} &\frac{m(x_2-x_1)^2}{4(\alpha+1)(\alpha+2)} \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\right]-\Pi\left(\frac{x_1+x_2}{2}\right)\\ \leq\ &\frac{M(x_2-x_1)^2}{4(\alpha+1)(\alpha+2)} \end{aligned}$$

*and*

$$\begin{aligned} &\frac{m(x_2-x_1)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)} \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}\\ &-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\right]\\ \leq\ &\frac{M(x_2-x_1)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}. \end{aligned}$$

**Theorem 28** ([36]). *Assume that $\Pi$ is as in Theorem 27. If $\Pi'(x_1+x_2-x) \geq \Pi'(x)$ for all $x \in \left[x_1, \frac{x_1+x_2}{2}\right]$, then fractional integral inequalities are given as:*

$$\Pi\left(\frac{x_1+x_2}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2) + J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\right] \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

Now, we present fractional H-H inequalities for functions whose $|\Pi''|$ or $|\Pi''|^q$ is a convex function .

**Theorem 29** ([37]). *Assume that $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $\Pi : \mathbb{I} \subset [0,\infty) \to \mathbb{R}$ is a twice differentiable function on $\mathbb{I}^{\circ}$ of an interval $\mathbb{I}$ such that $\Pi'' \in L_1[x_1, x_2]$. If $|\Pi''|^q$, $q \geq 1$ is a convex function on $[x_1, x_2]$, then fractional integral inequality is given as:*

$$\begin{aligned}
&\Bigg|(1-\lambda)\left\{\frac{(x_2-x)^{\alpha+1}-(x-x_1)^{\alpha+1}}{x_2-x_1}\right\}\Pi'(x) + (1+\alpha-\lambda)\left\{\frac{(x-x_1)^{\alpha}+(x_2-x)^{\alpha}}{x_2-x_1}\right\}\Pi(x) \\
&+\lambda\frac{(x-x_1)^{\alpha}\Pi(x_1)+(x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1} - \frac{\Gamma(\alpha+2)}{x_2-x_1}\left[J^{\alpha}_{x-}\Pi(x_1)+J^{\alpha}_{x+}\Pi(x_2)\right] \\
\leq\ & A_1^{1-\frac{1}{q}}\Bigg[\frac{(x-x_1)^{\alpha+2}}{x_2-x_1}\left\{A_2(\alpha,\lambda)|\Pi''(x)|^q + A_3(\alpha,\lambda)|\Pi''(x_1)|^q\right\}^{\frac{1}{q}} \\
&+\frac{(x_2-x)^{\alpha+2}}{x_2-x_1}\left\{A_2(\alpha,\lambda)|\Pi''(x)|^q + A_3(\alpha,\lambda)|\Pi''(x_2)|^q\right\}^{\frac{1}{q}}\Bigg],
\end{aligned}$$

*for any $x = tx_1 + (1-t)x_2$, $t \in [0,1]$, $\lambda \in [0,1]$ and $\alpha > 0$, where*

$$\begin{aligned}
A_1(\alpha,\lambda) &= \frac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2} - \frac{\lambda}{2}, \\
A_2(\alpha,\lambda) &= \frac{3-(\alpha+3)\lambda+2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}, \\
A_3(\alpha,\lambda) &= \frac{1}{(\alpha+2)(\alpha+3)} - \frac{\lambda}{6} + \left(\frac{\alpha}{\alpha+2}\right)\lambda^{1+\frac{2}{\alpha}}\left(1-\frac{2}{3}\lambda^{\frac{1}{\alpha}}\right).
\end{aligned}$$

**Theorem 30** ([37]). *Assume that $\Pi$ is as in Theorem 29. If $|\Pi''|^q$, $q \geq 1$, $\frac{1}{p}+\frac{1}{q}=1$, on $[x_1, x_2]$ then fractional integral inequality is given as:*

$$\begin{aligned}
&\Bigg|(1-\lambda)\left\{\frac{(x_2-x)^{\alpha+1}-(x-x_1)^{\alpha+1}}{x_2-x_1}\right\}\Pi'(x) + (1+\alpha-\lambda)\left\{\frac{(x-x_1)^{\alpha}+(x_2-x)^{\alpha}}{x_2-x_1}\right\}\Pi(x) \\
&+\lambda\frac{(x-x_1)^{\alpha}\Pi(x_1)+(x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1} - \frac{\Gamma(\alpha+2)}{x_2-x_1}\left[J^{\alpha}_{x-}\Pi(x_1)+J^{\alpha}_{x+}\Pi(x_2)\right] \\
\leq\ & B_1^{\frac{1}{p}}(\alpha,\lambda,p)\Bigg[\frac{(x-x_1)^{\alpha+2}}{x_2-x_1}\left\{\frac{|\Pi''(x)|^q+|\Pi''(x_1)|^q}{2}\right\}^{\frac{1}{q}} \\
&+\frac{(x_2-x)^{\alpha+2}}{x_2-x_1}\left\{\frac{|\Pi''(x)|^q+|\Pi''(x_2)|^q}{2}\right\}^{\frac{1}{q}}\Bigg],
\end{aligned}$$

*for any $x \in [x_1, x_2]$, $\lambda \in [0,1]$, $\alpha > 0$ and*

$$\begin{aligned}
B_1(\alpha,\lambda,p) = &\frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\left\{\Gamma(1+p)\Gamma\left(\frac{1+p+\alpha}{\alpha}\right){}_2F_1\left(1,1+p,2+p+\frac{1+p}{\alpha},1\right)\right. \\
&\left.+B\left(1+p,-\frac{1+p+\alpha p}{\alpha}\right) - B\left(\lambda,1+p,-\frac{1+p+\alpha p}{\alpha}\right)\right\},
\end{aligned}$$

*where* $B(.,.)$ *is Euler Beta function* $B(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}dt,\ x_1, x_2 > 0$ *and* $B(.,.,.)$ *is the incomplete Beta function* $B(x, x_1, x_2) = \int_0^x t^{x_1-1}(1-t)^{x_2-1}dt,\ x \in [0,1], x_1, x_2 > 0.$

**Theorem 31** ([38])**.** *Assume that* $x_1, x_2 \in \mathbb{I}^\circ$ *with* $x_1 < x_2$ *and* $\Pi : \mathbb{I} \to \mathbb{R}$ *is a twice differentiable function on* $\mathbb{I}^\circ$*. If* $\Pi'' \in L_1[x_1, x_2]$ *and* $|\Pi''|$ *is a convex function, then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2) + J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\right] - (\alpha+1)\Pi\left(\frac{x_1+x_2}{2}\right)\right| \\ \leq\ & \frac{(x_2-x_1)^2}{8(\alpha+2)}\Big(|\Pi''(x_1)| + |\Pi''(x_2)|\Big).\end{aligned}$$

**Theorem 32** ([38])**.** *Assume that* $\Pi$ *is as in Theorem 31. If* $\Pi'' \in L_1[x_1, x_2]$ *and* $|\Pi''|^q$*,* $q > 1$ *is a convex function on* $[x_1, x_2]$*, then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2) + J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\right] - (\alpha+1)\Pi\left(\frac{x_1+x_2}{2}\right)\right| \\ \leq\ & \frac{(x_2-x_1)^2}{2^{3+\frac{2}{q}}}\left(\frac{2}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left[\Big(|\Pi''(x_1)| + 3|\Pi''(x_2)|\Big)^{\frac{1}{q}} + \Big(3|\Pi''(x_1)| + |\Pi''(x_2)|\Big)^{\frac{1}{q}}\right],\end{aligned}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1.$

**Theorem 33** ([38])**.** *Assume that* $\Pi$ *is as in Theorem 31. If* $\Pi'' \in L_1[x_1, x_2]$ *and* $|\Pi''|^q$*,* $q \geq 1$ *is a convex function on* $[x_1, x_2]$*, then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2) + J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\right] - (\alpha+1)\Pi\left(\frac{x_1+x_2}{2}\right)\right| \\ \leq\ & \frac{(x_2-x_1)^2}{8}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2(\alpha+3)}|\Pi''(x_1)|^q + \left(\frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)}\right)|\Pi''(x_2)|^q\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\left(\frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)}\right)|\Pi''(x_2)|^q + \frac{1}{2(\alpha+3)}|\Pi''(x_1)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

**Theorem 34** ([39])**.** *Assume that* $\Pi$ *is as in Theorem 31. If* $|\Pi''|$ *is a convex function on* $[x_1, x_2]$*, then for each* $h \in (0,1)$*,* $\alpha > 0$*, fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(x_2-x_1)^\alpha}\left[J^\alpha_{w+}\Pi(x_2) + J^\alpha_{w-}\Pi(x_1)\right] + 2^{\alpha-1}(x_2-x_1)\Big[(1-h)^{\alpha+1} - h^{\alpha+1}\Big]\Pi'(w)\right. \\ &\left. -2^{\alpha-1}(\alpha+1)\Pi(w)\right| \\ \leq\ & \frac{(x_2-x_1)^2}{2^{1-\alpha}(\alpha+2)(\alpha+3)}\Big[(\alpha+2)\Big\{(1-h)^{\alpha+2} + h^{\alpha+2}\Big\}|\Pi''(w)| \\ &+(1-h)^{\alpha+2}|\Pi''(x_1)| + h^{\alpha+2}|\Pi''(x_2)|\Big],\end{aligned}$$

*where* $w = hx_1 + (1-h)x_2.$

**Theorem 35** ([39]). *Assume that $\Pi$ is as in Theorem 31. If $|\Pi''|^q$, $q > 1$ is a convex function on $[x_1, x_2]$, then for each $h \in (0,1)$, $\alpha > 0$, fractional integral inequalities are given as:*

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(x_2-x_1)^\alpha}\left[J^\alpha_{w+}\Pi(x_2)+J^\alpha_{w-}\Pi(x_1)\right]+2^{\alpha-1}(x_2-x_1)\left[(1-h)^{\alpha+1}-h^{\alpha+1}\right]\Pi'(w)\right.\\
&\left.-2^{\alpha-1}(\alpha+1)\Pi(w)\right|\\
\leq\ &\frac{(x_2-x_1)^2}{2^{1-\alpha}}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left[(1-h)^{\alpha+2}\left(\frac{|\Pi''(w)|^q+|\Pi''(x_1)|^q}{2}\right)^{\frac{1}{q}}\right.\\
&\left.+h^{\alpha+2}\left(\frac{|\Pi''(w)|^q+|\Pi''(x_2)|^q}{2}\right)^{\frac{1}{q}}\right]\\
\leq\ &\frac{(x_2-x_1)^2}{2^{2-\alpha}}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left[(1-h)^{\alpha+2}(|\Pi''(w)+|\Pi''(x_1)|)\right.\\
&\left.+h^{\alpha+2}(|\Pi''(w)+|\Pi''(x_2)|)\right],
\end{aligned}
$$

*where $w = hx_1 + (1-h)x_2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

The next theorems concern fractional H-H-type inequalities for three times differentiable functions.

**Theorem 36** ([40]). *Let $\Pi : \mathbb{I} \to \mathbb{R}$ be three times differentiable mapping on $\mathbb{I}^\circ$. If $\Pi''' \in L_1[x_1, x_2]$ and $|\Pi'''|$ is a convex function, then*

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)\right]\right.\\
&\left.-\frac{(x_2-x_1)^3}{4(\alpha+1)(\alpha+2)}\Pi''\left(\frac{x_1+x_2}{2}\right)-\Pi\left(\frac{x_1+x_2}{2}\right)\right|\\
\leq\ &\frac{(x_2-x_1)^3}{8(\alpha+1)(\alpha+2)(\alpha+3)}\left[|\Pi'''(x_1)|+|\Pi'''(x_2)|\right].
\end{aligned}
$$

**Theorem 37** ([40]). *Assume that $\Pi$ is as in Theorem 36 with the conditions $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p, q \geq 1$. Then*

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)\right]\right.\\
&\left.-\frac{(x_2-x_1)^3}{4(\alpha+1)(\alpha+2)}\Pi''\left(\frac{x_1+x_2}{2}\right)-\Pi\left(\frac{x_1+x_2}{2}\right)\right|\\
\leq\ &\frac{(x_2-x_1)^3}{16(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}\\
&\times\left[\left(\frac{3}{4}|\Pi'''(x_1)|^q+\frac{1}{2}|\Pi'''(x_2)|^q\right)^{\frac{1}{q}}+\left(\frac{1}{2}|\Pi'''(x_1)|^q+\frac{3}{4}|\Pi'''(x_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

**Theorem 38** ([40]). *Assume that $\Pi$ is as in Theorem 36 with the condition $q > 1$. Then*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)+J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)\right]\right.\\
&\left.-\frac{(x_2-x_1)^3}{4(\alpha+1)(\alpha+2)}\Pi''\left(\frac{x_1+x_2}{2}\right)-\Pi\left(\frac{x_1+x_2}{2}\right)\right|\\
\leq\ &\frac{(x_2-x_1)^3(\alpha+3)}{2^{q+4}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\\
&\times\left[\left(\frac{\alpha+5}{\alpha+3}|\Pi'''(x_1)|^q+|\Pi'''(x_2)|^q\right)^{\frac{1}{q}}+\left(|\Pi'''(x_1)|^q+\frac{\alpha+5}{\alpha+3}|\Pi'''(x_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

### 2.2. Fractional H-H-Type Inequalities for *m*-Convex Functions

**Definition 2** ([41]). *A function $\Pi : [0, x_2] \to \mathbb{R}$, $x_2 > 0$, is said to be m-convex, where $m \in [0, 1]$, if*

$$\Pi(tx + m(1-t)y) \leq t\Pi(x) + m(1-t)\Pi(y),$$

*for all $x, y \in [0, x_2]$, $t \in [0, 1]$.*

Fractional H-H inequalities for twice differentiable $m$-convex functions are given in the next theorems.

**Theorem 39** ([42]). *Assume that $\Pi : [0, x_2^*] \to \mathbb{R}$ is a twice differentiable function with $x_2^* > 0$. If $|\Pi''|^q$, $q \geq 1$ is measurable and m-convex on $\left[x_1, \frac{x_2}{m}\right]$, $0 \leq x_1 < x_2$ and $m \in (0, 1]$ with $\frac{x_2}{m} \leq x_2^*$, then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)]\right|\\
\leq\ &\frac{\alpha(x_2-x_1)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{|\Pi''(x_1)|^q+m\left|\Pi''\left(\frac{x_2}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

**Theorem 40** ([42]). *Assume that $\Pi$ is as in Theorem 39 with the condition $\frac{1}{p}+\frac{1}{q}=1$. Then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)]\right|\\
\leq\ &\frac{\alpha(x_2-x_1)^2}{2(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left[\frac{|\Pi''(x_1)|^q+m\left|\Pi''\left(\frac{x_2}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

**Theorem 41** ([43]). *Assume that $\Pi$ is as in t Theorem 39 with the condition $r > 0$. Then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(x_2)}{r(r+1)}+\frac{2}{r+1}\Pi\left(\frac{x_1+x_2}{2}\right)-\frac{\Gamma(\alpha+1)}{r(x_2-x_1)^{\alpha}}[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)]\right|\\
\leq\ &(x_2-x_1)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)\left[\frac{|\Pi''(x_1)|^q+m\left|\Pi''\left(\frac{x_2}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

**Theorem 42** ([43]). *Assume that $\Pi$ is as in Theorem 39 with the conditions $r > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{r(r+1)} + \frac{2}{r+1}\Pi\Big(\frac{x_1+x_2}{2}\Big) - \frac{\Gamma(\alpha+1)}{r(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\right|$$
$$\leq \frac{(x_2-x_1)^2}{r(\alpha+1)}\Big(1 - \frac{2}{p(\alpha+1)+1}\Big)^{\frac{1}{p}}\left[\frac{|\Pi''(x_1)|^q + m\left|\Pi''\left(\frac{x_2}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}.$$

*2.3. Fractional H-H-Type Inequalities for r-Convex Functions*

**Definition 3** ([44]). *A function $\Pi : [0, x_2] \to \mathbb{R}$ is said to be r-convex, where $r \geq 0$ and $x_2 > 0$, if for every $x, y \in [0, x_2]$ and $t \in [0, 1]$, we have*

$$\Pi(tx + (1-t)y) \leq \begin{cases} [t(\Pi(x))^r + (1-t)(\Pi(y))^r]^{1/r}, & r \neq 0, \\ [\Pi(x)]^t[\Pi(y)]^{(1-t)}, & r = 0. \end{cases}$$

Fractional H-H inequalities for $r$-convex functions are given in the next theorems.

**Theorem 43** ([45]). *Assume that $\Pi : [0, x_2] \to \mathbb{R}$ is a differentiable function with $x_2 > 0$. If $|\Pi'|$ is measurable and r-convex on $[x_1, x_2]$ for $0 \leq r < \infty$, $0 \leq x_1 < x_2$, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^2}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\right| \leq K_r,$$

*where*

$$K_r = \begin{cases} 2^{1/r-2}\dfrac{1-2^{-\alpha}}{1+\alpha}(x_2-x_1)(|\Pi'(x_1)| + |\Pi'(x_2)|), & 0 < r \leq 1, \\ 2^{-1/r}\dfrac{1-2^{-\alpha}}{1+\alpha}(x_2-x_1)(|\Pi'(x_1)| + |\Pi'(x_2)|), & r > 1, \end{cases}$$

$$K_0 = \frac{(x_2-x_1)|\Pi'(x_2)|}{2}\sum_{i=1}^{\infty}\Big[\frac{(\ln k)^{2i-1}}{(\alpha+1)_{2i}}(1-k) + \frac{(\ln k)^{2i-2}}{(\alpha+1)_{2i-1}}\Big(k+1-\frac{\sqrt{k}}{2^{\alpha+2i-3}}\Big)\Big],$$

*and $k = \dfrac{|\Pi'(x_1)|}{|\Pi'(x_2)|}$ and $(a)_i = a(a+1)\cdots(a+i-1)$.*

**Theorem 44** ([45]). *Assume that $\Pi$ is as in Theorem 43. If $|\Pi'|^q, q > 1$ is measurable and r-convex on $[x_1, x_2]$ for $0 \leq r < \infty$, $0 \leq x_1 < x_2$, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\right| \leq K_r,$$

*where*

$$K_r = \begin{cases} (x_2-x_1)2^{\frac{1-2r}{qr}}\Big(\dfrac{1-2^{-p\alpha}}{1+p\alpha}\Big)^{1/p}\Big[\dfrac{r(|\Pi'(x_1)|^q + |\Pi'(x_2)|^q)}{r+1}\Big]^{1/q}, & 0 < r \leq 1, \\ \dfrac{x_2-x_1}{2^{1-1/p}}\Big(\dfrac{1-2^{-p\alpha}}{1+p\alpha}\Big)^{1/p}\Big[\dfrac{r(|\Pi'(x_1)|^q + |\Pi'(x_2)|^q)}{r+1}\Big]^{1/q}, & r > 1, \end{cases}$$

$$K_0 = \begin{cases} \dfrac{x_2-x_1}{2^{1-1/p}}\Big(\dfrac{1-2^{-p\alpha}}{1+p\alpha}\Big)^{1/p}\Big[\dfrac{|\Pi'(x_1)|^q - |\Pi'(x_2)|^q}{q\ln|\Pi'(x_1)| - q\ln|\Pi'(x_2)|}\Big]^{1/q}, & |\Pi'(x_1)| \neq |\Pi'(x_2)|, \\ \dfrac{x_2-x_1}{2^{1-1/p}}\Big(\dfrac{1-2^{-p\alpha}}{1+p\alpha}\Big)^{1/p}|\Pi'(x_1)|, & |\Pi'(x_1)| = |\Pi'(x_2)|, \end{cases}$$

*and* $\frac{1}{p}+\frac{1}{q}=1$.

**Theorem 45** ([46]). *Assume that* $\Pi$ *is as in Theorem 43. If* $|\Pi''|$ *is integrable and r-convex on* $[x_1,x_2]$ *for* $0\le r<\infty$*, then fractional integral inequality is given as:*

$$\left|\frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\Big[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)\Big]-(1+\lambda)\frac{\Pi(x_1)+\Pi(x_2)}{2}-(1-\lambda)\Pi\Big(\frac{x_1+x_2}{2}\Big)\right|\le K_r,$$

*where*

$$K_r=\begin{cases}2^{\frac{1}{r}-1}(x_2-x_1)^2\Big\{\frac{|\Pi''(x_1)|+\Pi''(x_2)|}{\alpha+1}\Big[\frac{r}{r+1}-\frac{r}{\alpha r+2r+1}\\ \quad-B\Big(\frac{1}{r}+1,\alpha+2\Big)\Big]+\frac{1-\lambda}{2}|\Pi''(x_1)|\Big(\frac{r}{r+1}-\frac{r}{2r+1}\Big)\Big[1-\Big(\frac{1}{2}\Big)^{\frac{1}{r}+1}\Big]\\ \quad+\frac{1-\lambda}{2}|\Pi''(x_2)|\Big[B_{\frac{1}{2}}\Big(2,\frac{1}{r}+1\Big)+\frac{r}{2r+1}\Big(\frac{1}{2}\Big)^{\frac{1}{r}+2}\Big]\Big\}, & 0\le r\le 1,\\ (x_2-x_1)^2\Big\{\frac{|\Pi''(x_1)|+|\Pi''(x_2)|}{\alpha+1}\Big[\frac{r}{r+1}-\frac{r}{\alpha r+2r+1}-B\Big(\frac{1}{r}+1,\alpha+2\Big)\Big]\\ \quad+\frac{1-\lambda}{2}|\Pi''(x_1)|\Big(\frac{r}{r+1}-\frac{r}{2r+1}\Big)\Big[1-\Big(\frac{1}{2}\Big)^{\frac{1}{r}+1}\Big]\\ \quad+\frac{1-\lambda}{2}|\Pi''(x_2)|\Big[B_{\frac{1}{2}}\Big(2,\frac{1}{r}+1\Big)+\frac{r}{2r+1}\Big(\frac{1}{2}\Big)^{\frac{1}{r}+2}\Big]\Big\}, & r>1,\\ (x_2-x_1)^2|\Pi''(x_2)|\Big\{\frac{1}{\alpha+1}\Big[\frac{|k|-1}{\ln|k|}-|k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}\\ \quad-\sum_{i=1}^{\infty}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}\Big]+\frac{1-\lambda}{4}|k|^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\Big(-\frac{1}{2}\ln|k|\Big)^{i-1}}{(2)_i}\\ \quad+\frac{1-\lambda}{2}\Big[\frac{|k|-|k|^{\frac{1}{2}}}{\ln|k|}-|k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(2)_i}\Big]\Big\}, & r=0,\end{cases}$$

*where* $k=\frac{\Pi''(x_1)}{\Pi''(x_2)}$ *and B the incomplete Beta function* $B_x(x_1,x_2)=\int_0^x t^{x_1-1}(1-t)^{x_2-1}dt$, $x\in[0,1], x_1,x_2>0$.

### 2.4. Fractional H-H-Type Inequalities for $(\alpha,m)$-Convex Functions

**Definition 4** ([47]). *A function* $\Pi:[0,x_2]\to\mathbb{R}$ *is said to be* $(\alpha,m)$*-convex, where* $(\alpha,m)\in[0,1]^2$*, if for every* $x,y\in[0,x_2]$ *and* $t\in[0,1]$ *one has*

$$\Pi(tx+m(1-t)y)\le t^\alpha\Pi(x)+m(1-t^\alpha)\Pi(y).$$

Inequalities of fractional H-H-type for $(\alpha,m)$-convex derivative are now presented.

**Theorem 46** ([48]). *Assume that* $\Pi:[0,\infty)\to\mathbb{R}$ *is differentiable on* $[0,\infty)$ *and* $\Pi'\in L_1[x_1,x_2]$ *for* $0\le x_1<x_2$ *and* $\alpha>0$*. If* $|\Pi'|^q$ *is* $(\alpha_1,m)$*-convex on* $\Big[0,\frac{x_2}{m}\Big]$ *for some* $(\alpha_1,m)\in(0,1]\times(0,1]$ *and* $q\ge1$*, then fractional integral inequality is given as:*

$$\Bigg|\frac{1}{2}\Big[\frac{\Pi(x_1)+\Pi(x_2)}{2}+\Pi\Big(\frac{x_1+x_2}{2}\Big)\Big]-\frac{4^\alpha\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}$$
$$\times\Big[J^\alpha_{x_1+}\Pi\Big(\frac{3x_1+x_2}{4}\Big)+J^\alpha_{\left(\frac{3x_1+x_2}{2}\right)+}\Pi\Big(\frac{x_1+x_2}{2}\Big)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi\Big(\frac{3x_1+x_2}{4}\Big)$$
$$+J^\alpha_{\left(\frac{3x_1+x_2}{2}\right)+}\Pi(x_2)\Big]\Bigg|$$

$$\leq \frac{x_2-x_1}{16(\alpha+1)}\left[\frac{1}{(\alpha_1+1)(\alpha+\alpha_1+1)}\right]^{\frac{1}{q}}$$
$$\times\left[\left((\alpha+1)(\alpha_1+1)|\Pi'(x_1)|^q+m\alpha_1(\alpha_1+1)\left|\Pi'\left(\frac{3x_1+x_2}{4m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\alpha\left((\alpha+1)\left|\Pi'\left(\frac{3x_1+x_2}{4}\right)\right|^q+m\alpha_1(\alpha_1+\alpha+2)\left|\Pi'\left(\frac{x_1+x_2}{2m}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left((\alpha+1)(\alpha_1+1)\left|\Pi'\left(\frac{x_1+x_2}{2}\right)\right|^q+m\alpha_1(\alpha_1+1)\left|\Pi'\left(\frac{x_1+3x_2}{4m}\right)\right|^q\right)^{\frac{1}{q}}$$
$$\left.+\alpha\left((\alpha+1)\left|\Pi'\left(\frac{x_1+3x_2}{4}\right)\right|^q+m\alpha_1(\alpha_1+\alpha+2)\left|\Pi'\left(\frac{x_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

**Theorem 47** ([49]). *Assume that $x_1, x_2 \in \mathbb{I}$, $x_1 < x_2$ and $\Pi : \mathbb{I} \to \mathbb{R}$ is differentiable on $\mathbb{I}^\circ$ such that $\Pi' \in L_1[x_1, x_2]$. If $|\Pi'|$ is $(\alpha^*, m)$-convex on $[x_1, x_2]$, for $(\alpha^*, m) \in [0,1] \times (0,1]$, then fractional integral inequality is given as:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\right]-\Pi\left(\frac{x_1+x_2}{2}\right)\right|$$
$$\leq \frac{x_2-x_1}{4}\left\{\left[|\Pi'(x_1)|-m\left|\Pi'\left(\frac{x_2}{m}\right)\right|\right]\left[\frac{1}{2^\alpha(\alpha+\alpha^*+1}+2^{k+1}B\left(\frac{1}{2},k+1,\alpha^*+1\right)\right.\right.$$
$$\left.+\frac{2m}{\alpha^*+1}\left|\Pi'\left(\frac{x_2}{m}\right)\right|\right\},$$

*where $B(.,.,.)$ is the incomplete Beta function.*

**Theorem 48** ([49]). *] Assume that $\Pi$ is as in Theorem 47. If $|\Pi'|^q$ is $(\alpha^*, m)$-convex on $[x_1, x_2]$, for $(\alpha^*, m) \in [0,1] \times (0,1]$, then fractional integral inequality is given as:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)+J^\alpha_{\left(\frac{x_1+x_2}{2}\right)-}\Pi(x_1)\right]-\Pi\left(\frac{x_1+x_2}{2}\right)\right|$$
$$\leq \frac{x_2-x_1}{4}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left\{\left(\frac{|\Pi'(x_1)|^q}{2^{\alpha^*}(\alpha^*+1)}+m\left|\Pi'\left(\frac{x_2}{m}\right)\right|^q\left[1-\frac{1}{2^{\alpha^*}(\alpha^*+1)}\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left[\frac{2}{\alpha^*+1}-\frac{1}{2^{\alpha^*}(\alpha^*+1)}\right]\left[|\Pi'(x_1)|^q-m\left|\Pi'\left(\frac{x_2}{m}\right)\right|^q\right]+m\left|\Pi'\left(\frac{x_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\},$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

### 2.5. Fractional H-H-Type Inequalities for $(\alpha, m)$-Geometrically Convex Functions

**Definition 5** ([50]). *Let $\Pi : [0, x_2] \to \mathbb{R}$ be a positive function and $(\alpha, m) \in (0,1] \times (0,1]$. If*

$$\Pi(x^t y^{m(1-t)}) \leq [\Pi(x)]^{t^\alpha}[\Pi(y)]^{m(1-t^\alpha)}$$

*holds for all $x, y \in [0, x_2]$ and $t \in [0,1]$, then we say that the function $\Pi$ is $(\alpha, m)$-geometrically convex on $[0, b]$.*

We give in the next fractional H-H inequalities for $(\alpha, m)$-geometrically convex functions.

**Theorem 49** ([50]). *Assume that $x_1, x_2 \in \mathbb{I}$, $x_1 < x_2$ and $\Pi : \mathbb{I} \to \mathbb{R}$ is a differentiable function on $\mathbb{I}^\circ$ such that $\Pi' \in L_1[x_1, x_2]$. If $|\Pi'|^q, q \geq 1$ is non-increasing and $(\alpha, m)$-geometrically convex* *on* $[\min\{1,x_1\},x_2]$*, for* $x_2 \geq 1$*, and for* $(\alpha,m) \in (0,1]^2$*, then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{(x-x_1)^{\alpha}\Pi(x_1)+(x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1}-\frac{\Gamma(\alpha+1)}{x_2-x_1}\left[J^{\alpha}_{x-}\Pi(x_1)+J^{\alpha}_{x+}\Pi(x_2)\right]\right|\\ \leq&\left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left[\frac{(x-x_1)^{\alpha+1}|\Pi'(x)|^m}{x_2-x_1}\left(\int_0^1\left(\frac{|\Pi'(x_1)|}{|\Pi'(x)|^m}\right)^{q(1-t^{\alpha})}dt\right)^{\frac{1}{q}}\right.\\ &\left.+\frac{(x_2-x)^{\alpha+1}|\Pi'(x_2)|^m}{x_2-x_1}\left(\int_0^1\left(\frac{|\Pi'(x)|}{|\Pi'(x_2)|^m}\right)^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

**Theorem 50** ([50])**.** *Assume that* $\Pi$ *is as in Theorem 49. Then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{(x-x_1)^{\alpha}g(x_1)+(x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1}-\frac{\Gamma(\alpha+1)}{x_2-x_1}\left[J^{\alpha}_{x-}\Pi(x_1)+J^{\alpha}_{x+}\Pi(x_2)\right]\right|\\ \leq&\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{(x-x_1)^{\alpha+1}|\Pi'(x)|^m}{x_2-x_1}\left(\int_0^1(1-t^{\alpha})\left(\frac{|\Pi'(x_1)|}{|\Pi'(x)|^m}\right)^{q(1-t^{\alpha})}dt\right)^{\frac{1}{q}}\right.\\ &\left.+\frac{(x_2-x)^{\alpha+1}|\Pi'(x_2)|^m}{x_2-x_1}\left(\int_0^1(1-t^{\alpha})\left(\frac{|\Pi'(x)|}{|\Pi'(x_2)|^m}\right)^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

### 2.6. Fractional H-H-Type Inequalities for Harmonically Convex Functions

**Definition 6** ([51])**.** *Assume that* $\mathbb{I} \subset \mathbb{R}\setminus\{0\}$ *is a real interval. A function* $\Pi : \mathbb{I} \to \mathbb{R}$ *is harmonically convex, if*

$$\Pi\left(\frac{xy}{tx+(1-t)y}\right) \leq t\Pi(y)+(1-t)\Pi(x)$$

*for all* $x, y \in \mathbb{I}$ *and* $t \in [0,1]$*.*

Fractional H-H inequalities for harmonic convex functions are represented in the following theorems.

**Theorem 51** ([52])**.** *Assume that* $x_1, x_2 \in \mathbb{I}^{\circ}$ *with* $x_1 < x_2$ *and* $\Pi : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ *is a function such that* $\Pi \in L_1[x_1,x_2]$*. If* $\Pi$ *is harmonically convex function on* $\mathbb{I}$*, then fractional integral inequalities are given as:*

$$\begin{aligned}\Pi\left(\frac{2x_1x_2}{x_1+x_2}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{x_1x_2}{x_2-x_1}\right)^{\alpha}\left\{J^{\alpha}_{1/x_1-}(\Pi\circ g)(1/x_2)+J^{\alpha}_{1/x_2+}(\Pi\circ g)(1/x_1)\right\}\\ &\leq \frac{\Pi(x_1)+\Pi(x_2)}{2},\end{aligned}$$

*with* $\alpha > 0$ *and* $g(x) = \frac{1}{x}, x \in \left[\frac{1}{x_2}, \frac{1}{x_1}\right]$*.*

**Theorem 52** ([52])**.** *Assume that* $\Pi$ *is as in Theorem 51. If* $|\Pi'|^q$ *is harmonically convex function on* $\mathbb{I}$*, for* $q \geq 1$*, then fractional integral inequalities are given as:*

$$\begin{aligned}&\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{x_1x_2}{x_2-x_1}\right)^{\alpha}\left\{J^{\alpha}_{1/x_1-}(\Pi\circ g)(1/x_2)+J^{\alpha}_{1/x_2+}(\Pi\circ g)(1/x_1)\right.\\ \leq&\frac{x_1x_2(x_2-x_1)}{2}C_1^{1-\frac{1}{q}}(\alpha;x_1,x_2)\Big(C_2(\alpha;x_1,x_2)|\Pi'(x_2)|^q+C_3(\alpha;x_1,x_2)|\Pi'(x_1)|^q\Big)^{\frac{1}{q}},\end{aligned}$$

*where*

$$\begin{aligned} C_1(\alpha;x_1,x_2) &= \frac{x_2^{-2}}{\alpha+1}\Big[{}_2F_1\Big(2,1;\alpha+2;1-\frac{x_1}{x_2}\Big)+{}_2F_2\Big(2,\alpha+1;1-\frac{x_1}{x_2}\Big)\Big],\\ C_2(\alpha;x_1,x_2) &= \frac{x_2^{-2}}{\alpha+2}\Big[\frac{1}{\alpha+1}\,{}_2F_1\Big(2,2;\alpha+3;1-\frac{x_1}{x_2}\Big)+{}_2F_1\Big(2,\alpha+2;\alpha+3;1-\frac{x_1}{x_2}\Big)\Big],\\ C_3(\alpha;x_1,x_2) &= \frac{x_2^{-2}}{\alpha+1}\Big[{}_2F_1\Big(2,1;\alpha+3;1-\frac{x_1}{x_2}\Big)+\frac{1}{\alpha+1}\,{}_2F_1\Big(2,\alpha+1;\alpha+3;1-\frac{x_1}{x_2}\Big)\Big], \end{aligned}$$

*and* ${}_{(\cdot)}F_{(\cdot)}(a,b;c;z)$ *the hypergeometric function and* $g(x)=\frac{1}{x}, x\in\Big[\frac{1}{x_2},\frac{1}{x_1}\Big]$..

**Theorem 53** ([52])**.** *Assume that* $\Pi$ *is as in Theorem 51. If* $|\Pi'|^q$, $q\geq 1$ *is a harmonically convex function on* $\mathbb{I}$*, then fractional integral inequality is given as:*

$$\begin{aligned} &\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2}\Big(\frac{x_1x_2}{x_2-x_1}\Big)^{\alpha}\Big\{J^{\alpha}_{1/x_1-}(\Pi\circ g)(1/x_2)+J^{\alpha}_{1/x_2+}(\Pi\circ g)(1/x_1)\Big|\\ \leq\ &\frac{x_1x_2(x_2-x_1)}{2}C_1^{1-\frac{1}{q}}(\alpha;x_1,x_2)\Big(C_2(\alpha;x_1,x_2)|\Pi'(x_2)|^q+C_3(\alpha;x_1,x_2)|\Pi'(x_1)|^q\Big)^{\frac{1}{q}}, \end{aligned}$$

*where*

$$\begin{aligned} C_1(\alpha;x_1,x_2) &= \frac{x_2^{-2}}{\alpha+1}\Big[{}_2F_1\Big(2,\alpha+1;\alpha+2;1-\frac{x_1}{x_2}\Big)-{}_2F_1\Big(2,1;\alpha+2;1-\frac{x_1}{x_2}\Big)\Big]\\ &\quad+{}_2F_1\Big(2,1;\alpha+2;\frac{1}{2}\Big(1-\frac{x_1}{x_2}\Big)\Big)\Big],\\ C_2(\alpha;x_1,x_2) &= \frac{x_2^{-2}}{\alpha+2}\Big[\,{}_2F_1\Big(2,\alpha+2;\alpha+3;1-\frac{x_1}{x_2}\Big)-\frac{1}{\alpha+1}\,{}_2F_1\Big(2,2;\alpha+3;1-\frac{x_1}{x_2}\Big)\\ &\quad+\frac{1}{2(\alpha+1)}{}_2F_1\Big(2,2;\alpha+3;\frac{1}{2}\Big(1-\frac{x_1}{x_2}\Big)\Big)\Big],\\ C_3(\alpha;x_1,x_2) &= \frac{x_2^{-2}}{\alpha+2}\Big[\frac{1}{\alpha+1}{}_2F_1\Big(2,\alpha+1;\alpha+3;1-\frac{x_1}{x_2}\Big)-{}_2F_1\Big(2,1;\alpha+3;1-\frac{x_1}{x_2}\Big)\\ &\quad+{}_2F_1\Big(2,1;\alpha+3;\frac{1}{2}\Big(1-\frac{x_1}{x_2}\Big)\Big)\Big], \end{aligned}$$

*and* $0<\alpha\leq 1$ *and* $g(x)=\frac{1}{x}, x\in\Big[\frac{1}{x_2},\frac{1}{x_1}\Big]$.

**Theorem 54** ([52])**.** *Assume that* $\Pi$ *is as in Theorem 51. If* $|\Pi'|^q$, $q>1$ *is a harmonically convex function on* $\mathbb{I}$*, then fractional integral inequality is given as:*

$$\begin{aligned} &\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2x_2}-\frac{\Gamma(\alpha+1)}{2}\Big(\frac{x_1x_2}{x_2-x_1}\Big)^{\alpha}\Big[J^{\alpha}_{1/x_1-}(\Pi\circ g)(1/x_2)+J^{\alpha}_{1/x_2+}(\Pi\circ g)(1/x_1)\Big|\\ \leq\ &\frac{a(x_2-x_1)}{2x_2}\Big(\frac{1}{\alpha p+1}\Big)^{\frac{1}{p}}\Big(\frac{|\Pi'(x_2)|^q+|\Pi'(x_1)|^q}{2}\Big)^{\frac{1}{q}}\\ &\times\Big[{}_2F_1^{\frac{1}{p}}\Big(2p,1;\alpha p+2;1-\frac{x_1}{x_2}\Big)+{}_2F_1^{\frac{1}{p}}\Big(2p,\alpha p+1;\alpha p+2;1-\frac{x_1}{x_2}\Big)\Big], \end{aligned}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$ *and* $g(x)=\frac{1}{x}, x\in\Big[\frac{1}{x_2},\frac{1}{x_1}\Big]$.

**Theorem 55** ([53]). *Assume that $x_1, x_2 \in \mathbb{I}^\circ$ with $x_1 < x_2$ and $\Pi : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ is a harmonically convex function such that $\Pi \in L_1[x_1, x_2]$. Then right R-L fractional integral inequality is given as:*

$$\Pi\Big(\frac{(\alpha+1)x_1x_2}{x_1+\alpha x_2}\Big) \le \Gamma(\alpha+1)\Big(\frac{x_1x_2}{x_2-x_1}\Big)^{\alpha} J^{\alpha}_{\frac{1}{x_2}-}(\Pi\circ h)\Big(\frac{1}{x_1}\Big) \le \frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1},$$

*where* $h(x) = \frac{1}{x}, x \in \Big[\frac{1}{x_2}, \frac{1}{x_1}\Big]$ *and* $\alpha > 0$.

**Theorem 56** ([53]). *Assume that $\Pi$ is as in Theorem 55. If $\Pi \in L_1[x_1, x_2]$, then we have:*

$$\Pi\Big(\frac{(\alpha+1)x_1x_2}{\alpha x_1+x_2}\Big) \le \Gamma(\alpha+1)\Big(\frac{x_1x_2}{x_2-x_1}\Big)^{\alpha} J^{\alpha}_{\frac{1}{x_1}+}(\Pi\circ h)\Big(\frac{1}{x_2}\Big) \le \frac{\Pi(x_1)+\alpha\Pi(x_2)}{\alpha+1},$$

*where* $h(x) = \frac{1}{x}, x \in \Big[\frac{1}{x_2}, \frac{1}{x_1}\Big]$ *and* $\alpha > 0$.

**Theorem 57** ([53]). *Assume that $\Pi$ is as in Theorem 55. If $\Pi \in L_1[x_1, x_2]$. Then we have:*

$$\begin{aligned}\frac{\Pi\Big(\frac{(\alpha+1)x_1x_2}{x_1+\alpha x_2}\Big)+\Pi\Big(\frac{(\alpha+1)x_1x_2}{\alpha x_1+x_2}\Big)}{2} &\le \frac{\Gamma(\alpha+1)}{2}\Big[J^{\alpha}_{\frac{1}{x_2}-}(\Pi\circ h)\Big(\frac{1}{x_1}\Big)+J^{\alpha}_{\frac{1}{x_1}+}(\Pi\circ h)\Big(\frac{1}{x_2}\Big)\Big] \\ &\le \frac{\Pi(x_1)+\Pi(x_2)}{2},\end{aligned}$$

*where* $h(x) = \frac{1}{x}, x \in \Big[\frac{1}{x_2}, \frac{1}{x_1}\Big]$ *and* $\alpha > 0$.

**Theorem 58** ([54]). *Let $\Pi : \mathbb{I} = [x_1, x_2] \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\Pi \in L_1[x_1, x_2]$ with $0 < x_1 < x_2$. If $\Pi$ is a harmonically convex function on $\mathbb{I}$, then:*

$$\begin{aligned} &\Pi\left(\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}-\frac{x+y}{2xy}}\right) \\ \le\ &\Pi(x_1)+\Pi(x_2)-\frac{\Gamma(\alpha+1)}{2}\Big(\frac{xy}{x+y}\Big)^{\alpha}\Big\{J^{\alpha}_{\frac{1}{y}+}(\Pi\circ h)\Big(\frac{1}{x}\Big)+J^{\alpha}_{\frac{1}{x}-}(\Pi\circ h)\Big(\frac{1}{y}\Big)\Big\} \\ \le\ &\Pi(x_1)+\Pi(x_2)-\Pi\Big(\frac{2xy}{x+y}\Big),\end{aligned}$$

*and*

$$\begin{aligned} &\Pi\left(\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}-\frac{x+y}{2xy}}\right) \\ \le\ &\Pi(x_1)+\Pi(x_2)-\frac{\Gamma(\alpha+1)}{2}\Big(\frac{xy}{x+y}\Big)^{\alpha}\Big\{J^{\alpha}_{\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x}\right)+}(\Pi\circ h)\Big(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{y}\Big) \\ &+J^{\alpha}_{\left(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{y}\right)-}(\Pi\circ h)\Big(\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x}\Big)\Big\} \\ \le\ &\frac{1}{2}\left[\Pi\left(\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x}}\right)+\Pi\left(\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{y}}\right)\right] \\ \le\ &\Pi(x_1)+\Pi(x_2)-\frac{\Pi(x_1)+\Pi(x_2)}{2},\end{aligned}$$

*for all* $x, y \in [x_1, x_2]$, $\alpha > 0$ *and* $h(x) = \frac{1}{x}, x \in \Big[\frac{1}{x_2}, \frac{1}{x_1}\Big]$.

**Theorem 59** ([55]). *Assume that $\Pi : [x_1, x_2] \subseteq (0, \infty) \to \mathbb{R}$ is a positive, twice differentiable function with $x_1 < x_2$ and $\Pi \in L_1[x_1, x_2]$ and harmonically convex. If $\Phi''$ is bounded in $[x_1, x_2]$, where $\Phi(x) = \Pi\left(\frac{x_1 x_2}{x}\right)$, then:*

$$\begin{aligned}
&\frac{m\alpha}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}\left(\frac{x_1+x_2}{2}-x\right)^2\left[(x-x_1)^{\alpha-1}+(x_2-x)^{\alpha-1}\right]dx\\
\leq\ &\frac{\Gamma(\alpha+1)}{2}\left(\frac{x_1x_2}{x_2-x_1}\right)^\alpha\left\{J^\alpha_{1/x_1-}(\Pi\circ g)(1/x_2)+J^\alpha_{1/x_2+}(\Pi\circ g)(1/x_1)\right\}-\Pi\left(\frac{2x_1x_2}{x_1+x_2}\right)\\
\leq\ &\frac{M\alpha}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}\left(\frac{x_1+x_2}{2}-x\right)^2\left[(x-x_1)^{\alpha-1}+(x_2-x)^{\alpha-1}\right]dx,
\end{aligned}$$

*and*

$$\begin{aligned}
&\frac{-M\alpha}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}(x-x_1)(x_2-x)\left[(x-x_1)^{\alpha-1}+(x_2-x)^{\alpha-1}\right]dx\\
\leq\ &\frac{\Gamma(\alpha+1)}{2}\left(\frac{x_1x_2}{x_2-x_1}\right)^\alpha\left[J^\alpha_{1/x_1-}(\Pi\circ g)(1/x_2)+J^\alpha_{1/x_2+}(\Pi\circ g)(1/x_1)\right]-\Pi\left(\frac{2x_1x_2}{x_1+x_2}\right)\\
\leq\ &\frac{-m\alpha}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}(x-x_1)(x_2-x)\left[(x-x_1)^{\alpha-1}+(x_2-x)^{\alpha-1}\right]dx,
\end{aligned}$$

*where $m = \inf_{t\in[x_1,x_2]}\Phi''(t)$, $M = \sup_{t\in[x_1,x_2]}\Phi''(t)$ and $g(x) = \frac{1}{x}, x \in \left[\frac{1}{x_2}, \frac{1}{x_1}\right]$.*

**Theorem 60** ([55]). *Assume that $\Pi$ is as in Theorem 59. If $\Phi'(x_1 + x_2 - x) \geq \Phi'(x)$ for all $x \in \left[x_1, \frac{x_1+x_2}{2}\right]$, where $\Phi(x) = \Pi\left(\frac{x_1x_2}{x}\right)$, then:*

$$\begin{aligned}
\Pi\left(\frac{2x_1x_2}{x_1+x_2}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{x_1x_2}{x_2-x_1}\right)^\alpha\left\{J^\alpha_{1/x_1-}(\Pi\circ g)(1/x_2)+J^\alpha_{1/x_2+}(\Pi\circ g)(1/x_1)\right\}\\
&\leq \frac{\Pi(x_1)+\Pi(x_2)}{2},
\end{aligned}$$

*$\alpha > 0$ and $g(x) = \frac{1}{x}, x \in \left[\frac{1}{x_2}, \frac{1}{x_1}\right]$.*

### *2.7. Fractional H-H-Type Inequalities for Harmonically Symmetric Functions*

**Definition 7** ([56]). *A function $g : [x_1, x_2] \subseteq \mathbb{R}\setminus\{0\}$ is said to be harmonic symmetric with respect to $\frac{2x_1x_2}{x_1+x_2}$ if*

$$g(x) = g\left(\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}-\frac{1}{x}}\right)$$

*holds for all $x \in [x_1, x_2]$.*

Fractional H-H inequalities for harmonically symmetric functions are included in the following theorems.

**Theorem 61** ([57]). *Assume that $x_1, x_2 \in \mathbb{I}$ with $x_1 < x_2$ and $\Pi : [x_1, x_2] \to \mathbb{R}$ is a harmonic convex function and $\Pi \in L_1[x_1, x_2]$. If $g : [x_1, x_2] \to \mathbb{R}$ is harmonic symmetric with respect to $\frac{2x_1x_2}{x_1+x_2}$, integrable and non-negative then fractional integral inequalities are given as:*

$$\Pi\left(\frac{2x_1x_2}{x_1+x_2}\right)\left[J^\alpha_{\frac{1}{x_2}+}(g\circ h)\left(\frac{1}{x_1}\right)+J^\alpha_{\frac{1}{x_1}-}(g\circ h)\left(\frac{1}{x_2}\right)\right]$$

$$\begin{aligned}
&\leq \left[J^{\alpha}_{\frac{1}{x_2}+}(g\circ h)\left(\frac{1}{x_1}\right)+J^{\alpha}_{\frac{1}{x_1}-}(g\circ h)\left(\frac{1}{x_2}\right)\right]\\
&\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}\left[J^{\alpha}_{\frac{1}{x_2}+}(g\circ h)\left(\frac{1}{x_1}\right)+J^{\alpha}_{\frac{1}{x_1}-}(g\circ h)\left(\frac{1}{x_2}\right)\right],
\end{aligned}$$

*with* $\alpha>0$ *and* $h(x)=\frac{1}{x}, x\in\left[\frac{1}{x_2},\frac{1}{x_1}\right]$.

**Theorem 62** ([57]). *Assume that* $\Pi$ *is as in Theorem 61. If* $|\Pi'|$ *is harmonic convex on* $[x_1,x_2]$ *and* $g:[x_1,x_2]\to\mathbb{R}$ *is harmonically symmetric with respect to* $\frac{2x_1x_2}{x_1+x_2}$ *and continuous, then fractional integral inequalities are given as:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\left[J^{\alpha}_{\frac{1}{x_2}+}(g\circ h)\left(\frac{1}{x_1}\right)+J^{\alpha}_{\frac{1}{x_1}-}(g\circ h)\left(\frac{1}{x_2}\right)\right]\right.\\
&\left.-\left[J^{\alpha}_{\frac{1}{x_2}+}(\Pi g\circ h)\left(\frac{1}{x_1}\right)+J^{\alpha}_{\frac{1}{x_1}-}(\Pi g\circ h)\left(\frac{1}{x_2}\right)\right]\right|\\
\leq\ &\frac{\|g\|_\infty x_1x_2(x_2-x_1)}{\Gamma(\alpha+1)}\left(\frac{x_2-x_1}{x_1x_2}\right)^{\alpha}\left[C_1(a)|\Pi'(x_1)|+C_2(a)|\Pi'(x_2)|\right],
\end{aligned}$$

*where*

$$\begin{aligned}
C_1(a) &= \frac{x_2^{-2}}{\alpha+2}\,{}_2F_1\left(2,1;\alpha+3;1-\frac{x_1}{x_2}\right)-\frac{x_2^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{x_1}{x_2}\right)\\
&+\frac{2(x_1+x_2)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;\frac{x_2-x_1}{x_1+x_2}\right),\\
C_2(a) &= \frac{x_2^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,2;\alpha+3;1-\frac{x_1}{x_2}\right)-\frac{x_2^{-2}}{\alpha+2}\,{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{x_1}{x_2}\right)\\
&+\left(\frac{x_1+x_2}{2}\right)^{-2}\frac{1}{\alpha+1}\,{}_2F_1\left(2,\alpha+1;\alpha+2;\frac{x_2-x_1}{x_1+x_2}\right)\\
&-\frac{2(x_1+x_2)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;\frac{x_2-x_1}{x_1+x_2}\right),
\end{aligned}$$

*with* $0<\alpha\leq 1$ *and* $h(x)=\frac{1}{x}, x\in\left[\frac{1}{x_2},\frac{1}{x_1}\right]$.

*2.8. Fractional H-H-Type Inequalities for Harmonically* $(\theta,m)$*-Convex Functions*

**Definition 8** ([58]). *A function* $\Pi:(0,x_2]\to\mathbb{R}$, $x_2>0$, *is harmonically* $(\theta,m)$*-convex, if*

$$\Pi\left(\frac{mxy}{mty+(1-t)x}\right)\leq t^{\theta}\Pi(x)+m(1-t^{\theta})\Pi(y),$$

*fo rall* $x,y\in(0,x_2]$ *and* $t\in[0,1]$.

Fractional H-H-type inequalities for harmonically $(\theta,m)$-convex functions are given in the next theorems.

**Theorem 63** ([59]). *Assume that* $x_1,\frac{x_2}{m}\in\mathbb{I}^{\circ}$ *with* $x_1<x_2$, $m\in(0,1]$ *and* $\Pi:\mathbb{I}\subset(0,\infty)\to\mathbb{R}$ *is a differentiable on* $\mathbb{I}^{\circ}$ *such that* $\Pi'\in L_1[x_1,x_2]$. *If* $|\Pi'|^q$ *is harmonically* $(\theta,m)$*-convex on* $\left[x_1,\frac{x_2}{m}\right]$ *for* $q\geq 1$, *with* $\alpha\in[0,1]$, *then*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{x_1x_2}{x_2-x_1}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{x_1}-}(\Pi\circ g)\left(\frac{1}{x_2}\right)+J^{\alpha}_{\frac{1}{x_2}+}(\Pi\circ g)\left(\frac{1}{x_1}\right)\right]\right.$$

$$\leq \frac{x_1x_2(x_2-x_1)}{2}C_1^{1-\frac{1}{q}}(\alpha;x_1,x_2)\Big[C_2(\alpha;\theta;x_1,x_2)|\Pi'(x_1)|^q$$
$$+mC_3(\alpha;\theta;x_1,x_2)\Big|\Pi'\Big(\frac{x_2}{m}\Big)\Big|^q\Big]^{\frac{1}{q}},$$

*where*

$$\begin{aligned} C_1(\alpha;x_1,x_2) &= \frac{x_2^{-2}}{\alpha+1}\Big[\,{}_2F_1\Big(2,\alpha+1;\alpha+2;1-\frac{x_1}{x_2}\Big)+\,{}_2F_1\Big(2,1;\alpha+2;1-\frac{x_1}{x_2}\Big)\Big],\\ C_2(\alpha;\theta;x_1,x_2) &= \Big[\frac{B(\alpha+1;\theta+1)}{x_2^2}\,{}_2F_1\Big(2,\alpha+1;\alpha+\theta+2;1-\frac{x_1}{x_2}\Big)\\ &\quad+\frac{x_2^{-2}}{\alpha+\theta+1}\,{}_2F_1\Big((2,1;\alpha+\theta+2;1-\frac{x_1}{x_2}\Big)\Big],\\ C_3(\alpha;\theta;x_1,x_2) &= C_1(\alpha;x_1,x_2)-C_2(\alpha;\theta;x_1,x_2). \end{aligned}$$

### *2.9. Fractional H-H-Type Inequalities for M-Harmonic Harmonically Convex Functions*

**Definition 9** ([60]). *A function $\Pi:\mathbb{I}\subset(0,\infty)\to(0,\infty)$ is m-harmonic harmonically convex (briefly m-HH convex) function on $\mathbb{I}$, if for all $x,y\in\mathbb{I}$, $t\in[0,1]$*

$$\Pi\Big(\frac{1}{tx^{-1}+m(1-t)y^{-1}}\Big)\leq\frac{1}{t[\Pi(x)]^{-1}+n(1-t)[\Pi(y)]^{-1}},$$

*where $m\in(0,1]$.*

**Theorem 64** ([60]). *Let $\Pi:\mathbb{I}\subset(0,\infty)\to(0,\infty)$ be a differentiable and $\Pi'\in L_1[x_1,x_2]$. If $|\Pi'|^q$ is m-HH convex and monotonically non-increasing on $[mx_1,x_2]$ with $0\leq x_1<x_2$ for $m\in(0,1]$, $q\geq1$ and $|\Pi'(x)|\leq M$, then*

$$\begin{aligned} &\left|\frac{(x-x_1)^\alpha+(x_2-x)^\alpha}{x_2-x_1}\Pi(x)-\frac{\Gamma(\alpha+1)}{x_2-x_1}\Big[J^\alpha_{x-}\Pi(x_1)+J^\alpha_{x+}\Pi(x_2)\Big]\right|\\ \leq\;&\Big(\frac{1}{\alpha+1}\Big)^{1-\frac{1}{q}}\frac{M}{m(\alpha+1)}\,{}_2\Pi\Big(1,\alpha+1,\alpha+2,1-\frac{1}{m}\Big)\Big[\frac{(x-x_1)^{\alpha+1}}{x_2-x_1}+\frac{(x_2-x)^{\alpha+1}}{x_2-x_1}\Big]. \end{aligned}$$

**Theorem 65** ([60]). *Assume that $\Pi$ is as in Theorem 64. If $|\Pi'|^q$ is m-HH convex and monotonically non-increasing on $[mx_1,x_2]$ with $0\leq x_1<x_2$ for $m\in(0,1]$, and $q\geq1$, then*

$$\begin{aligned} &\left|\frac{(x-x_1)^\alpha+(x_2-x)^\alpha}{x_2-x_1}\Pi(x)-\frac{\Gamma(\alpha+1)}{x_2-x_1}\Big[J^\alpha_{x-}\Pi(x_1)+J^\alpha_{x+}\Pi(x_2)\Big]\right|\\ \leq\;&\Big(\frac{q-1}{(\alpha+1)q-1}\Big)^{1-\frac{1}{q}}\Big[\frac{(x-x_1)^{\alpha+1}}{x_2-x_1}[H^q(|\Pi'(mx_1))|,|\Pi'(x)|]^{\frac{1}{q}}\\ &+\frac{(x_2-x)^{\alpha+1}}{x_2-x_1}[H^q(|\Pi'(mx_1))|,|\Pi'(x)|]^{\frac{1}{q}}\Big], \end{aligned}$$

*where* $H^q(u,v)=\displaystyle\int_0^1\frac{dt}{tv^{-q}+m(1-t)u^{-q}}$.

### *2.10. Fractional H-H-Type Inequalities for $(S,R)$-Convex Functions*

**Definition 10** ([61]). *A function $\Pi:\mathbb{I}\subseteq[0,\infty)\to\mathbb{R}$ is said to be $(s,r)$-convex in the second sense, if we have*

$$\Pi(tx+(1-t)y)\leq\begin{cases}[t^sf^r(x)+(1-t)^sf^r(y)]^{\frac{1}{r}}, & r\neq0,\\ [\Pi(x)]^{t^s}[(\Pi(y)]^{(1-t)^s}, & r=0,\end{cases}$$

*holds, for every fixed $s\in(0,1]$, $x,y\in\mathbb{I}$ and $t\in[0,1]$.*

In the next theorems, we give fractional H-H-type inequalities for $(s,r)$-convex functions.

**Theorem 66** ([61]). *Assume that $\Pi : [x_1, x_2] \to (0, \infty)$ is a function twice differentiable with $x_1 < x_2$, such that $\Pi'' \in L_1[x_1, x_2]$. If $|\Pi''|$ is $(s,r)$-convex function in the second sense for some fixed $s \in (0,1]$, then fractional integral inequality is given as:*

$$\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\Big| \le T,$$

*where*

$$T = \begin{cases} \frac{c(r)(x_2-x_1)^2}{2(\alpha+1)}\Big[\frac{r^2(\alpha+1)}{(s+r)(s+r(\alpha+2))} \\ \quad - B\Big(\frac{s}{r}+1, \alpha+2\Big)\Big]\left(|\Pi''(x_1)| + |\Pi''(x_2)|\right), & r > 0, \\ \frac{(x_2-x_1)^2}{2(\alpha+1)}E(x_1,x_2,s)\Bigg[\frac{1}{s|\Pi''(x_2)|}\Big(\frac{|\Pi''(x_1)|^s - |\Pi''(x_2)|^s}{\ln|\Pi''(x_1)| - \ln|\Pi''(x_2)|}\Big) \\ \quad - \sum_{i=1}^{\infty}\frac{\left(\ln\left|\frac{\Pi''(x_1)|}{\Pi''(x_2)}\right|^s\right)^{i-1}}{(\alpha+2)_i} \\ \quad - \left|\frac{\Pi''(x_1)|}{\Pi''(x_2)}\right|^s\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left(\ln\left|\frac{\Pi''(x_1)|}{\Pi''(x_2)}\right|^s\right)^{i-1}}{(\alpha+2)_i}\Bigg], & r = 0 \text{ and } |\Pi''(x_1)| \ne |\Pi''(x_2)|, \\ \frac{\alpha(x_2-x_1)^2}{2(\alpha+1)(\alpha+2)}N(x_2,s), & r = 0 \text{ and } |\Pi''(x_1)| = |\Pi''(x_2)|, \end{cases}$$

*in which*

$$c(r) = \begin{cases} 1, & r \ge 1, \\ 2^{\frac{1}{r}-1}, & 0 < r \le 1, \end{cases}$$

$$E(x_1,x_2,s) = \begin{cases} |\Pi''(x_2)|^s, & |\Pi''(x_1)|, |\Pi''(x_2)| \le 1, \\ |\Pi''(x_2)|, & |\Pi''(x_1)| \le 1 \le |\Pi''(x_2)|, \\ |\Pi''(x_1)|^{1-s}|\Pi''(x_2)|^s, & |\Pi''(x_2)| \le 1 \le |\Pi''(x_1)|, \\ |\Pi''(x_1)|^{1-s}|\Pi''(x_2)|, & |\Pi''(x_1)|, |\Pi''(x_2)| \ge 1, \end{cases}$$

$$N(x_2,s) = \begin{cases} |\Pi''(x_2)|^s, & |\Pi''(x_1)| = |\Pi''(x_2)| \le 1, \\ |\Pi''(x_2)|^{2-s}, & |\Pi''(x_1)| = |\Pi''(x_2)| \ge 1, \end{cases}$$

*and* $(\alpha+2)_i = \prod_{j=1}^{i-1}(\alpha+2+j)$.

**Theorem 67** ([61]). *Assume that $\Pi$ is as in Theorem 66. If $|\Pi''|^q$ for $q > 1$ is $(s,r)$-convex function in the second sense for some fixed $s \in (0,1]$, then fractional integral inequality is given as for $r \ge 0$ and $\alpha > 0$:*

$$\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\Big| \le T_1,$$

*where*

$$T_1 = \begin{cases} \frac{[c(r)]^{\frac{1}{q}}(x_2-x_1)^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\left(|\Pi''(x_1)|^q+|\Pi''(x_2)|^q\right)^{\frac{1}{q}} \\ \quad\times\left[\frac{r^2(\alpha+1)}{(s+r)(s+r(\alpha+2))}-B\left(\frac{s}{r}+1,\alpha+2\right)\right]^{\frac{1}{q}}, & r>0, \\ \frac{(x_2-x_1)^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}(E(x_1,x_2,s,q))^{\frac{1}{q}} \\ \times\Bigg[\frac{|\Pi''(x_1)|^{qs}-|\Pi''(x_2)|^{qs}}{qs\ln|\Pi''(x_1)|-qs\ln|\Pi''(x_2)|} \\ \quad-|\Pi''(x_2)|^{qs}\sum_{i=1}^{\infty}\frac{\left(\ln\left|\frac{\Pi''(x_1)}{\Pi''(x_2)}\right|^{qs}\right)^{i-1}}{(\alpha+2)_i} \\ -|\Pi''(x_1)|^{qs}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left(\ln\left|\frac{\Pi''(x_1)}{\Pi''(x_2)}\right|^{s}\right)^{i-1}}{(\alpha+2)_i}\Bigg]^{\frac{1}{q}}, & r=0 \text{ and } |\Pi''(x_1)|\neq|\Pi''(x_2)|, \\ \frac{\alpha(x_2-x_1)^2}{2(\alpha+1)(\alpha+2)}(N(x_2,s,q))^{\frac{1}{q}}, & r=0 \text{ and } |\Pi''(x_1)|=|\Pi''(x_2)|, \end{cases}$$

*in which*

$$E(x_1,x_2,s,q) = \begin{cases} |\Pi''(x_2)|^{qs}, & |\Pi''(x_1)|,|\Pi''(x_2)|\leq 1, \\ |\Pi''(x_2)|^{q}, & |\Pi''(x_1)|\leq 1\leq|\Pi''(x_2)|, \\ |\Pi''(x_1)|^{q(1-s)}|\Pi''(x_2)|^{qs}, & |\Pi''(x_2)|\leq 1\leq|\Pi''(x_1)|, \\ |\Pi''(x_1)|^{q(1-s)}|\Pi''(x_2)|^{q}, & |\Pi''(x_1)|,|\Pi''(x_2)|\geq 1, \end{cases}$$

$$N(x_2,s) = \begin{cases} |\Pi''(x_2)|^{qs}, & |\Pi''(x_1)|=|\Pi''(x_2)|\leq 1, \\ |\Pi''(x_2)|^{q(2-s)}, & |\Pi''(x_1)|=|\Pi''(x_2)|\geq 1. \end{cases}$$

*and* $c(r)$, $(\alpha+2)_i$ *as in Theorem 66.*

**Theorem 68** ([61])**.** *Assume that* $\Pi$ *is as in Theorem 66. If* $|\Pi''|^q$, $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *is* $(s,r)$*-convex function in the second sense for some fixed* $s\in(0,1]$*, then fractional integral inequality is given as with* $\alpha>0$*:*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)]\right|\leq T_2,$$

*where*

$$T_2 = \begin{cases} \frac{[c(r)]^{\frac{1}{q}}(x_2-x_1)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \\ \quad\times\left(\frac{r}{s+r}\right)^{\frac{1}{q}}\left(|\Pi''(x_1)|^q+|\Pi''(x_2)|^q\right)^{\frac{1}{q}}, & r>0, \\ \frac{(x_2-x_1)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}(E(x_1,x_2,s,q))^{\frac{1}{q}} \\ \quad\times\left[\frac{|\Pi''(x_1)|^{qs}-|\Pi''(x_2)|^{qs}}{qs\ln|\Pi''(x_1)|-qs\ln|\Pi''(x_2)|}-|\Pi''(x_2)|^{qs}\right]^{\frac{1}{q}}, & r=0 \text{ and } |\Pi''(x_1)|\neq|\Pi''(x_2)| \\ \frac{(x_2-x_1)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}(N(b,s,q))^{\frac{1}{q}}, & r=0 \text{ and } |\Pi''(x_1)|=|\Pi''(x_2)|, \end{cases}$$

*in which* $c(r)$, $E(x_1,x_2,s,q)$ *and* $N(x_2,s,q)$ *are defined in Theorem 67.*

### 2.11. Fractional H-H-Type Inequalities for Arithmetic–Geometric Convex (Or $AG(\log)$-Convex) Functions

Now, we give H-H-type inequalities for $AG(\log)$-convex functions.

**Definition 11** ([62]). *A function $\Pi : \mathbb{I} \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is said to be arithmetic–geometric convex (or $AG(\log)$-convex), if for every $u, v \in \mathbb{I}$ and $t \in [0,1]$, we have*

$$\Pi((1-t)u + tv) \leq \Pi(u)^{1-t}\Pi(v)^t.$$

**Theorem 69** ([62]). *Let $\Pi : [x_1, x_2] \to \mathbb{R}$ be a differentiable on $(x_1, x_2)$ with $x_1 < x_2$. If $|\Pi'| \in L_1[x_1, x_2]$ and $|\Pi'|$ is $AG(log)$-convex function, $|\Pi'(x_1)| > 0$, $|\Pi'(x_1)| \neq |\Pi'(x_2)|$, then for any $0 < \alpha \leq 1, 0 < \lambda \leq 1, p > 1, q > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$, the fractional integral inequality is given as:*

$$\begin{aligned} &\Big|(1+\lambda(1-\alpha))\Pi(x_2) + (1-\lambda\alpha)\Pi(x_1) + \lambda(2\alpha-1)\Pi((1-\alpha)x_2 + \alpha x_1) \\ &-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right]\Big| \\ \leq\;& (x_2-x_1)\Big\{\frac{\lambda\alpha|\Pi'(x_1)|}{\ln|\Pi'(x_2)| - \ln|\Pi'(x_1)|}\Big\{\Big(\frac{|\Pi'(x_2)|}{|\Pi'(x_1)|}\Big)^{1-\alpha} - 1\Big\} \\ &+\frac{\lambda(1-\alpha)|\Pi'(x_1)|}{\ln|\Pi'(x_2)| - \ln|\Pi'(x_1)|}\Big(\frac{|\Pi'(x_2)|}{|\Pi'(x_1)|}\Big)^{1-\alpha}\Big\{\Big(\frac{|\Pi'(x_2)|}{|\Pi'(x_1)|}\Big)^{\alpha} - 1\Big\} \\ &+\Big(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p + 1}\Big)^{\frac{1}{p}}\Big\{1 + \Big(\frac{|\Pi'(x_2)|}{|\Pi'(x_1)|}\Big)^{\frac{1}{2}}\Big\}\Big[\frac{|\Pi'(x_1)|^q}{q(\ln|\Pi'(x_2)| - \ln|\Pi'(x_1)|)}\Big(\Big(\frac{|\Pi'(x_2)|}{|\Pi'(x_1)|}\Big)^{\frac{q}{2}} - 1\Big)\Big]^{\frac{1}{q}}\Big\}. \end{aligned}$$

**Theorem 70** ([62]). *Assume that $\Pi$ is as in Theorem 69. If $|\Pi'|$ is a convex function, then for any $0 < \alpha \leq 1, 0 < \lambda \leq 1$, the fractional integral inequality is given as:*
*Case 1: $\frac{1}{2} \leq \alpha \leq 1$.*

$$\begin{aligned} &\Big|(1+\lambda(1-\alpha))\Pi(x_2) + (1-\lambda\alpha)\Pi(x_1) + \lambda(2\alpha-1)\Pi((1-\alpha)x_2 + \alpha x_1) \\ &-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right]\Big| \\ \leq\;& (x_2-x_1)\Big\{|\Pi'(x_2)|\Big(\frac{\alpha\lambda}{2}(1-\alpha)(3-2\alpha) + \frac{1}{\alpha+1} - \frac{1}{\alpha+2}\Big(\frac{1}{2}\Big)^{\alpha+1}\Big) \\ &+|\Pi'(x_1)|\Big(\frac{\alpha\lambda}{2}(1-\alpha)(2\alpha+1) + \frac{1}{\alpha+1}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha}\Big)\Big\}. \end{aligned}$$

*Case 2: $0 < \alpha < \frac{1}{2}$.*

$$\begin{aligned} &\Big|(1+\lambda(1-\alpha))\Pi(x_2) + (1-\lambda\alpha)\Pi(x_1) + \lambda(2\alpha-1)\Pi((1-\alpha)x_2 + \alpha x_1) \\ &-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right]\Big| \\ \leq\;& (x_2-x_1)\Big\{|\Pi'(x_2)|\Big\{\frac{1}{\alpha+1}\Big(\frac{1}{\alpha+2} - \Big(\frac{1}{2}\Big)^{\alpha}\Big) + \frac{\alpha^{\alpha+1}(3\alpha+2)}{(\alpha+1)(\alpha+2)} + \frac{\alpha\lambda}{2}(1-\alpha)(3-2\alpha)\Big) \\ &+|\Pi'(x_1)|\Big(\frac{1}{\alpha+1}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha}\Big) + \frac{1-(1-\alpha)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{\alpha\lambda}{2}(1-\alpha)(1+2\alpha)\Big)\Big\}. \end{aligned}$$

**Theorem 71** ([62]). *Assume that $\Pi$ is as in Theorem 69. If $|\Pi'| \in L_1[x_1, x_2]$ and $|\Pi'|$ is an s-convex function, then for any $0 < \alpha \le 1$, $0 < \lambda \le 1$, the fractional integral inequality is given as:*

$$\begin{aligned}
&\bigg|(1+\lambda(1-\alpha))\Pi(x_2) + (1-\lambda\alpha)\Pi(x_1) + \lambda(2\alpha-1)\Pi((1-\alpha)x_2 + \alpha x_1) \\
&\quad - \frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right]\bigg| \\
\le\ & (x_2-x_1)\bigg\{\frac{\alpha\lambda}{x_2-x_1}|\Pi((1-\alpha)x_2+\alpha x_1)| + \frac{\lambda(1-\alpha)}{x_2-x_1}|\Pi(x_2) - \Pi((1-\alpha)x_2+\alpha x_1)| \\
&+ \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left[\frac{1}{8}|\Pi'(x_2)|^q + \frac{3}{8}|\Pi'(x_1)|^q\right]^{\frac{1}{q}} + \left[\frac{3}{8}|\Pi'(x_2)|^q + \frac{1}{8}|\Pi'(x_1)|^q\right]^{\frac{1}{q}}\right\}\bigg\},
\end{aligned}$$

*where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

### 2.12. H-H-Type Inequalities for Logarithmically Convex Functions

**Definition 12** ([63]). *A positive function $\Pi : \mathbb{I} \to \mathbb{R}$ is said to be logarithmically convex if*

$$\Pi(tx + (1-t)y) \le [\Pi(x)]^t[\Pi(y)]^{(1-t)}$$

*for all $x, y \in \mathbb{I}$ and $t \in [0, 1]$.*

In the next, we present H-H-type inequalities for functions whose $n$-th derivatives are logarithmically convex via R–L integral operators.

**Theorem 72** ([64]). *Let $\Pi : [x_1, x_2] \subseteq [0, \infty) \to \mathbb{R}$ be an $n$-times differentiable on $(x_1, x_2)$ where $n$ is a positive integer such that $\Pi^{(n)} \in L_1[x_1, x_2]$ with $\Pi^{(n)}(x_2) \ne 0$. If $|\Pi^{(n)}|$ is logarithmically convex, then fractional integral inequalities are given as:*

$$\left|\frac{\Gamma(\alpha+n)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) - J^\alpha_{x_2-}\Pi(x_1)] - \sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(x_2-x_1)^k}{2\Gamma(\alpha+k+1)}[\Pi^{(k)}(x_1) + (-1)^k\Pi^{(k)}(x_2)]\right|$$
$$\le \begin{cases}
\dfrac{(x_2-x_1)^n|\Pi^{(n)}(x_2)|}{\alpha+n}, & n \text{ even},\ \lambda = 1, \\
\dfrac{(x_2-x_1)^n(2^{\alpha+n-1}-1)|\Pi^{(n)}(x_2)|}{(\alpha+n)2^{\alpha+n-1}}, & n \text{ odd},\ \lambda = 1, \\
\dfrac{(x_2-x_1)^n|\Pi^{(n)}(x_2)|}{2}\left(\displaystyle\sum_{i=1}^{\infty}\frac{(\ln\lambda)^{i-1}}{(\alpha+n)_i} + \lambda\sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(\alpha+n)_i}\right), & n \text{ even},\ \lambda \ne 1, \\
\dfrac{(x_2-x_1)^n|\Pi^{(n)}(x_2)|}{2}\left(\displaystyle\sum_{i=1}^{\infty}\frac{(\ln\lambda)^{i-1}}{(\alpha+n)_i} - \frac{\sqrt{\lambda}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\frac{(\frac{1}{2}\ln\lambda)^{i-1}}{(\alpha+n)_i}\right. & \\
\quad - \sqrt{\lambda}\displaystyle\sum_{i=1}^{\infty}\frac{(-\ln\sqrt{\lambda})^{i-1}}{(\alpha+n)_i} + \lambda\sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(\alpha+n)_i} & \\
\quad - \sqrt{\lambda}\displaystyle\sum_{i=1}^{\infty}\frac{(-\ln\sqrt{\lambda})^{i-1}}{(\alpha+n)_i} - \frac{\sqrt{\lambda}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\frac{(\frac{1}{2}\ln\lambda)^{i-1}}{(\alpha+n)_i}, & n \text{ odd},\ \lambda \ne 1,
\end{cases}$$

*where $\lambda = \left|\dfrac{\Pi^{(n)}(x_1)}{\Pi^{(n)}(x_2)}\right|$ and $(\alpha)_i = \displaystyle\prod_{j=0}^{i-1}(\alpha + j)$.*

**Theorem 73** ([64]). *Assume that $\Pi$ is as in Theorem 72. If $|\Pi^{(n)}|^q$, $q > 1$ is logarithmically convex, then fractional integral inequalities are given as:*

$$\left|\frac{\Gamma(\alpha+n)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2)-J^\alpha_{x_2-}\Pi(x_1)]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(x_2-x_1)^k}{2\Gamma(\alpha+k+1)}[\Pi^{(k)}(x_1)+(-1)^k\Pi^{(k)}(x_2)]\right|$$
$$\leq\begin{cases}\dfrac{(x_2-x_1)^n|\Pi^{(n)}(x_2)|}{\alpha+n}, & n \text{ even}, \lambda=1,\\ \dfrac{(x_2-x_1)^n(2^{\alpha+n-1}-1)|\Pi^{(n)}(x_2)|}{(\alpha+n)2^{\alpha+n-1}}, & n \text{ odd}, \lambda=1,\\ \dfrac{(x_2-x_1)^n|\Pi^{(n)}(x_2)|}{2^{\frac{1}{q}}(\alpha+n)^{1-\frac{1}{q}}}\left(\sum_{i=1}^{\infty}\dfrac{(\ln\lambda)^{i-1}}{(\alpha+n)_i}+\lambda^q\sum_{i=1}^{\infty}\dfrac{(-\ln\lambda^q)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}, & n \text{ even}, \lambda\neq1,\\ \dfrac{(x_2-x_1)^n|\Pi^{(n)}(x_2)|}{2}\left(\dfrac{2^{\alpha+n-1}-1}{(\alpha+n)2^{\alpha+n-2}}\right)^{1-\frac{1}{q}}\left(\sum_{i=1}^{\infty}\dfrac{(\ln\lambda^q)^{i-1}}{(\alpha+n)_i}\right. & \\ -\dfrac{\lambda^{\frac{q}{2}}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\dfrac{(\frac{1}{2}\ln\lambda^q)^{i-1}}{(\alpha+n)_i}-\lambda^{\frac{q}{2}}\sum_{i=1}^{\infty}\dfrac{(-\ln\lambda^{\frac{q}{2}})^{i-1}}{(\alpha+n)_i}+\lambda^q\sum_{i=1}^{\infty}\dfrac{(-\ln\lambda^q)^{i-1}}{(\alpha+n)_i} & \\ \left.-\lambda^{\frac{q}{2}}\sum_{i=1}^{\infty}\dfrac{(-\ln\lambda^{\frac{q}{2}})^{i-1}}{(\alpha+n)_i}-\dfrac{\lambda^{\frac{q}{2}}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\dfrac{(\frac{1}{2}\ln\lambda^q)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}, & n \text{ odd}, \lambda\neq1,\end{cases}$$

*where* $\lambda=\left|\dfrac{\Pi^{(n)}(x_1)}{\Pi^{(n)}(x_2)}\right|.$

**Theorem 74** ([64]). *Assume that $\Pi$ is as in Theorem 72. If $|\Pi^{(n)}|^q$, $q>1$ is logarithmically convex, then fractional integral inequalities are given as:*

$$\left|\frac{\Gamma(\alpha+n)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2)-J^\alpha_{x_2-}\Pi(x_1)]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(x_2-x_1)^k}{2\Gamma(\alpha+k+1)}[\Pi^{(k)}(x_1)+(-1)^k\Pi^{(k)}(x_2)]\right|$$
$$\leq\begin{cases}\dfrac{2^{\frac{1}{p}}(x_2-x_1)^n|\Pi^{(n)}(x_2)|}{(p(\alpha+n-1)+1)^{\frac{1}{p}}}, & n \text{ even}, \lambda=1,\\ \dfrac{(x_2-x_1)^n(2^{p(\alpha+n)-1}-1)^{\frac{1}{p}}|\Pi^{(n)}(x_2)|}{(p(\alpha+n-1)+1)^{\frac{1}{p}}2^{\alpha+n-\frac{1}{p}}}, & n \text{ odd}, \lambda=1,\\ \dfrac{2^{\frac{1}{p}}(x_2-x_1)^n|\Pi^{(n)}(x_2)|}{(p(\alpha+n-1)+1)^{\frac{1}{p}}}\left(\dfrac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}, & n \text{ even}, \lambda\neq1,\\ -\dfrac{(2^{p(\alpha+n-1)}-1)^{\frac{1}{p}}(x_2-x_1)^n}{(p(\alpha+n-1)+1)^{\frac{1}{p}}2^{\alpha+n-\frac{1}{p}}}\left(\dfrac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}, & n \text{ odd}, \lambda\neq1,\end{cases}$$

*where* $\lambda=\left|\dfrac{\Pi^{(n)}(x_1)}{\Pi^{(n)}(x_2)}\right|$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1.$

### *2.13. Fractional H-H-Type Inequalities for $(\alpha,m)$-Logarithmically Convex Functions*

**Definition 13** ([65]). *A function $\Pi:[0,x_2]\to\mathbb{R}^+$ is said to be $(\alpha,m)$-logarithmically convex if for every $x,y\in[0,x_2]$, $(\alpha,m)\in(0,1]^2$, and $t\in[0,1]$, we have*

$$\Pi(tx+(1-t)y)\leq(\Pi(x))^{t^\alpha}+(\Pi(y))^{m(1-t^\alpha)}.$$

**Theorem 75** ([66]). *Let $\Pi:[0,x_2]\to\mathbb{R}$ be a differentiable mapping. If $|\Pi'|$ is measurable and $(\alpha,m)$-logarithmically convex on $[x_1,x_2]$ for some fixed $\alpha\in(0,1]$, $0\leq x_1<mx_2\leq x_2$, then fractional integral inequality is given as:*

$$\left|\frac{\Gamma(\alpha+1)}{2(mx_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(mx_2)+J^\alpha_{mx_2-}\Pi(x_1)]-\Pi\Big(\frac{x_1+mx_2}{2}\Big)\right|\leq I_k,$$

*where*

$$I_k=\begin{cases}(mx_2-x_1)|\Pi'(x_2)|^m(1+2^{1-\alpha})\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}\alpha^{i-1}(k2^{i\alpha+1-\alpha}-k^{2^{-\alpha}})}{[\alpha;i-1]2^{i\alpha+2-\alpha}} & \\ \quad+(mx_2-x_1)|\Pi'(x_1)|^m\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}\alpha^{i-1}(k^{2^{-\alpha}}-k2^{i\alpha})}{[\alpha;i]2^{i\alpha}}, & k\neq 1,\\ \frac{(mx_2-x_1)|\Pi'(x_2)|^m(\alpha2^{\alpha-1}+2^{\alpha-1}+1-2^\alpha)}{(\alpha+1)2^\alpha}, & k=1,\end{cases}$$

*and* $k=\frac{|\Pi'(x_1)|}{|\Pi'(x_2)|^m}$, $[\alpha;0]=1,\ [\alpha;i]=(\alpha+1)(2\alpha+1)\cdots(i\alpha+1),\ i\in\mathbb{N}$.

**Theorem 76** ([66])**.** *Assume that* $\Pi$ *is as in Theorem 75. If* $|\Pi'|^q$ *is measurable and* $(\alpha,m)$*-logarithmically convex, then fractional integral inequality is given as:*

$$\left|\frac{\Gamma(\alpha+1)}{2(mx_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(mx_2)+J^\alpha_{mx_2-}\Pi(x_1)]-\Pi\Big(\frac{x_1+mx_2}{2}\Big)\right|\leq I_k,$$

*where*

$$I_k=\begin{cases}\frac{(mx_2-x_1)|\Pi'(x_1)|}{2^\alpha(p\alpha+1)^{\frac{1}{p}}}\left[\sum_{i=1}^{\infty}\frac{(mq\ln|\Pi'(x_2)|-q\ln|\Pi'(x_1)|)^{i-1}\alpha^{i-1}}{[\alpha;i-1]}\right]^{\frac{1}{q}}, & k\neq 1,\\ \frac{(mx_2-x_1)|\Pi'(x_2)|^m}{2^\alpha(p\alpha+1)}, & k=1,\end{cases}$$

*and* $k=\frac{|\Pi'(x_1)|^q}{|\Pi'(x_2)|^{mq}}$, $\frac{1}{p}+\frac{1}{q}=1$.

**Theorem 77** ([67])**.** *Assume that* $\Pi$ *is as in Theorem 75. If* $|\Pi'|$ *is measurable and* $(\alpha,m)$*-logarithmically convex, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(mx_2)+\Pi(x_1)}{2}-\frac{\Gamma(\alpha+1)}{2(mx_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(mx_2)+J^\alpha_{mx_2-}\Pi(x_1)]\right|\leq I_k,$$

*where*

$$I_k=\begin{cases}\frac{(mx_2-x_1)|\Pi'(x_2)|^m}{\alpha}\sum_{i=1}^{\infty}\left[\frac{(-\ln k)^{i-1}\alpha^i}{[\alpha;i]}\Big(k-\frac{k^{2^{-\alpha}}}{2^{i\alpha}}\Big)\right. & \\ \quad\left.+\frac{\alpha^i(-\ln k)^{i-1}}{[\alpha;i-1]}\Big(\frac{k^{2^{-\alpha}}}{2^{i\alpha+1}}+\frac{k^{2^{-\alpha}}}{2^{i\alpha-\alpha+2}}-\frac{k}{2}\Big)\right], & k\neq 1,\\ \frac{|\Pi'(x_2)|^m(mx_2-x_1)(2^\alpha-1)}{(\alpha+1)2^\alpha}, & k=1,\end{cases}$$

*and* $k=\frac{|\Pi'(x_1)|}{|\Pi'(x_2)|^m}$, $[\alpha;0]=1,\ [\alpha;i]=(\alpha+1)(2\alpha+1)\cdots(i\alpha+1),\ i\in\mathbb{N}$.

**Theorem 78** ([67])**.** *Assume that* $\Pi$ *is as in Theorem 75. If* $|\Pi'|^q$ *is measurable and* $(\alpha,m)$*-logarithmically convex, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(mx_2)+\Pi(x_1)}{2}-\frac{\Gamma(\alpha+1)}{2(mx_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(mx_2)+J^\alpha_{mx_2-}\Pi(x_1)]\right|\leq I_k,$$

*where*

$$I_k = \begin{cases} \frac{(mx_2 - x_1)|\Pi'(x_1)|}{2^{\frac{1}{q}}\alpha^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[\sum_{i=1}^{\infty}\frac{(mq\ln|\Pi'(x_2)| - q\ln|\Pi'(x_1)|)^{i-1}\alpha^i}{[\alpha;i-1]}\right]^{\frac{1}{q}}, & k \neq 1, \\ \frac{(x_2 - x_1)|\Pi'(x_2)|^m}{2^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}, & k = 1, \end{cases}$$

*and* $k = \frac{|\Pi'(x_1)|^q}{|\Pi'(x_2)|^{mq}}, \frac{1}{p} + \frac{1}{q} = 1.$

Now we present H-H-type inequalities for twice differentiable $(\alpha, m)$-logarithmically convex functions.

**Theorem 79** ([67])**.** *Let $\Pi : [0, x_2] \to \mathbb{R}$ be differentiable mapping. If $|\Pi''|$ is measurable and $(\alpha, m)$-logarithmically convex on $[x_1, x_2]$ for some fixed $\alpha \in (0, 1]$, $0 \leq x_1 < mb \leq x_2$, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(mx_2) + \Pi(x_1)}{2} - \frac{\Gamma(\alpha+1)}{2(mx_2 - x_1)^\alpha}[J^\alpha_{x_1+}\Pi(mx_2) + J^\alpha_{mx_2-}\Pi(x_1)]\right| \leq I_k,$$

*where*

$$I_k = \begin{cases} \frac{|\Pi''(x_1)|(mx_2 - x_1)^2(2^\alpha - 1)}{2^{\alpha+1}(\alpha+1)}\sum_{i=1}^{\infty}\frac{(m\alpha\ln|\Pi''(x_2)| - \alpha\ln|\Pi''(x_1)|)^{i-1}}{[\alpha;i-1]}, & k \neq 1, \\ \frac{(mx_2 - x_1)^2|\Pi''(x_2)|^m}{2(\alpha+1)}\left(1 - \frac{2}{p\alpha+p+1}\right)^{\frac{1}{p}}, & k = 1, \end{cases}$$

*and* $k = \frac{|\Pi''(x_1)|}{|\Pi''(x_2)|^m}.$

**Theorem 80** ([67])**.** *Assume that $\Pi$ is as in Theorem 79. If $|\Pi''|^q$ is measurable and $(\alpha, m)$-logarithmically convex, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(mx_2) + \Pi(x_1)}{2} - \frac{\Gamma(\alpha+1)}{2(mx_2 - x_1)^\alpha}[J^\alpha_{x_1+}\Pi(mx_2) + J^\alpha_{mx_2-}\Pi(x_1)]\right| \leq I_k,$$

*where*

$$I_k = \begin{cases} \frac{|\Pi''(x_1)|(mx_2 - x_1)^2}{2(\alpha+1)}\left(1 - \frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left[\sum_{i=1}^{\infty}\frac{(mq\alpha\ln|\Pi'(x_2)| - q\alpha\ln|\Pi'(x_1)|)^{i-1}}{[\alpha;i-1]}, \right. & k \neq 1, \\ \frac{(mx_2 - x_1)^2|\Pi''(x_2)|^m}{2(\alpha+1)}\left(1 - \frac{1}{p\alpha}\right)^{\frac{1}{p}}, & k = 1, \end{cases}$$

*and* $k = \frac{|\Pi''(x_1)|^q}{|\Pi''(x_2)|^{mq}}, \frac{1}{p} + \frac{1}{q} = 1.$

*2.14. Fractional H-H-Type Inequalities for Geometric–Arithmetically s-Convex Functions (GA-s-Convex Functions)*

In this subsection we present fractional H-H-type inequalities for GA-$s$-convex functions.

**Definition 14** ([68])**.** *The function $\Pi : \mathbb{I} \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is said to be GA-s-convex on $\mathbb{I}$, if, for every $x, y \in \mathbb{I}$ and $t \in [0, 1]$ we have*

$$\Pi(x^t + y^{1-t}) \leq t^s(\Pi(x)) + (1-t)^s\Pi(y).$$

**Theorem 81** ([69]). *Let $\Pi : [0, x_2] \to \mathbb{R}$ be a differentiable mapping. If $|\Pi'|$ is measurable and $|\Pi'|$ is decreasing and GA-s-convex on $[0, x_2]$ for some fixed $\alpha > 0$, $s \in (0, 1]$, $0 \leq x_1 < x_2$, then fractional integral inequality is given as:*

$$\begin{aligned} & \left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^\alpha} [J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] \right| \\ \leq & \frac{(x_2 - x_1)(2^{\alpha+s}|\Pi'(x_2)| - |\Pi'(x_1)| - |\Pi'(x_2)|}{(\alpha+s+1)2^{\alpha+s+1}} \\ & +(x_2 - x_1)|\Pi'(x_1)|[0.5B(s+1, \alpha+1) - B_{0.5}(\alpha+1, s+1)] \\ & +(x_2 - x_1)|\Pi'(x_2)|[0.5B(s+1, \alpha+1) - 0.5B(s+1, \alpha+1)], \end{aligned}$$

*where $B(s+1, \alpha+1) = \int_0^1 t^s (1-t)^\alpha dt$ and $B_{0.5}(s+1, \alpha+1) = \int_0^{0.5} t^s (1-t)^\alpha dt$.*

**Theorem 82** ([69]). *Assume that $\Pi$ is as in Theorem 81. If $|\Pi'|^q$ is measurable and $|\Pi'|^q$ is decreasing and GA-s-convex on $[0, x_2]$, then fractional integral inequality is given as:*

$$\begin{aligned} & \left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^\alpha} [J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] \right| \\ \leq & \frac{x_2 - x_1}{2} \left( \frac{|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{s+1} \right)^{\frac{1}{q}} \left( \frac{2 - 2^{1-p\alpha}}{p\alpha+1} \right)^{\frac{1}{q}}, \end{aligned}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$.*

**Theorem 83** ([69]). *Assume that $\Pi$ is as in Theorem 81. If $|\Pi'|$ is measurable and $|\Pi'|$ is non-increasing and GA-s-convex on $[0, x_2]$, then fractional integral inequality is given as:*

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^\alpha} [J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] - \Pi\left(\frac{x_1 + x_2}{2}\right) \\ \leq & 0.5(x_2 - x_1)(|\Pi'(x_1)| + |\Pi'(x_2)|) \\ & \times \left( B_{0.5}(\alpha+1, s+1) - B_{0.5}(s+1, \alpha+1) + \frac{2^{-\alpha-1}}{\alpha+s+1} + \frac{1}{s+1} \right). \end{aligned}$$

**Theorem 84** ([69]). *Assume that $\Pi$ is as in Theorem 81. If $|\Pi'|^q$ is measurable and $|\Pi'|^q$ is non-increasing and GA-s-convex on $[0, x_2]$, then fractional integral inequality is given as:*

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^\alpha} [J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] - \Pi\left(\frac{x_1 + x_2}{2}\right) \\ \leq & \max\left\{ \frac{x_2 - x_1}{2} \left( \frac{|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{s+1} \right)^{\frac{1}{q}} \left( \frac{(1+2^{1-\alpha})^p}{2} - \frac{2^p(1-2^{-p\alpha})}{p\alpha+1} \right)^{\frac{1}{p}}, \right. \\ & \left. (x_2 - x_1) \left( \frac{|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{s+1} \right)^{\frac{1}{q}} \left( 2^{p-1} - \frac{2^p(1-2^{-p\alpha-1})}{p\alpha+1} \right)^{\frac{1}{p}} \right\}, \end{aligned}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$.*

**Theorem 85** ([69]). *Assume that $\Pi$ is as in Theorem 81. If $|\Pi'|$ is measurable and $|\Pi'|$ is non-increasing and GA-s-convex on $[0, x_2]$, then fractional integral inequality is given as:*

$$\frac{(x - x_1)^\alpha + (x_2 - x)^\alpha}{x_2 - x_1} \Pi(x) - \frac{\Gamma(\alpha+1)}{(x_2 - x_1)} [J^\alpha_{x-}\Pi(x_1) + J^\alpha_{x+}\Pi(x_2)]$$

$$\leq \frac{(x-x_1)^{\alpha+1}+(x_2-x)^{\alpha+1}}{x_2-x_1}\Big[\frac{|\Pi'(x_1)|}{\alpha+s+1}+|\Pi'(x_2)|B(\alpha+1,s+1)\Big],$$

*where* $B(s+1,\alpha+1)=\int_0^1 t^s(1-t)^\alpha dt.$

**Theorem 86** ([69]). *Assume that* $\Pi$ *is as in Theorem 81. If* $|\Pi'|^q$ *is measurable and* $|\Pi'|^q$ *is non-increasing and GA-s-convex on* $[0,x_2]$*, then fractional integral inequality is given as:*

$$\frac{(x-x_1)^\alpha+(x_2-x)^\alpha}{x_2-x_1}\Pi(x)-\frac{\Gamma(\alpha+1)}{(x_2-x_1)}[J^\alpha_{x-}\Pi(x_1)+J^\alpha_{x+}\Pi(x_2)]$$
$$\leq \frac{x_2-x_1}{2(p\alpha+1)}\Big(\frac{|\Pi'(x_1)|^q+|\Pi'(x_2)|^q}{s+1}\Big)^{\frac{1}{q}}.$$

**Theorem 87** ([70]). *Let* $\Pi:[0,x_2]\to\mathbb{R}$ *be a differentiable mapping. If* $|\Pi''|$ *is measurable and* $|\Pi''|$ *is decreasing and GA-s-convex on* $[0,x_2]$ *for some fixed* $\alpha>0, s\in(0,1], 0\leq x_1<x_2$*, then fractional integral inequality is given as:*

$$\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)]\Big|$$
$$\leq \frac{(x_2-x_1)^2(|\Pi''(x_1)|+|\Pi''(x_2)|)}{2(\alpha+1)}\Big(\frac{1}{s+1}-\frac{1}{\alpha+s+2}\Big).$$

**Theorem 88** ([70]). *Assume that* $\Pi$ *is as in Theorem 87. If* $|\Pi''|^q$ *is measurable and* $|\Pi''|^q$ *is non-increasing and GA-s-convex on* $[0,x_2]$*, then fractional integral inequality is given as:*

$$\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)]\Big|$$
$$\leq \frac{(x_2-x_1)^2\max\{1-2^{1-\alpha},2^{1-\alpha}-1\}}{2(\alpha+1)}\Big(\frac{|\Pi''(x_1)|^q+|\Pi''(x_2)|^q}{s+1}\Big)^{\frac{1}{q}}.$$

**Theorem 89** ([70]). *Assume that* $\Pi$ *is as in Theorem 87. If* $|\Pi''|$ *is measurable and* $|\Pi''|$ *is non-increasing and GA-s-convex on* $[0,x_2]$*, then fractional integral inequality is given as:*

$$\Big|\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)]-\Pi\Big(\frac{x_1+x_2}{2}\Big)\Big|$$
$$\leq \frac{(x_2-x_1)^2|\Pi''(x_1)|}{2(\alpha+1)}\Big[\frac{\alpha-\alpha 2^{-s-1}-2^{-s-1}}{s+1}-\frac{\alpha+1}{s+2}+2B(s+1,\alpha+2)+\frac{1}{\alpha+s+2}\Big]$$
$$+\frac{(x_2-x_1)^2|\Pi''(x_2)|}{2(\alpha+1)}\Big[\frac{\alpha 2^{-s-1}+2^{-s-1}}{s+1}+\frac{1}{\alpha+s+2}+2B(\alpha+2,s+1)\Big],$$

*where* $B(s+1,\alpha+2)=\int_0^1 t^s(1-t)^{\alpha+1}dt.$

**Theorem 90** ([70]). *Assume that* $\Pi$ *is as in Theorem 87. If* $|\Pi''|$ *is measurable and* $1<q<\infty$*. If* $|\Pi''|^q$ *is non-increasing and GA-s-convex on* $[0,x_2]$*, then fractional integral inequality is given as:*

$$\Big|\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)]-\Pi\Big(\frac{x_1+x_2}{2}\Big)\Big|$$
$$\leq \frac{(x_2-x_1)^2}{2(\alpha+1)}\Big(\frac{|\Pi''(x_1)|^q+|\Pi''(x_2)|^q}{s+1}\Big)^{\frac{1}{q}}\Big(\frac{(\alpha+1)2^{-p-1}+(\alpha+0.5)^{p+1}-\alpha^{p+1}}{p+1}\Big)^{\frac{1}{p}},$$

*where* $\frac{1}{p}+\frac{1}{q}=1.$

### 2.15. Fractional H-H-Type Inequalities for s-Convex Functions

Fractional H-H-type inequalities for $s$-convex functions are presented in this subsection.

**Definition 15** ([71])**.** *A function $\Pi : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$\Pi(tx + (1-t)y) \leq t^s \Pi(x) + (1-t)^s \Pi(y),$$

*for all $x, y \in [0, \infty)$, $t \in [0,1]$ and for some fixed $s \in (0,1]$.*

**Theorem 91** ([72])**.** *Let $\Pi : [x_1, x_2] \to \mathbb{R}$ be a positive function with $0 \leq x_1 < x_2$ and $\Pi \in L_1[x_1, x_2]$. If $\Pi$ is a s-convex mapping in the second sense on $[x_1, x_2]$, then fractional integral inequalities are given as*

$$\begin{aligned} 2^{s-1}\Pi\Big(\frac{x_1+x_2}{2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] \\ &\leq \Big[\frac{1}{\alpha+s} + B(\alpha, s+1)\Big]\frac{\Pi(x_1)+\Pi(x_2)}{2}, \end{aligned} \tag{3}$$

*with $\alpha > 0$ and $s \in (0,1)$ and $B$ represents the Euler Beta function.*

**Theorem 92** ([72])**.** *Let $\Pi : [x_1, x_2] \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $(x_1, x_2)$ with $x_1 < x_2$ such that $\Pi' \in L_1[x_1, x_2]$. If $|\Pi|^q$ is a s-convex in the second sense on $[x_1, x_2]$, for some fixed $s \in (0,1)$ and $q \geq 1$, then fractional integral inequality is given as:*

$$\begin{aligned} &\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)]\Big| \\ \leq\;& \frac{x_2-x_1}{2}\Big[\frac{2}{\alpha+1}\Big(1-\frac{1}{2^\alpha}\Big)\Big]^{\frac{q-1}{q}}\Big\{B\Big(\frac{1}{2}; s+1, \alpha+1\Big) - \Big(\frac{1}{2}; \alpha+1, s+1\Big) \\ &+ \frac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\Big\}\Big(|\Pi'(x_1)|^q + |\Pi'(x_2)|^q\Big)^{\frac{1}{q}}. \end{aligned}$$

**Theorem 93** ([73])**.** *Assume that $\Pi$ is as in Theorem 92. If $|\Pi'|^q$ is an s-convex function on $[x_1, x_2]$, then fractional integral inequality is given as:*

$$\begin{aligned} &\Big|(1-\mu)\Big[\frac{(x-x_1)^\alpha + (x_2-x)^\alpha}{x_2-x_1}\Big]\Pi(x) + \mu\Big[\frac{(x-x_1)^\alpha\Pi(x_1) + (x_2-x)^\alpha\Pi(x_2)}{x_2-x_1}\Big] \\ &- \frac{\Gamma(\alpha+1)}{x_2-x_1}[J^\alpha_{x-}\Pi(x_1) + J^\alpha_{x+}\Pi(x_2)]\Big| \\ \leq\;& A_1^{1-\frac{1}{q}}(\alpha,\mu)\Big\{\frac{(x-x_1)^{\alpha+1}}{x_2-x_1}\Big(|\Pi'(x)|^q A_2(\alpha,\mu,s) + |\Pi'(x_1)|^q A_3(\alpha,\mu,s)\Big)^{\frac{1}{q}} \\ &+ \frac{(x_2-x)^{\alpha+1}}{x_2-x_1}\Big(|\Pi'(x)|^q A_2(\alpha,\mu,s) + |\Pi'(x_2)|^q A_3(\alpha,\mu,s)\Big)^{\frac{1}{q}}\Big\}, \end{aligned}$$

*where*

$$\begin{aligned} A_1(\alpha,\mu) &= \frac{2\alpha\mu^{1+\frac{1}{\alpha}}+1}{\alpha+1} - \mu, \\ A_2(\alpha,\mu,s) &= \frac{2\alpha\mu^{1+\frac{s+1}{\alpha}}+s+1}{(s+1)(\alpha+s+1)} - \frac{\mu}{s+1}, \end{aligned}$$

$$A_3(\alpha,\mu,s) = \mu\left[\frac{1-2\left(1-\mu^{\frac{1}{\alpha}}\right)^{s+1}}{s+1}\right] + B(\alpha+1,s+1) - 2B\left(\mu^{\frac{1}{\alpha}},\alpha+1,s+1\right).$$

**Theorem 94** ([74]). *Let $\Pi : [x_1,x_2] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on $(x_1,x_2)$ with $x_1 < x_2$ such that $|\Pi'| \in L_1[x_1,x_2]$ and $|\Pi'|$ is an s-convex function. Then for some $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)(1-\alpha(1-\lambda)) + \Pi(x_2)(1+\alpha(1-\lambda)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}[J^{\alpha}_{x_1+}\Pi(x_2) + J^{\alpha}_{x_2-}\Pi(x_1)]\right|$$
$$\leq \frac{x_2-x_1}{2}\left[|\Pi'(x_2)|\left(\frac{1}{\alpha+s+1} + \frac{\alpha(1-\lambda)}{s+1} - \frac{\Gamma(s+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)}\right)\right.$$
$$\left. + |\Pi'(x_1)|\left(\frac{\Gamma(s+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)} + \frac{\alpha(1-\lambda)}{s+1} - \frac{1}{\alpha+s+1}\right)\right].$$

**Theorem 95** ([74]). *Assume that $\Pi$ is as in Theorem 94. If $|\Pi'|^q \in L_1[x_1,x_2]$ and $|\Pi'|^q$ $(q>1)$ is an s-convex function, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)(1-\alpha(1-\lambda)) + \Pi(x_2)(1+\alpha(1-\lambda)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}[J^{\alpha}_{x_1+}\Pi(x_2) + J^{\alpha}_{x_2-}\Pi(x_1)]\right|$$
$$\leq \frac{x_2-x_1}{2}\left(\alpha(1-\lambda) + 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\right)\left(\frac{|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{1+sq}\right)^{\frac{1}{q}},$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 96** ([75]). *Let $\Pi : \mathbb{I} \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on $\mathbb{I}^{\circ}$ such that $\Pi' \in L_1[x_1,x_2]$, where $x_1,x_2 \in \mathbb{I}$ with $x_1 < x_2$. If $|\Pi'|$ is an s-convex on $[x_1,x_2]$ for some fixed $s \in (0,1]$ and $x \in [x_1,x_2]$, then fractional integral inequality is given with $\alpha > 0$ as:*

$$\left|\frac{(x-x_1)^{\alpha}\Pi(x_1) + (x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1} - \frac{\Gamma(\alpha+1)}{x_2-x_1}\left[J^{\alpha}_{x-}\Pi(x_1) + J^{\alpha}_{x+}\Pi(x_2)\right]\right|$$
$$\leq \frac{\alpha}{(s+1)(\alpha+s+1)}\left[\frac{(x-x_1)^{\alpha+1} + (x_2-x)^{\alpha+1}}{x_2-x_1}\right]|\Pi'(x)|$$
$$+\left[\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left[\frac{(x-x_1)^{\alpha+1}|\Pi'(x_1)| + (x_2-x)^{\alpha+1}|\Pi'(x_2)|}{x_2-x_1}\right].$$

**Theorem 97** ([76]). *Assume that $\Pi$ is as in Theorem 96. If $|\Pi'|$ is an s-convex on $[x_1,x_2]$, then fractional integral inequality is given as:*

$$\left|\Pi\left(\frac{x_1+x_2}{2}\right)\left[J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-}g(x_1) + J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}g(x_2)\right]\right.$$
$$\left. - \left[J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-}(\Pi g)(x_1) + J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}(\Pi g)(x_2)\right]\right|$$
$$\leq \frac{(x_2-x_1)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+1)}\left\{B_{1/2}(\alpha+1,s+1) + \frac{1}{2^{\alpha+s+1}(\alpha+s+1)}\right\}[|\Pi'(x_1)| + |\Pi'(x_2)|].$$

**Theorem 98** ([76]). *Assume that $\Pi$ is as in Theorem 96 and $g : [x_1, x_2] \to \mathbb{R}$ is continuous. If $|\Pi'|^q$ is an s-convex on $[x_1, x_2]$, then fractional integral inequality is given as:*

$$\begin{aligned}
&\Bigg|\Pi\Big(\frac{x_1+x_2}{2}\Big)\Big[J^{\alpha}_{(\frac{x_1+x_2}{2})_-}g(x_1) + J^{\alpha}_{(\frac{x_1+x_2}{2})_+}g(x_2)\Big] \\
&-\Big[J^{\alpha}_{(\frac{x_1+x_2}{2})_-}(\Pi g)(x_1) + J^{\alpha}_{(\frac{x_1+x_2}{2})_+}(\Pi g)(x_2)\Big]\Bigg| \\
\le\ &\frac{(x_2-x_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+2)^{\frac{1}{q}}(\alpha+s+q)^{\frac{1}{q}}\Gamma(\alpha+1)} \\
&\times\Big\{\Big((\alpha+s+1)(\alpha+3)|\Pi'(x_1)|^q + 2^{1-s}(\alpha+1)(\alpha+2)|\Pi'(x_2)|^q\Big)^{\frac{1}{q}} \\
&+\Big(2^{1-s}(\alpha+1)(\alpha+2)|\Pi'(x_1)|^q + (\alpha+s+1)(\alpha+3)|\Pi'(x_2)|^q\Big)^{\frac{1}{q}}\Big\}.
\end{aligned}$$

**Theorem 99** ([77]). *Let $\Pi : \mathbb{I} \subset (0,\infty) \to \mathbb{R}$ be twice differentiable function on $\mathbb{I}^\circ$ such that $\Pi'' \in L_1[x_1, x_2]$, where $x_1, x_2 \in \mathbb{I}$ with $x_1 < x_2$. If $|\Pi''|^q, q \ge 1$ is an s-convex function on $[x_1, x_2]$ for some fixed $s \in (0,1]$, then for $x \in [x_1, x_2]$, $\lambda \in [0,1]$, $\alpha > 0$, and fractional integral inequality is given as:*

$$\begin{aligned}
&\Bigg|(1-\lambda)\Big[\frac{(x-x_1)^\alpha + (x_2-x)^\alpha}{x_2-x_1}\Big]\Pi(x) + \lambda\Big[\frac{(x-x_1)^\alpha\Pi(x_1) + (x_2-x)^\alpha\Pi(x_2)}{x_2-x_1}\Big] \\
&+\Big(\frac{1}{\alpha}-\lambda\Big)\Big[\frac{(x_2-x)^{\alpha+1}-(x-x_1)^{\alpha+1}}{x_2-x_1}\Big]\Pi'(x) - \frac{\Gamma(\alpha+1)}{x_2-x_1}\Big[J^{\alpha}_{x-}\Pi(x_1) + J^{\alpha}_{x+}\Pi(x_2)\Big]\Bigg| \\
\le\ &C_1^{1-\frac{1}{q}}(\alpha,\lambda)\Big\{\frac{(x-x_1)^{\alpha+2}}{(\alpha+1)(x_2-x_1)}\big(|\Pi''(x)|^q C_2(\alpha,\lambda,s) + |\Pi''(x_1)|^q C_3(\alpha,\lambda,s)\big)^{\frac{1}{q}} \\
&+\Big\{\frac{(x_2-x)^{\alpha+2}}{(\alpha+1)(x_2-x_1)}\big(|\Pi''(x)|^q C_2(\alpha,\lambda,s) + |\Pi''(x_2)|^q C_3(\alpha,\lambda,s)\big)^{\frac{1}{q}}\Big\},
\end{aligned}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$ and*

$$C_1(\alpha,\lambda) = \begin{cases} \dfrac{\alpha[(\alpha+1)\lambda]^{1+\frac{2}{\alpha}}}{\alpha+2} - \dfrac{(\alpha+1)\lambda}{2} + \dfrac{1}{\alpha+2}, & 0 \le \lambda \le \dfrac{1}{\alpha+1}, \\ \dfrac{(\alpha+1)\lambda}{2} - \dfrac{1}{\alpha+2}, & \dfrac{1}{\alpha+1} \le \lambda \le 1, \end{cases}$$

$$C_2(\alpha,\lambda,s) = \begin{cases} \dfrac{2\alpha[(\alpha+1)\lambda]^{\frac{\alpha+s+2}{\alpha}}}{(\alpha+2)(\alpha+s+2)} - \dfrac{(\alpha+1)\lambda}{s+2} + \dfrac{1}{\alpha+s+2}, & 0 \le \lambda \le \dfrac{1}{\alpha+1}, \\ \dfrac{(\alpha+1)\lambda}{s+2} - \dfrac{1}{\alpha+s+2}, & \dfrac{1}{\alpha+1} \le \lambda \le 1, \end{cases}$$

$$C_3(\alpha,\lambda,s) = \begin{cases} B(\alpha+2,s+1) - (\alpha+1)\lambda x_2(2,s+1) \\ +2(\alpha+1)\lambda B\big([(\alpha+1)\lambda]^{\frac{1}{\alpha}},2,s+1\big) \\ -2B\big([(\alpha+1)\lambda]^{\frac{1}{\alpha}},\alpha+2,s+1\big), & 0 \le \lambda \le \dfrac{1}{\alpha+1}, \\ (\alpha+1)\lambda B(2,s+1) - B(\alpha+2,s+1), & \dfrac{1}{\alpha+1} \le \lambda \le 1. \end{cases}$$

**Theorem 100** ([77]). *Assume that $\Pi$ is as in Theorem 99. Then fractional integral inequality is given as:*

$$\begin{aligned}&\bigg|(1-\lambda)\Big[\frac{(x-x_1)^\alpha+(x_2-x)^\alpha}{x_2-x_1}\Big]\Pi(x)+\lambda\Big[\frac{(x-x_1)^\alpha\Pi(x_1)+(x_2-x)^\alpha\Pi(x_2)}{x_2-x_1}\Big]\\&+\Big(\frac{1}{\alpha}-\lambda\Big)\Big[\frac{(x_2-x)^{\alpha+1}-(x-x_1)^{\alpha+1}}{x_2-x_1}\Big]\Pi'(x)-\frac{\Gamma(\alpha+1)}{x_2-x_1}\Big[J^\alpha_{x-}\Pi(x_1)+J^\alpha_{x+}\Pi(x_2)\Big]\bigg|\\\leq\;& C_4^{\frac{1}{p}}(\alpha,\lambda,p)\Big\{\frac{(x-x_1)^{\alpha+2}}{(\alpha+1)(x_2-x_1)}\Big(\frac{|\Pi''(x)|^q+|\Pi''(x_1)|^q}{s+1}\Big)^{\frac{1}{q}}\\&+\frac{(x_2-x)^{\alpha+2}}{(\alpha+1)(x_2-x_1)}\Big(\frac{|\Pi''(x)|^q+|\Pi''(x_2)|^q}{s+1}\Big)^{\frac{1}{q}}\Big\},\end{aligned}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$C_4(\alpha,\lambda,p)=\begin{cases}\dfrac{1}{p(\alpha+1)+1}, & \lambda=0,\\ \dfrac{[(\alpha+1)\lambda]^{\frac{1+(\alpha+1)p}{\alpha}}}{\alpha}B\Big(\dfrac{1+p}{\alpha},1+p\Big) & \\ +\dfrac{[1-(\alpha+1)\lambda]^{p+1}}{\alpha(p+1)} & \\ \times\,{}_2F_1\Big(1-\frac{1+p}{\alpha},1;p+2;1-(\alpha+1)\lambda\Big), & 0\leq\lambda\leq\dfrac{1}{\alpha+1},\\ \dfrac{[(\alpha+1)\lambda]^{\frac{p(\alpha+1)+1}{\alpha}}}{\alpha}B\Big(\dfrac{1}{(\alpha+1)\lambda};\dfrac{1+p}{\alpha},1+p\Big), & \dfrac{1}{\alpha+1}\leq\lambda\leq 1.\end{cases}$$

**Theorem 101** ([78]). *For $n\in\mathbb{N}$ and $x_1,x_2\in[0,\infty)$ with $x_1<x_2$, let $\Pi:[0,\infty)\to\mathbb{R}$ be an $n$-times differentiable function on $[0,\infty)$ such that $\Pi^{(n)}\in L_1[x_1,x_2]$. If $|\Pi^{(n)}|^q$ is an s-convex function on $[x_1,x_2]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and some fixed $s\in(0,1]$, then*

$$\begin{aligned}&\bigg|\frac{\Gamma(\alpha+n)}{2(x_2-x_1)^\alpha}\Big[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)\Big]\\&-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(x_2-x_1)^k}{2\Gamma(\alpha+k+1)}\Big]\Pi^{(k)}(x_1)+(-1)^k\Pi^{(k)}(x_2)\Big]\bigg|\\\leq\;&\frac{(x_2-x_1)^n}{2(\alpha+n)^{1-\frac{1}{q}}}\Big\{\Big[B(s+1,\alpha+n)|\Pi^{(n)}(x_1)|^q+\frac{1}{\alpha+n+s}|\Pi^{(n)}(x_2)|^q\Big]^{\frac{1}{q}}\\&+\Big[\frac{1}{\alpha+n+s}|\Pi^{(n)}(x_2)|^q+B(s+1,\alpha+n)|\Pi^{(n)}(x_1)|^q\Big]^{\frac{1}{q}}\Big\},\end{aligned}$$

*where $\alpha>0$ and $B$ is the classical Beta function.*

**Theorem 102** ([78]). *Assume that $\Pi$ is as in Theorem 101, then*

$$\begin{aligned}&\bigg|\frac{\Gamma(\alpha+n)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)]\\&-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(x_2-x_1)^k}{2\Gamma(\alpha+k+1)}[\Pi^{(k)}(x_1)+(-1)^k\Pi^{(k)}(x_2)]\bigg|\end{aligned}$$

$$\leq \frac{(x_2-x_1)^n}{2}\Big[\frac{q-1}{q(\alpha+n)-r-1}\Big]^{1-\frac{1}{q}}\Bigg\{\Big[B(s+1,r+1)|\Pi^{(n)}(x_1)|^q + \frac{1}{r+s+1}|\Pi^{(n)}(x_2)|^q\Big]^{\frac{1}{q}} + \Big[\frac{1}{r+s+1}|\Pi^{(n)}(x_1)|^q + B(s+1,r+n)|\Pi^{(n)}(x_2)|^q\Big]^{\frac{1}{q}}\Bigg\},$$

*where* $\alpha > 0$ *and* $0 \leq r \leq q(\alpha+n-1)$.

### 2.16. Fractional H-H-Type Inequalities for Godunova–Levin-Convex Functions

**Definition 16** ([79]). *A function* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is said to be Godunova–Levin (G-L) function if*

$$\Pi(tx+(1-t)y) \leq \frac{\Pi(x)}{t} + \frac{\Pi(y)}{1-t},$$

*for all* $x, y \in [x_1, x_2]$ *and* $t \in [0,1]$.

**Definition 17** ([80]). *A function* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is said to be s-G-L function of the first kind, where* $s \in (0,1]$, *if*

$$\Pi(tx+(1-t)y) \leq \frac{\Pi(x)}{t^s} + \frac{\Pi(y)}{1-t^s},$$

*for all* $x, y \in [x_1, x_2]$ *and* $t \in [0,1]$.

**Definition 18** ([80]). *A function* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is said to be s-G-L function of the second kind, where* $s \in (0,1]$, *if*

$$\Pi(tx+(1-t)y) \leq \frac{\Pi(x)}{t^s} + \frac{\Pi(y)}{(1-t)^s},$$

*for all* $x, y \in [x_1, x_2]$ *and* $t \in [0,1]$.

**Theorem 103** ([80]). *Let* $\Pi : [x_1, x_2] \to \mathbb{R}$ *be a twice differentiable mapping on* $(x_1, x_2)$ *with* $x_1 < x_2$ *and* $\Pi'' \in L_1[x_1, x_2]$. *If* $|\Pi''|$ *is an s-G-L function of first kind, then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right]\right|\\ \leq\ & \frac{(x_2-x_1)^2}{2(\alpha+1)}\Bigg[\Big(\frac{1}{1-s} - \frac{1}{\alpha-s+2} - \frac{\Gamma(\alpha+2)\Gamma(1-s)}{\Gamma(\alpha-s+3)}\Big)|\Pi''(x_1)|\\ &+\Big(\frac{s+2}{s+1} - \frac{1}{\alpha+2} - \frac{s\Gamma(s)\Gamma(\alpha+2)}{\Gamma(\alpha+s+3)} - \frac{2(\alpha+2)+s}{(\alpha+2)(\alpha+s+2)}\Big)|\Pi''(x_2)|\Bigg].\end{aligned}$$

**Theorem 104** ([80]). *Assume that* $\Pi$ *is as in Theorem 103. If* $|\Pi''|$ *is an s-G-L function of second kind, then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\right]\right|\\ \leq\ & \frac{(x_2-x_1)^2}{2}\Big(\frac{1}{1-s} - \frac{1}{\alpha-s+2} - \frac{\Gamma(\alpha+2)\Gamma(1-s)}{\Gamma(\alpha-s+3)}\Big)\Big[|\Pi''(x_1)| + |\Pi''(x_2)|\Big].\end{aligned}$$

**Theorem 105** ([81]). *Let* $\Pi : [x_1, x_2] \subset \mathbb{R} \to \mathbb{R}$ *be differentiable function such that* $\Pi' \in L_1[x_1, x_2]$ *and* $n \in \mathbb{N}^*$, $\alpha \geq 0$. *If* $|\Pi'|$ *is an s-G-L function where* $0 < s < 1$, *then:*

$$\left|\frac{n+1}{2}\Big[\frac{(x-x_1)^\alpha + (x_2-x)^\alpha}{x_2-x_1}\Pi(x) + \frac{(x-x_1)^\alpha\Pi(x_1) + (x_2-x)^\alpha\Pi(x_2)}{x_2-x_1}\right.$$

$$
\begin{aligned}
&-\frac{(n+1)^{\alpha+1}\Gamma(\alpha+1)}{2(x_2-x_1)}\Big[J^{\alpha}_{x-}\Pi\Big(\frac{n}{n+1}x+\frac{1}{n+1}x_1\Big)+J^{\alpha}_{x_1+}\Pi\Big(\frac{1}{n+1}x+\frac{n}{n+1}x_1\Big)\\
&+J^{\alpha}_{x+}\Pi\Big(\frac{n}{n+1}x+\frac{1}{n+1}x_2\Big)+J^{\alpha}_{x_2-}\Pi\Big(\frac{1}{n+1}x+\frac{n}{n+1}x_1\Big)\Big]\Bigg|\\
\leq\;&\frac{(n+1)^s(A+B)}{2(x_2-x_1)}\Big[(x-x_1)^{\alpha+1}|\Pi'(x_1)|+(x_2-x)^{\alpha+1}|\Pi'(x_2)|\Big],
\end{aligned}
$$

*where*

$$A=\frac{\Gamma(1-s)\Gamma(1+\alpha)}{\Gamma(2-s+\alpha)}\ \text{ and }\ B=\frac{n^{-s}\,{}_2F_1\Big[s,1+\alpha,2+\alpha,-\frac{1}{n}\Big]}{1+\alpha}.$$

**Theorem 106** ([81])**.** *Assume that the $\Pi$ is as in Theorem 105. If $|\Pi'|^q$, $q>1$ is an s-G-L function, then*

$$
\begin{aligned}
&\Bigg|\frac{n+1}{2}\Big[\frac{(x-x_1)^{\alpha}+(x_2-x)^{\alpha}}{x_2-x_1}\Pi(x)+\frac{(x-x_1)^{\alpha}\Pi(x_1)+(x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1}\\
&-\frac{(n+1)^{\alpha+1}\Gamma(\alpha+1)}{2(x_2-x_1)}\Big[J^{\alpha}_{x-}\Pi\Big(\frac{n}{n+1}x+\frac{1}{n+1}x_1\Big)+J^{\alpha}_{x_1+}\Pi\Big(\frac{1}{n+1}x+\frac{n}{n+1}x_1\Big)\\
&+J^{\alpha}_{x+}\Pi\Big(\frac{n}{n+1}x+\frac{1}{n+1}x_2\Big)+J^{\alpha}_{x_2-}\Pi\Big(\frac{1}{n+1}x+\frac{n}{n+1}x_1\Big)\Big]\Bigg|\\
\leq\;&\frac{(n+1)^{\frac{s}{q}}}{2(x_2-x_1)}\Big(\frac{1}{\alpha+1}\Big)^{1-\frac{1}{q}}\Big\{(x-x_1)^{\alpha+1}\big\{\big(B|\Pi'(x)|^q+A|\Pi'(x_1)|^q\big)^{\frac{1}{q}}\\
&+\big(A|\Pi'(x)|^q+B|\Pi'(x_1)|^q\big)^{\frac{1}{q}}\big\}+(x_2-x)^{\alpha+1}\big\{\big(B|\Pi'(x)|^q+A|\Pi'(x_2)|^q\big)^{\frac{1}{q}}\\
&+\big(A|\Pi'(x)|^q+B|\Pi'(x_2)|^q\big)^{\frac{1}{q}}\big\}\Big\},
\end{aligned}
$$

*where A and B are given in the previous theorem.*

**Theorem 107** ([82])**.** *Let $\Pi:[x_1,x_2]\to\mathbb{R}$ be a differentiable mapping on $(x_1,x_2)$ with $x_1<x_2$ if $|\Pi'|\in L_1[x_1,x_2]$ and $|\Pi'|$ is an s-convex function. Then for any $0<\alpha\leq 1$ and $a<x<b$, the fractional integral inequality is given as:*

$$
\begin{aligned}
&\Bigg|\frac{(x-x_1)](x_1+x)\Pi(x)-x\Pi(x_1)]+(x_2-x)[(x+x_2)\Pi(x)-x\Pi(x_2)]}{(x_2-x_1)^2}\\
&-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^2}\Big(\frac{x_1}{x-x_1)^{\alpha-1}}J^{\alpha}_{x-}\Pi(x_1)+\frac{x_2}{x_2-x)^{\alpha-1}}J^{\alpha}_{x+}\Pi(x_2)\Big)\Bigg|\\
\leq\;&\frac{(x-x_1)^2}{(x_2-x_1)^2}\Big[\Big(\frac{x_1}{\alpha+s+1}+\frac{x}{s+1}\Big)|\Pi'(x)|+\Big(\frac{x_1\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}+\frac{x}{s+1}\Big)|\Pi'(x_1)|\Big]\\
&+\frac{(x_2-x)^2}{(x_2-x_1)^2}\Big[\Big(\frac{x_2\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}+\frac{x}{s+1}\Big)|\Pi'(x_2)|+\Big(\frac{x_2}{\alpha+s+1}+\frac{x}{s+1}\Big)|\Pi'(x)|\Big].
\end{aligned}
$$

**Theorem 108** ([82])**.** *Assume that $\Pi$ is as in Theorem 107. If $|\Pi'|\in L_1[x_1,x_2]$ and $|\Pi'|$ is an s-G-L function, then fractional integral inequality is given as:*

$$
\begin{aligned}
&\Bigg|\frac{(x-x_1)[(x_1+x)\Pi(x)-x\Pi(x_1)]+(x_2-x)[(x+x_2)\Pi(x)-x\Pi(x_2)]}{(x_2-x_1)^2}\\
&-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^2}\Big(\frac{x_1}{x-x_1)^{\alpha-1}}J^{\alpha}_{x-}\Pi(x_1)+\frac{x_2}{x_2-x)^{\alpha-1}}J^{\alpha}_{x+}\Pi(x_2)\Big)\Bigg|
\end{aligned}
$$

$$\leq \frac{(x-x_1)^2}{(x_2-x_1)^2}\Big[\Big(\frac{x_1}{\alpha-s+1}+\frac{x}{1-s}\Big)|\Pi'(x)|+\Big(\frac{x_1\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}+\frac{x}{1-s}\Big)|\Pi'(x_1)|\Big]$$
$$+\frac{(x_2-x)^2}{(x_2-x_1)^2}\Big[\Big(\frac{x_2\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}+\frac{x}{1-s}\Big)|\Pi'(x_2)|+\Big(\frac{x_2}{\alpha-s+1}+\frac{x}{1-s}\Big)|\Pi'(x)|\Big].$$

**Theorem 109** ([82])**.** *Assume that $\Pi$ is as in Theorem 107. If $|\Pi'|^q \in L_1[x_1,x_2]$ and $|\Pi'|^q, q>1$, is an s-convex function, then fractional integral inequality is given as:*

$$\Bigg|\frac{(x-x_1)[(x_1+x)\Pi(x)-x\Pi(x_1)]+(x_2-x)[(x+x_2)\Pi(x)-x\Pi(x_2)]}{(x_2-x_1)^2}$$
$$-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^2}\Big(\frac{x_1}{x-x_1)^{\alpha-1}}J^{\alpha}_{x-}\Pi(x_1)+\frac{x_2}{x_2-x)^{\alpha-1}}J^{\alpha}_{x+}\Pi(x_2)\Big)\Bigg|$$
$$\leq \frac{(x-x_1)^2}{(x_2-x_1)^2}\Big(\frac{2^{p-1}x_1^p}{p\alpha+1}+2^{p-1}x^p\Big)^{\frac{1}{p}}\Big(\frac{|\Pi'(x_1)|^q+|\Pi'(x)|^q}{s+1}\Big)^{\frac{1}{q}}$$
$$+\frac{(x_2-x)^2}{(x_2-x_1)^2}\Big(\frac{2^{p-1}x_2^p}{p\alpha+1}+2^{p-1}x^p\Big)^{\frac{1}{p}}\Big(\frac{|\Pi'(x_2)|^q+|\Pi'(x)|^q}{s+1}\Big)^{\frac{1}{q}}.$$

**Theorem 110** ([82])**.** *Assume that $\Pi$ is as in Theorem 107. If $|\Pi'|^q \in L_1[x_1,x_2]$ and $|\Pi'|^q, q>1$, is an s-G-L function, then the fractional integral inequality is given as:*

$$\Bigg|\frac{(x-x_1)[(x_1+x)\Pi(x)-x\Pi(x_1)]+(x_2-x)[(x+x_2)\Pi(x)-x\Pi(x_2)]}{(x_2-x_1)^2}$$
$$-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^2}\Big(\frac{x_1}{x-x_1)^{\alpha-1}}J^{\alpha}_{x-}\Pi(x_1)+\frac{x_2}{x_2-x)^{\alpha-1}}J^{\alpha}_{x+}\Pi(x_2)\Big)\Bigg|$$
$$\leq \frac{(x-x_1)^2}{(x_2-x_1)^2}\Big(\frac{x_1^p}{p\alpha+1}\Big)^{\frac{1}{p}}\Big(\frac{|\Pi'(x)|^q}{1-s}+\frac{|\Pi'(x_1)|^q}{1-s}\Big)^{\frac{1}{q}}$$
$$+\frac{(x_2-x)^2}{(x_2-x_1)^2}\Big(\frac{x_2^p}{p\alpha+1}\Big)^{\frac{1}{p}}\Big(\frac{|\Pi'(x_2)|^q}{1-s}+\frac{|\Pi'(x)|^q}{1-s}\Big)^{\frac{1}{q}}.$$

**Theorem 111** ([83])**.** *Let $\Pi:\mathbb{I}\subset[0,\infty)\to\mathbb{R}$ be a three times differentiable function on the interior $\mathbb{I}^\circ$ of $\mathbb{I}$ such that $\Pi'''\in L_1[x_1,x_2]$, where $x_1,x_2\in\mathbb{I}^\circ$ with $x_1<x_2$. If $|\Pi'''|$ is an s-convex function in the first sense on $[x_1,x_2]$, then fractional integral inequality is given as:*

$$\Bigg|\frac{\Gamma(\alpha+1)}{2(1+\alpha)}[\Pi'(x_1)-\Pi'(x_2)]+\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)]$$
$$\leq \frac{(x_2-x_1)^3}{2(\alpha+1)(\alpha+2)}\Big[|\Pi'''(x_1)|\Big(\frac{1}{s+1}-\frac{1}{\alpha+s+3}-B(s+1,\alpha+3)\Big)$$
$$+|\Pi'''(x_2)|\Big(\frac{\alpha+1}{\alpha+3}-\frac{1}{s+1}-\frac{1}{\alpha+s+3}-B(s+1,\alpha+3)\Big)\Big].$$

**Theorem 112** ([83])**.** *Assume that $\Pi$ is as in Theorem 111. If $|\Pi'''|^q$ is an s-convex function in the first sense on $[x_1,x_2]$, then fractional integral inequality is given as:*

$$\Bigg|\frac{\Gamma(\alpha+1)}{2(1+\alpha)}[\Pi'(x_1)-\Pi'(x_2)]+\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)]$$
$$\leq \frac{(x_2-x_1)^3}{2(\alpha+1)(\alpha+2)}\Big(\frac{p(\alpha+2)-1}{p(\alpha+2)+1}\Big)^{\frac{1}{p}}\Big(\frac{|\Pi'''(x_1)|^q+|\Pi'''(x_2)|^q}{s+1}\Big)^{\frac{1}{q}}.$$

**Theorem 113** ([83]). *Assume that $\Pi$ is as in Theorem 111. If $|\Pi'''|$ is an s-convex function in the second sense on $[x_1, x_2]$, then fractional integral inequality is given as:*

$$\left|\frac{\Gamma(\alpha+1)}{2(1+\alpha)}[\Pi'(x_1)-\Pi'(x_2)]+\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)\right]\right.$$
$$\leq \frac{(x_2-x_1)^3}{2(\alpha+1)(\alpha+2)}[|\Pi'''(x_1)|+\Pi'''(x_2)|]\Big(\frac{1}{s+1}-\frac{1}{\alpha+s+3}-B(s+1,\alpha+3)\Big).$$

**Theorem 114** ([83]). *Assume that $\Pi$ is as in Theorem 111. If $|\Pi'''|^q$ is an s-convex in the second sense on $[x_1, x_2]$ for a fixed $q \geq 1$, then fractional integral inequality is given as:*

$$\left|\frac{\Gamma(\alpha+1)}{2(1+\alpha)}[\Pi'(x_1)-\Pi'(x_2)]+\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)\right]\right.$$
$$\leq \frac{(x_2-x_1)^3}{2(\alpha+1)(\alpha+2)}\Big(\frac{\alpha+1}{\alpha+2}\Big)^{1-\frac{1}{q}}[|\Pi'''(x_1)|^q+\Pi'''(x_2)|^q]$$
$$\times\Big(\frac{1}{s+1}-\frac{1}{\alpha+s+3}-B(s+1,\alpha+3)\Big)^{\frac{1}{q}}.$$

**Theorem 115** ([83]). *Assume that $\Pi$ is as in Theorem 111. If $|\Pi'''|$ is an s-G-L function of the first kind on $[x_1, x_2]$, then fractional integral inequality is given as:*

$$\left|\frac{\Gamma(\alpha+1)}{2(1+\alpha)}[\Pi'(x_1)-\Pi'(x_2)]+\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)\right]\right.$$
$$\leq \frac{(x_2-x_1)^3}{2(\alpha+1)(\alpha+2)}|\Pi'''(x_1)|\Big(\frac{1}{1-s}-\frac{1}{\alpha-s+3}-B(\alpha+3,1-s)\Big)$$
$$+\Pi'''(x_2)|\Big(\frac{s+2}{s+1}-\frac{1}{\alpha+3}-\frac{2(\alpha+3)+s}{(\alpha+3)(\alpha+s+3)}-B(s+1,\alpha+3)\Big).$$

**Theorem 116** ([83]). *Assume that $\Pi$ is as in Theorem 111. If $|\Pi'''|^q$ is an s-G-L function of first kind on $[x_1, x_2]$ for a fixed $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then fractional integral inequality is given as:*

$$\left|\frac{\Gamma(\alpha+1)}{2(1+\alpha)}[\Pi'(x_1)-\Pi'(x_2)]+\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)\right]\right.$$
$$\leq \frac{(x_2-x_1)^3}{2(\alpha+1)(\alpha+2)}\Big(\frac{p(\alpha+1)-1}{p(\alpha+2)+1}\Big)^{1-\frac{1}{q}}\Big(\frac{1}{1-s}|\Pi'''(x_1)|^q+\frac{s+2}{s+1}|\Pi'''(x_2)|^q\Big)^{\frac{1}{q}}.$$

**Theorem 117** ([83]). *Assume that $\Pi$ is as in Theorem 111. If $|\Pi'''|$ is an s-G-L function of second kind on $[x_1, x_2]$, then fractional integral inequality is given as:*

$$\left|\frac{\Gamma(\alpha+1)}{2(1+\alpha)}[\Pi'(x_1)-\Pi'(x_2)]+\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2)+J^\alpha_{x_2-}\Pi(x_1)\right]\right.$$
$$\leq \frac{(x_2-x_1)^3}{2(\alpha+1)(\alpha+2)}[|\Pi'''(x_1)|+\Pi'''(x_2)|]\Big(\frac{1}{s+1}-\frac{1}{\alpha+s+3}-B(s+1,\alpha+3)\Big).$$

**Theorem 118** ([84])**.** *Let $\Pi : \mathbb{I} \subset \mathbb{R} \to [0,\infty)$ be a differentiable mapping on $I$, $x_1, x_2 \in \mathbb{I}$ and $x_1 < x_2$. If $|\Pi|^q$ is s-G-L-type function, then for $\alpha \in [0,1]$, the fractional integral inequality is given as:*

$$\begin{aligned} &\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_2-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_2)\right]\right. \\ \leq\ &\frac{x_2-x_1}{2(\alpha p+1)^{\frac{1}{p}}(1-s)^{\frac{1}{q}}}\left[|\Pi'(x_1)|^q + |\Pi'(x_2)|^q\right]^{\frac{1}{q}}, \end{aligned}$$

*where* $\dfrac{1}{p}+\dfrac{1}{q}=1$.

*2.17. Fractional H-H-Type Inequalities for Differentiable $\phi$-Convex Functions*

**Definition 19** ([85])**.** *Let $\phi : [x_1, x_2] \subset \mathbb{R} \to [x_1, x_2]$. A function $\Pi : [x_1, x_2] \to \mathbb{R}$ is said to be $\phi$-convex on $[x_1, x_2]$ if, for every $x, y \in [x_1, x_2]$ and $\lambda \in [0,1]$, the following inequality holds:*

$$\Pi(\lambda\phi(x) + (1-\lambda)\phi(y)) \leq \lambda\Pi(\phi(x)) + (1-\lambda)\Pi(\phi(y)).$$

H-H inequality in fractional integral forms for $\phi$-convex functions is given in the next theorem.

**Theorem 119** ([85])**.** *Let $J$ be an interval $x_1, x_2 \in J$ with $x_1 < x_2$ and $\phi : J \to \mathbb{R}$ a continuous increasing function. Let $\Pi : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a $\phi$-convex function on $\mathbb{I} = [x_1, x_2]$, then fractional integral inequalities are given as:*

$$\begin{aligned} \Pi\Big(\frac{\phi(x_1)+\phi(x_2)}{2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2(\phi(x_2)-\phi(x_1))^\alpha}\Big[J^a_{\phi(x_1)+}\Pi(\phi(x_2)) + J^a_{\phi(x_2)-}\Pi(\phi(x_1))\Big] \\ &\leq \frac{\Pi(\phi(x_1))+\Pi(\phi(x_2))}{2}. \end{aligned}$$

*2.18. Fractional H-H-Type Inequalities for Symmetric Functions*

**Definition 20** ([86])**.** *A function $\Pi : [x_1, x_2] \to [0,\infty)$ is said to be symmetric with respect to $\dfrac{x_1+x_2}{2}$ if*

$$\Pi(x_1+x_2-k) = \Pi(k), \ \text{for all } \ k \in [x_1, x_2].$$

**Theorem 120** ([87])**.** *Let $\Pi : [x_1, x_2] \to \mathbb{R}$ be convex function with $x_1 < x_2$ and $\Pi \in L_1[x_1, x_2]$. If $g : [x_1, x_2] \to \mathbb{R}$ is non-negative, integrable and symmetric to $\dfrac{x_1+x_2}{2}$, then the fractional integral inequalities are given as:*

$$\begin{aligned} &\Pi\Big(\frac{x_1+x_2}{2}\Big)\Big[J^\alpha_{x_1+}g(x_2) + J^\alpha_{x_2-}g(x_1)\Big] \leq \Big[J^\alpha_{x_1+}(\Pi g)(x_2) + J^\alpha_{x_2-}(\Pi g)(x_1)\Big] \\ \leq\ &\frac{\Pi(x_1)+\Pi(x_2)}{2}\Big[J^\alpha_{x_1+}g(x_2) + J^\alpha_{x_2-}g(x_1)\Big], \quad \alpha > 0. \end{aligned}$$

**Theorem 121** ([87])**.** *Let $\Pi : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ$ and $\Pi' \in L_1[x_1, x_2]$ with $x_1 < x_2$. If $|\Pi'|$ is a convex function on $[x_1, x_2]$ and $g : [x_1, x_2] \to \mathbb{R}$ is continuous and symmetric to $\dfrac{x_1+x_2}{2}$, then the fractional integral inequality is given as:*

$$\begin{aligned} &\left|\Big(\frac{\Pi(x_1)+\Pi(x_2)}{2}\Big)\Big[J^\alpha_{x_1+}g(x_2) + J^\alpha_{x_2-}g(x_1)\Big] - \Big[J^\alpha_{x_1+}(\Pi g)(x_2) + J^\alpha_{x_2-}(\Pi g)(x_1)\Big]\right| \\ \leq\ &\frac{(x_2-x_1)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)\Big[|\Pi'(x_1)| + |\Pi'(x_2)|\Big], \quad \alpha > 0. \end{aligned}$$

**Theorem 122** ([88]). *Let $\Pi : [x_1, x_2] \to \mathbb{R}$ be a positive, differentiable function with $x_1 < x_2$ and $\Pi \in L_1[x_1, x_2]$. If $\Pi'(x_1 + x_2 - x) \geq \Pi'(x)$ for all $x \in \left[x_1, \frac{x_1+x_2}{2}\right]$, then*

$$\Pi\left(\frac{x_1+x_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}, \quad \alpha > 0.$$

**Theorem 123** ([88]). *Let $\Pi : [x_1, x_2] \to \mathbb{R}$ be a positive, twice differentiable function with $x_1 < x_2$ and $\Pi \in L_1[x_1, x_2]$. If $\Pi''$ is bounded in $[x_1, x_2]$, then we have*

$$\begin{aligned}
&\frac{m\alpha}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}\left(\frac{x_1+x_2}{2}-x\right)^2\left[(x-x_1)^{\alpha-1}+(x_2-x)^{\alpha-1}\right]dx \\
\leq\; &\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] - \Pi\left(\frac{x_1+x_2}{2}\right) \\
\leq\; &\frac{M\alpha}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}\left(\frac{x_1+x_2}{2}-x\right)^2\left[(x-x_1)^{\alpha-1}+(x_2-x)^{\alpha-1}\right]dx,
\end{aligned}$$

*and*

$$\begin{aligned}
&\frac{-M\alpha}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}(x-x_1)(x_2-x)\left[(x-x_1)^{\alpha-1}+(x_2-x)^{\alpha-1}\right]dx \\
\leq\; &\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] - \frac{\Pi(x_1)+\Pi(x_2)}{2} \\
\leq\; &\frac{-m\alpha}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}(x-x_1)(x_2-x)\left[(x-x_1)^{\alpha-1}+(x_2-x)^{\alpha-1}\right]dx
\end{aligned}$$

*with $\alpha > 0$, where $m = \inf_{t\in[x_1,x_2]}\Pi''(t)$, $M = \sup_{t\in[x_1,x_2]}\Pi''(t)$.*

### *2.19. Fractional H-H-Type Inequalities for MT-Convex Functions*

**Definition 21** ([89] ). *A function $\Pi : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be a MT-convex function, If $\Pi$ is non-negative function and for all $x, y \in \mathbb{I}$ and $t \in (0,1)$ we have*

$$\Pi(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}}.$$

**Theorem 124** ([90]). *Let $\Pi : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathbb{I}^\circ$ such that $\Pi' \in L_1[x_1, x_2]$ where $x_1, x_2 \in \mathbb{I}$ with $x_1 < x_2$. If $|\Pi'|$ is MT-convex on $[x_1, x_2]$ and $|\Pi'(x)| \leq M, x \in [x_1, x_2]$, then the fractional integral inequality with $\alpha > 0$ is given as:*

$$\begin{aligned}
&\left|\frac{(x-x_1)^\alpha\Pi(x_1)+(x_2-x)^\alpha\Pi(x_2)}{x_2-x_1} - \frac{\Gamma(\alpha+1)}{x_2-x_1}\left[J^\alpha_{x-}\Pi(x_1) + J^\alpha_{x+}\Pi(x_2)\right]\right| \\
\leq\; &\frac{M\left[(x-x_1)^{\alpha+1}+(x_2-x)^{\alpha+1}\right]}{2(x_2-x_1)}\left[\pi - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right].
\end{aligned}$$

**Theorem 125** ([90]). *Assume that $\Pi$ is as in Theorem 124. If $|\Pi'|^q$ is MT-convex on $[x_1, x_2]$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, then the fractional integral inequality with $\alpha > 0$ is given as:*

$$\left|\frac{(x-x_1)^\alpha\Pi(x_1)+(x_2-x)^\alpha\Pi(x_2)}{x_2-x_1} - \frac{\Gamma(\alpha+1)}{x_2-x_1}\left[J^\alpha_{x-}\Pi(x_1) + J^\alpha_{x+}\Pi(x_2)\right]\right|$$

$$\leq \frac{M\Big[(x-x_1)^{\alpha+1}+(x_2-x)^{\alpha+1}\Big]}{2(x_2-x_1)}\Big(\frac{\pi}{2}\Big)^{\frac{1}{q}}\left(\frac{\Gamma(1+p)\Gamma\Big(\frac{1}{\alpha}\Big)}{\alpha\Gamma\Big(1+p+\frac{1}{\alpha}\Big)}\right)^{\frac{1}{p}}.$$

**Theorem 126** ([90]). *Assume that $\Pi$ is as in Theorem 124. If $|\Pi'|^q$ is MT-convex on $[x_1,x_2]$, $q\geq 1$, then the fractional integral inequality with $\alpha>0$ is given as:*

$$\left|\frac{(x-x_1)^{\alpha}\Pi(x_1)+(x_2-x)^{\alpha}\Pi(x_2)}{x_2-x_1}-\frac{\Gamma(\alpha+1)}{x_2-x_1}\Big[J^{\alpha}_{x-}\Pi(x_1)+J^{\alpha}_{x+}\Pi(x_2)\Big]\right|$$
$$\leq \frac{M\Big[(x-x_1)^{\alpha+1}+(x_2-x)^{\alpha+1}\Big]}{(x_2-x_1)}\Big(\frac{\alpha}{\alpha+1}\Big)^{1-\frac{1}{q}}\left[\frac{\pi}{2}-\frac{\Gamma\Big(\alpha+\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big)}{2\Gamma(\alpha+1)}\right]^{\frac{1}{q}}.$$

**Theorem 127** ([91]). *Suppose that $\mathbb{I}\subset[0,\infty)$ is an interval, $\Pi:\mathbb{I}\to\mathbb{R}$ is twice differentiable on $\mathbb{I}^{\circ}$, $x_1,x_2\in\mathbb{I}^{\circ}$ with $x_1<x_2$ and $\Pi''\in L_1[x_1,x_2]$. If $|\Pi''|$ is MT-convex on $[x_1,x_2]$ such that $|\Pi''(x)|\leq K$ for all $x\in[x_1,x_2]$, then the fractional integral inequality is given as:*

$$\left|\frac{1}{(\alpha+1)(x_2-x_1)}\Big[\Pi'(x)[(x-x_1)^{\alpha+1}-(x_2-x)^{\alpha+1}]+(\alpha+1)\Pi(x_1)(x-x_1)^{\alpha}\right.$$
$$\left.+(\alpha+1)\Pi(x_2)(x_2-x)^{\alpha}\Big]-\frac{\Gamma(\alpha+1)}{x_2-x_1}\Big[J^{\alpha}_{x_1+}\Pi(x)+J^{\alpha}_{x_2-}\Pi(x)\Big]\right|$$
$$\leq \frac{K[(x-x_1)^{\alpha+2}+(x_2-x)^{\alpha+2}]}{2(\alpha+1)(x_2-x_1)}\Big[\pi-\frac{\Gamma(\alpha+3/2)\Gamma(1/2)}{\Gamma(\alpha+2)}\Big].$$

**Theorem 128** ([91]). *Assume that $\Pi$ is as in Theorem 127. If $|\Pi''|^q$, $q>1$, and $\dfrac{1}{p}+\dfrac{1}{q}=1$, is MT-convex on $[x_1,x_2]$, then the fractional integral inequality is given as:*

$$\left|\frac{1}{(\alpha+1)(x_2-x_1)}\Big[\Pi'(x)[(x-x_1)^{\alpha+1}-(x_2-x)^{\alpha+1}]+(\alpha+1)\Pi(x_1)(x-x_1)^{\alpha}\right.$$
$$\left.+(\alpha+1)\Pi(x_2)(x_2-x)^{\alpha}\Big]-\frac{\Gamma(\alpha+1)}{x_2-x_1}\Big[J^{\alpha}_{x_1+}\Pi(x)+J^{\alpha}_{x_2-}\Pi(x)\Big]\right|$$
$$\leq \frac{K[(x-x_1)^{\alpha+2}+(x_2-x)^{\alpha+2}]}{(\alpha+1)(x_2-x_1)}\Big(\frac{\pi}{2}\Big)^{\frac{1}{q}}\Big(\frac{\Gamma(1+p)\Gamma(1/(\alpha+1)}{(\alpha+1)\Gamma(1+p+1/(\alpha+1)}\Big)^{\frac{1}{q}}.$$

### *2.20. Fractional H-H-Type Inequalities for $(s,m)$-Convex Functions*

**Definition 22** ([92]). *A mapping $\Pi:\mathbb{I}\subseteq\mathbb{R}^{+}\to\mathbb{R}$ is said to be $(s,m)$-convex in the second sense on $\mathbb{I}$ if the inequality*

$$\Pi(tx+(1-t)y)\leq t^{s}\Pi(x)+m(1-t^{s})\Pi(y)$$

*holds for all $x,y\in\mathbb{I}$ and for some fixed $(s,m)\in(0,1]^2$.*

**Theorem 129** ([93]). *Let $\Pi:\mathbb{I}\to\mathbb{R}$ be defined on $\mathbb{I}$ and differentiable on the interior $\mathbb{I}^{\circ}$ of $I$, such that $x_1,x_2\in\mathbb{I}$ with $x_1<x_2$, $\Pi'\in L_1[x_1,x_2]$. If $|\Pi'|$ is $(s,m)$-convex in the second sense on $[x_1,x_2]$ for $(s,m)\in(0,1]^2$, then the fractional integral inequality is given as:*

$$\left|\Pi\Big(\frac{x_1+x_2}{2}\Big)+\frac{\Gamma(\alpha+1)}{x_2-x_1}\left[\Big(\frac{2}{x_1-x_2}\Big)^{\alpha-1}J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_1)\right.\right.$$

$$-\left(\frac{2}{x_2-x_1}\right)^{\alpha-1} J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)\Bigg]\Bigg|$$
$$\leq \frac{x_2-x_1}{4}\left[\frac{2}{s+\alpha+1}\left|\Pi'\left(\frac{x_1+x_2}{2}\right)\right| + mB(\alpha+1,s+1)\left(\left|\Pi'\left(\frac{x_1}{m}\right)\right| + \left|\Pi'\left(\frac{x_2}{m}\right)\right|\right)\right],$$

*where B is the Euler Beta function.*

**Theorem 130** ([93]). *Assume that $\Pi$ is as in Theorem 129. If $|\Pi'|$ is m-convex function in the second sense on $[x_1, x_2]$ for $m \in (0, 1]$, then the fractional integral inequality is given as:*

$$\left|\Pi\left(\frac{x_1+x_2}{2}\right) + \frac{\Gamma(\alpha+1)}{x_2-x_1}\left[\left(\frac{2}{x_1-x_2}\right)^{\alpha-1} J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_1)\right.\right.$$
$$\left.\left.-\left(\frac{2}{x_2-x_1}\right)^{\alpha-1} J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)\right]\right|$$
$$\leq \frac{x_2-x_1}{4}\left[\frac{1}{2(\alpha+2)}(|\Pi'(x_1)| + |\Pi'(x_2)|) + \frac{m(\alpha+3)}{2(\alpha+1)(\alpha+2)}\left(\left|\Pi'\left(\frac{x_1}{m}\right)\right| + \left|\Pi'\left(\frac{x_2}{m}\right)\right|\right)\right].$$

**Theorem 131** ([93]). *Assume that $\Pi$ is as in Theorem 129. If $|\Pi'|^q$ is $(s, m)$-convex function in the second sense on $[x_1, x_2]$ and if $q > 1$, $q \geq r \geq 0$, then the fractional integral inequality is given as:*

$$\left|\Pi\left(\frac{x_1+x_2}{2}\right) + \frac{\Gamma(\alpha+1)}{x_2-x_1}\left[\left(\frac{2}{x_1-x_2}\right)^{\alpha-1} J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_1)\right.\right.$$
$$\left.\left.-\left(\frac{2}{x_2-x_1}\right)^{\alpha-1} J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+}\Pi(x_2)\right]\right|$$
$$\leq \frac{x_2-x_1}{4}\left[\frac{q-1}{\alpha(q-r)+q-1}\right]^{1-\frac{1}{q}}\left[2\left(\frac{1}{s+\alpha r+1}\right)^{\frac{1}{q}}\left|\Pi'\left(\frac{x_1+x_2}{2}\right)\right|\right.$$
$$\left.+m^{\frac{1}{q}}B(\alpha r+1,s+1)^{\frac{1}{q}}\left(\left|\Pi'\left(\frac{x_1}{m}\right)\right| + \left|\Pi'\left(\frac{x_2}{m}\right)\right|\right)\right].$$

### *2.21. Fractional H-H-Type Inequalities via Green Functions*

**Theorem 132** ([94]). *Let $\alpha > 0$, $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $\Pi(x) = \Pi(x_1) + (x - x_1)\Pi'(x_2) + \int_{x_1}^{x_2} G(x,\mu)\Pi''(\mu)d\mu$, $G(\lambda,\mu) = \begin{cases} x_1-\mu, & x_1 \leq \mu \leq \lambda, \\ x_1-\lambda, & \lambda \leq \mu \leq x_2. \end{cases}$ Then, we have:*

$$\Pi\left(\frac{x_1+\alpha x_2}{\alpha+1}\right) \leq \frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}} J^{\alpha}_{x_2-}\Pi(x_1) \leq \frac{\Pi(x_1)+\alpha\Pi(x_2)}{\alpha+1}.$$

**Theorem 133** ([94]). *Let $\alpha > 0$, $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $\Pi \in C^2([x_1, x_2])$ as in previous theorem. If $|\Pi''|$ is a convex function , then:*

$$\left|\Pi\left(\frac{x_1+\alpha x_2}{\alpha+1}\right) - \frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}} J^{\alpha}_{x_2-}\Pi(x_1)\right|$$
$$\leq \frac{(x_2-x_1)^2}{(1+\alpha)^{\alpha+3}(\alpha+2)}\left[\max\left\{|\Pi''(x_1)|, \left|\Pi''\left(\frac{x_1+\alpha x_2}{\alpha+1}\right)\right|\right\}\alpha^{\alpha+2}\right.$$
$$\left.+\max\left\{\left|\Pi''\left(\frac{x_1+\alpha x_2}{\alpha+1}\right)\right|, |\Pi''(x_2)|\right\}\frac{\alpha(-2\alpha^{1+\alpha}+(1+\alpha)^{1+\alpha})}{2}\right].$$

**Theorem 134** ([94]). *Assume that $\Pi$ is as in Theorem 133. Then, we have:*

$$\left|\frac{\Pi(x_1)+\alpha\Pi(x_2)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}J^{\alpha}_{x_2-}\Pi(x_1)\right|\leq\frac{\alpha(x_2-x_1)^2}{3(1+\alpha)(\alpha+3)}\left[\frac{\alpha+5}{2(\alpha+2)}|\Pi''(x_1)|+|\Pi''(x_2)|\right].$$

**Theorem 135** ([95]). *Let $\Pi\in C^2([x_1,x_2])$ be a convex function such that $\Pi(x)=\Pi(x_1)+(x-x_1)\Pi'(x_2)+\int_{x_1}^{x_2}G(x,\mu)\Pi''(\mu)d\mu$, $G(\lambda,\mu)=\begin{cases}x_1-\mu, & x_1\leq\mu\leq\lambda,\\ x_1-\lambda, & \lambda\leq\mu\leq x_2.\end{cases}$ Then for any $\alpha>0$, the following fractional integral inequalities hold:*

$$\Pi\left(\frac{x_1+x_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\right]\leq\frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

**Theorem 136** ([95]). *Let $\Pi\in C^2([x_1,x_2])$ be a convex function as in previous theorem. Then we have:*

*(i) If $|\Pi''|$ is an increasing function, then*

$$\Pi\left(\frac{x_1+x_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\right]\leq\frac{|\Pi''(x_2)|\alpha(x_2-x_1)^2}{2(\alpha+1)(\alpha+2)}.$$

*(ii) If $|\Pi''|$ is a decreasing function, then*

$$\Pi\left(\frac{x_1+x_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\right]\leq\frac{|\Pi''(x_1)|\alpha(x_2-x_1)^2}{2(\alpha+1)(\alpha+2)}.$$

*(iii) If $|\Pi''|$ is a convex function, then*

$$\begin{aligned}\Pi\left(\frac{x_1+x_2}{2}\right)&\leq\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\right]\\&\leq\frac{\max\{|\Pi''(x_1)|,|\Pi''(x_2)|\}\alpha(x_2-x_1)^2}{2(\alpha+1)(\alpha+2)}.\end{aligned}$$

**Theorem 137** ([96]). *Let $\alpha>0$, $x_1,x_2\in\mathbb{R}$ with $x_1<x_2$ and $\Pi(x)=\Pi(x_1)+(x-x_1)\Pi'(x_2)+\int_{x_1}^{x_2}G(x,\mu)\Pi''(\mu)d\mu$, $G(\lambda,\mu)=\begin{cases}x_1-\mu, & x_1\leq\mu\leq\lambda,\\ x_1-\lambda, & \lambda\leq\mu\leq x_2.\end{cases}$ Then, we have:*

$$\Pi\left(\frac{\alpha x_1+x_2}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}J^{\alpha}_{x_1+}\Pi(x_2)\leq\frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1}.$$

**Theorem 138** ([96]). *Let $\Pi\in C^2([x_1,x_2])$ be a twice differentiable function as in the previous theorem. Then we have:*

*(i) If $|\Pi''|$ is an increasing function, then*

$$\left|\frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}J^{\alpha}_{x_1+}\Pi(x_2)\right|\leq\frac{\alpha|\Pi''(x_2)|(x_2-x_1)^2}{2(\alpha+1)(\alpha+2)}.$$

*(ii) If $|\Pi''|$ is a decreasing function, then*

$$\left|\frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}J^{\alpha}_{x_1+}\Pi(x_2)\right|\leq\frac{\alpha|\Pi''(x_1)|(x_2-x_1)^2}{2(\alpha+1)(\alpha+2)}.$$

*(iii)* *If* $|\Pi''|$ *is a convex function, then*

$$\left|\frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}J^\alpha_{x_1+}\Pi(x_2)\right|\leq\frac{\alpha(x_2-x_1)^2\max\{|\Pi''(x_1)|,|\Pi''(x_2)|\}}{2(1+\alpha)(\alpha+2)}.$$

**Theorem 139** ([96]). *Let* $\Pi\in C^2([x_1,x_2])$ *be a twice differentiable function as in the previous theorem. Then we have:*

*(i)* *If* $|\Pi''|$ *is an increasing function, then*

$$\begin{aligned}&\left|\Pi\left(\frac{\alpha x_1+x_2}{\alpha+1}\right)-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}J^\alpha_{x_1+}\Pi(x_2)\right|\\ \leq\;&\frac{(x_2-x_1)^2}{(\alpha+1)^{\alpha+3}(\alpha+2)}\left[\left|\Pi''\left(\frac{\alpha x_1+x_2}{\alpha+1}\right)\right|\left(\frac{\alpha[(\alpha+1)^{\alpha+1}-2\alpha^{\alpha+1}]}{2}\right)\Pi''(x_2)|\alpha^{\alpha+2}\right].\end{aligned}$$

*(ii)* *If* $|\Pi''|$ *is a decreasing function, then*

$$\begin{aligned}&\left|\Pi\left(\frac{\alpha x_1+x_2}{\alpha+1}\right)-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}J^\alpha_{x_1+}\Pi(x_2)\right|\\ \leq\;&\frac{(x_2-x_1)^2}{(\alpha+1)^{\alpha+3}(\alpha+2)}\left[\Pi''(x_2)\left(\frac{\alpha[(\alpha+1)^{\alpha+1}-2\alpha^{\alpha+1}]}{2}\right)+\left|\Pi''\left(\frac{\alpha x_1+x_2}{\alpha+1}\right)\right|\alpha^{\alpha+2}\right].\end{aligned}$$

*(iii)* *If* $|\Pi''|$ *is a convex function, then*

$$\begin{aligned}&\left|\Pi\left(\frac{x_1+\alpha x_2}{\alpha+1}\right)-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}J^\alpha_{x_1+}\Pi(x_2)\right|\leq\frac{(x_2-x_1)^2}{(\alpha+1)^{\alpha+3}(\alpha+2)}\\ &\times\left[\max\left\{|\Pi''(x_1)|,\left|\Pi''\left(\frac{\alpha x_1+x_2}{\alpha+1)}\right)\right|\right\}\left(\frac{\alpha[(\alpha+1)^{\alpha+1}-2\alpha^{\alpha+1}]}{2}\right)\right.\\ &\left.+\max\left\{\left|\Pi''\left(\frac{\alpha x_1+x_2}{\alpha+1}\right)\right|,|\Pi''(x_2)|\right\}\alpha^{\alpha+2}\right].\end{aligned}$$

### 2.22. H-H-Type Inequalities for p-Convex Functions

**Definition 23** ([97]). *A function* $\Pi:\mathbb{I}\subset(0,\infty)\to\mathbb{R}$ *is said to be p-convex, if*

$$\Pi\left([tx^p+(1-t)y^p]^{\frac{1}{p}}\right)\leq t\Pi(x)+(1-t)\Pi(y),$$

*for all* $x,y\in\mathbb{I}$ *and* $t\in[0,1]$.

**Theorem 140** ([98]). *Assume that* $\Pi:\mathbb{I}\subset(0,\infty)\to\mathbb{R}$ *is a p-convex function,* $p\in\mathbb{R}\setminus\{0\}$, $\alpha>0$ *and* $x_1,x_2\in\mathbb{I}$ *with* $x_1<x_2$. *If* $\Pi\in L_1[x_1,x_2]$, *then the fractional integral inequalities are given as:*

$$\begin{aligned}&\Pi\left(\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(x_2^p-x_1^p)^\alpha}\right)\left[J^\alpha_{\frac{x_1^p+x_2^p}{2}+}(\Pi\circ g)(x_2^p)+J^\alpha_{\frac{x_1^p+x_2^p}{2}-}(\Pi\circ g)(x_1^p)\right]\\ \leq\;&\frac{\Pi(x_1)+\Pi(x_2)}{2},\quad p>0,\end{aligned}$$

*with* $g(x)=x^{\frac{1}{p}}$, $x\in[x_1^p,x_2^p]$, *and*

$$\Pi\left(\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(x_2^p-x_1^p)^\alpha}\right)\left[J^\alpha_{\frac{x_1^p+x_2^p}{2}+}(\Pi\circ g)(x_1^p)+J^\alpha_{\frac{x_1^p+x_2^p}{2}-}(\Pi\circ g)(p^p)\right]$$

$$\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}, \quad p<0,$$

*with* $g(x)=x^{\frac{1}{p}},\ x\in[x_2^p,x_1^p]$.

**Theorem 141** ([98]). *Assume that* $\Pi:\mathbb{I}\subset(0,\infty)\to\mathbb{R}$ *is a function differentiable on* $\mathbb{I}^\circ$ *with* $\Pi'\in L_1[x_1,x_2]$, $x_1,x_2\in\mathbb{I}$ *with* $x_1<x_2$. *If* $|\Pi'|$ *is p-convex function on* $[x_1,x_2]$, *for* $p\in\mathbb{R}\setminus\{0\}$, $\alpha>0$, *then the fractional integral inequalities are given as:*

$$\begin{aligned}
&\left|\Pi\left(\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(x_2^p-x_1^p)^\alpha}\right)\left[J^\alpha_{\frac{x_1^p+x_2^p}{2}+}(\Pi\circ g)(x_2^p)+J^\alpha_{\frac{x_1^p+x_2^p}{2}-}(\Pi\circ g)(x_1^p)\right]\right|\\
\leq\ &\frac{x_2^p-x_1^p}{2^{1-\alpha}}\left[C_1(\alpha,p)|\Pi'(x_1)|+C_2(\alpha,p)|\Pi'(x_2)|\right] \quad p>0,
\end{aligned}$$

*with* $g(x)=x^{\frac{1}{p}},\ x\in[x_1^p,x_2^p]$, *and*

$$\begin{aligned}
&\left|\Pi\left(\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(x_2^p-x_1^p)^\alpha}\right)\left[J^\alpha_{\frac{x_1^p+x_2^p}{2}+}(\Pi\circ g)(x_1^p)+J^\alpha_{\frac{x_1^p+x_2^p}{2}-}(\Pi\circ g)(x_2^p)\right]\right|\\
\leq\ &\frac{x_2^p-x_1^p}{2^{1-\alpha}}\left[-C_1(\alpha,p)|\Pi'(x_1)|-C_2(\alpha,p)|\Pi'(x_2)|\right], \quad p<0,
\end{aligned}$$

*with* $g(x)=x^{\frac{1}{p}},\ x\in[x_2^p,x_1^p]$, *where*

$$\begin{aligned}
C_1(\alpha,p) &= \int_0^{1.2}\frac{u^{\alpha+1}}{p[ux_1^p+(1-u)x_2^p]^{1-(1/p)}}du+\int_{1/2}^1\frac{(1-u)^\alpha u}{p[ux_1^p+(1-u)x_2^p]^{1-(1/p)}}du,\\
C_2(\alpha,p) &= \int_0^{1.2}\frac{u^{\alpha}(1-u)}{p[ux_1^p+(1-u)x_2^p]^{1-(1/p)}}du+\int_{1/2}^1\frac{(1-u)^{\alpha+1}}{p[ux_1^p+(1-u)x_2^p]^{1-(1/p)}}du.
\end{aligned}$$

**Definition 24** ([99]). *Let* $p\in\mathbb{R}\setminus\{0\}$. *A function* $\Pi:[x_1,x_2]\subset(0,\infty)\to\mathbb{R}$ *is said to be p-symmetric with respect to* $\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}$ *if*

$$\Pi(x)=\Pi\left([x_1^p+x_2^p-x^p]^{\frac{1}{p}}\right),$$

*holds for all* $x\in[x_1,x_2]$.

**Theorem 142** ([99]). *Assume that* $\Pi:\mathbb{I}\subset(0,\infty)\to\mathbb{R}$ *is a p-convex function,* $p\in\mathbb{R}\setminus\{0\}$, $\alpha>0$ *and* $x_1,x_2\in\mathbb{I}$ *with* $x_1<x_2$. *If* $\Pi\in L_1[x_1,x_2]$ *and* $w:[x_1,x_2]\to\mathbb{R}$ *is non-negative, integrable and p-symmetric with respect to* $\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}$, *then the fractional integral inequalities are given as:*

$$\begin{aligned}
&\Pi\left(\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}\right)\left[J^\alpha_{x_1^p+}(w\circ g)(x_2^p)+J^\alpha_{x_2^p-}(w\circ g)(x_1^p)\right]\\
\leq\ &\left[J^\alpha_{x_1^p+}(\Pi w\circ g)(x_2^p)+J^\alpha_{x_2^p-}(\Pi w\circ g)(x_1^p)\right]\\
\leq\ &\frac{\Pi(x_1)+\Pi(x_2)}{2}\left[J^\alpha_{x_1^p+}(w\circ g)(x_2^p)+J^\alpha_{x_2^p-}(w\circ g)(x_1^p)\right], \quad p>0,
\end{aligned}$$

*with* $g(x)=x^{\frac{1}{p}},\ x\in[x_1^p,x_2^p]$, *and*

$$\Pi\left(\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}\right)\left[J^\alpha_{x_2^p+}(w\circ g)(x_1^p)+J^\alpha_{x_1^p-}(w\circ g)(x_2^p)\right]$$

$$\leq \left[J^{\alpha}_{x_2^p+}(\Pi w\circ g)(x_1^p)+J^{\alpha}_{x_1^p-}(\Pi w\circ g)(x_2^p)\right]$$
$$\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}\left[J^{\alpha}_{x_2^p+}(w\circ g)(x_1^p)+J^{\alpha}_{x_1^p-}(w\circ g)(x_2^p)\right],\ p<0,$$

*with* $g(x)=x^{\frac{1}{p}},\ x\in[x_2^p,x_1^p]$.

**Definition 25** ([100]). *A mapping $\Pi:\mathbb{R}\to\mathbb{R}$ is called uniformly p-convex function with modulus $\psi:[0,\infty)\to[0,\infty)$ if $\psi$ is increasing, $\psi$ vanishes only at 0, and*

$$\Pi(tx+(1-t)y)+t(1-t)\psi(|x-y|)\leq \Pi(x)+\Pi(y),$$

*for each $x,y\in[0,\infty)$ and $t\in[0,1]$.*

**Theorem 143** ([101]). *Assume that $\Pi:[x_1,x_2]\to\mathbb{R}$ is uniformly p-convex function with modulus $\psi$. Then for each $\alpha>0$ the fractional integral inequalities are given as:*

$$\begin{aligned}&\Pi\left(\frac{x_1+x_2}{2}\right)+\frac{\Gamma(\alpha+1)}{2^{\alpha+2}(x_2-x_1)^{\alpha}}J^{\alpha}_{(x_1-x_2)+}\psi(|x_1-x_2|)\\ \leq\ &\frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\right]\\ \leq\ &2(\Pi(x_1)+\Pi(x_2))-2\alpha B(\alpha+1,2)\Pi(|x_1-x_2|).\end{aligned}$$

**Theorem 144** ([101]). *Assume that $\Pi:\mathbb{I}^{\circ}\subset\mathbb{R}\to\mathbb{R}$ is a twice differentiable function on $\mathbb{I}^{\circ}$ and let $x_1,x_2\in\mathbb{I}^{\circ}$ with $x_1<x_2$. If $|\Pi''|$ is a convex function on $[x_1,x_2]$, then:*

$$\left|\frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}J^{\alpha}_{x_1+}\Pi(x_2)\right|\leq\frac{\alpha(x_2-x_1)^2}{3(\alpha+1)(\alpha+3)}\left[|\Pi''(x_1)|+|\Pi''(x_2)|\right].$$

**Theorem 145** ([101]). *Assume that $\Pi$ is as in Theorem 144. If $|\Pi''|^q$, $q>1$ is a convex function on $[x_1,x_2]$, then the fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{\alpha\Pi(x_1)+\Pi(x_2)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}J^{\alpha}_{x_1+}\Pi(x_2)\right|\\ \leq\ &\frac{(x_2-x_1)^2}{\alpha(\alpha+1)}\left[B\left(\frac{p+1}{\alpha},p+1\right)\right]^{\frac{1}{p}}\left[\frac{|\Pi''(x_1)|+|\Pi''(x_2)|}{2}\right]^{\frac{1}{q}},\end{aligned}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

### *2.23. Fractional H-H-Type Inequalities for h-Convex Functions*

**Definition 26** ([102]). *Let $h:\mathbb{I}\to\mathbb{R}$ be a non-negative function. We say that $\Pi:\mathbb{I}\to\mathbb{R}$ is an h-convex function, If $\Pi$ is non-negative and for all $x,y\in\mathbb{I}$, $\lambda\in(0,1)$ we have*

$$\Pi(\lambda x+(1-\lambda)y)\leq h(\lambda)\Pi(x)+h(1-\lambda)\Pi(y).$$

**Theorem 146** ([103]). *Assume that $\Pi:\mathbb{I}\to\mathbb{R}$ is three times differentiable function on the interior $\mathbb{I}^{\circ}$ of $\mathbb{I}$. If $\Pi'''\in L_1[x_1,x_2]$ and $|\Pi'''|$ is h-convex function, then*

$$\begin{aligned}&\left|\frac{(n+1)^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\right)-}\Pi(x_1)+J^{\alpha}_{\left(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\right)+}\Pi(x_2)\right]\right.\\ &-(x_2-x_1)^2(n+1)^3(\alpha+1)(\alpha+2)\left[\Pi''\left(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\right)+\Pi''\left(\frac{1}{n+1}x_1+\frac{n}{n+1}x_2\right)\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{x_2-x_1}{(n+1)^2(\alpha+1)}\Big[\Pi'\Big(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\Big)+\Pi'\Big(\frac{1}{n+1}x_1+\frac{n}{n+1}x_2\Big)\Big]\\
&-\frac{1}{n+1}\Big[\Pi\Big(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\Big)+\Pi\Big(\frac{1}{n+1}x_1+\frac{n}{n+1}x_2\Big)\Big]\Bigg|\\
\leq\ &\frac{(x_2-x_1)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\big[|\Pi'''(x_1)|+|\Pi'''(x_2)|\big]\int_0^1(1-t)^{\alpha+2}\Big[h\Big(\frac{n+t}{n+1}\Big)+h\Big(\frac{1-t}{n+1}\Big)\Big]dt.
\end{aligned}
$$

**Theorem 147** ([103]). *Assume that $\Pi$ is as in Theorem 146. If $|\Pi'''|$ is h-convex function with $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$
\begin{aligned}
&\Bigg|\frac{(n+1)^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}}\Big[J^{\alpha}_{\left(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\right)-}\Pi(x_1)+J^{\alpha}_{\left(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\right)+}\Pi(x_2)\Big]\\
&-(x_2-x_1)^2(n+1)^3(\alpha+1)(\alpha+2)\Big[\Pi''\Big(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\Big)+\Pi''\Big(\frac{1}{n+1}x_1+\frac{n}{n+1}x_2\Big)\Big]\\
&+\frac{x_2-x_1}{(n+1)^2(\alpha+1)}\Big[\Pi'\Big(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\Big)+\Pi'\Big(\frac{1}{n+1}x_1+\frac{n}{n+1}x_2\Big)\Big]\\
&-\frac{1}{n+1}\Big[\Pi\Big(\frac{n}{n+1}x_1+\frac{1}{n+1}x_2\Big)+\Pi\Big(\frac{1}{n+1}x_1+\frac{n}{n+1}x_2\Big)\Big]\Bigg|\\
\leq\ &\frac{(x_2-x_1)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\Big(\frac{1}{\alpha+3}\Big)^{1-\frac{1}{q}}\\
&\times\Big[\Big(\int_0^1(1-t)^{\alpha+2}\Big\{h\Big(\frac{n+t}{n+1}\Big)|\Pi'''(x_1)|^q+h\Big(\frac{1-t}{n+1}\Big)|\Pi'''(x_2)|^q\Big\}dt\Big)^{\frac{1}{q}}\\
&+\Big(\int_0^1(1-t)^{\alpha+2}\Big\{h\Big(\frac{1-t}{n+1}\Big)|\Pi'''(x_1)|^q+h\Big(\frac{n+t}{n+1}\Big)|\Pi'''(x_2)|^q\Big\}dt\Big)^{\frac{1}{q}}\Big].
\end{aligned}
$$

**Theorem 148** ([104]). *Assume that $\Pi:\mathbb{I}\to\mathbb{R}$ is h-convex function on $[x_1,x_2]$, $x_1,x_2\in\mathbb{I}$, $x_1<x_2$ and $\Pi\in L_1[x_1,x_2]$. Then we have:*

$$
\begin{aligned}
\frac{\Gamma(\alpha)}{(x_2-x_1)^{\alpha}}\big[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\big]&\leq[\Pi(x_1)+\Pi(x_2)]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]dt\\
&\leq\frac{2[\Pi(x_1)+\Pi(x_2)]}{(p\alpha+1)^{\frac{1}{p}}}\Big(\int_0^1(h(t))^q dt\Big)^{\frac{1}{q}},
\end{aligned}
$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**Theorem 149** ([104]). *Assume that $\Pi$ is as in Theorem 148. If $|\Pi'|$ is an h-convex mapping on $[x_1,x_2]$, then the fractional integral inequality is given as:*

$$
\begin{aligned}
&\Bigg|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\big[J^{\alpha}_{x_1+}\Pi(x_2)+J^{\alpha}_{x_2-}\Pi(x_1)\big]\Bigg|\\
\leq\ &\frac{(x_2-x_1)[|\Pi'(x_1)|+|\Pi'(x_2)|]}{2}\Big[\Big(\frac{2^{p\alpha+1}-1}{2^{p\alpha+1}(p\alpha+1)}\Big)^{\frac{1}{p}}-\Big(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\Big)^{\frac{1}{p}}\Big]\\
&\times\Big[\Big(\int_0^{1/2}(h(t))^q dt\Big)^{\frac{1}{q}}+\Big(\int_{1/2}^1(h(t))^q dt\Big)^{\frac{1}{q}}\Big],
\end{aligned}
$$

*where $\alpha>0$, $p>1$ and* $\frac{1}{p}+\frac{1}{q}=1$.

### 2.24. H-H-Type Inequalities for Modified $(p,h)$-Convex Functions

**Definition 27** ([105]). *Assume that $h : \mathbb{J} \subseteq \mathbb{R} \to \mathbb{R}$ is a non-negative and nonzero function. A function $\Pi : \mathbb{I} \to \mathbb{R}$, where $\mathbb{I}$ is p-convex set in $\mathbb{R}$ is called modified $(p,h)$-convex function, if $\Pi$ is non-negative and*

$$\Pi([tr^p + (1-t)s^p]^{\frac{1}{p}}) \leq h(t)\Pi(r) + (1-h(t))\Pi(s)$$

*for $t \in (0,1)$ and for all $r,s \in \mathbb{I}$, where $p > 0$.*

In the following theorem, we present fractional H-H-type inequality via modified $(p,h)$-convex functions.

**Theorem 150** ([105]). *Assume that $\Pi$ is modified $(p,h)$-convex function and $\Pi \in L_1[x_1,x_2]$, with $x_1 < x_2$. Then*

$$\frac{1}{\alpha}\Pi\left(\frac{x_1^p + x_2^p}{2}\right)^{\frac{1}{p}} \leq \frac{\Gamma(\alpha)}{(x_2^p - x_1^p)^\alpha}\left[\left(1 - h\left(\frac{1}{2}\right)\right)J^\alpha_{a+}\Pi(x_2) + h\left(\frac{1}{2}\right)J^\alpha_{x_2^-}\Pi(x_1)\right],$$
$$\frac{\Gamma(\alpha)}{(x_2^p - x_1^p)^\alpha}\left[J^\alpha_{a+}\Pi(x_2) + J^\alpha_{x_2^-}\Pi(x_1)\right] \leq \frac{\Pi(x_1)+\Pi(x_2)}{\alpha}.$$

### 2.25. Fractional H-H-Type Inequalities for $\sigma$-Convex Functions

**Definition 28** ([106]). *A set $Q \subset \mathbb{R}$ is said to be $\sigma$-convex set with respect to strictly monotonic continuous function $\sigma$ if*

$$\sigma^{-1}((1-t)\sigma(x) + t\sigma(x)) \in Q, \ \ \forall x,y \in Q, \ t \in [0,1].$$

**Definition 29** ([106]). *A function $\Pi : Q \subset \mathbb{R}$ is said to be $\sigma$-convex function with respect to strictly monotonic continuous function $\sigma$ if*

$$\Pi(\sigma^{-1}((1-t)\sigma(x) + t\sigma(x))) \leq (1-t)\Pi(x) + t\Pi(y), \ \ \forall x,y \in Q, \ t \in [0,1].$$

**Theorem 151** ([107]). *Let $\Pi : [x_1,x_2] \subseteq \mathbb{R} \to \mathbb{R}$ be an integrable $\sigma$-convex function and $\Pi \in L_1(x_1,x_2)$ with $0 \leq x_1 < x_2$. If the function $\sigma$ is increasing and positive on $[x_1,x_2]$ and $\sigma'$ is continuous on $(x_1,x_2)$, the for $\alpha > 0$*

$$\Pi\left(\sigma^{-1}\left(\frac{\sigma(x_1)+\sigma(x_2)}{2}\right)\right) \leq \frac{\Gamma(\alpha+1)}{2(\sigma(x_2)-\sigma(x_1))^\alpha}\left[I^{\alpha,\rho}_{x_1+}\Pi(x_2) + I^{\alpha,\rho}_{x_2-}\Pi(x_2)\right] \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

**Theorem 152** ([107]). *Assume that $\Pi$ is as in above Theorem 151. Then we have*

$$\begin{aligned}
&\Pi\left(\sigma^{-1}\left(\frac{\sigma(x_1)+\sigma(x_2)}{2}\right)\right) \\
\leq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma(x_2)-\sigma(x_1))^\alpha}\left[I^{\alpha,\rho}_{\left(\sigma^{-1}\left(\frac{\sigma(x_1)+\sigma(x_2)}{2}\right)\right)+}\Pi(x_2) + I^{\alpha,\rho}_{\left(\sigma^{-1}\left(\frac{\sigma(x_1)+\sigma(x_2)}{2}\right)\right)-}\Pi(x_2)\right] \\
\leq\ & \frac{\Pi(x_1)+\Pi(x_2)}{2}.
\end{aligned}$$

### 2.26. Fractional H-H-Type Inequalities for Exponential-Convex Functions

**Definition 30** ([108]). *A function $\Pi : [x_1,x_2] \subset \mathbb{R} \to \mathbb{R}$ is said to be exponential-convex function, if*

$$e^{\Pi(tx+(1-t)y)} \leq te^{\Pi(x)} + (1-t)e^{\Pi(y)},$$

*for all $t \in [0,1]$ and all $x,y \in [x_1,x_2]$.*

**Definition 31** ([109]). *The logarithmic-exponential mean of a given function* $\Pi$ *on* $[x_1, x_2]$ *is defined as*

$$LE(x) = \ln\left[\frac{e^{\Pi(x)} + e^{\Pi(x_1+x_2-x)}}{2}\right], \quad x \in [x_1, x_2].$$

**Theorem 153** ([109]). *For* $\alpha > 0$*, If* $\Pi$ *is an exp-convex function and continuous on* $[x_1, x_2]$*, then the fractional integral inequalities are given as:*

$$\Pi\left(\frac{x_1+x_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}} I^{\alpha}_{x_1+} LE(x_2) \leq \ln\left[\frac{e^{\Pi(x_1)} + e^{\Pi(x_2)}}{2}\right].$$

**Theorem 154** ([109]). *For* $\alpha > 0$*, let* $\Pi$ *be differentiable on* $[x_1, x_2]$ *and* $|LE'|$ *be convex, then*

$$\left|\ln\left[\frac{e^{\Pi(x_1)} + e^{\Pi(x_2)}}{2}\right] - \frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}} I^{\alpha}_{x_1+} LE(x_2)\right| \leq \frac{(x_2-x_1)|\Pi'(x_2)e^{\Pi(x_2)} - \Pi'(x_1)e^{\Pi(x_1)}|}{(\alpha+1)(e^{\Pi(x_1)} + e^{\Pi(x_2)})}.$$

*2.27. Fractional H-H-Type Inequalities for Refined Exponential-Type Convex Functions*

**Definition 32** ([110]). *A function* $\Pi : \mathbb{I} \to \mathbb{R}$ *is said to be refined exponential-type convex function if for every* $x, y \in \mathbb{I}$ *and* $t \in (0,1)$

$$\Pi(tx + (1-t)y) \leq (e^t - 1)(e^{1-t} - 1)[\Pi(x) + \Pi(y)].$$

**Theorem 155** ([110]). *Let* $\mathrm{I}^{\alpha}_{x_1+} f$ *and* $\mathrm{I}^{\alpha}_{x_2-} f$ *be R-L fractional operator. In addition, suppose* $\Pi : [x_1, x_2] \to \mathbb{R}$*is a mapping with* $0 \leq x_1 \leq x_2$ *and* $\Pi \in L_1[x_1, x_2]$*. If* $\Pi$ *is a refined exponential-type convex function then the following inequality holds:*

$$\begin{aligned} &\frac{1}{\left(e^{\frac{1}{2}} - 1\right)^2} \Pi\left(\frac{x_1+x_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(x_2-x_1)^{\alpha}} \left[\mathrm{I}^{\alpha}_{x_1+}\Pi(x_2) + \mathrm{I}^{\alpha}_{x_2-}\Pi(x_1)\right] \\ &\leq 2[\Pi(x_1) + \Pi(x_2)][e - {}_1F_1(\alpha, \alpha+1, 1) - {}_1F_1(1, \alpha+1, 1) + 1]. \end{aligned} \tag{4}$$

**Theorem 156** ([110]). *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is a differentiable mapping with* $0 \leq x_1 \leq x_2$ *and* $\mathrm{I}^{\alpha}_{x_1+}\Pi$*,* $\mathrm{I}^{\alpha}_{x_2-}\Pi$ *be R-L fractional operators. If* $|\Pi'|$ *is a refined exponential-type convex function then the following inequality holds:*

$$\begin{aligned} &\frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}} \left[\mathrm{I}^{\alpha}_{x_1+}\Pi(x_2) + \mathrm{I}^{\alpha}_{x_2-}\Pi(x_1)\right] \\ &\leq \frac{(x_2-x_1)[|\Pi'(x_1)| + |\Pi'(x_2)|]}{2} \left[\frac{[e - {}_1F_1(\alpha+1, \alpha+2, 1) - {}_1F_1(1, \alpha+1, 1) + 1]}{\alpha+1}\right]. \end{aligned}$$

**Theorem 157** ([110]). *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is a differentiable mapping with* $0 \leq x_1 \leq x_2$ *and* $\mathrm{I}^{\alpha}_{x_1+}\Pi$*,* $\mathrm{I}^{\alpha}_{x_2-}\Pi$ *be R-L fractional operators. For* $q > 1$*, if* $|\Pi'|^q$ *is a refined exponential-type convex function then the following inequality holds:*

$$\begin{aligned} &\frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}} \left[\mathrm{I}^{\alpha}_{x_1+}\Pi(x_2) + \mathrm{I}^{\alpha}_{x_2-}\Pi(x_1)\right] \\ &\leq (x_2-x_1)\left[\frac{1}{p(\alpha p+1)} \frac{(3-e)[|\Pi'(x_1)|^q + |\Pi'(x_2)|^q]}{q}\right]. \end{aligned}$$

*2.28. Fractional H-H-Type Inequalities for* $\sigma - s$*-Convex Functions*

**Definition 33** ([111]). *A function* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is said to be* $\sigma - s$*-convex function iff*

$$\Pi(\sigma^{-1}(t\sigma(x_1) + (1-t)\sigma(x_2)) \leq t^s\Pi(x_1) + (1-t)^s\Pi(x_2),$$

*holds for some fixed* $s \in (0,1)$ *and* $\sigma$ *is a strictly monotone and continuous function and* $t \in [0,1]$*.*

**Theorem 158** ([111]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is an integrable $\sigma - s$-convex function and $\Pi \in L_1(x_1, x_2)$ with $0 \le x_1 < x_2$. If the function $\sigma$ is increasing and positive on $[x_1, x_2]$ and $\sigma'$ is continuous on $(x_1, x_2)$, then for $\alpha > 0$*

$$\begin{aligned} \Pi\Big(\sigma^{-1}\Big(\frac{\sigma(x_1)+\sigma(x_2)}{2}\Big)\Big) &\le \frac{\Gamma(\alpha+1)}{2^s(\sigma(x_2)-\sigma(x_1))^\alpha}\Big[I^{\alpha,\sigma}_{x_1+}\Pi(x_2)+I^{\alpha,\sigma}_{x_2-}\Pi(x_1)\Big] \\ &\le \frac{\alpha[\Pi(x_1)+\Pi(x_2)]}{2^s}\Big[\frac{1}{s+\alpha}+B(\alpha,s+1)\Big]. \end{aligned}$$

**Theorem 159** ([111]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is an integrable $\sigma - s$-convex function and $\Pi \in L_1(x_1, x_2)$ with $0 \le x_1 < x_2$. If the function $\sigma$ is increasing and positive on $[x_1, x_2]$ and $\sigma'$ is continuous on $(x_1, x_2)$, then for $\alpha > 0$ we have:*

$$\begin{aligned} &\Pi\Big(\sigma^{-1}\Big(\frac{\sigma(x_1)+\sigma(x_2)}{2}\Big)\Big) \\ \le\ & \frac{2^{\alpha-s}\Gamma(\alpha+1)}{(\sigma(x_2)-\sigma(x_1))^\alpha}\Big[I^{\alpha,\sigma}_{\left(\sigma^{-1}\left(\frac{\sigma(x_1)+\sigma(x_2)}{2}\right)\right)+}\Pi(x_2)+I^{\alpha,\sigma}_{\left(\sigma^{-1}\left(\frac{\sigma(x_1)+\sigma(x_2)}{2}\right)\right)-}\Pi(x_1)\Big] \\ \le\ & \frac{\alpha[\Pi(x_1)+\Pi(x_2)]}{2^s}\Big[\frac{1}{2^s(s+\alpha)}+2^\alpha B_{1/2}(\alpha,s+1)\Big]. \end{aligned}$$

### *2.29. Fractional H-H-Type Inequalities for Co-Ordinated-Convex Functions*

**Definition 34** ([112]). *A function $\Pi : \Delta = [x_1, x_2] \times [y_1, y_2] \to \mathbb{R}$ will be called co-ordinated convex on $\Delta$, for all $t, s \in [0, 1]$ and $(x, y), (u, w) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} &\Pi(tx+(1-t)y, su+(1-s)w) \\ \le\ & ts\Pi(x,u)+s(1-t)\Pi(y,u)+t(1-s)\Pi(x,w)+(1-t)(1-s)\Pi(y,w). \end{aligned}$$

**Definition 35** ([113]). *Assume that $\Pi \in L_1([x_1, x_2] \times [y_1, y_2])$. The R-L integrals $J^{\alpha,\beta}_{x_1+,y_1+}\Pi$, $J^{\alpha,\beta}_{x_1+,y_2-}\Pi$, $J^{\alpha,\beta}_{x_2-,y_1+}\Pi$, and $J^{\alpha,\beta}_{x_2-,y_2-}\Pi$ of order $\alpha, \beta > 0$ with $a, c \ge 0$ are defined by*

$$\begin{aligned} J^{\alpha,\beta}_{x_1+,y_1+}\Pi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x_1}^{x}\int_{y_1}^{y}(x-t)^{\alpha-1}(y-s)^{\beta-1}\Pi(s,t)dsdt, \ \ x>x_1, y>y_1, \\ J^{\alpha,\beta}_{x_1+,y_2-}\Pi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x_1}^{x}\int_{y}^{y_2}(x-t)^{\alpha-1}(s-y)^{\beta-1}\Pi(s,t)dsdt, \ \ x>x_1, y<y_2, \\ J^{\alpha,\beta}_{x_2-,y_1+}\Pi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{x_2}\int_{y_1}^{y}(t-x)^{\alpha-1}(y-s)^{\beta-1}\Pi(s,t)dsdt, \ \ x<x_2, y>y_1, \\ J^{\alpha,\beta}_{x_2-,y_2-}\Pi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{x_2}\int_{y}^{y_2}(t-x)^{\alpha-1}(s-y)^{\beta-1}\Pi(s,t)dsdt, \ \ x<x_2, y<y_2, \end{aligned}$$

*respectively. In addition, we introduce the following fractional integrals:*

$$\begin{aligned} J^{\alpha}_{x_1+}\Pi\Big(x,\frac{y_1+y_2}{2}\Big) &= \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x}(x-t)^{\alpha-1}\Pi\Big(t,\frac{y_1+y_2}{2}\Big)dt, \ \ x>x_1, \\ J^{\alpha,}_{x_2-}\Pi\Big(x,\frac{y_1+y_2}{2}\Big) &= \frac{1}{\Gamma(\alpha)}\int_{x}^{x_2}(t-x)^{\alpha-1}\Pi\Big(t,\frac{y_1+y_2}{2}\Big)dt, \ \ x<x_2, \\ J^{\beta}_{y_1+}\Pi\Big(\frac{x_1+x_2}{2},y\Big) &= \frac{1}{\Gamma(\beta)}\int_{y_1}^{y}(y-s)^{\beta-1}\Pi\Big(\frac{x_1+x_2}{2},s\Big)ds, \ \ y>y_1, \\ J^{\beta}_{y_2-}\Pi\Big(\frac{x_1+x_2}{2},y\Big) &= \frac{1}{\Gamma(\beta)}\int_{y}^{y_2}(s-y)^{\beta-1}\Pi\Big(\frac{x_1+x_2}{2},s\Big)ds, \ \ y<y_2. \end{aligned}$$

**Theorem 160** ([113]). *Assume that $\Pi : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is a co-ordinated convex on $\Delta = [x_1, x_2] \times [y_1, y_2]$ with $0 \le x_1 < x_2, 0 \le y_1 < y_2$ and $\Pi \in L_1(\Delta)$. Then one has the inequalities:*

$$\begin{aligned}
&\Pi\Big(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\Big) \\
\le\ & \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\Big[J^\alpha_{x_1+}\Pi\Big(x_2, \frac{y_1+y_2}{2}\Big) + J^\alpha_{x_2-}\Pi\Big(x_1, \frac{y_1+y_2}{2}\Big)\Big] \\
&+\frac{\Gamma(\beta+1)}{2(y_2-y_1)^\beta}\Big[J^\beta_{y_1+}\Pi\Big(\frac{x_1+x_2}{2}, y_2\Big) + J^\beta_{y_2-}\Pi\Big(\frac{x_1+x_2}{2}, y_1\Big)\Big] \\
\le\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(x_2-x_1)^\alpha(y_2-y_1)^\beta}\Big[J^{\alpha,\beta}_{x_1+,y_1+}\Pi(x_2,y_2) + J^{\alpha,\beta}_{x_1+,y_2-}\Pi(x_2,y_1) \\
&+J^{\alpha,\beta}_{y_2-,y_1+}\Pi(x_1,y_2) + J^{\alpha,\beta}_{x_2-,y_2-}\Pi(x_1,y_1)\Big] \\
\le\ & \frac{\Gamma(\alpha+1)}{4(x_2-x_1)^\alpha}\Big[J^\alpha_{x_1+}\Pi(x_2,y_1) + J^\alpha_{x_1+}\Pi(x_2,y_2) + J^\alpha_{x_2-}\Pi(x_1,y_1) + J^\alpha_{x_2-}\Pi(x_1,y_2) \\
&+\frac{\Gamma(\beta+1)}{4(y_2-y_1)^\beta}\Big[J^\beta_{y_1+}\Pi(x_1,y_2) + J^\beta_{y_1+}\Pi(x_2,y_2) + J^\beta_{y_2-}\Pi(x_1,y_1) + J^\beta_{y_2-}\Pi(x_2,y_1) \\
\le\ & \frac{\Pi(x_1,y_1)+\Pi(x_1,y_2)+\Pi(x_2,y_1)+\Pi(x_2,y_2)}{4}.
\end{aligned}$$

**Theorem 161** ([114]). *Assume that $\Pi : \Delta \to \mathbb{R}$ is a co-ordinated convex function such that $\Pi \in L_1(\Delta)$. If $g : \Delta \to \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}$ on the co-ordinates, then for any $\alpha, \beta > 0$ with $x_1, y_1 \ge 0$, the following integral inequalities holds:*

$$\begin{aligned}
&\Pi\Big(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\Big)\Big[J^{\alpha,\beta}_{x_1+,y_1+}g(x_2,y_2) + J^{\alpha,\beta}_{x_1+,y_2-}g(x_2,y_1) + J^{\alpha,\beta}_{x_2-,y_1+}g(x_1,y_2) + J^{\alpha,\beta}_{x_2-,y_2-}g(x_1,y_1)\Big] \\
\le\ & \frac{1}{4}\Big[J^{\alpha,\beta}_{x_1+,y_1+}(\Pi g)(x_2,y_2) + J^{\alpha,\beta}_{x_1+,y_2-}(\Pi g)(x_2,y_1) + J^{\alpha,\beta}_{x_2-,y_1+}(\Pi g)(x_1,y_2) + J^{\alpha,\beta}_{x_2-,y_2-}(\Pi g)(x_1,y_1)\Big] \\
\le\ & \frac{\Pi(x_1,y_1)+\Pi(x_1,y_2)+\Pi(x_2,y_1)+\Pi(x_2,y_2)}{4}\Big[J^{\alpha,\beta}_{x_1+,y_1+}g(x_2,y_2) + J^{\alpha,\beta}_{x_1+,y_2-}g(x_2,y_1) \\
&+J^{\alpha,\beta}_{x_2-,y_1+}g(x_1,y_2) + J^{\alpha,\beta}_{x_2-,y_2-}g(x_1,y_1)\Big].
\end{aligned}$$

**Definition 36** ([115]). *A function $\Pi : \Delta \to \mathbb{R}^+_I$ ($\mathbb{R}_I$ is the family of all positive intervals of $I$) is said to be interval-valued co-ordinated convex function, if the following inequality holds:*

$$\begin{aligned}
&\Pi(tx+(1-t)y, su+(1-s)w) \\
\supseteq\ & ts\Pi(x,u) + s(1-t)\Pi(y,u) + t(1-s)\Pi(x,w) + (1-t)(1-s)\Pi(y,w),
\end{aligned}$$

*for all $(x, y), (u, w) \in \Delta$ and $t, s \in [0, 1]$.*

The H-H integral inequalities for interval-valued co-ordinated convex functions are given in the following theorem.

**Theorem 162** ([116]). *Let $\Pi : \Delta \to \mathbb{R}^+_I$ be an interval co-ordinated convex function on $\Delta$ such that $\Pi(t) = [\underline{F}(t), \overline{F}(t)]$, then the following inequalities hold:*

$$\begin{aligned}
&\Pi\Big(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\Big) \\
\supseteq\ & \Bigg[\frac{\Gamma(\alpha+1)}{4(x_2-x_1)^\alpha}\Big[J^\alpha_{x_1+}\Pi\Big(x_2, \frac{y_1+y_2}{2}\Big) + J^\alpha_{x_2-}\Pi\Big(x_1, \frac{y_1+y_2}{2}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(\beta+1)}{4(y_2-y_1)^\beta}\left[J^\beta_{y_1+}\Pi\left(\frac{x_1+x_2}{2},y_2\right)+J^\beta_{y_2-}\Pi\left(\frac{x_1+x_2}{2},y_1\right)\right]\Bigg]\\
\supseteq\ &\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(x_2-x_1)^\alpha(y_2-y_1)^\beta}\left[J^{\alpha,\beta}_{x_1+,y_1+}\Pi(x_2,y_2)+J^{\alpha,\beta}_{x_1+,y_2-}\Pi(x_2,y_1)\right.\\
&\left.+J^{\alpha,\beta}_{x_2-,y_1+}\Pi(x_1,y_2)+J^{\alpha,\beta}_{x_2-,y_2-}\Pi(x_1,y_1)\right]\\
\supseteq\ &\frac{\Gamma(\alpha+1)}{8(x_2-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(x_2,y_1)+J^\alpha_{x_1+}\Pi(x_2,y_2)+J^\alpha_{x_2-}\Pi(x_1,y_1)+J^\alpha_{x_2-}\Pi(x_1,y_2)\right]\\
&+\frac{\Gamma(\beta+1)}{4(y_2-y_1)^\beta}\left[J^\beta_{y_1+}\Pi(x_1,y_2)+J^\beta_{y_1+}\Pi(x_2,y_2)+J^\beta_{y_2-}\Pi(x_1,y_1)+J^\beta_{y_2-}\Pi(x_2,y_1)\right]\\
\supseteq\ &\frac{\Pi(x_1,y_1)+\Pi(x_1,y_2)+\Pi(x_2,y_1)+\Pi(x_2,y_2)}{4}.
\end{aligned}
$$

### *2.30. Fractional H-H-Type Inequalities for Relative-Convex Functions*

**Definition 37** ([117])**.** *(i) A set $K_g$ is said to be relative convex with respect to an arbitrary function $g:\mathbb{R}\to\mathbb{R}$ such that*

$$(1-t)x_1+tg(x_2)\in K_g,\ \ \forall x_1,x_2\in\mathbb{R},\ \ x_1,g(x_2)\in K_g,\ t\in[0,1].$$

*(ii) A function $\Pi:K_g\to\mathbb{R}$ is said to be relative convex, if there exists an arbitrary function $g:\mathbb{R}\to\mathbb{R}$ such that*

$$\Pi((1-t)x_1+tg(x_2))\le(1-t)\Pi(x_1)+t\Pi(g(x_2)),\ \ \forall x_1,x_2\in\mathbb{R},\ \ x_1,g(x_2)\in K_g,\ t\in[0,1].$$

*(iii) A function $\Pi:K_g\to(0,\infty)$ is said to be relative logarithmic convex, if there exists an arbitrary function $g:\mathbb{R}\to\mathbb{R}$ such that*

$$\Pi((1-t)x_1+tg(x_2))\le[\Pi(x_1)^{1-t}[\Pi(g(x_2))]^t,\ \forall x_1,x_2\in\mathbb{R},\ \ x_1,g(x_2)\in K_g,\ t\in[0,1].$$

*(iv) A function $\Pi:K_g\to\mathbb{R}$ is said to be relative quasi convex, if there exists an arbitrary function $g:\mathbb{R}\to\mathbb{R}$ such that*

$$\Pi((1-t)x_1+tg(x_2))\le\max\{\Pi(x_1),\Pi(g(x_2))\},\ \ \forall x_1,x_2\in\mathbb{R},\ \ x_1,g(x_2)\in K_g,\ t\in[0,1].$$

**Theorem 163** ([117])**.** *Let $\Pi$ be positive and relative convex function and $\Pi\in L_1[x_1,g(x_2)]$. Then we have:*

$$\Pi\left(\frac{x_1+g(x_2)}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(g(x_2)-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(g(x_2))+J^\alpha_{x_2-}\Pi(x_1)\right]\le\frac{\Pi(x_1)+\Pi(g(x_2))}{2}.$$

**Theorem 164** ([117])**.** *Assume that $\Pi:K_g\to\mathbb{R}$ is a differentiable function on $K_g^\circ$ and $\Pi'\in L_1[x_1,g(x_2)]$. If $|\Pi'|$ is relative convex on $K_g$, then:*

$$
\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(g(x_2))}{2}-\frac{\Gamma(\alpha+1)}{2(g(x_2)-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(g(x_2))+J^\alpha_{x_2-}\Pi(x_1)\right]\right|\\
\le\ &\frac{g(x_2)-x_1}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|\Pi'(x_1)|+\Pi'(g(x_2))|].
\end{aligned}
$$

**Theorem 165** ([117])**.** *Assume that $\Pi$ is as in Theorem 164. If $|\Pi''|$ is relative convex on $K_g$, then:*

$$
\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(g(x_2))}{2}-\frac{\Gamma(\alpha+1)}{2(g(x_2)-x_1)^\alpha}\left[J^\alpha_{x_1+}\Pi(g(x_2))+J^\alpha_{x_2-}\Pi(x_1)\right]\right|\\
\le\ &\frac{\alpha(g(x_2)-x_1)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{|\Pi'(x_1)|+\Pi'(g(x_2))|}{2}\right].
\end{aligned}
$$

*2.31. Fractional H-H-Type Inequalities for Quasi-Convex Functions*

**Definition 38** ([118]). *A function $\Pi : \mathbb{I} \to \mathbb{R}$ is said to be quasi convex, if*

$$\Pi(tx + (1-t)y) \leq \max\{\Pi(x), \Pi(y)\},$$

*for all $x, y \in \mathbb{I}$ and $t \in [0,1]$.*

In the following we give fractional H-H-type inequalities for quasi-convex functions.

**Theorem 166** ([118]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is a differentiable mapping on $(x_1, x_2)$ with $x_1 < x_2$. If $|\Pi'|$ is quasi-convex on $[x_1, x_2]$ and $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{(x_2 - x_1)^\alpha} [J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] \right| \\ \leq\ & \frac{x_2 - x_1}{\alpha + 1} \Big( 1 - \frac{1}{2^\alpha} \Big) \max\{|\Pi'(x_1)|, |\Pi'(x_2)|\}. \end{aligned}$$

**Theorem 167** ([118]). *Assume that $\Pi$ is as in Theorem 166. If $|\Pi'|^q$, $q > 1$ is quasi-convex on $[x_1, x_2]$, then:*

$$\begin{aligned} &\left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^\alpha} [J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)] \right| \\ \leq\ & \frac{x_2 - x_1}{2(\alpha p + 1)^{\frac{1}{p}}} \Big( \max\{|\Pi'(x_1)|^q, |\Pi'(x_2)|^q\} \Big)^{\frac{1}{q}}, \end{aligned}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > 0$.*

**Theorem 168** ([119]). *Assume that $\Pi$ is as in Theorem 166. If $|\Pi'|^q$, $q \geq 1$ is quasi-convex function on $[x_1, x_2]$, then we have:*

$$\begin{aligned} &\Bigg| \frac{1}{2} \Big[ \frac{\Pi(x_1) + \Pi(x_2)}{2} + \Pi\Big(\frac{x_1 + x_2}{2}\Big) \Big] - \frac{4^\alpha \Gamma(\alpha+1)}{(x_2 - x_1)^\alpha} \\ &\times \Big[ J^\alpha_{x_1+}\Pi\Big(\frac{3x_1 + x_2}{4}\Big) + J^\alpha_{\left(\frac{3x_1+x_2}{2}\right)+}\Pi\Big(\frac{x_1 + x_2}{2}\Big) \\ &+ J^\alpha_{\left(\frac{x_1+x_2}{2}\right)+}\Pi\Big(\frac{3x_1 + x_2}{4}\Big) + {}^\alpha_{\left(\frac{3x_1+x_2}{2}\right)+}\Pi(x_2) \Bigg| \\ \leq\ & \frac{x_2 - x_1}{16} \Bigg[ \frac{\Big( \max\Big\{|\Pi'(x_1)|^q, \Big|\Pi'\Big(\frac{3x_1+x_2}{4}\Big)\Big|^q\Big\} + \max\Big\{\Big|\Pi'\Big(\frac{x_1+x_2}{2}\Big)\Big|^q, \Big|\Pi'\Big(\frac{x_1+3x_2}{4}\Big)\Big|^q\Big\}\Big)^{\frac{1}{q}}}{\alpha + 1} \\ &+ \frac{\alpha}{\alpha+1} \Big( \max\Big\{\Big|\Pi'\Big(\frac{3x_1 + x_2}{4}\Big)\Big|^q, \Big|\Pi'\Big(\frac{x_1 + x_2}{2}\Big)\Big|^q\Big\} \\ &+ \max\Big\{\Big|\Pi'\Big(\frac{x_1 + 3x_2}{4}\Big)\Big|^q, |\Pi'(x_2)|^q\Big\}\Big)^{\frac{1}{q}} \Bigg]. \end{aligned}$$

**Theorem 169** ([119]). *Assume that $\Pi$ is as in Theorem 166. If $|\Pi'|^q$, $q \geq 1$ is quasi-convex function on $[x_1, x_2]$, and $\frac{1}{p} + \frac{1}{q} = 1$, then we have:*

$$\left| \frac{1}{2} \Big[ \frac{\Pi(x_1) + \Pi(x_2)}{2} + \Pi\Big(\frac{x_1 + x_2}{2}\Big) \Big] - \frac{4^\alpha \Gamma(\alpha+1)}{(x_2 - x_1)^\alpha} \right.$$

$$\times\Big[J^{\alpha}_{x_1+}\Pi\Big(\frac{3x_1+x_2}{4}\Big)+J^{\alpha}_{\big(\frac{3x_1+x_2}{2}\big)+}\Pi\Big(\frac{x_1+x_2}{2}\Big)$$
$$+J^{\alpha}_{\big(\frac{x_1+x_2}{2}\big)+}\Pi\Big(\frac{3x_1+x_2}{4}\Big)+J^{\alpha}_{\Big(\frac{3x_1+x_2}{2}\Big)+}\Pi(x_2)$$
$$\leq \frac{x_2-x_1}{16}\bigg[\Big(\frac{1}{\alpha p+1}\Big)^{\frac{1}{p}}\Big(\max\Big\{|\Pi'(x_1)|^q,\Big|\Pi'\Big(\frac{3x_1+x_2}{4}\Big)\Big|^q\Big\}$$
$$+\max\Big\{\Big|\Pi'\Big(\frac{x_1+x_2}{2}\Big)\Big|^q,\Big|\Pi'\Big(\frac{x_1+3x_2}{4}\Big)\Big|^q\Big\}\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{\Gamma(1+1/\alpha)\Gamma(p+1)}{\Gamma(1+(1/p)+1}\Big)^{\frac{1}{p}}\Big(\max\Big\{\Big|\Pi'\Big(\frac{3x_1+x_2}{4}\Big)\Big|^q,\Big|\Pi'\Big(\frac{x_1+x_2}{2}\Big)\Big|^q\Big\}$$
$$+\max\Big\{\Big|\Pi'\Big(\frac{x_1+3x_2}{4}\Big)\Big|^q,|\Pi'(x_2)|^q\Big\}\Big)^{\frac{1}{q}}\bigg].$$

**Definition 39** ([120]). *A function $\Pi:\mathbb{I}\subseteq(0,\infty)\to[0,\infty)$ is said to be harmonically quasi-convex, if*

$$\Pi\Big(\frac{xy}{tx+(1-t)y}\Big)\leq\max\{\Pi(x,\Pi(y)\},$$

*for all $x,y\in\mathbb{I}$ and $t\in[0,1]$.*

**Theorem 170** ([121]). *Assume that $\Pi:\mathbb{I}\subset(0,\infty)\to\mathbb{R}$ is a differentiable function on $\mathbb{I}^\circ$ such that $\Pi'\in L_1[x_1,x_2]$, where $x_1,x_2\in\mathbb{I}$ and $x_1<x_2$. If $|\Pi'|$ is harmonically quasi-convex function on $[x_1,x_2]$, $g:[x_1,x_2]\to\mathbb{R}$ is continuous and harmonically symmetric with respect to $\dfrac{2x_1x_2}{x_1+x_2}$, then the following inequality for fractional integrals holds:*

$$\bigg|\frac{\Pi(x_1)+\Pi(x_2)}{2}\Big[J^{\alpha}_{\frac{1}{x_2}+}(g\circ h)\Big(\frac{1}{x_1}\Big)+J^{\alpha}_{\frac{1}{x_1}-}(g\circ h)\Big(\frac{1}{x_2}\Big)\Big]$$
$$-\Big[J^{\alpha}_{\frac{1}{x_2}+}(\Pi g\circ h)\Big(\frac{1}{x_1}\Big)+J^{\alpha}_{\frac{1}{x_1}-}(\Pi g\circ h)\Big(\frac{1}{x_2}\Big)\Big]\bigg|$$
$$\leq \frac{\|g\|_\infty x_1x_2(x_2-x_1)}{\Gamma(\alpha+1)}\Big(\frac{x_2-x_1}{a}\Big)^{\alpha}\sup\{|\Pi'(x_1)|,|\Pi'(x_2)|\}\Big[\frac{x_2^{-2}}{\alpha+1}\,{}_2F_1\Big(2,1,\alpha+2,1-\frac{x_1}{x_2}\Big)$$
$$-\frac{x_2^{-2}}{\alpha+1}\,{}_2F_1\Big(2,\alpha+1,\alpha+2,1-\frac{x_1}{x_2}\Big)+\frac{4(x_1+x_2)^{-2}}{\alpha+1}\,{}_2F_1\Big(2,\alpha+1,\alpha+2,1-\frac{x_2-x_1}{x_1+x_2}\Big)\Big],$$

*with $0<\alpha\leq1$ and $h(x)=\dfrac{1}{x}, x\in\Big[\dfrac{1}{x_2},\dfrac{1}{x_1}\Big]$.*

**Theorem 171** ([121]). *Assume that $\Pi$ is as in Theorem 170. If $|\Pi'|^q, q>1$ is harmonically quasi-convex function on $[x_1,x_2]$, $g:[x_1,x_2]\to\mathbb{R}$ is continuous and harmonically symmetric with respect to $\dfrac{2x_1x_2}{x_1+x_2}$, then the following inequality for fractional integrals holds:*

$$\bigg|\frac{\Pi(x_1)+\Pi(x_2)}{2}\Big[J^{\alpha}_{\frac{1}{x_2}+}(g\circ h)\Big(\frac{1}{x_1}\Big)+J^{\alpha}_{\frac{1}{x_1}-}(g\circ h)\Big(\frac{1}{x_2}\Big)\Big]$$
$$-\Big[J^{\alpha}_{\frac{1}{x_2}+}(\Pi g\circ h)\Big(\frac{1}{x_1}\Big)+J^{\alpha}_{\frac{1}{x_1}-}(\Pi g\circ h)\Big(\frac{1}{x_2}\Big)\Big]\bigg|$$
$$\leq \frac{\|g\|_\infty x_1x_2(x_2-x_1)}{2^{\frac{1}{q}}\Gamma(\alpha+1)}\Big(\frac{x_2-x_1}{a}\Big)^{\alpha}\sup\{|\Pi'(x_1)|^q,|\Pi'(x_2)|^q\}$$
$$\times\Big[\Big(\frac{x_1+x_2}{2}\Big)^{-2p}\frac{1}{2(\alpha p+1)}\,{}_2F_1\Big(2p,\alpha p+1,\alpha p+2,1-\frac{x_2-x_1}{x_1+x_2}\Big)$$

$$+x_2^{-2p}\frac{1}{2(\alpha p+1)}\ {}_2F_1\Big(2p,1,\alpha p+2,\frac{1}{2}\Big(1-\frac{x_1}{x_2}\Big)\Big)\Big],$$

*with* $0<\alpha\leq 1$ *and* $h(x)=\frac{1}{x}, x\in\Big[\frac{1}{x_2},\frac{1}{x_1}\Big]$ *and* $\frac{1}{p}+\frac{1}{q}=1.$

### 2.32. Fractional H-H-Type Inequalities for $(\theta, h-m)-p$-Convex Functions

**Definition 40** ([122]). *Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h:J\to\mathbb{R}$ be a non-negative function. Let $\mathbb{I}\subset(0,\infty)$ be an interval and $p\in\mathbb{R}\setminus\{0\}$. A function $\Pi:\mathbb{I}\to\mathbb{R}$ is said to be $(\theta,h-m)-p$-convex, if*

$$\Pi\Big(\big(tx_1^p+m(1-t)x_2^p\big)^{\frac{1}{p}}\Big)\leq h(t^\theta)\Pi(x_1)+mh(1-t^\theta)\Pi(x_2),$$

*holds, provided $\big(tx_1^p+m(1-t)x_2^p\big)^{\frac{1}{p}}\in\mathbb{I}$ for $t\in[0,1]$ and $(\theta,m)\in[0,1]^2$.*

**Theorem 172** ([122]). *Let $\Pi:\mathbb{I}\to\mathbb{R}$ be a positive $(\theta,h-m)-p$-convex function with $\big(tx_2^p+m(1-t)(x_1^p/m)\big)^{\frac{1}{p}}\in\mathbb{I}$, $m\neq 0$, $x_1^p<mx_2^p$. Then, the following inequality for fractional integral operators holds:*

$$\begin{aligned}
&\Pi\Big(\Big(\frac{x_1^p+mx_2^p}{2}\Big)^{\frac{1}{p}}\Big)\\
\leq\ &\frac{\Gamma(\alpha+1)}{(mx_2^p-x_1^p)^\alpha}\Big(h\Big(\frac{1}{2^\theta}\Big)(I^\alpha_{x_1^p+}\Pi\circ\xi)(mx_2^p)+m^{\alpha+1}h\Big(\frac{2^\theta-1}{2^\theta}\Big)(I^\alpha_{x_2^p-}\Pi\circ\xi)\Big(\frac{x_1^p}{m}\Big)\Big)\\
\leq\ &\alpha\Big\{\Big(h\Big(\frac{1}{2^\theta}\Big)\Pi(x_1)+mh\Big(\frac{2^\theta-1}{2^\theta}\Big)\Pi(x_2)\Big)\int_0^1 t^{\alpha-1}h(t^\theta)dt\\
&+m\alpha\Big(h\Big(\frac{1}{2^\theta}\Big)\Pi(x_2)+mh\Big(\frac{2^\theta-1}{2^\theta}\Big)\Pi\Big(\frac{x_1}{m^2}\Big)\Big)\int_0^1 t^{\alpha-1}h(1-t^\theta)dt\Big\},\ \ \xi(t)=t^{\frac{1}{p}}.
\end{aligned}$$

**Theorem 173** ([122]). *Assume that $\Pi$ as in Theorem 172. Then, the following inequalities hold:*

$$\begin{aligned}
&\Pi\Big(\Big(\frac{x_1^p+mx_2^p}{2}\Big)^{\frac{1}{p}}\Big)\\
\leq\ &\Gamma(\alpha+1)\Big(\frac{2}{mx_2^p-x_1^p}\Big)^\alpha\Big(h\Big(\frac{1}{2^\theta}\Big)(I^\alpha_{(x_1^p+mx_2^p/2)+}\Pi\circ\xi)(mx_2^p)\\
&+mh\Big(\frac{2^\theta-1}{2^\theta}\Big)(I^\alpha_{(x_1^p+mx_2^p/2)-}\Pi\circ\xi)\Big(\frac{x_1^p}{m}\Big)\Big)\\
\leq\ &\alpha\Big\{\Big(h\Big(\frac{1}{2^\theta}\Big)\Pi(x_1)+mh\Big(\frac{2^\theta-1}{2^\theta}\Big)\Pi(x_2)\Big)\int_0^1 t^{\alpha-1}h\Big(\Big(\frac{t}{2}\Big)^\theta\Big)dt\\
&+m\Big(h\Big(\frac{1}{2^\theta}\Big)\Pi(x_2)+mh\Big(\frac{2^\theta-1}{2^\theta}\Big)\Pi\Big(\frac{x_1}{m^2}\Big)\Big)\int_0^1 t^{\alpha-1}h\Big(\Big(\frac{2-t}{2}\Big)^\theta\Big)dt\Big\},\ \ \xi(t)=t^{\frac{1}{p}}.
\end{aligned}$$

### 2.33. Fractional H-H-Type Inequalities for Preinvex Functions

**Definition 41** ([123]). *A set $K\subseteq\mathbb{R}^n$ is said an invex with respect to the bifunction $\eta:K\times K\to\mathbb{R}^n$, if for all $x,y\in K$, we have*

$$x+t\eta(y,x)\in K.$$

**Definition 42** ([123]). *A function $\Pi:K\to\mathbb{R}$ is said to be preinvex with respect to $\eta$ if*

$$\Pi(x+t\eta(y,x))\leq(1-t)\Pi(x)+t\Pi(y),$$

*for all $x,y\in K$, and all $t\in[0,1]$.*

**Definition 43** ([124])**.** *A non-negative function $\Pi : K \subset (0,\infty) \to \mathbb{R}$ is said to be s-preinvex in the second sense with respect to $\eta$ for some $s \in (0,1]$, if*

$$\Pi(x + t\eta(y,x)) \leq (1-t)^s \Pi(x) + t^s \Pi(y),$$

*for all $x, y \in K$, and all $t \in [0,1]$.*

**Theorem 174** ([125])**.** *Let $\Pi : [x_1, x_1 + \eta(x_2,x_1)] \subset \mathbb{R} \to \mathbb{R}$ be a positive function on $[x_1, x_1+\eta(x_2,x_1)]$ with $\eta(x_2,x_1) > 0$ and $\Pi \in L_1[x_1, x_1+\eta(x_2,x_1)]$. If $|\Pi'|$ is an s-preinvex function, where $s \in (0,1]$, then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|\Pi\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big) - \frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^\alpha}\Big[J^\alpha_{(x_1+\eta(x_2,x_1))-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_1+\eta(x_2,x_1))\Big]\right| \\
\leq\ & \frac{\eta(x_2,x_1)}{2}\Big(\frac{1}{s+1} + \frac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1,s+1) + B_{\frac{1}{2}}(s+1,\alpha+1)\Big)\Big[|\Pi'(x_1)| + \Pi'(x_2)|\Big],
\end{aligned}$$

*where $B_{\frac{1}{2}}(\cdot,\cdot)$ is the incomplete Beta function.*

**Theorem 175** ([125])**.** *Assume that $\Pi$ is as in Theorem 174. If $|\Pi'|^q$ ,$q > 1$ is an s-preinvex function, and $\frac{1}{p} + \frac{1}{q} = 1$, then we have:*

$$\begin{aligned}
&\left|\Pi\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big) - \frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^\alpha}\Big[J^\alpha_{(x_1+\eta(x_2,x_1))-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_1+\eta(x_2,x_1))\Big]\right| \\
\leq\ & \frac{\eta(x_2,x_1)}{2}\left(1 + \frac{\Big(1-\left(\frac{1}{2}\right)^{\alpha p}\Big)^{\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}}\right)\left[\left(\frac{\Big(1-\left(\frac{1}{2}\right)^{s+1}\Big)|\Pi'(x_1)|^q + \left(\frac{1}{2}\right)^{s+1}|\Pi'(x_2)|^q}{s+1}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{\left(\frac{1}{2}\right)^{s+1}|\Pi'(x_1)|^q + \Big(1-\left(\frac{1}{2}\right)^{s+1}|\Pi'(x_2)|^q}{s+1}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

**Theorem 176** ([126])**.** *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta$ and $x_1, x_2 \in K$ with $x_1 < x_1 + \eta(x_2,x_1)$. If $\Pi : K \to \mathbb{R}$ is a differentiable function such that $\Pi' \in L_1[x_1, x_1 + \eta(x_2,x_1)]$ and $|\Pi'|$ is a preinvex function on $[x_1, x_1+\eta(x_2,x_1)]$, then fractional integral inequality with $\alpha > 0$ is given as:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(x_1+\eta(x_2,x_1))}{2} - \frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^\alpha}\Big[J^\alpha_{(x_1+\eta(x_2,x_1))-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_1+\eta(x_2,x_1))\Big]\right| \\
\leq\ & \frac{\eta(x_2,x_1)}{2(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)\Big[|\Pi'(x_1)| + |\Pi'(x_2)|\Big].
\end{aligned}$$

**Theorem 177** ([126])**.** *Assume that $\Pi$ is as in Theorem 176 with the condition $q > 1$. Then fractional integral inequality with $\alpha > 0$ is given as:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(x_1+\eta(x_2,x_1))}{2} - \frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^\alpha}\Big[J^\alpha_{(x_1+\eta(x_2,x_1))-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_1+\eta(x_2,x_1))\Big]\right| \\
\leq\ & \frac{\eta(x_2,x_1)}{2(\alpha+1)^{\frac{1}{p}}}\Big[\frac{|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{2}\Big]^{\frac{1}{q}},
\end{aligned}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.*

**Definition 44** ([127]). *A function $\Pi : K \to \mathbb{R}$ is said to be s-preinvex of first kind with respect to the bifunction $\eta(\cdot,\cdot)$, if*

$$\Pi(x_1 + t\eta(x_2, x_1)) \leq (1 - t^s)\Pi(x_1) + t^s\Pi(x_2), \;\; x_1, x_2 \in K, \; t \in [0,1], \; s \in [0,1].$$

**Theorem 178** ([127]). *Let $\Pi : K \to \mathbb{R}$ be a twice differentiable function on $(x_1, x_2)$ such that $\Pi'' \in L_1[x_1, x_1 + \eta(x_2, x_1)]$ and $|\Pi''|$ is an s-preinvex function of first kind. Then:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1) + \Pi(x_1 + \eta(x_2, x_1))}{2} - \frac{\Gamma(\alpha+1)}{2(\eta(x_2, x_1))^\alpha}\left[J^\alpha_{(x_1+\eta(x_2,x_1))-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_1 + \eta(x_2, x_1))\right]\right| \\
\leq\; & \frac{\eta^2(x_2, x_1)}{2(\alpha+1)}\Big[|\Pi'(x_1)|\Big(\frac{\alpha}{\alpha+2} - \frac{1}{s+1} + \frac{1}{s+\alpha+2} + \frac{\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(s+\alpha+3)}\Big) \\
&+ |\Pi'(x_2)|\Big(\frac{1}{s+1} - \frac{1}{s+\alpha+2} + \frac{\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(s+\alpha+3)}\Big)\Big].
\end{aligned}$$

**Theorem 179** ([127]). *Assume that $\Pi$ is as in Theorem 178. If $|\Pi''|^q$ for $q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1$ is an s-preinvex function of first kind. Then*

$$\begin{aligned}
&\left|\frac{\Pi(x_1) + \Pi(x_1 + \eta(x_2, x_1))}{2} - \frac{\Gamma(\alpha+1)}{2(\eta(x_2, x_1))^\alpha}\left[J^\alpha_{(x_1+\eta(x_2,x_1))-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_1 + \eta(x_2, x_1))\right]\right| \\
\leq\; & \frac{\eta^2(x_2, x_1)}{2(\alpha+1)}\Big(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\Big)^{\frac{1}{p}}\Big(\frac{s|\Pi''(x_1)|^q + |\Pi''(x_2)|^q}{s+1}\Big)^{\frac{1}{q}}.
\end{aligned}$$

**Theorem 180** ([127]). *Assume that $\Pi$ is as in Theorem 178. If $|\Pi'|^q$ for $q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1$ is an s-preinvex function of second kind. Then:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1) + \Pi(x_1 + \eta(x_2, x_1))}{2} - \frac{\Gamma(\alpha+1)}{2(\eta(x_2, x_1))^\alpha}\left[J^\alpha_{(x_1+\eta(x_2,x_1))-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_1 + \eta(x_2, x_1))\right]\right| \\
\leq\; & \frac{\eta(x_2, x_1)}{2(\alpha+1)}\Big(\frac{1}{p\alpha+1}\Big)^{\frac{1}{p}}\Big(\frac{|\Pi''(x_1)|^q + |\Pi''(x_2)|^q}{s+1}\Big)^{\frac{1}{q}}.
\end{aligned}$$

**Theorem 181** ([127]). *Assume that $\Pi$ is as in Theorem 178. If $|\Pi''|$ is an s-preinvex function of second kind, then:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1) + \Pi(x_1 + \eta(x_2, x_1))}{2} - \frac{\Gamma(\alpha+1)}{2(\eta(x_2, x_1))^\alpha}\left[J^\alpha_{x_1+\eta(x_2,x_1))-}\Pi(x_1) + J^\alpha_{x_1+}\Pi(x_1 + \eta(x_2, x_1))\right]\right| \\
\leq\; & \frac{\eta^2(x_2, x_1)}{2}\left[|\Pi''(x_1)| + |\Pi''(x_2)|\right]\frac{(\alpha+1) - (s+1)(s+\alpha+2)B(s+1, \alpha+2)}{(\alpha+1)(s+1)(s+\alpha+2)}.
\end{aligned}$$

**Definition 45** ([128]). *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $\Pi : K \to \mathbb{R}$ is said to be $(s, m)$-preinvex with respect to $\eta$ for every $x, y \in K$, $t \in [0,1]$ and $m \in (0,1]$, if*

$$\Pi(x_1 + t\eta(x_2, x_1)) \leq m(1-t)^s\Pi\Big(\frac{x_1}{m}\Big) + t^s\Pi(x_2).$$

**Theorem 182** ([128]). *Let $K \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $x_1, x_2 \in K$ with $x_1 < x_1 + \eta(x_2, x_1)$ where $\eta(x_2, x_1) \neq 0$. Suppose $\Pi : K \to \mathbb{R}$ is a differentiable mapping on $K^\circ$ such that $\Pi' \in L_1[x_1, x_1 + \eta(x_2, x_1)]$. If $w : [x_1, x_1 + \eta(x_2, x_1)] \to [0, \infty)$ is a continuous mapping and symmetric to $x_1 + \frac{1}{2}\eta(x_2, x_1)$, and $\Pi'$ is $(s, m)$-preinvex on $K$, then fractional integral inequality is given as:*

$$\left|\frac{1}{\eta(x_2,x_1)}\int_{x_1}^{x_1+\eta(x_2,x_1)}\Pi(x)w(x)dx-\frac{1}{\eta(x_2,x_1)}\Pi\Big(x_1+\frac{1}{2}\eta(x_2,x_1)\Big)\int_{x_1}^{x_1+\eta(x_2,x_1)}w(x)dx\right|$$
$$\leq \frac{\|w\|_\infty}{(s+1)(s+2)}\Big[m\Big|\Pi'\Big(\frac{x_1}{m}\Big)\Big|+|\Pi'(x_2)|\Big]\Big(1-\frac{1}{2^{s+1}}\Big).$$

**Theorem 183** ([128]). *Assume that $\Pi$ is as in Theorem 182 and $|\Pi'|^q, q>1$ is $(s,m)$-preinvex on K, then fractional integral inequality is given as:*

$$\left|\frac{1}{\eta(x_2,x_1)}\int_{x_1}^{x_1+\eta(x_2,x_1)}\Pi(x)w(x)dx-\frac{1}{\eta(x_2,x_1)}\Pi\Big(x_1+\frac{1}{2}\eta(x_2,x_1)\Big)\int_{x_1}^{x_1+\eta(x_2,x_1)}w(x)dx\right|$$
$$\leq \eta(x_2,x_1)\Big(\frac{1}{(\eta(x_2,x_1))^2}\int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)}\Big[\frac{\eta(x_2,x_1)}{2}-(x-x_1)\Big]w^p(x)dx\Big)^{\frac{1}{p}}$$
$$\times\Big[\Big(m\Big|\Pi'\Big(\frac{x_1}{m}\Big)\Big|^q\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}+|\Pi'(x_2)|^q\frac{1}{2^{s+2}(s+2)}\Big)^{\frac{1}{q}}$$
$$+\Big(m\Big|\Pi'\Big(\frac{x_1}{m}\Big)\Big|^q\frac{1}{2^{s+2}(s+2)}+|\Pi'(x_2)|^q\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}\Big)^{\frac{1}{q}}\Big],$$

*where* $\frac{1}{p}+\frac{1}{q}=1.$

**Definition 46** ([129]). *Let $K\subseteq\mathbb{R}^n$ be an open m-invex set with respect to $\eta: K\times K\times(0,1]\to\mathbb{R}^n$. For $\Pi: K\to R$ and some fixed $\alpha, m\in(0,1]$, if*

$$\Pi(mx+t\eta(y,x,m))\leq m(1-t^\alpha)\Pi(x)+t^\alpha\Pi(y) \tag{5}$$

*is valid $\forall x,y\in K, t\in[0,1]$, then we say that $\Pi$ is a generalized $(\alpha,m)$-preinvex function with respect to $\eta$.*

H-H-type inequalities via generalized ($\alpha$, m)-preinvex functions via the R-L fractional integral are presented in the following theorems.

**Theorem 184** ([129]). *Let $A\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: A\times A\times(0,1]\to\mathbb{R}$ for some fixed $m\in(0,1]$ and let $x_1,x_2\in A$, $x_1<x_2$ with $\eta(x_2,x_1,m)>0$. Assume that $\Pi: A\to\mathbb{R}$ is a twice differentiable function, $|\Pi''|$ is a generalized $(\alpha,m)$-preinvex function on A for some fixed $\alpha,m\in(0,1]$ and $x\in[mx_1, mx_1+\eta(x_2,x_1,m)]$, then for $0<\alpha\leq 1$ we have:*

$$\left|\frac{\Pi(mx_1)+\Pi(mx_1+\eta(x_2,x_1),m))}{2}-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(x_2,x_1,m)}\Big[J^\alpha_{mx_1+}\Pi(mx_1+\eta(x_2,x_1),m))\right.$$
$$\left.+J^\alpha_{(mx_1+\eta(x_2,x_1),m))-}\Pi(mx_1)\Big]\right|$$
$$\leq\frac{\eta^2(x_2,x_1,m)}{2(\alpha+1)}\Big[m\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+B(\alpha+1,\alpha+2)\Big)|\Pi''(x_1)|$$
$$+\Big(\frac{1}{2\alpha+2}-B(\alpha+1,\alpha+2)\Big)|\Pi''(x_2)|\Big].$$

**Theorem 185** ([129]). *Let $\Pi$ be defined as in Theorem 184. If $|\Pi''|$ is a generalized $(\alpha,m)$-preinvex function on $A$ for some fixed $\alpha, m \in (0,1]$ and $x \in [mx_1, mx_1+\eta(x_2,x_1),m)]$, then for $0<\alpha\le 1$ we have:*

$$\left|\frac{\Pi(mx_1)+\Pi(mx_1+\eta(x_2,x_1),m))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(x_2,x_1,m)}\left[J^{\alpha}_{mx_1+}\Pi(mx_1+\eta(x_2,x_1),m))\right.\right.$$
$$\left.\left.+J^{\alpha}_{(mx_1+\eta(x_2,x_1),m))-}\Pi(mx_1)\right]\right|$$
$$\le \frac{\eta^2(x_2,x_1,m)}{2(\alpha+1)}\left\{m\left[\frac{q\alpha+\alpha+1}{(\alpha+1)(q+1)} - \frac{2}{q(\alpha+1)+1} + B(\alpha+1,q(\alpha+1)+1)\right]|\Pi''(x_1)|^q\right.$$
$$\left.+\left[\frac{q}{(\alpha+1)(q+1)} - B(\alpha+1,q(\alpha+1)+1)\right]|\Pi''(x_2)|^q\right\}^{\frac{1}{q}}.$$

**Theorem 186** ([129]). *Let $\Pi$ be defined as in Theorem 184 with $\frac{1}{p}+\frac{1}{q}=1$, $q\ge 1$. If $|\Pi''|$ is a generalized $(\alpha,m)$-preinvex function on $A$ for some fixed $\alpha, m \in (0,1]$ and $x\in[x_1,x_1+\eta(x_2,x_1),m)]$, then for $0<\alpha\le 1$ we have:*

$$\left|\frac{\Pi(mx_1)+\Pi(mx_1+\eta(x_2,x_1),m))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(x_2,x_1,m)}\left[J^{\alpha}_{m_1+}\Pi(mx_1+\eta(x_2,x_1),m))\right.\right.$$
$$\left.\left.+J^{\alpha}_{(mx_1+\eta(x_2,x_1),m))-}\Pi(mx_1)\right]\right|$$
$$\le \frac{\eta^2(x_2,x_1,m)}{2(\alpha+1)}\left(\frac{p\alpha+p-1}{p\alpha+p+1}\right)^{\frac{1}{p}}\left(\frac{m\alpha}{\alpha+1}|\Pi''(x_1)|^q + \frac{1}{\alpha+1}|\Pi''(x_2)|^q\right)^{\frac{1}{q}}.$$

**Theorem 187** ([129]). *Suppose all the assumptions of Theorem 185 are satisfied. Then the following inequality for the R-L fractional integral with $0<\alpha\le 1$ holds:*

$$\left|\frac{\Pi(m_1)+\Pi(mx_1+\eta(x_2,x_1),m))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(x_2,x_1,m)}\left[J^{\alpha}_{ma+}\Pi(mx_1+\eta(x_2,x_1),m))\right.\right.$$
$$\left.\left.+J^{\alpha}_{(mx_1+\eta(x_2,x_1),m))-}\Pi(mx_1)\right]\right|$$
$$\le \frac{\eta^2(x_2,x_1,m)}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\left[m\left(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)} + B(\alpha+1,\alpha+2)\right)|\Pi''(x_1)|^q\right.$$
$$\left.+\left(\frac{1}{2\alpha+2} - B(\alpha+1,\alpha+2)\right)|\Pi''(x_2)|^q\right]^{\frac{1}{q}}.$$

## 3. H-H-Type Inequalities via Katugampola Fractional Integral

**Definition 47** ([130]). *Let $[x_1,x_2]\subset\mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integral of order $\alpha>0$ of $\Pi\in X^p_c(x_1,x_2)$ are defined by*

$${}^{\rho}I^{\alpha}_{x_1+}\Pi(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x_1}^{x}\frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}g(t)dt \ \ and \ \ {}^{\rho}I^{\alpha}_{x_2-}\Pi(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{x_2}\frac{t^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}g(t)dt,$$

*with $x_1<x<x_2$ and $\rho>0$, if the integrals exist. Here $X^p_c(x_1,x_2)$, $c\in\mathbb{R}$, $1\le p\le\infty$ denote the space of those complex-valued Lebesque measurable functions $\Pi$ on $[x_1,x_2]$ for which $\|f\|_{X^p_c}<\infty$, where $\|f\|_{X^p_c}=\left(\int_{x_1}^{x_2}|t^c\Pi(t)|^p\frac{dt}{t}\right)^{1/p}<\infty$ for $1\le p<\infty$ and $\|f\|_{X^p_c}=ess\sup_{x_1\le t\le x_2}[t^c|\Pi(t)|]$, if $p=\infty$.*

We present H-H-type inequalities for convex functions via Katugampola fractional integral in the following theorems.

**Theorem 188** ([131]). *Assume that $\Pi : [x_1^\rho, x_2^\rho] \to \mathbb{R}$ is a convex function on $[x_1^\rho, x_2^\rho]$ with $\Pi \in X_c^p(x_1^\rho, x_2^\rho)$. Then fractional integral inequality is given as:*

$$\Pi\Big(\frac{\alpha x_1^\rho + x_2^\rho}{\alpha+1}\Big) \le \frac{\rho^\alpha \Gamma(\alpha+1)}{(x_2^\rho - x_1^\rho)^\alpha}\, {}^\rho I_{x_1+}^\alpha \Pi(x_2^\rho) \le \frac{\alpha\Pi(x_1^\rho) + \Pi(x_2^\rho)}{\alpha+1},$$

*where $-\infty < x_1^\rho < x_2^\rho < \infty$, $\alpha$ and $\rho > 0$.*

**Theorem 189** ([131]). *Assume that $\Pi$ is as in Theorem 188. Then fractional integral inequality is given as:*

$$\Pi\Big(\frac{x_1^\rho + \alpha x_2^\rho}{\alpha+1}\Big) \le \frac{\rho^\alpha \Gamma(\alpha+1)}{(x_2^\rho - x_1^\rho)^\alpha}\, {}^\rho I_{x_2-}^\alpha \Pi(x_1^\rho) \le \frac{\Pi(x_1^\rho) + \alpha\Pi(x_2^\rho)}{\alpha+1},$$

*where $-\infty < x_1^\rho < x_2^\rho < \infty$, $\alpha$ and $\rho > 0$.*

**Theorem 190** ([131]). *Assume that $\Pi$ is as in Theorem 188. Then fractional integral inequalities are given as:*

$$\begin{aligned}\frac{1}{2}\left[\Pi\Big(\frac{x_1^\rho + \alpha x_2^\rho}{\alpha+1}\Big) + \Pi\Big(\frac{x_1^\rho + \alpha x_2^\rho}{\alpha+1}\Big)\right] &\le \frac{\rho^\alpha\Gamma(\alpha+1)}{2(x_2^\rho - x_1^\rho)^\alpha}\Big[{}^\rho I_{x_2-}^\alpha \Pi(x_1^\rho) + {}^\rho I_{x_2-}^\alpha \Pi(x_1^\rho)\Big] \\ &\le \frac{\Pi(x_1^\rho) + \alpha\Pi(x_2^\rho)}{2},\end{aligned}$$

*where $-\infty < x_1^\rho < x_2^\rho < \infty$, $\alpha$ and $\rho > 0$.*

**Theorem 191** ([132]). *Let $\Pi : [x_1^\rho, x_2^\rho] \to \mathbb{R}$ be a function with $\rho > 0$ and $0 \le x_1^\rho < x_2^\rho$, and $\Pi \in X_c^p(x_1^\rho, x_2^\rho)$. If $\Pi$ is a convex function on $[x_1^\rho, x_2^\rho]$, then for any $\alpha > 0$ and any $\xi \in [x_1, x_2]$,*

$$\begin{aligned}\Pi\Big(\frac{1}{\alpha+1}\frac{x_1^\rho + x_2^\rho}{2} + \frac{\alpha}{\alpha+1}\xi^\rho\Big) &\le \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho I_{x_1+}^\alpha \Pi(x_2^\rho)}{(x_2^\rho - \xi^\rho)^\alpha} + \frac{{}^\rho I_{x_2-}^\alpha \Pi(x_1^\rho)}{(\xi^\rho - x_1^\rho)^\alpha}\Big] \\ &\le \frac{1}{\alpha+1}\Big[\frac{\Pi(x_1^\rho) + \Pi(x_2^\rho)}{2} + \alpha\Pi(\xi^\rho)\Big].\end{aligned}$$

**Theorem 192** ([132]). *Assume that $\Pi$ is as in Theorem 191 and differentiable. If $\Pi'$ is a convex function on $[x_1^\rho, x_2^\rho]$, then for any $\alpha > 0$ and any $\xi \in [x_1, x_2]$,*

$$\begin{aligned}&\left|\frac{\Pi(x_1^\rho) + \Pi(x_2^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho I_{x_1+}^\alpha \Pi(x_2^\rho)}{(x_2^\rho - \xi^\rho)^\alpha} + \frac{{}^\rho I_{x_2-}^\alpha \Pi(x_1^\rho)}{(\xi^\rho - x_1^\rho)^\alpha}\Big]\right| \\ \le\; & \frac{\alpha}{4(\alpha+1)(\alpha+2)}\Big[(\alpha+3)(\xi^\rho - x_1^\rho)|\Pi'(x_1^\rho)| + (\alpha+1)(x_2^\rho - x_1^\rho)|\Pi'(\xi^\rho)| \\ & + (\alpha+3)(x_2^\rho - \xi^\rho)|\Pi'(x_2^\rho)|\Big].\end{aligned}$$

**Theorem 193** ([132]). *Assume that $\Pi$ is as in Theorem 191 and differentiable. If $|\Pi'|^q, q > 1$ is a convex function on $[x_1^\rho, x_2^\rho]$, then for any $\alpha > 0$ and any $\xi \in [x_1, x_2]$,*

$$\left|\frac{\Pi(x_1^\rho) + \Pi(x_2^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho I_{x_1+}^\alpha \Pi(x_2^\rho)}{(x_2^\rho - \xi^\rho)^\alpha} + \frac{{}^\rho I_{x_2-}^\alpha \Pi(x_1^\rho)}{(\xi^\rho - x_1^\rho)^\alpha}\Big]\right|$$

$$\leq \frac{\rho^{\frac{1}{q}}}{2\alpha^{1-\frac{1}{q}}[(\rho r-r+1)(\rho(r+1)-r+1)]^{\frac{1}{q}}}\left[B\left(\frac{2q-1}{q-1},\frac{(\rho-1)(q-r)+q-1}{\rho\alpha(q-1)}\right)\right]^{\frac{1}{q}}$$
$$\times\Big\{(\xi^{\rho}-x_1^{\rho})[(\rho r-r+1)|\Pi'(\xi^{\rho})|^q+\rho|\Pi'(x_1^{\rho})|^q]^{\frac{1}{q}}$$
$$+(x_2^{\rho}-\xi^{\rho})[(\rho r-r+1)|\Pi'(\xi^{\rho})|^q+\rho|\Pi'(x_2^{\rho})|^q]^{\frac{1}{q}}\Big\},\quad 0\leq r\leq q.$$

**Theorem 194** ([133]). *Let $\alpha > 0$ and $\rho > 0$ and $F(x) = \Pi(x) + \Pi(x_1 + x_2 - x)$. If $\Pi$ is a convex function on $[x_1, x_2]$, then:*

$$\Pi\left(\frac{x_1+x_2}{2}\right)\leq\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(x_2^{\rho}-x_1^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{x_1+}F(x_2)+{}^{\rho}I^{\alpha}_{x_2-}F(x_1)\right]\leq\frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

**Theorem 195** ([133]). *Let $\alpha > 0$ and $\rho > 0$ and $F(x) = \Pi(x) + \Pi(x_1 + x_2 - x)$. If $\Pi \in C^1(x_1, x_2)$ and $|\Pi'|$ is convex on $[x_1, x_2]$, then:*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(x_2^{\rho}-x_1^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{x_1+}F(x_2)+{}^{\rho}I^{\alpha}_{x_2-}F(x_1)\right]\right|$$
$$\leq\frac{I^{\alpha}_{\rho}(x_1,x_2)}{4(x_2^{\rho}-x_1^{\rho})^{\alpha}(x_2-x_1)}\Big[|\Pi'(x_1)|+\Pi'(x_2)|\Big],$$

*where*

$$I^{\alpha}_{\rho}(x_1,x_2)=\mathcal{L}^{\alpha}(\rho,x_2,x_2)+\mathcal{L}^{\alpha}(\rho,x_1,x_2)-\mathcal{L}^{\alpha}(\rho,x_2,x_1)-\mathcal{L}^{\alpha}(\rho,x_1,x_1)$$

*and*

$$\mathcal{L}^{\alpha}(\rho,x,y)=\int_{x_1}^{\frac{x_1+x_2}{2}}|x-u||y^{\rho}-u^{\rho}|^{\alpha}du-\int_{\frac{x_1+x_2}{2}}^{x_2}|x-u||y^{\rho}-u^{\rho}|^{\alpha}du,\ \rho>0,\ x,y\in[x_1,x_2].$$

**Theorem 196** ([134]). *Let $\alpha > 0$ and $\rho > 0$. Let $\Pi : [x_1^{\rho}, x_2^{\rho}] \to \mathbb{R}$ be a positive convex function with $[x_1^{\rho}, x_2^{\rho}]$ and $\Pi \in X_c^p(x_1^{\rho}, x_2^{\rho})$. Then fractional integral inequalities are given as:*

$$\Pi\left(\frac{x_1^{\rho}+x_2^{\rho}}{2}\right)\leq\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(x_2^{\rho}-x_1^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{x_1+}\Pi(x_2^{\rho})+{}^{\rho}I^{\alpha}_{x_2-}\Pi(x_1^{\rho})\right]\leq\frac{\Pi(x_1^{\rho})+\Pi(x_2^{\rho})}{2}.$$

**Theorem 197** ([134]). *Assume that $\Pi$ is as in Theorem 196. If $\Pi'$ is differentiable on $(x_1^{\rho}, x_2^{\rho})$, then fractional integral inequalities are given as:*

$$\left|\frac{\Pi(x_1^{\rho})+\Pi(x_2^{\rho})}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(x_2^{\rho}-x_1^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{x_1+}\Pi(x_2^{\rho})+^{\rho}I^{\alpha}_{x_2-}\Pi(x_1^{\rho})\right]\right|$$
$$\leq\frac{(x_2^{\rho}-x_1^{\rho})^2}{2(\alpha+1)(\alpha+2)}\left(\alpha+\frac{1}{2^{\alpha}}\right)\sup_{\xi\in[x_1^{\rho},x_2^{\rho}]}|\Pi''(\xi)|.$$

**Theorem 198** ([134]). *Assume that $\Pi$ is as in Theorem 196. If $|\Pi'|$ is a convex function on $[x_1^{\rho}, x_2^{\rho}]$, then fractional integral inequalities are given as:*

$$\left|\frac{\Pi(x_1^{\rho})+\Pi(x_2^{\rho})}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(x_2^{\rho}-x_1^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{x_1+}\Pi(x_2^{\rho})+^{\rho}I^{\alpha}_{x_2-}\Pi(x_1^{\rho})\right]\right|$$
$$\leq\frac{(x_2^{\rho}-x_1^{\rho})^2}{2(\alpha+1)}\Big[|\Pi'(x_1^{\rho})|+|\Pi'(x_2^{\rho})|\Big].$$

**Theorem 199** ([134])**.** *Assume that* $\Pi$ *is as in Theorem 196 and* $F(x) = \Pi(x) + \Pi(x_1 + x_2 - x)$*. Then* $\Pi$ *is integrable and the following inequalities hold:*

$$\Pi\Big(\frac{x_1 + x_2}{2}\Big) \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2(x_2^\rho - x_1^\rho)^\alpha}\Big[{}^\rho I^\alpha_{x_1+} F(x_2) + {}^\rho I^\alpha_{x_2-} F(x_1)\Big] \leq \frac{\Pi(x_1) + \Pi(x_2)}{2},$$

*with* $\alpha > 0$ *and* $\rho > 0$.

Next, we present H-H inequalities for quasi-convex functions via Katugampola fractional integrals.

**Theorem 200** ([135])**.** *Let* $\alpha, \rho > 0$ *and* $\Pi : [x_1^\rho, x_2^\rho] \to \mathbb{R}$ *be a positive function with* $0 \leq x_1 < x_2$*. If* $\Pi$ *is quasi-convex on* $[x_1^\rho, x_2^\rho]$*, then fractional integral inequality is given as:*

$$\frac{\rho^\alpha \Gamma(\alpha+1)}{2(x_2^\rho - x_1^\rho)^\alpha}\Big[{}^\rho I^\alpha_{x_1+}\Pi(x_2^\rho) +^\rho I^\alpha_{x_2-}\Pi(x_1^\rho)\Big] \leq \max\Big\{|\Pi(x_1^\rho)|, |\Pi(x_2^\rho)|\Big\}.$$

**Theorem 201** ([135])**.** *Assume that* $\Pi$ *is as in Theorem 200. If* $\Pi$ *is differentiable and* $|\Pi'|$ *is quasi-convex on* $[x_1^\rho, x_2^\rho]$*, then fractional integral inequality is given as:*

$$\begin{aligned} &\left|\frac{\Pi(x_1^\rho) + \Pi(x_2^\rho)}{2} - \frac{\alpha\rho^\alpha \Gamma(\alpha+1)}{2(x_2^\rho - x_1^\rho)^\alpha}\Big[{}^\rho I^\alpha_{x_1+}\Pi(x_2^\rho) +^\rho I^\alpha_{x_2-}\Pi(x_1^\rho)\Big]\right| \\ \leq\ & \frac{(x_2^\rho - x_1^\rho)^2}{\rho(\alpha+1)}\Big(1 - \frac{1}{2^{\rho(\alpha+1)}}\Big)\max\Big\{|\Pi'(x_1^\rho)|, |\Pi'(x_2^\rho)|\Big\}. \end{aligned}$$

**Definition 48** ([136])**.** *A function* $\Pi : \mathbb{I} \to \mathbb{R}$ *is said to be strongly convex function in the classical sense with modulus* $\mu \geq 0$*, if*

$$\Pi(tx + (1-t)y) \leq t\Pi(x) + (1-t)\Pi(y) - \mu t(1-t)(y-x)^2, \quad \forall x, y \in \mathbb{I}, \ \ t \in [0,1].$$

In the following theorem, we present the fractional H-H inequality via strongly convex functions.

**Theorem 202** ([137])**.** *Let* $\Pi : [x_1, x_2] \to \mathbb{R}$ *be a strongly convex function with* $0 < x_1 < x_2$ *and* $\Pi \in L_1[x_1, x_2]$*. Then R-L fractional integral inequality is given as:*

$$\begin{aligned} &\Pi\Big(\frac{x_1 + x_2}{2}\Big) + \frac{\mu}{4}\Big(\frac{2 + \alpha^2 - \alpha}{(\alpha+1)(\alpha+2)}\Big)(x_2 - x_1)^2 \\ \leq\ & \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^\alpha}\Big[J^\alpha_{x_1+}\Pi(x_2) + J^\alpha_{x_2-}\Pi(x_1)\Big] \\ \leq\ & \frac{\Pi(x_1) + \Pi(x_2)}{2} - \mu\Big(\frac{\alpha}{(\alpha+1)(\alpha+2)}\Big)(x_2 - x_1)^2. \end{aligned}$$

We now give a more generalized refinement of fractional H-H inequality involving Katugampola fractional integrals.

**Theorem 203** ([137])**.** *Assume that* $\Pi$ *is as in Theorem 202. Then fractional integral inequality is given as:*

$$\begin{aligned} &\Pi\Big(\frac{x_1^\rho + x_2^\rho}{2}\Big) + \frac{\mu}{4}\Big(\frac{2\rho^2 + \alpha^2\rho^2 - \alpha\rho^2}{(\alpha\rho + \rho)(\alpha\rho + 2\rho)}\Big)(x_2^\rho - x_1^\rho)^2 \\ \leq\ & \frac{\rho^\alpha \Gamma(\alpha+1)}{2(x_2^\rho - x_1^\rho)^\alpha}\Big[{}^\rho I^\alpha_{x_1+}\Pi(x_2^\rho) +^\rho I^\alpha_{x_2-}\Pi(x_1^\rho)\Big] \end{aligned}$$

$$\leq \frac{\Pi(x_1^\rho)+\Pi(x_2^\rho)}{2} - \mu\Big(\frac{\alpha\rho}{(\alpha\rho+1)(\alpha\rho+2\rho)}\Big)(x_2^\rho - x_1^\rho)^2,$$

*where the fractional integrals are considered for the function $\Pi(x^\rho)$ and evaluated at $x_1$ and $x_2$, respectively.*

**Theorem 204** ([137]). *Assume that $\Pi$ is as in Theorem 202. If $\Pi$ is differentiable and $|\Pi'|$ is a strongly convex function on $[x_1^\rho, x_2^\rho]$, then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\Pi\Big(\frac{x_1^\rho+x_2^\rho}{2}\Big) - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(x_2^\rho-x_1^\rho)^\alpha}\Big[{}^\rho I^\alpha_{x_1+}\Pi(x_2^\rho) + {}^\rho I^\alpha_{x_2-}\Pi(x_1^\rho)\Big]\right| \\ \leq\ & \frac{x_2^\rho-x_1^\rho}{2}\left[\Big(\frac{1-2^{-\alpha}}{\rho(\alpha+1)}\Big)\Big[|\Pi'(x_1)|+\Pi'(x_2)|\Big] - \mu(x_2^\rho-x_1^\rho)^2\Big(\frac{2^{-1-\alpha}(2^{2+\alpha}-\alpha-4)}{\rho(\alpha+2)(\alpha+3)}\Big)\right].\end{aligned}$$

H-H inequalities for harmonically convex functions can be represented in Katugampola fractional integral forms as follows:

**Theorem 205** ([138]). *Let $\alpha$ and $\rho > 0$. Let $\Pi : \mathbb{I} \subset (0,\infty) \to \mathbb{R}$ be a function such that $\Pi \in X_c^p(x_1^\rho, x_2^\rho)$, where $x_1^\rho, x_2^\rho \in \mathbb{I}$ with $x_1 < x_2$. If $\Pi$ is harmonically convex function on $[x_1, x_2]$, then fractional integral inequalities are given as:*

$$\begin{aligned}\Pi\Big(\frac{2x_1^\rho x_2^\rho}{x_1^\rho+x_2^\rho}\Big) &\leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big(\frac{x_1^\rho x_2^\rho}{x_2^\rho-x_1^\rho}\Big)^\alpha\Big[{}^\rho I^\alpha_{1/a-}(\Pi\circ g)\Big(\frac{1}{x_2}\Big) + {}^\rho I^\alpha_{1/b+}(\Pi\circ g)\Big(\frac{1}{x_1}\Big) \\ &\leq \frac{\Pi(x_1^\rho)+\Pi(x_2^\rho)}{2},\end{aligned}$$

*where $g(x) = \frac{1}{x^\rho}$, $x \in \Big[\frac{1}{x_2}, \frac{1}{x_1}\Big]$.*

## 4. H-H-Type Inequalities via *k*-R-L Fractional Integral

**Definition 49** ([139]). *Let $\Pi \in L_1[x_1, x_2]$, $a \geq 0$, and $k > 0$. The $k$-R-L fractional integrals $I^\alpha_{x_1+,k}\Pi$ and $I^\alpha_{x_2-,k}\Pi$ of order $\alpha > 0$ for a real-valued function $\Pi$ are defined by*

$$I^\alpha_{x_1+,k}\Pi(t) = \frac{1}{k\Gamma_k(\alpha)}\int_{x_1}^t (t-s)^{\frac{\alpha}{k}-1}\Pi(s)ds, \ \ t > x_1,$$

*and*

$$I^\alpha_{x_2-,k}\Pi(t) = \frac{1}{k\Gamma_k(\alpha)}\int_t^b (s-t)^{\frac{\alpha}{k}-1}\Pi(s)ds, \ \ t < x_2,$$

*respectively, where $\Gamma_k$ is the k-Gamma function $Gamma_k(t) = \int_0^\infty s^{t-1}e^{-\frac{s^k}{k}}ds$.*

We give in the following theorems H-H-type inequalities for GA-*s*-convex functions via *k*-R-L fractional integrals.

**Theorem 206** ([140]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is a differentiable function. If $|\Pi''|$ is measurable and $|\Pi''|$ is decreasing and GA-s-convex on $[x_1, x_2]$ for some fixed $\alpha > 0$, $s \in (0,1]$, $0 \leq x_1 < x_2$, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma_k(\alpha+k)}{2(x_2-x_1)^{\frac{\alpha}{k}}}\Big[J^\alpha_{x_1+,k}\Pi(x_2) + J^\alpha_{x_2-,k}\Pi(x_1)\Big]\right|$$

$$\leq \frac{(x_2-x_1)^2\Big[|\Pi''(x_1)|+|\Pi''(x_2)|\Big]}{2\Big(\frac{\alpha}{k}+1\Big)}\Bigg(\frac{1}{s+1}-\frac{1}{\frac{\alpha}{k}+s+2}\Bigg).$$

**Theorem 207** ([140]). *Assume that $\Pi$ is as in Theorem 206, then fractional integral inequality is given as:*

$$\Big|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+k)}{2(x_2-x_1)^{\frac{\alpha}{k}}}[J^{\alpha}_{x_1+,k}\Pi(x_2)+J^{\alpha}_{x_2-,k}\Pi(x_1)]\Big|$$
$$\leq \frac{(x_2-x_1)^2\max\{1-2^{1-\frac{\alpha}{k}},2^{1-\frac{\alpha}{k}}-1\}}{2\Big(\frac{\alpha}{k}+1\Big)}\Big(\frac{|\Pi''(x_1)|^q+|\Pi''(x_2)|^q}{s+1}\Big)^{\frac{1}{q}}.$$

H-H-type inequalities for convex functions via $k$-R-L fractional integrals are given now.

**Theorem 208** ([141]). *Assume that $\Pi:[x_1,x_2]\to\mathbb{R}$ is a positive mapping with $0\leq x_1<x_2$, $\Pi\in L_1[x_1,x_2]$. If $\Pi$ is a convex function on $[x_1,x_2]$, then*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big)\leq\frac{\Gamma_k(\alpha+k)}{2(x_2-x_1)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{x_1+,k}\Pi(x_2)+I^{\alpha}_{x_2-,k}\Pi(x_1)\Big]\leq\frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

**Theorem 209** ([141]). *Assume that $\Pi$ is as in Theorem 208 and differentiable on $(x_1,x_2)$ $x_1<x_2$. If $|\Pi'|$ is a convex function on $[x_1,x_2]$, then*

$$\Bigg|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma_k(\alpha+k)}{2(x_2-x_1)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{x_1+,k}\Pi(x_2)+I^{\alpha}_{x_2-,k}\Pi(x_1)\Big]\Bigg|$$
$$\leq\frac{x_2-x_1}{2\Big(\frac{\alpha}{k}+1\Big)}\Big(1-\frac{1}{2^{\frac{\alpha}{k}}}\Big)\Big(|\Pi'(x_1)|+|\Pi'(x_2)|\Big).$$

**Theorem 210** ([141]). *Assume that $k>0$, and $\Pi:[x_1,x_2]\to\mathbb{R}$ is a positive mapping with $0\leq x_1<x_2$, $\Pi\in L_1[x_1,x_2]$. If $\Pi$ is a convex function on $[x_1,x_2]$, then*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big)\leq\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(x_2-x_1)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+,k}\Pi(x_2)+I^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-,k}\Pi(x_1)\Big]\leq\frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

**Theorem 211** ([141]). *Assume that $\Pi$ is as in Theorem 210 and differentiable. If $|\Pi'|^q$, $q\geq1$ is a convex function on $[x_1,x_2]$, then:*

$$\Bigg|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(x_2-x_1)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+,k}\Pi(x_2)+I^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-,k}\Pi(x_1)\Big]-\Pi\Big(\frac{x_1+x_2}{2}\Big)\Bigg|$$
$$\leq\Big(\frac{x_2-x_1}{4}\Big)\Big(\frac{1}{\frac{\alpha}{k}+1}\Big)\Bigg(\frac{1}{2\Big(\frac{\alpha}{k}+2\Big)}\Bigg)^{\frac{1}{q}}\Bigg[\Big(\Big(\frac{\alpha}{k}+1\Big)|\Pi'(x_1)|^q+\Big(\frac{\alpha}{k}+3\Big)|\Pi'(x_2)|^q\Big)^{\frac{1}{q}}$$
$$+\Big(\Big(\frac{\alpha}{k}+3\Big)|\Pi'(x_1)|^q+\Big(\frac{\alpha}{k}+1\Big)|\Pi'(x_2)|^q\Big)^{\frac{1}{q}}\Bigg].$$

Next we present H-H-type inequalities for $(h,m)$-convex functions via $k$-R-L fractional integrals.

**Definition 50** ([142]). *Let $h:\mathbb{I}\to\mathbb{R}$ be a non-negative function. We say that $\Pi:\mathbb{I}\to\mathbb{R}$ is a $(h,m)$-convex function, If $\Pi$ is non-negative and for all $x,y\in\mathbb{I}$, $m\in(0,1)$ and $\lambda\in(0,1)$ we have*

$$\Pi(\lambda x + m(1-\lambda)y) \leq h(a)\Pi(x) + mh(1-\lambda)\Pi(y).$$

**Theorem 212** ([143]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is $(h,m)$-convex function with $0 \leq x_1 \leq x_2$, $m \in (0,1]$. If $\Pi \in L_1[x_1, x_2]$, then we have:*

$$\frac{1}{h\left(\frac{1}{2}\right)}\Pi\left(\frac{x_1 + mx_2}{2}\right) \leq \frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(mx_2 - x_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\left(\frac{x_1+mx_2}{2}\right)+,k}\Pi(mx_2) + m^{\frac{\alpha}{k}+1}I^{\alpha}_{\left(\frac{x_1+mx_2}{2}\right)-,k}\Pi\left(\frac{x_1}{m}\right)\right]$$
$$\leq \frac{\alpha\left[\Pi(x_1) + \Pi(mx_2)\right]}{k}\int_0^1 s^{\frac{\alpha}{k}-1}h\left(\frac{s}{2}\right)ds + \frac{\alpha\left[m\Pi(x_2) + m^2\Pi\left(\frac{x_1}{m^2}\right)\right]}{k}\int_0^1 s^{\frac{\alpha}{k}-1}h\left(\frac{2-s}{2}\right)ds.$$

**Theorem 213** ([143]). *Assume that $\Pi : \mathbb{I} \to \mathbb{R}$ is a twice differentiable mapping on $\mathbb{I}^\circ$, where $x_1, x_2 \in \mathbb{I}^\circ$ with $0 \leq x_1 < x_2$ and $\Pi'' \in L_1[x_1, x_2]$. If $\Pi''$ is $(h,m)$-convex function, then we have:*

$$\left|\frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma_k(\alpha+k)}{2(mx_2 - x_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{x_1+,k}\Pi(mx_2) + I^{\alpha}_{mx_2-}\Pi(x_1)\right]\right|$$
$$\leq \frac{k(mx_2 - x_1)^2}{2(\alpha+k)}\left\{|\Pi''(x_1)|\int_0^1\left[1 - (1-s)^{\frac{\alpha}{k}+1} - s^{\frac{\alpha}{k}+1}\right]h(s)ds\right.$$
$$\left.+|\Pi''(mx_2)|\int_0^1\left[1 - (1-s)^{\frac{\alpha}{k}+1} - s^{\frac{\alpha}{k}+1}\right]h(1-s)ds\right\}.$$

**Theorem 214** ([143]). *Assume that $\Pi : \mathbb{I} \to \mathbb{R}$ is a twice differentiable mapping on $\mathbb{I}^\circ$, where $x_1, x_2 \in \mathbb{I}^\circ$ with $0 \leq x_1 < x_2$ and $\Pi'' \in L_1[x_1, x_2]$. If $|\Pi''|^q$, $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ is $(h,m)$-convex function, then we have:*

$$\left|\frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma_k(\alpha+k)}{2(mx_2 - x_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{x_1+,k}\Pi(mx_2) + I^{\alpha}_{mx_2-}\Pi(x_1)\right]\right|$$
$$\leq \frac{k(mx_2 - x_1)^2}{2(\alpha+k)}\left(1 - \frac{2k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left\{|\Pi''(x_1)|^q\int_0^1 h(s)ds + |\Pi''(mx_2)|^q\int_0^1 h(1-s)ds\right)^{\frac{1}{q}}.$$

Now we give H-H-type inequalities for $p$-convex functions via $k$-R-L fractional integrals.

**Theorem 215** ([144]). *Assume that $\Pi : S \subseteq \mathbb{R} \to \mathbb{R}$ is a function and $x_1, x_2 \in S, \Pi' \in L_1[x_1, x_2]$ for all $x_1 \leq u < v \leq x_2$ and $\alpha, k > 0$. If $\Pi'$ is $p$-convex on $[u,v]$, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(v) + \Pi(u)}{v-u} + \frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I^{\alpha}_{u+,k}\Pi(v) + I^{\alpha}_{v-,k}\Pi(v)\right]\right| \leq \frac{2k}{\alpha+k}\left[|\Pi'(u) + \Pi'(v)|\right].$$

H-H-type inequalities for quasi-convex functions via $k$-R-L fractional integrals concerns the following theorems.

**Theorem 216** ([145]). *Assume that $\Pi : \mathbb{I} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ$ and $\Pi' \in L_1[x_1, x_2]$ with $x_1 < x_2$ and $g : [x_1, x_2] \to \mathbb{R}$ is continuous. If $|\Pi'|^q$ is quasi-convex on $[x_1, x_2]$, $q > 1$, then $k$-fractional integral inequality is given as:*

$$\left|\Pi\left(\frac{x_1+x_2}{2}\right)\left[J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-,k}g(x_1) + J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+,k}g(x_2)\right]\right.$$
$$\left.-\left[J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)-,k}(\Pi g)(x_1) + J^{\alpha}_{\left(\frac{x_1+x_2}{2}\right)+,k}(\Pi g)(x_2)\right]\right|$$

$$\leq \frac{(x_2-x_1)^{\frac{\alpha}{k}+1}\|g\|_\infty}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left[\max\{|\Pi'(x_1)|^q,|\Pi'(x_2)|^q\}\right]^{\frac{1}{q}},$$

*where* $\frac{\alpha}{k}>0$.

**Theorem 217** ([145]). *Assume that* $\Pi$ *is as in Theorem 216. Then k-fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2}\left[J^{\alpha}_{x_2-,k}g(x_1)+J^{\alpha}_{x_1+,k}g(x_2)\right]-\left[J^{\alpha}_{x_2-,k}(\Pi g)(x_1)+J^{\alpha}_{x_1+,k}(\Pi g)(x_2)\right]\right|$$
$$\leq \frac{2(x_2-x_1)^{\frac{\alpha}{k}+1}\|g\|_\infty}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\max\{|\Pi'(x_1)|^q,|\Pi'(x_2)|^q\},$$

*where* $\frac{\alpha}{k}>0$.

We give in the next, H-H-type inequalities for $h$-convex functions via $k$-R-L fractional integrals.

**Theorem 218** ([146]). *Let a function* $\Pi:[x_1,x_2]\to\mathbb{R}$ *be h-convex with* $0\leq x_1\leq x_2$. *If* $\Pi\in L_1[x_1,x_2]$, *then fractional integral inequalities are given as:*

$$\frac{\Gamma_k(\alpha+k)}{(x_2-x_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{x_1+,k}\Pi(x_2)+I^{\alpha}_{x_2-,k}\Pi(x_1)\right]\leq\frac{\alpha[\Pi(x_1)+\Pi(x_2)]}{k}\int_0^1 s^{\frac{\alpha}{k}}[h(s)+h(1-s)]ds$$
$$\leq \frac{\alpha[\Pi(x_1)+\Pi(x_2)]}{k^{\frac{p-1}{p}}}\left(\frac{1}{p\alpha-pk+k}\right)^{\frac{1}{p}}\left[\left(\int_0^1(h(1-s))^r ds\right)^{\frac{1}{r}}+\left(\int_0^1 h(s)^r ds\right)^{\frac{1}{r}}\right],$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**Theorem 219** ([146]). *Assume that* $\Pi:\mathbb{I}\to\mathbb{R}$ *is a differentiable function on* $\mathbb{I}^\circ$, *where* $x_1,\frac{x_2}{m}\in\mathbb{I}^\circ$ *with* $0\leq x_1\leq\frac{x_2}{m}$, $0<m<1$ *and* $\Pi''\in\left[x_1,\frac{x_2}{m}\right]$. *If* $|\Pi''|$ *is an h-convex function, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)+\Pi\left(\frac{x_2}{m}\right)}{2}-\frac{\Gamma_k(\alpha+k)}{2\left(\frac{x_2}{m}-a\right)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{x_1+,k}\Pi\left(\frac{x_2}{m}\right)+I^{\alpha}_{\frac{x_2}{m}-,k}\Pi(x_1)\right]\right|$$
$$\leq \frac{k\left(\frac{x_2}{m}-a\right)^{\frac{\alpha}{k}}}{2(\alpha+k)}\left\{|\Pi''(x_1)|\int_0^1\left[1-(1-s)^{\frac{\alpha}{k}+1}-s^{\frac{\alpha}{k}+1}\right]h(s)ds\right.$$
$$\left.+\left|\Pi''\left(\frac{x_2}{m}\right)\right|\int_0^1\left[1-(1-s)^{\frac{\alpha}{k}+1}-s^{\frac{\alpha}{k}+1}\right]h(1-s)ds\right\}.$$

Now we present H-H-type inequalities for exponential-type convex functions for $k$-fractional integral operators.

**Definition 51** ([147]). *A non-negative function* $\Pi:\mathbb{I}\to\mathbb{R}$ *is called exponential-type convex function if, for every* $x,y\in\mathbb{I}$ *and* $t\in[0,1]$,

$$\Pi(tx+(1-t)y)\leq(e^t-1)\Pi(x)+(e^{1-t}-1)\Pi(y).$$

**Definition 52** ([148]). *The non-negative function $\Pi : \mathbb{I} \to \mathbb{R}$, is said to be a modified exponential-type convex if*

$$\Pi\Big(tx + m(1-t)y\Big) \le (e^t - 1)\Pi(x) + m(e^{1-t} - 1)\Pi(y),$$

*holds for all $x, y \in \mathbb{I}$, $m \in [0,1]$, and $t \in [0,1]$.*

**Theorem 220** ([148]). *Assume that $0 < w \le 1$, and $\Pi : \big(0, \frac{x_2}{mw}\big] \to \mathbb{R}$ is a differentiable function on $\big(0, \frac{x_2}{mw}\big)$ with $0 < x_1 < x_2$. If $|\Pi'|^q$ is a modified exponential-type convex function on $\big(0, \frac{x_2}{mw}\big]$ for $q > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then for some fixed $m \in (0,1]$ the following inequality for $k-$fractional integral holds true:*

$$\begin{aligned} &\left| \frac{\Pi(mwx_1) + \frac{\alpha}{k}\Pi(x_2)}{\frac{\alpha}{k}+1} - \frac{\Gamma_k(\alpha+k)}{(x_2 - mwx_1)^{\frac{\alpha}{k}}}\, {}^kJ^{\alpha}_{x_2-}\Pi(mwx_1) \right| \\ \le\ & \left( \frac{x_2 - mwx_1}{\frac{\alpha}{k}+1} \right) [U_1(\alpha,k,p) + U_2(\alpha,k,p)]^{\frac{1}{p}} \Big[ (e-2)\Big( m|f'(wx_1)|^q + |f'(x_2)|^q \Big) \Big]^{\frac{1}{q}}, \end{aligned}$$

*where*

$$\begin{aligned} U_1(\alpha,k,p) &= \int_0^{\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}} \left( 1 - \left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}} \right)^p d\varrho, \\ U_2(\alpha,k,p) &= \int_{\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}^1 \left( \left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}} - 1 \right)^p d\varrho. \end{aligned}$$

**Theorem 221** ([148]). *Assume that $0 < w \le 1$, and $\Pi : \big(0, \frac{x_2}{m}\big] \to \mathbb{R}$ is a differentiable function on $\big(0, \frac{x_2}{m}\big)$ with $0 < x_1 < x_2$. If $|\Pi'|^q$ is an modified exponential-type convex function on $\big(0, \frac{x_2}{m}\big]$ for $q > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then for some fixed $m \in (0,1]$ the following inequality for $k-$fractional integral holds true:*

$$\begin{aligned} &\left| \frac{\Pi(mwx_1) + \Pi(x_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(x_2 - mwx_1)^{\frac{\alpha}{k}}} \left\{ {}^kJ^{\alpha}_{x_1+}\Pi(x_2) + {}^kJ^{\alpha}_{x_2-}\Pi(mwx_1) \right\} \right| \\ \le\ & \frac{2(x_2 - mwx_1)}{w+1} \left( \frac{k}{\alpha p + k} \right)^{\frac{1}{p}} \Big[ (e-2)\Big( m|\Pi'(wx_1)|^q + |f'(x_2)|^q \Big) \Big]^{\frac{1}{q}}. \end{aligned}$$

**Theorem 222** ([149]). *Assume that $\Pi : K_g \to \mathbb{R}$ is a differentiable function on $K_g^\circ$ and $\Pi' \in L_1[x_1, g(x_2)]$, where $K_g$ is defined in Definition 37. Then the following inequalities for k-fractional integrals hold:*

$$\Pi\Big( \frac{x_1 + g(x_2)}{2} \Big) \le \frac{\Gamma_k(\alpha+1)}{2(g(x_2) - x_1)^{\frac{\alpha}{k}}} \Big[ J^{\alpha}_{x_1+,k}\Pi(g(x_2)) + J^{\alpha}_{x_2-,k}\Pi(x_1) \Big] \le \frac{\Pi(x_1) + \Pi(g(x_2))}{2},$$

*with $\alpha, k > 0$.*

**Theorem 223** ([149]). *Assume that $\Pi$ is as in Theorem 164. If $|\Pi'|$ is relative convex on $K_g$, then:*

$$\begin{aligned} &\left| \frac{\Pi(x_1) + \Pi(g(x_2))}{2} - \frac{\Gamma_k(\alpha+1)}{2(g(x_2) - x_1)^{\frac{\alpha}{k}}} \Big[ J^{\alpha}_{x_1+,k}\Pi(g(x_2)) + J^{\alpha}_{x_2-,k}\Pi(x_1) \Big] \right| \\ \le\ & \frac{g(x_2) - x_1}{2\left(\frac{\alpha}{k}+1\right)} \left( 1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) [|\Pi'(x_1)| + \Pi'(g(x_2))|]. \end{aligned}$$

Now we present H-H inequalities for $k$-fractional operators using the Green function.

**Theorem 224** ([150]). *Let $\Pi \in C^2([x_1,x_2])$ be a convex twice differentiable function on $(x_1,x_2)$ such that*

$$\Pi(x) = \Pi(x_1) + (x - x_1)\Pi'(x_2) + \int_{x_1}^{x_2} G(x,\mu)\Pi''(\mu)d\mu, \ G(x,u) = \begin{cases} x_1 - u, & x_1 \le u \le x, \\ x_1 - x, & x \le u \le x_2. \end{cases}$$

*Then fractional integral inequalities are given as:*

$$\Pi\left(\frac{x_1+x_2}{2}\right) \le \frac{\Gamma_k(\alpha+k)}{2(x_2-x_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{x_1+,k}\Pi(x_2) + I^{\alpha}_{x_2-,k}\Pi(x_1)\right] \le \frac{\Pi(x_1)+\Pi(x_2)}{2},$$

*with $\alpha, k > 0$.*

**Theorem 225** ([150]). *Let $\Pi \in C^2([x_1,x_2])$ and $\alpha, k > 0$. If $|\Pi''|$ is a convex function on $[x_1,x_2]$, then fractional integral inequality is given as:*

$$\begin{aligned} &\left|\frac{\Gamma_k(\alpha+k)}{2(x_2-x_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{x_1+,k}\Pi(x_2) + I^{\alpha}_{x_2-,k}\Pi(x_1)\right] - \Pi\left(\frac{x_1+x_2}{2}\right)\right| \\ \le\ & \frac{(x_2-x_1)^2\left(\left(\frac{\alpha}{k}\right)^2 - \frac{\alpha}{k} + 2\right)}{16\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+1\right)}\left[|\Pi''(x_1)| + |\Pi''(x_2)|\right]. \end{aligned}$$

**Theorem 226** ([151]). *Let $\Pi \in C^2([x_1,x_2])$ be a twice differentiable function as in the previous theorem. Then we have:*

*(i)* *If $|\Pi''|$ is an increasing function, then*

$$\left|\frac{\alpha\Pi(x_1) + k\Pi(x_2)}{\alpha+k} - \frac{\Gamma_k(\alpha+k)}{(x_2-x_1)^{\frac{\alpha}{k}}} J^{\alpha}_{x_1+,k}\Pi(x_2)\right| \le \frac{\alpha k|\Pi''(x_2)|(x_2-x_1)^2}{2(\alpha+k)(\alpha+2k)}.$$

*(ii)* *If $|\Pi''|$ is a decreasing function, then*

$$\left|\frac{\alpha\Pi(x_1) + k\Pi(x_2)}{\alpha+k} - \frac{\Gamma_k(\alpha+k)}{(x_2-x_1)^{\frac{\alpha}{k}}} J^{\alpha}_{x_1+,k}\Pi(x_2)\right| \le \frac{\alpha k|\Pi''(x_1)|(x_2-x_1)^2}{2(\alpha+k)(\alpha+2k)}.$$

*(iii)* *If $|\Pi''|$ is a convex function, then*

$$\left|\frac{\alpha\Pi(x_1) + k\Pi(x_2)}{\alpha+k} - \frac{\Gamma_k(\alpha+k)}{(x_2-x_1)^{\frac{\alpha}{k}}} J^{\alpha}_{x_1+,k}\Pi(x_2)\right| \le \frac{\max\{|\Pi''(x_1)|, |\Pi''(x_2)|\}\alpha k(x_2-x_1)^2}{2(\alpha+k)(\alpha+2k)}.$$

**Theorem 227** ([151]). *Let $\Pi \in C^2([x_1,x_2])$ be a twice differentiable function such that $\Pi(x) = \Pi(x_1) + (x-x_1)\Pi'(x_2) + \int_{x_1}^{x_2} G(x,\mu)\Pi''(\mu)d\mu, G(\lambda,\mu) = \begin{cases} x_1 - \mu, & x_1 \le \mu \le \lambda, \\ x_1 - \lambda, & \lambda \le \mu \le x_2. \end{cases}$ Then for any $\alpha, k > 0$, the fractional integral inequalities are given as:*

$$\Pi\left(\frac{\alpha a + kb}{\alpha+k}\right) \le \frac{\Gamma_k(\alpha+k)}{(x_2-x_1)^{\frac{\alpha}{k}}} J^{\alpha}_{x_1+,k}\Pi(x_2) \le \frac{\alpha\Pi(x_1) + k\Pi(x_2)}{\alpha+k}.$$

## 5. H-H-Type Inequalities via $(k,s)$-R-L Fractional Integral

**Definition 53** ([152]). *For a real valued continuous function $\Pi$, the $(k,s)$-R-L fractional integral ${}^{s}_{k}J^{\alpha}_{x_1}\Pi$ of order $\alpha > 0$ is defined as*

$${}^{s}_{k}J^{\alpha}_{x_1}\Pi(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_{x_1}^{x} (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s \Pi(t)dt,$$

*where $k > 0$ and $s \in \mathbb{R} \setminus \{-1\}$.*

**Theorem 228** ([153]). *Let $\alpha, k > 0$ and $s \in \mathbb{R} \setminus \{-1\}$ and $F(x) = \Pi(x) + \Pi(x_1 + x_2 - x)$. If $\Pi$ is a convex function on $[x_1, x_2]$, then we have:*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(x_2^{s+1}-x_1^{s+1})^{\frac{\alpha}{k}}}\Big[{}^s_k J^{\alpha}_{x_1+}F(x_2) + {}^s_k J^{\alpha}_{x_2-}F(x_1)\Big] \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

**Theorem 229** ([153]). *Assume that $\Pi$ is as in Theorem 228. If $\Pi$ is a differentiable function on $I^{\circ}$ such that $\Pi' \in L_1[x_1, x_2]$ with $x_1 < x_2$ and $|\Pi'|$ is a convex function on $[x_1, x_2]$, then:*

$$\begin{aligned}
&\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(x_2^{s+1}-x_1^{s+1})^{\frac{\alpha}{k}}}\Big[{}^s_k J^{\alpha}_{x_1+}F(x_2) + {}^s_k J^{\alpha}_{x_2-}F(x_1)\Big]\right| \\
\leq\ & \frac{\Phi(s,\alpha,x_1,x_2)}{4(x_2^{s+1}-x_1^{s+1})^{\frac{\alpha}{k}}(x_2-x_1)}\Big(|\Pi'(x_1)| + |\Pi'(x_2)|\Big),
\end{aligned}$$

*where $\Phi(s,\alpha,x_1,x_2) = \mathcal{J}(s,x_2,x_2) + \mathcal{J}(s,x_1,x_2) - \mathcal{J}(s,x_2,x_1) - \mathcal{J}(s,x_1,x_1)$ and*

$$\mathcal{J}(s,x,y) := \int_{x_1}^{\frac{x_1+x_2}{2}} |x-u||y^{s+1}-u^{s+1}|^{\frac{\alpha}{k}}du - \int_{\frac{x_1+x_2}{2}}^{x_2} |x-u||y^{s+1}-u^{s+1}|^{\frac{\alpha}{k}}du, \quad x, y \in [x_1, x_2].$$

## 6. H-H-Type Inequalities via C-F Fractional Integral

**Definition 54** ([154]). *Let $\Pi \in H^1(x_1, x_2) = \{y \in L^2(x_1, x_2) : y' \in L^2(x_1, x_2)\}$, $x_1 < x_2$, $\alpha \in [0, 1]$, then the definition of the left fractional integral in the sense of C-F becomes*

$${}^{CF}I^{\alpha}_{x_1}\Pi(t) = \frac{1-\alpha}{B(\alpha)}\Pi(t) + \frac{\alpha}{B(\alpha)}\int_{x_1}^{t}\Pi(x)dx,$$

*the right fractional integral*

$${}^{CF}I^{\alpha}_{x_2}\Pi(t) = \frac{1-\alpha}{B(\alpha)}\Pi(t) + \frac{\alpha}{B(\alpha)}\int_{t}^{b}\Pi(x)dx,$$

*where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.*

Now we give H-H-type inequalities for convex functions via C-F fractional integrals.

**Theorem 230** ([155]). *Assume that $\Pi : [x_1, x_2] \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on $[x_1, x_2]$ and $\Pi \in L_1[x_1, x_2]$. If $\alpha \in [0, 1]$, then the following double inequality holds:*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{B(\alpha)}{\alpha(x_2-x_1)}\Big[{}^{CF}I^{\alpha}_{x_1}\Pi(k) + {}^{CF}I^{\alpha}_{x_2}\Pi(k) - \frac{2(1-\alpha)}{B(\alpha)}\Pi(k)\Big] \leq \frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

**Theorem 231** ([155]). *Assume that $\Pi$ is as in Theorem 230 and differentiable. If $\Pi' \in L_1[x_1, x_2]$ and $\alpha \in [0, 1]$, then fractional integral inequality is given as:*

$$\begin{aligned}
&\frac{\Pi(x_1)+\Pi(x_2)}{2} + \frac{2(1-\alpha)}{\alpha(x_2-x_1)}\Pi(k) - \frac{B(\alpha)}{\alpha(x_2-x_1)}\Big[{}^{CF}I^{\alpha}_{x_1}\Pi(k) + {}^{CF}I^{\alpha}_{x_2}\Pi(k)\Big] \\
\leq\ & \frac{(x_2-x_1)[|\Pi'(x_1)| + |\Pi'(x_2)|]}{8}.
\end{aligned}$$

**Theorem 232** ([156]). *Assume that $\Pi$ is as in Theorem 230. If $|\Pi'|$ is a convex function, then for $\alpha \in [0,1]$, then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{B(\alpha)}{\alpha(x_2-x_1)}\left[{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}-}\Pi(x_1)+{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}+}\Pi(x_2)\right]-\Pi\left(\frac{x_1+x_2}{2}\right)\right|\\ \leq\ & \frac{x_2-x_1}{4}\left\{\frac{|\Pi'(x_1)|+|\Pi'(x_2)|}{2}\right\}+\frac{1-\alpha}{\alpha(x_2-x_1)}\left[\Pi(x_1)+\Pi(x_2)\right].\end{aligned}$$

**Theorem 233** ([156]). *Assume that $\Pi$ is as in Theorem 230. If $|\Pi'|^q$, $q>1$ is a convex function, then for $\alpha \in [0,1]$, and $\dfrac{1}{p}+\dfrac{1}{q}=1$, the fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{B(\alpha)}{\alpha(x_2-x_1)}\left[{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}-}\Pi(x_1)+{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}+}\Pi(x_2)\right]-\Pi\left(\frac{x_1+x_2}{2}\right)\right.\\ &\left.-\frac{1-\alpha}{\alpha(x_2-x_1)}\left[\Pi(x_1)+\Pi(x_2)\right]\right|\\ \leq\ & \frac{x_2-x_1}{4}\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\Pi'(x_1)|^q}{4}+\frac{3|\Pi'(x_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Pi'(x_1)|^q}{4}+\frac{|\Pi'(x_2)|^q}{4}\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}$$

**Theorem 234** ([156]). *Assume that $\Pi:\mathbb{I}\to\mathbb{R}$ is a convex function on $\mathbb{I}$ such that $x_1,x_2\in\mathbb{I}$ and $\Pi\in L_1[x_1,x_2]$. Then, for $\alpha\in[0,1]$ and $B(\alpha)$ as the normalizaton function, we have:*

$$\begin{aligned}\Pi\left(\frac{x_1+x_2}{2}\right) &\leq \frac{B(\alpha)}{\alpha(x_2-x_1)}\left[{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}-}\Pi(x_1)+{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}+}\Pi(x_2)-\frac{1-\alpha}{B(\alpha)}[\Pi(x_1)+\Pi(x_2)]\right]\\ &\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}.\end{aligned}$$

**Theorem 235** ([156]). *Assume that $\Pi:\mathbb{I}=[x_1,x_2]\to\mathbb{R}$ is a differentiable function on $\mathbb{I}$ such that $x_1,x_2\in\mathbb{I}$ with $x_1<x_2$ and $\Pi\in L_1[x_1,x_2]$. If $|\Pi'|$ is a convex function, then, for $\alpha\in[0,1]$ and $B(\alpha)$ as the normalization function, we have:*

$$\begin{aligned}&\left|\frac{B(\alpha)}{\alpha(x_2-x_1)}\left[{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}-}\Pi(x_1)+{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}+}\Pi(x_2)-\Pi\left(\frac{x_1+x_2}{2}\right)\right|\right.\\ \leq\ & \frac{x_2-x_1}{4}\left[\frac{\Pi'(x_1)+\Pi'(x_2)}{2}\right]+\frac{1-\alpha}{\alpha(x_2-x_1)}[\Pi(x_1)+\Pi(x_2)].\end{aligned}$$

**Theorem 236** ([156]). *Assume that $\Pi:\mathbb{I}=[x_1,x_2]\to\mathbb{R}$ is a differentiable function on $\mathbb{I}$ such that $x_1,x_2\in\mathbb{I}$ with $x_1<x_2$ and $\Pi\in L_1[x_1,x_2]$. If $|\Pi'|^q$, $q>1$ is a convex function, then, for $\alpha\in[0,1]$, $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $B(\alpha)$ as the normalization function, we have:*

$$\begin{aligned}&\left|\frac{B(\alpha)}{\alpha(x_2-x_1)}\left[{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}-}\Pi(x_1)+{}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}+}\Pi(x_2)-\Pi\left(\frac{x_1+x_2}{2}\right)\right.\right.\\ &\left.-\frac{1-\alpha}{\alpha(x_2-x_1)}[\Pi(x_1)+\Pi(x_2)]\right|\\ \leq\ & \frac{x_2-x_1}{4}\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\Pi'(x_1)|^q}{4}+\frac{3|\Pi'(x_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Pi'(x_1)|^q}{4}+\frac{|\Pi'(x_2)|^q}{4}\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}$$

H-H-type inequalities for $s$-convex functions via C-F fractional integrals are presented in the following.

**Theorem 237** ([157]). *Let $\Pi : [x_1, x_2] \to \mathbb{R}$ be $s$-convex on $[x_1, x_2]$ for $s \in (0, 1)$ and $\Pi \in L_1[x_1, x_2]$. If $0 \le \alpha \le 1$, then fractional integral inequality is given as:*

$$\begin{aligned} 2^{s-1}\Pi\Big(\frac{x_1+x_2}{2}\Big) &\le \frac{B(\alpha)}{\alpha(x_2-x_1)}\Big[({}^{CF}I^{\alpha}_{x_1}\Pi)(k) + ({}^{CF}I^{\alpha}_{x_2}\Pi)(k) - \frac{2(1-\alpha)}{B(\alpha)}\Pi(k)\Big] \\ &\le \frac{\Pi(x_1)+\Pi(x_2)}{2}, \end{aligned}$$

*where $k \in [x_1, x_2]$ and $B(\alpha) > 0$ is a normalization function.*

**Theorem 238** ([157]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is a positive differentiable mapping on $(x_1, x_2)$ and $|\Pi'|$ is an $s$-convex function on $[x_1, x_2]$ for $s \in (0, 1)$ and $\Pi' \in L_1[x_1, x_2]$. If $0 \le \alpha \le 1$, then fractional integral inequality is given as:*

$$\begin{aligned} &\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{B(\alpha)}{\alpha(x_2-x_1)}\Big[({}^{CF}I^{\alpha}_{x_1}\Pi)(k) + ({}^{CF}I^{\alpha}_{x_2}\Pi)(k) - \frac{2(1-\alpha)}{B(\alpha)}\Pi(k)\Big]\right| \\ \le\;& \frac{x_2-x_1}{2}\Big(\frac{2^{-s}+s}{(s+1)(s+2)}\Big)(|\Pi'(x)| + |\Pi'(x_2)|), \end{aligned}$$

*where $k \in [x_1, x_2]$ and $B(\alpha) > 0$ is a normalization function.*

H-H-type inequality for $n$-polynomial $s$-type convex function via C-F integral operator is given now.

**Definition 55** ([158]). *A function $f : \mathrm{T} \to \mathbb{R}$ is said to be $n$-polynomial $s$-type convex function for $n \in \mathbb{N}$, if for $x_1, x_2 \in \mathrm{T}$ with $\ell, s \in [0, 1]$, the following inequality holds*

$$\Pi(\ell x_1 + (1-\ell)x_2) \le \frac{1}{n}\sum_{\wp=1}^{n}\Big[1-(s(1-\ell))^{\wp}\Big]\Pi(x_1) + \frac{1}{n}\sum_{\wp=1}^{n}\Big[1-(s\ell)^{\wp}\Big]\Pi(x_2).$$

**Theorem 239** ([159]). *Assume that $\Pi : \mathrm{T} \to \mathbb{R}$ is $n$-polynomial $s$-type convex function on $\mathrm{T}$ with $x_1 < x_2$ and $x_1, x_2 \in \mathrm{T}$. If $\Pi$ is Lebesgue integrable function on $[x_1, x_2]$ then,*

$$\begin{aligned} \frac{2^{-1}n}{\sum_{\wp=1}^{n}\Big[1-\big(\frac{s}{2}\big)^{\wp}\Big]}\Pi\Big(\frac{x_1+x_2}{2}\Big) &\le \frac{B(\alpha)}{\alpha(x_2-x_1)}\Big[{}^{CF}\mathrm{I}^{\alpha}_{x_1}\Pi(r) + {}^{CF}\mathrm{I}^{\alpha}_{x_2}\Pi(r) - \frac{2(1-\alpha)}{B(\alpha)}\Pi(r)\Big] \\ &\le \frac{\Pi(x_1)+\Pi(x_2)}{n}\sum_{\wp=1}^{n}\frac{p+1-s^p}{p+1}, \end{aligned}$$

*where $\alpha \in [0, 1], s \in [0, 1], r \in [0, 1]$ and $B(\alpha) > 0$ is a normalization function.*

The following H-H-type inequalities concern preinvex functions.

**Theorem 240** ([160]). *Assume that $\Pi : [x_1, x_1 + \eta(x_2, x_1)] \to (0, \infty)$ be a preinvex function on $(x_1, x_1 + \eta(x_2, x_1))$ and $\Pi \in L_1[x_1, x_1 + \eta(x_2, x_1)]$. If $\alpha \in [0, 1]$, then we have:*

$$\begin{aligned} \Pi\Big(\frac{2x_1+\eta(x_2,x_1)}{2}\Big) &\le \frac{B(\alpha)}{\alpha\eta(x_2,x_1)}\Big[{}^{CF}I^{\alpha}_{x_1}\Pi(k) + {}^{CF}I^{\alpha}_{(x_1+\eta(x_2,x_1))}\Pi(k) - \frac{2(1-\alpha)}{B(\alpha)}\Pi(k)\Big] \\ &\le \frac{\Pi(x_1)+\Pi(x_2)}{2}, \end{aligned}$$

*where* $k \in [x_1, x_1 + \eta(x_2, x_1)]$ *and* $B(\alpha)$ *is a normalization function.*

**Theorem 241** ([160]). *Assume that* $\Pi : I = [x_1, x_1 + \eta(x_2, x_1)] \to (0, \infty)$ *is a differentiable mapping on* $I^\circ$ *and* $|\Pi'|$ *be a preinvex function on* $[x_1, x_1 + \eta(x_2, x_1)]$*. If* $\Pi' \in L_1[x_1, x_1 + \eta(x_2, x_1)]$ *where* $x_1, x_2 \in I$ *with* $x_1 < x_1 + \eta(x_2, x_1)$*, then we have:*

$$\begin{aligned} &\left| -\frac{\Pi(x_1) + \Pi(x_1 + \eta(x_2, x_1))}{2} - \frac{2(1-\alpha)}{B(\alpha)}\Pi(k) \right. \\ &\left. + \frac{B(\alpha)}{\alpha\eta(x_2, x_1)}\left[ {}^{CF}I^{\alpha}_{x_1}\Pi(k) + {}^{CF}I^{\alpha}_{(x_1+\eta(x_2,x_1))}\Pi(k) \right] \right| \\ \le \;& \frac{\eta(x_2, x_1)(|\Pi'(x_1)| + |\Pi'(x_2)|)}{8}, \end{aligned}$$

*where* $k \in [x_1, x_1 + \eta(x_2, x_1)]$ *and* $B(\alpha)$ *is a normalizaton function.*

**Theorem 242** ([160]). *Assume that* $\Pi : I = [x_1, x_1 + \eta(x_2, x_1)] \to (0, \infty)$ *is a differentiable function on* $I^\circ$ *and* $|\Pi'|$ *is a preinvex function on* $[x_1, x_1 + \eta(x_2, x_1)]$*, where* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$*. If* $\Pi' \in L_1[x_1, x_1 + \eta(x_2, x_1)]$ *where* $x_1, x_2 \in I$ *with* $x_1 < x_1 + \eta(x_2, x_1)$*, then we have:*

$$\begin{aligned} &\left| -\frac{\Pi(x_1) + \Pi(x_1 + \eta(x_2, x_1))}{2} - \frac{2(1-\alpha)}{B(\alpha)}\Pi(k) \right. \\ &\left. + \frac{B(\alpha)}{\alpha\eta(x_2, x_1)}\left[ {}^{CF}I^{\alpha}_{x_1}\Pi(k) + {}^{CF}I^{\alpha}_{(x_1+\eta(x_2,x_1))}\Pi(k) \right] \right| \\ \le \;& \frac{\eta(x_2, x_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(|\Pi'(x_1)|^q + |\Pi'(x_2)|^q)}{2}\right)^{\frac{1}{q}}, \end{aligned}$$

*where* $k \in [x_1, x_1 + \eta(x_2, x_1)]$ *and* $B(\alpha)$ *is a normalization function.*

H-H-type inequalities for $(s, m)$-convex functions via C-F fractional integrals are given in the next theorems.

**Theorem 243** ([161]). *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is* $(s, m)$*-convex function and integrable on* $[x_1, x_2]$*. Then*

$$\begin{aligned} 2^s\Pi\left(\frac{x_1 + mx_2}{2}\right) \le \;& \frac{B(\alpha)}{\alpha(mx_2 - x_1)}\Big(\left[({}^{CF}I^{\alpha}_{x_1}\Pi)(x) + ({}^{CF}I^{\alpha}_{mx_2}\Pi)(x)\right] \\ & + m^2\left[({}^{CF}I^{\alpha}_{x_1/m}\Pi)(x) + ({}^{CF}I^{\alpha}_{x_2}\Pi)(x)\right] - \frac{2(1+m^2)(1-\alpha)}{B(\alpha)}\Pi(x)\Big) \\ \le \;& \left[\frac{\Pi(x_1) + m\Pi(x_2)}{s+1} + m\frac{\Pi(x_2) + m\Pi\left(\frac{x_1}{m^2}\right)}{s+1}\right], \end{aligned}$$

*where* $\alpha > 0$ *and* $B(\alpha)$ *is a normalization function.*

**Theorem 244** ([161]). *Assume that* $\Pi : [x_1, mx_2] \to \mathbb{R}$ *is a differentiable function on* $(x_1, mx_2)$*. If* $\Pi'$ *is an* $(s, m)$*-convex function and integrable on* $[x_1, mx_2]$*, then*

$$\begin{aligned} & \frac{\Pi(x_1) + \Pi(mx_2)}{2} - \frac{B(\alpha)}{\alpha(mx_2 - x_1)}\left[({}^{CF}I^{\alpha}_{x_1}\Pi)(x) + ({}^{CF}I^{\alpha}_{mx_2}\Pi)(x)\right] + \frac{2(1-\alpha)}{\alpha(mx_2 - x_1)}\Pi(x) \\ \le \;& \frac{mx_2 - x_1}{2}[m\Pi'(x_2) - \Pi'(x_1)]\frac{s}{(s+1)(s+2)}. \end{aligned}$$

**Theorem 245** ([161]). *Assume that $\Pi : [x_1, mx_2] \to \mathbb{R}$ is a twice differentiable function on $(x_1, mx_2)$. If $\Pi''$ is an $(s, m)$-convex function and integrable on $[x_1, mx_2]$, then*

$$\begin{aligned} &\frac{\Pi(x_1) + \Pi(mx_2)}{2} - \frac{B(\alpha)}{\alpha(mx_2 - x_1)}\left[({}^{CF}I^{\alpha}_{x_1}\Pi)(x) + ({}^{CF}I^{\alpha}_{mx_2}\Pi)(x)\right] + \frac{2(1-\alpha)}{\alpha(mx_2 - x_1)}\Pi(x) \\ \le\; &\frac{(mx_2 - x_1)^2}{2}\frac{[\Pi''(x_1) + m\Pi''(x_2)]}{(s+3)(s+2)}. \end{aligned}$$

## 7. H-H–Mercer (H-H-M)-Type Inequalities via R-L Fractional Integrals

In the next theorems, we give H-H-M-type inequalities for convex functions via R-L fractional integrals.

**Theorem 246** ([162]). *Assume that $\Pi : \mathbb{I} \to \mathbb{R}$ is a function such that $\Pi \in L_1[x_1, x_2]$, where $x_1, x_2 \in \mathbb{I}$ with $x_1 < x_2$. If $\Pi$ is a convex function on $[x_1, x_2]$ then the fractional integral inequalities are given as:*

$$\begin{aligned} &\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \\ \le\; &\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{(x_1+x_2-y)+}\Pi(x_1 + x_2 - x) + J^{\alpha}_{(x_1+x_2-x)-}\Pi(x_1 + x_2 - y)\right] \\ \le\; &\frac{1}{2}[\Pi(x_1 + x_2 - x) + \Pi(x_1 + x_2 - y)] \\ \le\; &\Pi(x_1) + \Pi(x_2) - \frac{\Pi(x) + \Pi(y)}{2,} \end{aligned}$$

*for all $x, y \in [x_1, x_2]$ with $x < y$ and $\alpha > 0$.*

**Theorem 247** ([162]). *Assume that $\Pi$ is as in Theorem 246. Then the fractional integral inequalities are given as:*

$$\begin{aligned} &\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \\ \le\; &\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{(x_1+x_2-y)+}\Pi(x_1 + x_2 - x) + J^{\alpha}_{(x_1+x_2-x)-}\Pi(x_1 + x_2 - y)\right] \\ \le\; &\Pi(x_1) + \Pi(x_2) - \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J^{\alpha}_{y-}\Pi(x) + J^{\alpha}_{x+}\Pi(y)\right] \\ \le\; &\Pi(x_1) + \Pi(x_2) - \frac{\Pi(x) + \Pi(y)}{2}, \end{aligned}$$

*for all $x, y \in [x_1, x_2]$ with $x < y$ and $\alpha > 0$.*

In the next theorem, we give H-H-M-type inequalities for strongly convex functions via R-L fractional integrals.

**Theorem 248** ([163]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is a strongly convex function with modulus $c$. Then, for all $x, z \in [x_1, x_2]$ with $x < z$, $x_1 \ge 0$ and $\alpha > 0$, we have:*

$$\begin{aligned} &\Pi\Big(x_1 + x_2 - \frac{x+z}{2}\Big) \\ \le\; &\Pi(x_1) + \Pi(x_2) - \frac{\Gamma(\alpha+1)}{2(z-x)^{\alpha}}\left[J^{\alpha}_{x+}\Pi(z) + J^{\alpha}_{z-}\Pi(x)\right] - c(x_1 - x_2)^2[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] \\ &- \frac{c(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)}(z-x)^2 \\ \le\; &\Pi(x_1) + \Pi(x_2) - \Pi\Big(\frac{x+z}{2}\Big) - c(x_1 - x_2)^2[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)] \end{aligned}$$

$$-\frac{c(\alpha^2-\alpha+2)}{4(\alpha+1)(\alpha+2)}(z-x)^2.$$

We now give H-H-M-type inequalities for convex functions via R-L fractional integrals.

**Theorem 249** ([163])**.** *Assume that $\Pi$ is as in Theorem 248 and differentiable such that $|\Pi'|$ is a convex function on $[x_1,x_2]$. Then, for any $x,z\in[x_1,x_2], x<z$, fractional integral inequality is given as:*

$$\begin{aligned}&\Bigg|\frac{\Pi(x_1+x_2-x)+\Pi(x_1+x_2-z)}{2}-\frac{\Gamma(\alpha+1)}{2(z-x)^\alpha}\Big[J^\alpha_{(x_1+x_2-z)+}\Pi(x_1+x_2-x)\\&+J^\alpha_{(x_1+x_2-x)-}\Pi(x_1+x_2-z)\Big]-\frac{\alpha c}{(\alpha+1)(\alpha+2)}(z-x)^2\Bigg|\\\leq\;&\frac{z-x}{\alpha+1}\Big(1-\frac{1}{2^\alpha}\Big)\Big[|\Pi'(x_1)|+|\Pi'(x_2)|-\frac{|\Pi'(x_1)|+|\Pi'(x_2)|}{2}+\frac{2^\alpha\alpha c(z-x)}{(2^\alpha-1)(\alpha+2)}\Big].\end{aligned}$$

**Theorem 250** ([164])**.** *Assume that $\Pi:[x_1,x_2]\to\mathbb{R}$ is a convex function. Then fractional integral inequalities are given as:*

$$\begin{aligned}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)&\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{2(y-x)^\alpha}\Big[J^\alpha_{(x_1+x_2-x)-}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\\&\quad+J^\alpha_{(x_1+x_2-y)+}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\Big]\\&\leq\Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2},\end{aligned}$$

$$\begin{aligned}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)&\leq\Pi(x_1)+\Pi(x_2)-\frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\Big[J^\alpha_{x_1+}\Pi\Big(\frac{x+y}{2}\Big)+J^\alpha_{x_2-}\Pi\Big(\frac{x+y}{2}\Big)\Big]\\&\leq\Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2},\end{aligned}$$

*for all $x,y\in[x_1,x_2]$ and $\alpha>0$.*

**Theorem 251** ([164])**.** *Suppose that $\Pi:[x_1,x_2]\to\mathbb{R}$ is a differentiable function on $(x_1,x_2)$ with $0\leq x_1<x_2$ and $\Pi\in L_1[x_1,x_2]$. If $|\Pi'|$ is a convex function on $[x_1,x_2]$, then R-L fractional integral inequality is given as:*

$$\begin{aligned}&\Bigg|\frac{\Pi(x_1+x_2-x)+\Pi(x_1+x_2-y)}{2}-\frac{2^\alpha\Gamma(\alpha+1)}{2(z-x)^\alpha}\Big[J^\alpha_{(x_1+x_2-y)+}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\\&+J^\alpha_{(x_1+x_2-x)-}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\Big]\Bigg|\\\leq\;&\frac{(y-x)^2}{2(\alpha+1)}\Big[|\Pi'(x_1)|+|\Pi'(x_2)|-\frac{|\Pi'(x)|+|\Pi'(y)|}{2}\Big],\end{aligned}$$

*for all $x,y\in[x_1,x_2]$ and $\alpha>0$.*

**Theorem 252** ([164])**.** *Assume that $\Pi$ is as in Theorem 251. If $|\Pi''|$ is a convex function on $[x_1,x_2]$, then R-L fractional integral inequality is given as:*

$$\Bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(z-x)^\alpha}\Big[J^\alpha_{\left(x_1+x_2-\frac{x+y}{2}\right)+}\Pi(x_1+x_2-x)+J^\alpha_{\left(x_1+x_2-\frac{x+y}{2}\right)-}\Pi(x_1+x_2-y)\Big]$$

$$-\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big)\Bigg|$$
$$\leq \frac{(y-x)^2}{4(\alpha+1)(\alpha+2)}\Big[|\Pi''(x_1)| + |\Pi''(x_2)| - \frac{|\Pi''(x)| + |\Pi''(y)|}{2}\Big],$$

*for all* $x, y \in [x_1, x_2]$ *and* $\alpha > 0$.

**Theorem 253** ([165])**.** *For a convex function* $\Pi : [x_1, x_2] \subseteq \mathbb{R} \to \mathbb{R}$ *with* $x, y \in [x_1, x_2]$*, we have:*

$$\begin{aligned}\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) &\leq \Pi(x_1) + \Pi(x_2) - \frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\Big[J^\alpha_{x+}\Pi(y) + J^\alpha_{y-}\Pi(x)\Big) \\ &\leq \Pi(x_1) + \Pi(x_2) - \frac{\Pi(x) + \Pi(y)}{2},\end{aligned}$$

*and*

$$\begin{aligned}&\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \\ \leq\ & \frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\Big[J^\alpha_{(x_1+x_2-y)+}\Pi(x_1 + x_2 - x) + J^\alpha_{(x_1+x_2-x)-}\Pi(x_1 + x_2 - y)\Big] \\ \leq\ & \frac{\Pi(x_1 + x_2 - x) + \Pi(x_1 + x_2 - y)}{2} \leq \Pi(x_1) + \Pi(x_2) - \frac{\Pi(x) + \Pi(y)}{2}.\end{aligned}$$

**Theorem 254** ([165])**.** *Assume that* $\Pi$ *is as in Theorem 253. Then, we have:*

$$\begin{aligned}&\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \\ \leq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{2(y-x)^\alpha}\Big[J^\alpha_{\left(x_1+x_2-\frac{x+y}{2}\right)+}\Pi(x_1 + x_2 - x) + J^\alpha_{\left(x_1+x_2-\frac{x+y}{2}\right)-}\Pi(x_1 + x_2 - y)\Big] \\ \leq\ & \Pi(x_1) + \Pi(x_2) - \frac{\Pi(x) + \Pi(y)}{2}.\end{aligned}$$

**Theorem 255** ([165])**.** *Assume that* $\Pi$ *is as in Theorem 251. If* $|\Pi'|$ *is a convex function on* $[x_1, x_2]$*, then fractional integral inequality is given as:*

$$\begin{aligned}&\Bigg|\frac{\Pi(x_1 + x_2 - x) + \Pi(x_1 + x_2 - y)}{2} - \frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\Big[J^\alpha_{(x_1+x_2-y)+}\Pi(x_1 + x_2 - x) \\ &+ J^\alpha_{(x_1+x_2-x)-}\Pi(x_1 + x_2 - y)\Big] \\ \leq\ & \frac{y-x}{\alpha+1}\Big(1 - \frac{1}{2^\alpha}\Big)\Big[|\Pi'(x_1)| + |\Pi'(x_2)| - \frac{|\Pi'(x)| + |\Pi'(y)|}{2}\Big],\end{aligned}$$

*for all* $x, y \in [x_1, x_2]$ *and* $\alpha > 0$.

**Theorem 256** ([165])**.** *Assume that* $\Pi$ *is as in Theorem 251. If* $|\Pi'|$ *is a convex function on* $[x_1, x_2]$*, then fractional integral inequality is given as:*

$$\begin{aligned}&\Bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^\alpha}\Big[J^\alpha_{\left(x_1+x_2-\frac{x+y}{2}\right)+}\Pi(x_1 + x_2 - x) + J^\alpha_{\left(x_1+x_2-\frac{x+y}{2}\right)-}\Pi(x_1 + x_2 - y)\Big] \\ &-\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big)\Bigg| \\ \leq\ & \frac{y-x}{2(\alpha+1)}\Big[|\Pi'(x_1)| + |\Pi'(x_2)| - \frac{|\Pi'(x)| + |\Pi'(y)|}{2}\Big],\end{aligned}$$

*for all* $x, y \in [x_1, x_2]$ *and* $\alpha > 0$.

**Theorem 257** ([165])**.** *Assume that* $\Pi$ *is as in Theorem 251. If* $|\Pi'|^q$ *is a convex function on* $[x_1, x_2]$, $q > 1$, *then fractional integral inequality is given as:*

$$\begin{aligned}
&\bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\Big[J^{\alpha}_{\left(x_1+x_2-\frac{x+y}{2}\right)+}\Pi(x_1+x_2-x)+J^{\alpha}_{\left(x_1+x_2-\frac{x+y}{2}\right)-}\Pi(x_1+x_2-y)\Big]\\
&-\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\bigg|\\
\leq\ &\frac{y-x}{4(\alpha p+1)^{\frac{1}{p}}}\bigg[\Big(|\Pi'(x_1)|^q+|\Pi'(x_2)|^q-\frac{3|\Pi'(x)|^q+|\Pi'(y)|^q}{4}\Big)^{\frac{1}{q}}\\
&+\Big(|\Pi'(x_1)|^q+|\Pi'(x_2)|^q-\frac{|\Pi'(x)|^q+3|\Pi'(y)|^q}{4}\Big)^{\frac{1}{q}}\bigg],
\end{aligned}$$

*for all* $x, y \in [x_1, x_2]$ *and* $\alpha > 0$, *where* $\frac{1}{p}+\frac{1}{q}=1$.

**Theorem 258** ([166])**.** *Assume that* $\Pi$ *is as in Theorem 253. Then, we have:*

$$\begin{aligned}
\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\ &\leq\ \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\Big[J^{\alpha}_{(x_1+x_2-y)+}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\\
&\quad+J^{\alpha}_{(x_1+x_2-x)-}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\Big]\\
&\leq\ \Pi(x_1)+\Pi(x_2)-\frac{\Pi(x_1)+\Pi(x_2)}{2}.
\end{aligned}$$

**Theorem 259** ([166])**.** *Assume that* $\Pi$ *is as in Theorem 251. If* $|\Pi'|$ *is a convex function on* $[x_1, x_2]$ *with* $x_1 \leq x_2$ *and* $x, y \in [x_1, x_2]$, *then, we have:*

$$\begin{aligned}
&\bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\Big[J^{\alpha}_{(x_1+x_2-y)+}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)+J^{\alpha}_{(x_1+x_2-x)-}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\Big]\\
&-\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\bigg|\\
\leq\ &\frac{y-x}{2(1+\alpha)}\Big[|\Pi'(x_1)|+|\Pi'(x_2)|-\frac{|\Pi'(x_1)|+|\Pi'(x_2)|}{2}\Big].
\end{aligned}$$

We present the next H-H-M inequalities via $k$-fractional integral.

**Theorem 260** ([167])**.** *Assume that* $\Pi : [x_1, x_2] \subseteq \mathbb{R} \to \mathbb{R}$ *is a convex function with* $x, y \in [x_1, x_2]$ *and* $\frac{\alpha}{k} > 0$. *Then fractional integral inequalities are given as:*

$$\begin{aligned}
&\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\\
\leq\ &\frac{\Gamma_k(\alpha+1)}{2(y-x)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{(x_1+x_2-x)+,k}\Pi(x_1+x_2-x)+I^{\alpha}_{(x_1+x_2-y)-,k}\Pi(x_1+x_2-y)\Big]\\
\leq\ &\frac{\Pi(x_1+x_2-x)+\Pi(x_1+x_2-y)}{2}\leq\Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2}.
\end{aligned}$$

**Theorem 261** ([167]). *Assume that* $\Pi$ *is as in Theorem 260. Then fractional integral inequalities are given as:*

$$\begin{aligned}
&\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \\
\leq\ & \frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}{2(y-x)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\left(x_1+x_2-\frac{x+y}{2}\right)+,k}\Pi(x_1+x_2-x) + I^{\alpha}_{\left(x_1+x_2-\frac{x+y}{2}\right)-,k}\Pi(x_1+x_2-y)\Big] \\
\leq\ & \frac{\Pi(x_1+x_2-x)+\Pi(x_1+x_2-y)}{2} \leq \Pi(x_1)+\Pi(x_2) - \frac{\Pi(x)+\Pi(y)}{2}.
\end{aligned}$$

**Theorem 262** ([167]). *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is a differentiable function on* $(x_1, x_2)$ *with* $x_1 < x_2$*. If* $|\Pi'|$ *is a convex function on* $[x_1, x_2]$*, then k-fractional integral inequality is given as:*

$$\begin{aligned}
&\Bigg|\frac{\Pi(x_1+x_2-x)+\Pi(x_1+x_2-y)}{2} - \frac{\Gamma_k(\alpha+1)}{2(y-x)^{\frac{\alpha}{k}}}\Big[J^{\alpha}_{(x_1+x_2-y)+,k}\Pi(x_1+x_2-x) \\
&+J^{\alpha}_{(x_1+x_2-x)-,k}\Pi(x_1+x_2-y)\Big] \\
\leq\ & \frac{y-x}{\alpha+k}\Big(k - \frac{k}{2^{\frac{\alpha}{k}}}\Big)\Big[|\Pi'(x_1)| + |\Pi'(x_2)| - \frac{|\Pi'(x)|+|\Pi'(y)|}{2}\Big],
\end{aligned}$$

*for all* $x, y \in [x_1, x_2]$ *and* $\frac{\alpha}{k} > 0$.

We present H-H-M-type inequalities for convex functions via C-F fractional integrals.

**Theorem 263** ([168]). *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is a convex function on* $[x_1, x_2]$ *such that* $x, y \in [x_1, x_2]$ *and* $\Pi \in L_1[x_1, x_2]$*. Then, for* $\alpha \in [0, 1]$ *and* $B(\alpha)$ *as the normalization function, then the fractional integral inequalities are given as:*

$$\begin{aligned}
&\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \\
\leq\ & \frac{B(\alpha)}{\alpha(y-x)}\Big[{}^{CF}I^{\alpha}_{\left(x_1+x_2-\frac{x+y}{2}\right)-}\Pi(x_1+x_2-y) + {}^{CF}I^{\alpha}_{\left(x_1+x_2-\frac{x+y}{2}\right)+}\Pi(x_1+x_2-x) \\
&-\frac{1-\alpha}{B(\alpha)}\left[\Pi(x_1+x_2-x)+\Pi(x_1+x_2-y)\right]\Big] \\
\leq\ & \Pi(x_1)+\Pi(x_2) - \frac{\Pi(x)+\Pi(y)}{2}.
\end{aligned}$$

**Theorem 264** ([168]). *Assume that* $\Pi$ *is as in Theorem 263 and differentiable. If* $|\Pi'|$ *is a convex function, then fractional integral inequalities are given as:*

$$\begin{aligned}
&\Bigg|\frac{B(\alpha)}{\alpha(y-x)}\Big[{}^{CF}I^{\alpha}_{\left(x_1+x_2-\frac{x+y}{2}\right)-}\Pi(x_1+x_2-y) + {}^{CF}I^{\alpha}_{\left(x_1+x_2-\frac{x+y}{2}\right)+}\Pi(x_1+x_2-x)\Big] \\
&-\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big) - \frac{1-\alpha}{\alpha(y-x)}\left[\Pi(x_1+x_2-x)+\Pi(x_1+x_2-y)\right]\Bigg| \\
\leq\ & \frac{y-x}{4}\Big[|\Pi'(x_1)| + |\Pi'(x_2)| - \frac{|\Pi'(x)|+|\Pi'(y)|}{2}\Big].
\end{aligned}$$

H-H–Mercer inequalities via Katugampola fractional integral operator are presented in the next theorems.

**Theorem 265** ([169]). *Let $\alpha, \rho > 0$, $x_1, x_2 \in [0, \infty)$ with $x_1 < x_2$ and $\Pi : [x_1^\rho, x_2^\rho] \to (0, \infty)$ be a positive convex function such that $\Pi \in X_c^p(x_1^\rho, x_2^\rho)$. Then fractional integral inequality is given as:*

$$\begin{aligned}\Pi\Big(x_1^\rho + x_2^\rho - \frac{x^\rho + y^\rho}{2}\Big) &\leq \Pi(x_1^\rho) + \Pi(x_2^\rho) - \frac{\rho\Gamma(\alpha+1)}{2(y^\rho - x^\rho)}\Big[{}^\rho I_{x+}^\alpha \Pi(y^\rho) + {}^\rho I_{y-}^\alpha \Pi(x^\rho)\Big] \\ &\leq \Pi(x_1^\rho) + \Pi(x_2^\rho) - \Pi\Big(\frac{x^\rho + y^\rho}{2}\Big),\end{aligned}$$

*and*

$$\begin{aligned}\Pi\Big(x_1^\rho + x_2^\rho - \frac{x^\rho + y^\rho}{2}\Big) &\leq \frac{\rho\Gamma(\alpha+1)}{2(y^\rho - x^\rho)^\alpha}\Big[{}^\rho I^\alpha_{(x_1^\rho + x_2^\rho - y^\rho)+}\Pi(x_1^\rho + x_2^\rho - x^\rho) \\ &\quad + {}^\rho I^\alpha_{(x_1^\rho + x_2^\rho - x^\rho)-}\Pi(x_1^\rho + x_2^\rho - y^\rho)\Big] \\ &\leq \frac{\Pi(x_1^\rho + x_2^\rho - x^\rho) + \Pi(x_1^\rho + x_2^\rho - y^\rho)}{2} \\ &\leq \Pi(x_1^\rho) + \Pi(x_2^\rho) - \Pi\Big(\frac{x^\rho + y^\rho}{2}\Big),\end{aligned}$$

*for all $x, y \in [x_1, x_2]$.*

**Theorem 266** ([169]). *Assume that $\Pi$ is as in Theorem 265 and differentiable such that $\Pi' \in L_1[x_1^\rho, x_2^\rho]$. Then fractional integral inequality is given as:*

$$\begin{aligned}&\Bigg|\frac{\Pi(x_1^\rho + x_2^\rho - x^\rho) + \Pi(x_1^\rho + x_2^\rho - y^\rho)}{2} - \frac{\rho\Gamma(\alpha+1)}{2(y^\rho - x^\rho)}\Big[{}^\rho I^\alpha_{(x_1^\rho + x_2^\rho - y^\rho)+}\Pi(x_1^\rho + x_2^\rho - x^\rho) \\ &+ {}^\rho I^\alpha_{(x_1^\rho + x_2^\rho - x^\rho)-}\Pi(x_1^\rho + x_2^\rho - y^\rho)\Big]\Bigg| \\ \leq\ & \frac{x_2^\rho - x_1^\rho}{\alpha+1}\Big[1 - \Big(1 - \frac{1}{2^\rho}\Big)^{\alpha+1} - \frac{1}{2^{\rho(\alpha+1)}}\Big]\Big[|\Pi'(x_1^\rho)| + |\Pi'(x_2^\rho)| - \frac{|\Pi'(x^\rho)| + |\Pi'(y^\rho)|}{2}\Big],\end{aligned}$$

*for all $x, y \in [x_1, x_2]$ if $|\Pi'$ is a convex function on $[x_1^\rho, x_2^\rho]$.*

**Theorem 267** ([170]). *Let $\Pi : [x_1, x_2] \to \mathbb{R}$ be a convex function and $x, y \in [x_1, x_2]$ such that $x < y$. Then*

$$\begin{aligned}\Pi\Big(x_1 + x_2 - \frac{\alpha x + y}{\alpha+1}\Big) &\leq \Pi(x_1) + \Pi(x_2) - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha}J^\alpha_{x+}\Pi(y) \\ &\leq \Pi(x_1) + \Pi(x_2) - \Pi\Big(\frac{\alpha x + y}{\alpha+1}\Big),\end{aligned}$$

*and*

$$\begin{aligned}\Pi\Big(x_1 + x_2 - \frac{\alpha x + y}{\alpha+1}\Big) &\leq \frac{\Gamma(\alpha+1)}{(y-x)^\alpha}J^\alpha_{(x_1+x_2-x)+}\Pi(x_1 + x_2 - y) \\ &\leq \frac{\alpha\Pi(x_1 + x_2 - x) + \Pi(x_1 + x_2 - y)}{\alpha+1} \\ &\leq \Pi(x_1) + \Pi(x_2) - \Pi\Big(\frac{\alpha\Pi(x) + \Pi(y)}{\alpha+1}\Big).\end{aligned}$$

**Theorem 268** ([170]). *Assume that $\Pi$ is as in Theorem 267. Then*

$$\Bigg|\frac{\alpha\Pi(x_1 + x_2 - x) + \Pi(x_1 + x_2 - y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha}J^\alpha_{(x_1+x_2-x)+}\Pi(x_1 + x_2 - y)\Bigg|$$

$$\leq \frac{y-x}{\alpha+1}\Big[\frac{2\alpha}{(\alpha+1)^{1-\frac{1}{\alpha}}}\Big(|\Pi'(x_1)|+|\Pi'(x_2)|\Big)-\frac{\alpha\Big[2+(\alpha+1)^{\frac{2}{\alpha}}\Big]}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}|\Pi'(x)|$$
$$-\frac{\alpha\Big[4(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1}-(\alpha+1)^{\frac{2}{\alpha}}\Big]}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}|\Pi'(y)|\Big].$$

## 8. H-H-Type Inequalities via R-L Fractional Integrals of a Function with Respect to Another Function

**Definition 56** ([22,171]). *Let $(x_1,x_2)$ $(-\infty\leq x_1<x_2\leq\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha>0$. In addition, let $\psi(x)$ be an increasing and positive monotone function on $(x_1,x_2]$, having a continuous derivative $\psi'(x)$ on $(x_1,x_2)$. The left- and right-sided $\Pi$-R-L fractional integrals of a function $g$ with respect to another function $\Pi$ on $[x_1,x_2]$ are defined by*

$$I^{\alpha;\psi}_{x_1+}g(x)=\frac{1}{\Gamma(\alpha)}\int_{x_1}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}g(t)dt,$$

$$I^{\alpha;\psi}_{x_2-}g(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{x_2}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}g(t)dt,$$

*respectively.*

We present H-H-type inequalities for $s$-convex functions via R-L fractional integrals of a function with respect to another function.

**Theorem 269** ([172]). *Assume that $\psi:[x_1,x_2]\to\mathbb{R}$ is an increasing and positive monotone function on $(x_1,x_2]$, having a continuous derivative on $(x_1,x_2)$ and let $\alpha>0$. If $\Pi$ is an $s$-convex in the second sense on $[x_1,x_2]$ for some fixed $s\in(0,1]$, then fractional integral inequalities are given as:*

$$2^{s-1}\Pi\Big(\frac{x_1+x_2}{2}\Big)$$
$$\leq \frac{\Gamma(\alpha+1)}{2[\Pi(x_2)-\Pi(x_1)]^{\alpha}}\frac{1}{2}\big[I^{\alpha,\psi}_{x_1+}[\Pi(x_1)+\Pi(x_2)]+I^{\alpha,\psi}_{x_2-}[\Pi(x_1)+\Pi(x_2)]\big]$$
$$\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}\frac{\alpha(x_2-x_1)}{[\Pi(x_2)-\Pi(x_1)]^{\alpha}}\frac{1}{2}\Big[\int_0^1\frac{[t^s+(1-t)^s]\psi'((1-t)x_1+tx_2)}{[\psi(x_2)-\psi((1-t)x_1+tx_2)]^{1-\alpha}}dt$$
$$+\int_0^1\frac{[t^s+(1-t)^s]\psi'((1-t)x_1+tx_2)}{[\psi((1-t)x_1+tx_2)-\psi(x_1)]^{1-\alpha}}dt\Big].$$

**Theorem 270** ([172]). *Let $\psi$ be as the above. If $\Pi\in C^1(x_1,x_2)$ and $|\Pi'|$ is an $s$-convex function in the second sense on $[x_1,x_2]$ for $s\in(0,1]$, then fractional integral inequality is given as:*

$$\left|\frac{\Pi(x_1)+\Pi(x_2)}{2}-\frac{\Gamma(\alpha+1)}{4[\Pi(x_2)-\Pi(x_1)]^{\alpha}}\big[I^{\alpha,\psi}_{x_1+}[\Pi(x_1)+\Pi(x_2)]+I^{\alpha,\psi}_{x_2-}[\Pi(x_1)+\Pi(x_2)]\big]\right|$$
$$\leq \frac{[|\Pi'(x_1)|+|\Pi'(x_2)|]}{4[\psi(x_2)-\psi(x_1)]^{\alpha}}I^{\alpha,s}_{\psi}(x_1,x_2),$$

*where*

$$I^{\alpha,s}_{\psi}(x_1,x_2)=L^{\alpha,s}_{\psi}(x_2,x_2)+L^{\alpha,s}_{\psi}(x_1,x_2)+L^{\alpha,s}_{\psi}(x_2,x_1)+L^{\alpha,s}_{\psi}(x_1,x_1)$$

*and*

$$L^{\alpha,s}_{\psi}(x,y)=\int_{x_1}^{\frac{x_1+x_2}{2}}|x-u|^s|\psi(y)-\psi(u)|^{\alpha}du-\int_{\frac{x_1+x_2}{2}}^{x_2}|x-u|^s|\psi(y)-\psi(u)|^{\alpha}du,\quad x,y\in[x_1,x_2].$$

Now we present H-H-type inequalities for convex functions via R-L fractional integrals of a function with respect to another function.

**Theorem 271** ([173]). *Let $0 \leq x_1 < x_2$, $\Pi : [x_1, x_2] \to \mathbb{R}$ be a positive function and $\Pi \in L_1[x_1, x_2]$. In addition, suppose that $\Pi$ is a convex function on $[x_1, x_2]$, $\psi$ is an increasing and positive monotone function on $(x_1, x_2]$, having a continuous derivative $\psi'$ on $(x_1, x_2)$ and $\alpha \in (0,1)$. Then the fractional integral inequalities are given as:*

$$\begin{aligned}\Pi\Big(\frac{x_1+x_2}{2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\Big[I^{\alpha;\psi}_{\psi^{-1}(x_1)+}(\Pi\circ\psi)(\psi^{-1}(x_2)) + I^{\alpha;\psi}_{\psi^{-1}(x_2)-}(\Pi\circ\psi)(\psi^{-1}(x_1))\Big] \\ &\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}.\end{aligned}$$

**Theorem 272** ([173]). *Assume that $\Pi$ is as in Theorem 271 and suppose that $|\Pi'|$ is a convex function on $[x_1, x_2]$. Then fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\Big[I^{\alpha;\psi}_{\psi^{-1}(x_1)+}(\Pi\circ\psi)(\psi^{-1}(x_2)) + I^{\alpha;\psi}_{\psi^{-1}(x_2)-}(\Pi\circ\psi)(\psi^{-1}(x_1))\Big]\right| \\ \leq\ & \frac{x_2-x_1}{2(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)\Big[|\Pi'(x_1)|+|\Pi'(x_2)|\Big].\end{aligned}$$

**Theorem 273** ([173]). *Assume that $\Pi$ is as in Theorem 271. Then the fractional integral inequality is given as:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\Big[I^{\alpha;\psi}_{\psi^{-1}(x_1)+}(\Pi\circ\psi)(\psi^{-1}(x_2)) + I^{\alpha;\psi}_{\psi^{-1}(x_2)-}(\Pi\circ\psi)(\psi^{-1}(x_1))\Big] - \Pi\Big(\frac{x_1+x_2}{2}\Big)\right| \\ \leq\ & \frac{\Pi(x_2)-\Pi(x_1)}{2} + \frac{x_2-x_1}{2(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)\Big[|\Pi'(x_1)|+|\Pi'(x_2)|\Big].\end{aligned}$$

**Theorem 274** ([174]). *Let $\Pi : [x_1, x_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function and $\Pi' \in L_1[x_1, x_2]$ with $0 \leq x_1 < x_2$. Suppose that $|\Pi'|^q$ is a convex function on $[x_1, x_2]$, $\psi$ is an increasing and positive monotone function on $(x_1, x_2]$ and its derivative $\psi'$ is continuous on $(x_1, x_2)$. Then for $\alpha \in (0,1)$ and $q \geq 1$, we have*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\Big[I^{\alpha;\psi}_{\psi^{-1}(\frac{x_1+x_2}{2})+}(\Pi\circ\psi)(\psi^{-1}(x_2)) + I^{\alpha;\psi}_{\psi^{-1}(\frac{x_1+x_2}{2})-}(\Pi\circ\psi)(\psi^{-1}(x_1))\Big]\right. \\ &\left.-\Pi\Big(\frac{x_1+x_2}{2}\Big)\right| \\ \leq\ & \frac{x_2-x_1}{4(\alpha+1)}\Big[\Big(\frac{\alpha+1}{2(\alpha+2)}|\Pi'(x_1)|^q + \frac{\alpha+3}{2(\alpha+2)}|\Pi'(x_2)|^q\Big)^{\frac{1}{q}} \\ &+\Big(\frac{\alpha+3}{2(\alpha+2)}|\Pi'(x_1)|^q + \frac{\alpha+1}{2(\alpha+2)}|\Pi'(x_2)|^q\Big)^{\frac{1}{q}}\Big].\end{aligned}$$

**Theorem 275** ([174]). *Assume that $\Pi$ is as in Theorem 274. Then for $\alpha \in (0,1)$ we have*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^\alpha}\Big[I^{\alpha;\psi}_{\psi^{-1}(\frac{x_1+x_2}{2})+}(\Pi\circ\psi)(\psi^{-1}(x_2)) + I^{\alpha;\psi}_{\psi^{-1}(\frac{x_1+x_2}{2})-}(\Pi\circ\psi)(\psi^{-1}(x_1))\Big]\right. \\ &\left.-\Pi\Big(\frac{x_1+x_2}{2}\Big)\right|\end{aligned}$$

$$\leq \frac{x_2 - x_1}{4}\Big(\frac{1}{\alpha p+1}\Big)^{\frac{1}{p}}\Big[\Big(\frac{|\Pi'(x_1)|^q + 3|\Pi'(x_2)|^q}{4}\Big)^{\frac{1}{q}} + \Big(\frac{3|\Pi'(x_1)|^q + |\Pi'(x_2)|^q}{4}\Big)^{\frac{1}{q}}\Big]$$
$$\leq \frac{x_2 - x_1}{4}\Big(\frac{1}{\alpha p+1}\Big)^{\frac{1}{p}}\Big[|\Pi'(x_1)| + |\Pi'(x_2)|\Big],$$

*where* $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 276** ([175])**.** *Let* $\alpha > 0$, $\Pi$ *be a convex function on* $[x_1, x_2]$ *and* $F(x) = \Pi(x) + \Pi(x_1 + x_2 - x)$. *Then*

$$\Pi\Big(\frac{x_1 + x_2}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{4(\psi(x_2) - \psi(x_1))^\alpha}\Big[I^{\alpha;\psi}_{x_1+}F(x_2) + I^{\alpha;\psi}_{x_2-}F(x_1)\Big] \leq \frac{\Pi(x_1) + \Pi(x_2)}{2}.$$

**Theorem 277** ([175])**.** *Let* $\alpha > 0$. *If* $\Pi \in C^1(x_1, x_2)$ *and* $|\Pi'|$ *is a convex function on* $[x_1, x_2]$, *then*

$$\Big|\frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{4(\psi(x_2) - \psi(x_1))^\alpha}\Big[I^{\alpha;\psi}_{x_1+}F(x_2) + I^{\alpha;\psi}_{x_2-}F(x_1)\Big]\Big|$$
$$\leq \frac{Q^\alpha_\psi(x_1, x_2)}{4(\psi(x_2) - \psi(x_1))^\alpha(x_2 - x_1)}\Big[|\Pi'(x_1)| + |\Pi'(x_2)|\Big],$$

*where*

$$Q^\alpha_\psi(x_1, x_2) = \mathcal{L}^\alpha_\psi(x_2, x_2) + \mathcal{L}^\alpha_\psi(x_1, x_2) - \mathcal{L}^\alpha_\psi(x_2, x_1) - \mathcal{L}^\alpha_\psi(x_1, x_1)$$

*and*

$$\mathcal{L}^\alpha_\psi(x, y) = \int_{x_1}^{\frac{x_1+x_2}{2}} |x - u||\psi(y) - \psi(u)|^\alpha du - \int_{\frac{x_1+x_2}{2}}^{x_2} |x - u||\psi(y) - \psi(u)|^\alpha du, \quad x, y \in [x_1, x_2].$$

**Theorem 278** ([176])**.** *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is a positive function and* $\Pi \in L_1[x_1, x_2]$, *where* $0 \leq x_1 < x_2$. *In addition, suppose that* $\psi$ *is an increasing and positive monotone function on* $(x_1, x_2)$, *having a continuous derivative* $\psi'$ *on* $(x_1, x_2)$, *and* $\alpha > 0$. *If* $\Pi$ *is a convex function on* $[x_1, x_2]$, *then*

$$\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \leq \Pi(x_1) + \Pi(x_2) - \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^\alpha}\Big[\big(I^{\alpha,\psi}_{\psi^{-1}(x_1)+}\big)(\Pi \circ \psi)(\psi^{-1}(x_2))$$
$$+\big(I^{\alpha,\psi}_{\psi^{-1}(x_2)-}\big)(\Pi \circ \psi)(\psi^{-1}(x_1))\Big]$$
$$\leq \Pi(x_1) + \Pi(x_2) - \Pi\Big(\frac{x_1 + x_2}{2}\Big),$$

$$\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2 - x_1)^\alpha}\Big[\big(I^{\alpha,\psi}_{\psi^{-1}(x_1+x_2-y)+}\big)(\Pi \circ \psi)(\psi^{-1}(x_1 + x_2 - x))$$
$$+\big(I^{\alpha,\psi}_{\psi^{-1}(x_1+x_2-x)-}\big)(\Pi \circ \psi)(\psi^{-1}(x_1 + x_2 - y))\Big]$$
$$\leq \Pi(x_1) + \Pi(x_2) - \frac{\Pi(x) + \Pi(y)}{2},$$

*for all* $x, y \in [x_1, x_2]$.

**Theorem 279** ([176])**.** *Assume that* $\Pi$ *is as in Theorem 278. Then*

$$\Pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(x_2 - x_1)^\alpha}\Big[\big(I^{\alpha,\psi}_{\psi^{-1}\left(x_1+x_2-\frac{x+y}{2}\right)+}\big)(\Pi \circ \psi)(\psi^{-1}(x_1 + x_2 - y))$$

$$+\left(I^{\alpha,\psi}_{\psi^{-1}\left(x_1+x_2-\frac{x+y}{2}\right)-}\right)(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-x))\Big]$$
$$\leq \Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2},$$

*for all* $x,y\in[x_1,x_2]$.

**Theorem 280** ([176]). *Assume that* $\Pi$ *is as in Theorem 278. If* $|\Pi'|$ *is a convex function on* $[x_1,x_2]$*, then*

$$\left|\frac{\Pi(x_1+x_2-x)+\Pi(x_1+x_2-y)}{2}-\frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\Big[(I^{\alpha,\psi}_{\psi^{-1}(x_1+x_2-y)+})\Pi(x_1+x_2-x)\right.$$
$$\left.+(I^{\alpha,\psi}_{\psi^{-1}(x_1+x_2-x)-})\Pi(x_1+x_2-y)\Big]\right|$$
$$\leq \frac{x_2-x_1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left[|\Pi'(x_1)|+|\Pi'(x_2)|-\frac{|\Pi'(x)|+|\Pi'(y)|}{2}\right],$$

*for all* $x,y\in[x_1,x_2]$.

**Condition C.** [177] *Let* $A\subseteq\mathbb{R}^n$ *be an invex subset with respect to* $\eta: A\times A\to\mathbb{R}^n$*. We say that the function* $\eta$ *satisfies the condition C if for any* $x,y\in A$ *and* $t\in[0,1]$*,*

$$\eta(y,y+t\eta(x,y))=-t\eta(x,y),\quad \eta(x,y+t\eta(x,y))=(1-t)\eta(x,y).$$

**Theorem 281** ([177]). *Let* $K\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta: K\times K\to\mathbb{R}$ *and* $x_1,x_2\in K$ *with* $x_1<x_1+\eta(x_2,x_1)$*. Suppose that* $\Pi:[x_1,x_1+\eta(x_2,x_1)]\to(0,\infty)$ *is a preinvex function,* $\Pi\in L_1[x_1,x_1+\eta(x_2,x_1)]$ *and* $\eta$ *satisfies Condition C. In addition, suppose that* $\psi$ *is an increasing and positive monotone function on* $(x_1,x_1+\eta(x_2,x_1))$*, having a continuous derivative* $\psi'$ *on* $(x_1,x_1+\eta(x_2,x_1))$ *and* $\alpha\in(0,1)$*. Then,*

$$\begin{aligned}\Pi\Big(x_1+\frac{1}{2}\eta(x_2,x_1)\Big) &\leq \frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^{\alpha}}\Big[J^{\alpha,\psi}_{\psi^{-1}(x_1)+}(\Pi\circ\psi)\psi^{-1}(x_1,x_1+\eta(x_2,x_1))\\ &\quad +J^{\alpha,\psi}_{\psi^{-1}(x_1,x_1+\eta(x_2,x_1))-}(\Pi\circ\psi)\psi^{-1}(x_1)\Big]\\ &\leq \frac{\Pi(x_1)+\Pi(x_1,x_1+\eta(x_2,x_1))}{2}\leq\frac{\Pi(x_1)+\Pi(x_2)}{2}.\end{aligned}$$

**Theorem 282** ([177]). *Assume that* $\psi$ *is as in Theorem 281. If* $|\Pi'|$ *is a preinvex function on* $K$*, then*

$$\left|\frac{\Pi(x_1)+\Pi(x_1,x_1+\eta(x_2,x_1))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^{\alpha}}\Big[J^{\alpha,\psi}_{\psi^{-1}(x_1)+}(\Pi\circ\psi)\psi^{-1}(x_1,x_1+\eta(x_2,x_1))\right.$$
$$\left.+J^{\alpha,\psi}_{\psi^{-1}(x_1,x_1+\eta(x_2,x_1))-}(\Pi\circ\psi)\psi^{-1}(x_1)\Big]\right|$$
$$\leq \frac{\eta(x_2,x_1)}{2(\eta(x_2,x_1))^{\alpha}}\left(1-\frac{1}{2^{\alpha}}\right)\Big[|\Pi'(x_1)|+|\Pi'(x_2)|\Big].$$

**Theorem 283** ([177]). *Assume that* $\psi$ *is as in Theorem 281. If* $|\Pi'|$ *is a preinvex function, then*

$$\left|\frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^{\alpha}}\Big[J^{\alpha,\psi}_{\psi^{-1}(x_1)+}(\Pi\circ\psi)\psi^{-1}(x_1,x_1+\eta(x_2,x_1))\right.$$

$$+J^{\alpha,\psi}_{\psi^{-1}(x_1,x_1+\eta(x_2,x_1))-}(\Pi\circ\psi)\psi^{-1}(x_1)\Big]\Big|-\Pi\Big(x_1+\frac{1}{2}\eta(x_2,x_1)\Big)\Big|$$
$$\leq \frac{|\Pi(x_2)-\Pi(x_1)|}{2}+\frac{\eta(x_2,x_1)}{2(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)\Big[|\Pi'(x_1)|+|\Pi'(x_2)|\Big].$$

**Theorem 284** ([177]). *Assume that $\psi$ is as in Theorem 281. If $|\Pi'|^q$, $q>1$ is preinvex function on K, then:*

$$\Bigg|\frac{\Pi(x_1)+\Pi(x_1,x_1+\eta(x_2,x_1))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^\alpha}\Big[J^{\alpha,\psi}_{\psi^{-1}(x_1)+}(\Pi\circ\psi)\psi^{-1}(x_1,x_1+\eta(x_2,x_1))$$
$$+J^{\alpha,\psi}_{\psi^{-1}(x_1,x_1+\eta(x_2,x_1))-}(\Pi\circ\psi)\psi^{-1}(x_1)\Big]\Bigg|$$
$$\leq \frac{\eta(x_2,x_1)}{2(\alpha p+1)^{\frac{1}{p}}}\Big(\frac{|\Pi'(x_1)|^q+|\Pi'(x_2)|^q}{2}\Big)^{\frac{1}{q}},$$

*where* $\frac{1}{p}+\frac{1}{q}=1.$

## 9. Fractional H-H Inequalities via Weighted Symmetric Function

**Definition 57** ([178]). *Let $(x_1,x_2)\subset\mathbb{R}$ and $\psi$ be an increasing positive monotonic function on the interval $(x_1,x_2]$ with a continuous derivative $\psi'(x)$ on the interval $(x_1,x_2)$ with $\psi(0)=0$, $0\in[x_1,x_2]$. Then, the left-side and right-side of the weighted fractional integrals of a function $\Pi$ with respect to another function $\psi(x)$ on $[x_1,x_2]$ are defined by*

$$({}_{x_1+}J^{\alpha,\psi}_{w}\Pi)(x)=\frac{w^{-1}(x)}{\Gamma(\alpha)}\int_{x_1}^{x}\psi'(s)(\psi(x)-\psi(s))^{\alpha-1}\Pi(s)w(s)ds,$$

*and*

$$({}_{x_2-}J^{\alpha,\psi}_{w}\Pi)(x)=\frac{w^{-1}(x)}{\Gamma(\alpha)}\int_{x}^{x_2}\psi'(s)(\psi(s)-\psi(x))^{\alpha-1}\Pi(s)w(s)ds,$$

*where $\alpha>0$ and $w^{-1}(x)=\frac{1}{w(x)}$, $w(x)\neq 0$.*

**Theorem 285** ([178]). *Assume that $\Pi:[x_1,x_2]\subset[0,\infty)\to\mathbb{R}$ is a convex function with $0\leq x_1<x_2$ and $w:[x_1,x_2]\to\mathbb{R}$ be an integrable, positive and weighted symmetric function with respect to $\frac{x_1+x_2}{2}$. If $\psi$ is an increasing and positive function on $[x_1,x_2)$ and $\psi'(x)$ is continuous on $(x_1,x_2)$, then, we have for $\alpha>0$:*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big)\Big[\Big({}_{\psi^{-1}(x_1)+}J^{\alpha,\psi}(w\circ\psi)\Big)(\psi^{-1}(x_2))+\Big({}_{\psi^{-1}(x_2)-}J^{\alpha,\psi}(w\circ\psi)\Big)(\psi^{-1}(x_1))\Big]$$
$$\leq w(x_2)\Big({}_{\psi^{-1}(x_1)+}J^{\alpha,\psi}_{w\circ\psi}(\Pi\circ\psi)\Big)(\psi^{-1}(x_2))+w(x_1)\Big({}_{\psi^{-1}(x_2)-}J^{\alpha,\psi}_{w\circ\psi}(\Pi\circ\psi)\Big)(\psi^{-1}(x_1))$$
$$\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}\Big[\Big({}_{\psi^{-1}(x_1)+}J^{\alpha,\psi}(w\circ\psi)\Big)(\psi^{-1}(x_2))+\Big({}_{\psi^{-1}(x_2)-}J^{\alpha,\psi}(w\circ\psi)\Big)(\psi^{-1}(x_1))\Big].$$

**Theorem 286** ([179]). *Assume that $\Pi$ is as in Theorem 285. Then, we have for $\alpha>0$:*

$$\Pi\Big(\frac{x_1+x_2}{2}\Big)\Big[\Big({}_{\psi^{-1}(\frac{x_1+x_2}{2})+}J^{\alpha,\psi}(w\circ\psi)\Big)(\psi^{-1}(x_2))+\Big({}_{\psi^{-1}(\frac{x_1+x_2}{2})-}J^{\alpha,\psi}(w\circ\psi)\Big)(\psi^{-1}(x_1))\Big]$$
$$\leq w(x_2)\Big({}_{\psi^{-1}(\frac{x_1+x_2}{2})+}J^{\alpha,\psi}_{w\circ\psi}(\Pi\circ\psi)\Big)(\psi^{-1}(x_2))+w(x_1)\Big({}_{\psi^{-1}(x_2)-}J^{\alpha,\psi}_{w\circ\psi}(\Pi\circ\psi)\Big)(\psi^{-1}(x_1))$$
$$\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}\Big[\Big({}_{\psi^{-1}(\frac{x_1+x_2}{2})+}J^{\alpha,\psi}(w\circ\psi)\Big)(\psi^{-1}(x_2))+\Big({}_{\psi^{-1}(\frac{x_1+x_2}{2})-}J^{\alpha,\psi}(w\circ\psi)\Big)(\psi^{-1}(x_1))\Big].$$

## 10. H-H-Type Inequalities via Hadamard Fractional Integral

**Definition 58** ([22]). *The left-sided and right-sided Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ of function $\Pi$ are defined by*

$$({}_HJ^{\alpha}_{x_1+}\Pi)(x) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} \left(\ln\frac{x}{t}\right)^{\alpha-1} \Pi(t)\frac{dt}{t}, \ 0 < x_1 < x \le x_2,$$

*and*

$$({}_HJ^{\alpha}_{x_2-}\Pi)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{x_2} \left(\ln\frac{t}{x}\right)^{\alpha-1} \Pi(t)\frac{dt}{t}, \ 0 < x_1 \le x < x_2.$$

We present H-H-type inequalities for convex functions via Hadamard fractional integral.

**Theorem 287** ([180]). *Assume that $\alpha > 0$ and the function $\Pi : [x_1, x_2] \to \mathbb{R}$ is a convex function. Then we have*

$$\Pi\left(\frac{x_1+x_2}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^{\alpha}}\left[{}_HJ^{\alpha}_{e^{x_1}+}(\Pi\circ\ln)(e^{x_2}) + {}_HJ^{\alpha}_{e^{x_2}-}(\Pi\circ\ln)(e^{x_1})\right] \le \frac{\Pi(x_1)+\Pi(x_2)}{2}.$$

**Theorem 288** ([180]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ and $\Pi' \in L_1[x_1, x_2]$. Suppose that $|\Pi'|^p$ is a convex function for some fixed $p \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|(z-x_1)^{\alpha+1}\int_0^1 (s^{\alpha}-\mu)\Pi'(sz+(1-s)x_1)ds - (x_2-z)^{\alpha+1}\int_0^1 (s^{\alpha}-\mu)\Pi'(sz+(1-s)x_2)ds\right| \\
\le\ & A_1^{1-\frac{1}{p}}(\alpha,\mu)\left\{(z-x_1)^{\alpha+1}\left[A_2(\alpha,\mu)|\Pi'(z)|^p + A_3(\alpha,\mu)|\Pi'(x_1)|^p\right]^{\frac{1}{p}}\right. \\
&\left.+(x_2-z)^{\alpha+1}\left[A_2(\alpha,\mu)|\Pi'(z)|^p + A_3(\alpha,\mu)|\Pi'(x_2)|^p\right]^{\frac{1}{p}}\right\},
\end{aligned}$$

*where $z \in [x_1, x_2]$, $\mu \in [0,1]$, $\alpha > 0$ and*

$$\begin{aligned}
A_1(\alpha,\mu) &= \frac{2\alpha\mu^{1+\frac{1}{\alpha}}+1}{\alpha+1} - \mu, \\
A_2(\alpha,\mu) &= \frac{\alpha}{\alpha+1}\mu^{1+\frac{2}{\alpha}} + \frac{1}{\alpha+2} - \frac{\mu}{2}, \\
A_3(\alpha,\mu) &= \frac{2\alpha}{\alpha+1}\mu^{1+\frac{1}{\alpha}} - \frac{2}{\alpha+2}\mu^{1+\frac{2}{\alpha}} + \frac{1}{(\alpha+2)(\alpha+1)} - \frac{\mu}{2}.
\end{aligned}$$

**Theorem 289** ([181]). *Assume that $\Pi : (0, x_2) \to \mathbb{R}$ is a differentiable function. If $|\Pi'|$ is measurable and $|\Pi'|$ is a convex function and $|\Pi'|$ is a non-decreasing function on $(0, x_2)$ for $\alpha > 0$, $0 < x_1 < x_2$, then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|\frac{(\ln x - \ln x_1)^{\alpha} + (\ln x_2 - \ln x)^{\alpha}}{\ln x_2 - \ln x_1}\Pi(x) - \frac{\Gamma(\alpha+1)}{\ln x_2 - \ln x_1}\left[{}_HJ^{\alpha}_{x_2-}\Pi(x_1) + {}_HJ^{\alpha}_{x_1+}\Pi(x_2)\right]\right| \\
\le\ & \frac{(\ln x - \ln x_1)^{\alpha+1}}{\ln x_2 - \ln x_1}\left(x(|\Pi'(x)| - |\Pi'(x_1)|)\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(xx_1^{-1}))^{i-1}}{(\alpha+2)_i}\right. \\
&\left.+x|\Pi'(x_1)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(xx_1^{-1}))^{i-1}}{(\alpha+2)_i}\right) \\
&+\frac{(\ln x_2 - \ln x)^{\alpha+1}}{\ln x_2 - \ln x_1}\left(x(|\Pi'(x)| - |\Pi'(x_2)|)\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(xx_2^{-1}))^{i-1}}{(\alpha+2)_i}\right. \\
&\left.+x|\Pi'(x_2)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(xx_2^{-1}))^{i-1}}{(\alpha+2)_i}\right),
\end{aligned}$$

*for any* $x \in (x_1, x_2)$.

**Theorem 290** ([181]). *Assume that* $\Pi$ *is as in Theorem 289. If* $|\Pi'|^q$, $q > 1$ *is measurable and convex, then fractional integral inequality is given as:*

$$\begin{aligned} &\left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln x_2 - \ln x_1)^\alpha} \left[ {}_H J^\alpha_{x_2-}\Pi(x_1) + {}_H J^\alpha_{x_1+}\Pi(x_2) \right] \right| \\ \leq\; & \frac{(\ln x_2 - \ln x_1)}{2} x_2 \left( \frac{2 - 2\left(\frac{1}{2}\right)^{\alpha p+1}}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ |\Pi'(x_1)| \left( \left(\frac{x_1}{x_2}\right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\left(q \ln \frac{x_1}{x_2}\right)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right. \\ & \left. + |\Pi'(x_2)| \left( \frac{\left(\frac{x_1}{x_2}\right)^q - 1}{q \ln \frac{x_1}{x_2}} - \left(\frac{x_1}{x_2}\right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\left(q \ln \frac{x_1}{x_2}\right)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right], \end{aligned}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 291** ([182]). *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is a positive function with* $0 < x_1 < x_2$ *and* $\Pi \in L_1[x_1, x_2]$. *If* $\Pi$ *is a increasing and convex function on* $[x_1, x_2]$, *then fractional integral inequality is given as:*

$$\Pi(\sqrt{x_1 x_2}) \leq \frac{\Gamma(\alpha+1)}{2(\ln x_2 - \ln x_1)^\alpha} \left[ {}_H J^\alpha_{x_1+}\Pi(x_2) + {}_H J^\alpha_{x_2-}\Pi(x_1) \right] \leq \Pi(x_2).$$

**Theorem 292** ([182]). *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is a differentiable function on* $(x_1, x_2)$ *with* $0 < x_1 < x_2$. *If* $\alpha \in (0, 1]$, $\Pi' \in L_1[x_1, x_2]$ *and is increasing, then fractional integral inequality is given as:*

$$\begin{aligned} &\left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln x_2 - \ln x_1)^\alpha} \left[ {}_H J^\alpha_{x_1+}\Pi(x_2) + {}_H J^\alpha_{x_2-}\Pi(x_1) \right] \right| \\ \leq\; & \frac{x_2(\ln x_2 - \ln x_1)}{2} \left[ \frac{\alpha+2}{\alpha+1} \left( \frac{\ln x_2 - \ln x_1}{2} \right)^{-1} + \frac{\sqrt{\frac{x_1}{x_2}}}{2(\alpha+1)} \right] |\Pi'(x_2)|. \end{aligned}$$

**Theorem 293** ([182]). *Assume that* $\Pi$ *is as in Theorem 292. Then fractional integral inequality is given as:*

$$\begin{aligned} &\left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln x_2 - \ln x_1)^\alpha} \left[ {}_H J^\alpha_{x_1+}\Pi(x_2) + {}_H J^\alpha_{x_2-}\Pi(x_1) \right] \right| \\ \leq\; & \frac{x_2(\ln x_2 - \ln x_1)}{\alpha+1} \left( 1 - \frac{1}{2^\alpha} \right) |\Pi'(x_2)|. \end{aligned}$$

In the next theorems we present H-H inequalities for GA-convex and GG-convex functions via Hadamard fractional integral.

**Definition 59** ([183]). *A function* $\Pi : \mathbb{I} \subseteq (0, \infty) \to \mathbb{R}$ *is said to be geometric–arithmetically convex (GA-convex) on* $\mathbb{I}$, *if*

$$\Pi\left(x^t y^{1-t}\right) \leq t\Pi(x) + (1-t)\Pi(y),$$

*for any* $x, y \in \mathbb{I}$ *and* $t \in [0, 1]$.

**Definition 60** ([183]). *A function $\Pi : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ is said to be multiplicatively convex function (GG-convex) on $\mathbb{I}$, if*

$$\Pi\Big(x^t y^{1-t}\Big) \le [\Pi(x)]^t [\Pi(y)]^{(1-t)},$$

*for any $x, y \in \mathbb{I}$ and $t \in [0,1]$.*

**Theorem 294** ([184]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is a differentiable function on $(x_1, x_2)$ with $0 \le x_1 < x_2$. If $|\Pi'|$ is integrable and GA-convex on $[x_1, x_2]$, then for $0 \le \lambda \le 1$, $x \in (x_1, x_2)$ and $\alpha > 0$, then fractional integral inequality is given as:*

$$\begin{aligned}
&\Bigg|\lambda\Big[\frac{(\ln x_2 - \ln x)^{\alpha-1}\Pi(x_2) + (\ln x - \ln x_1)^{\alpha-1}\Pi(x_1)}{\ln x_2 - \ln x_1}\Big] \\
&-(1+\lambda)\Big[\frac{(\ln x_2 - \ln x)^{\alpha-1} + (\ln x - \ln x_1)^{\alpha-1}}{\ln x_2 - \ln x_1}\Big]\Pi(x) \\
&+\frac{\Gamma(\alpha+1)}{\ln x_2 - \ln x_1}\Big[{}_H J^{\alpha}_{x-}\Pi(x_1) + {}_H J^{\alpha}_{x+}\Pi(x_2)\Big]\Bigg| \\
\le\ & \frac{(\ln x_2 - \ln x)^{\alpha}}{\ln x_2 - \ln x_1}\bigg|x|\Pi'(x_2)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\big(\ln\frac{x}{x_2}\big)^{i-1}}{(\alpha+1)_i} \\
&-x\big[|\Pi'(x_2)| - |\Pi'(x)|\big]\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\big(\ln\frac{x}{x_2}\big)^{i-1}}{(\alpha+2)_i} \\
&+\frac{\lambda}{\ln x - \ln x_1}\big[x|\Pi'(x)| - 2\lambda|\Pi'(x_2)| + x_2|\Pi'(x)|\big]\bigg| \\
&+\frac{(\ln x - \ln x_1)^{\alpha}}{\ln x_2 - \ln x_1}\bigg|x|\Pi'(x_1)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\big(\ln\frac{x}{x_1}\big)^{i-1}}{(\alpha+1)_i} \\
&-x\big[|\Pi'(x_1)| - |\Pi'(x)|\big]\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\big(\ln\frac{x}{x_1}\big)^{i-1}}{(\alpha+2)_i} \\
&+\frac{\lambda}{\ln x - \ln x_1}\big[x|\Pi'(x)| - 2\lambda|\Pi'(x_1)| + x_1|\Pi'(x)|\big]\bigg|.
\end{aligned}$$

**Theorem 295** ([184]). *Assume that $\Pi$ is as in Theorem 294. If $|\Pi'|$ is integrable and GG-convex on $[x_1, x_2]$, the fractional integral inequality is given as:*

$$\begin{aligned}
&\Bigg|\lambda\Big[\frac{(\ln x_2 - \ln x)^{\alpha-1}\Pi(x_2) + (\ln x - \ln x_1)^{\alpha-1}\Pi(x_1)}{\ln x_2 - \ln x_1}\Big] \\
&-(1+\lambda)\Big[\frac{(\ln x_2 - \ln x)^{\alpha-1} + (\ln x - \ln x_1)^{\alpha-1}}{\ln x_2 - \ln x_1}\Big]\Pi(x) \\
&+\frac{\Gamma(\alpha+1)}{\ln x_2 - \ln x_1}\Big[{}_H J^{\alpha}_{x-}\Pi(x_1) + {}_H J^{\alpha}_{x+}\Pi(x_2)\Big]\Bigg| \\
\le\ & \frac{(\ln x_2 - \ln x)^{\alpha}}{\ln x_2 - \ln x_1}\bigg|x|\Pi'(x)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\Big(\ln\frac{x|\Pi'(x)|}{x_2|\Pi'(x_2)|}\Big)^{i-1}}{(\alpha+1)_i} \\
&+\frac{\lambda}{\ln\frac{x|\Pi'(x)|}{x_2|\Pi'(x_2)|}}\big[x|\Pi'(x)| - x_2|\Pi'(x_2)|\big]\bigg| \\
&+\frac{(\ln x - \ln x_1)^{\alpha}}{\ln x_2 - \ln x_1}\bigg|x|\Pi'(x)|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\Big(\ln\frac{x|\Pi'(x)|}{x_1|\Pi'(x_1)|}\Big)^{i-1}}{(\alpha+1)_i}
\end{aligned}$$

$$+\frac{\lambda}{\ln\frac{x|\Pi'(x)|}{x_1|\Pi'(x_1)|}}\left[x|\Pi'(x)|-x_1|\Pi'(x_1)|\right]\Big|.$$

**Theorem 296** ([185]). *Assume that $\Pi$ is as in Theorem 294. If $|\Pi'|$ is measurable and $|\Pi'|$ is GG-convex on $[0, x_2]$, then fractional integral inequality is given as:*

$$\left|\frac{(\ln x-\ln x_1)^\alpha+(\ln x_2-\ln x)^\alpha}{\ln x_2-\ln x_1}\Pi(x)-\frac{\Gamma(\alpha+1)}{\ln x_2-\ln x_1}\left[{}_HJ^\alpha_{x-}\Pi(x_1)+{}_HJ^\alpha_{x-}\Pi(x_2)\right]\right|$$
$$\le \frac{(\ln x-\ln x_1)^{\alpha+1}}{\ln x_2-\ln x_1}x_1|\Pi'(x_1)|\frac{x|\Pi'(x)|}{x_1|\Pi'(x_1)|}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left(\ln\frac{x|\Pi'(x)|}{x_1|\Pi'(x_1)|}\right)^{i-1}}{(\alpha+1)_i}$$
$$+\frac{(\ln x_2-\ln x)^{\alpha+1}}{\ln x_2-\ln x_1}x_2|\Pi'(x_2)|\frac{x|\Pi'(x)|}{x_2|\Pi'(x_2)|}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left(\ln\frac{x|\Pi'(x)|}{x_2|\Pi'(x_2)|}\right)^{i-1}}{(\alpha+1)_i},$$

*for any $x \in (x_1, x_2)$.*

**Theorem 297** ([185]). *Assume that $\Pi$ is as in Theorem 294. If $|\Pi'|^q$, $q > 1$ is measurable and GG-convex on $[0, x_2]$, then the fractional integral inequality is given as:*

$$\left|\frac{(\ln x-\ln x_1)^\alpha+(\ln x_2-\ln x)^\alpha}{\ln x_2-\ln x_1}\Pi(x)-\frac{\Gamma(\alpha+1)}{\ln x_2-\ln x_1}\left[{}_HJ^\alpha_{x-}\Pi(x_1)+{}_HJ^\alpha_{x-}\Pi(x_2)\right]\right|$$
$$\le \frac{(\ln x-\ln x_1)^{\alpha+1}}{\ln x_2-\ln x_1}x_1|\Pi'(x_1)|\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left(\frac{x|\Pi'(x)|}{x_1|\Pi'(x_1)|}\right)^q-1}{q\ln\left(\frac{x|\Pi'(x)|}{x_1|\Pi'(x_1)|}\right)}\right)^{\frac{1}{q}}$$
$$+\frac{(\ln x_2-\ln x)^{\alpha+1}}{\ln x_2-\ln x_1}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left(\frac{x|\Pi'(x)|}{x_2|\Pi'(x_2)|}\right)^q-1}{q\ln\left(\frac{x|\Pi'(x)|}{b|\Pi'(x_2)|}\right)}\right)^{\frac{1}{q}},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**Theorem 298** ([186]). *Assume that $\Pi : \mathbb{I} \subseteq (0, \infty) \to \mathbb{R}$ is a function such that $\Pi \in L_1[x_1, x_2]$, where $x_1, x_2 \in \mathbb{I}$ with $x_1 < x_2$. If $\Pi$ is a GA-convex function on $[x_1, x_2]$, then the fractional integral inequalities are given as:*

$$\Pi(\sqrt{x_1x_2}) \le \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{x_2}{x_1}\right)^\alpha}\left[{}_HJ^\alpha_{x_1+}\Pi(x_2)+{}_HJ^\alpha_{x_2-}\Pi(x_1)\right] \le \frac{\Pi(x_1)+\Pi(x_2)}{2}, \quad \alpha>0.$$

In the next theorems we present H-H inequalities for quasi-geometrically convex functions via Hadamard fractional integral.

**Definition 61** ([186]). *A function $\Pi : \mathbb{I} \subseteq (0, \infty) \to \mathbb{R}$ is said quasi-geometrically convex on $\mathbb{I}$ if*

$$\Pi\left(x^ty^{1-t}\right) \le \sup\{\Pi(x), \Pi(y)\},$$

*for any $x, y \in \mathbb{I}$ and $t \in [0, 1]$.*

**Theorem 299** ([186]). *Assume that $\Pi : \mathbb{I} \subset (0, \infty) \to \mathbb{R}$ is a differentiable function on $\mathbb{I}^\circ$ such that $\Pi' \in L_1[x_1, x_2]$, where $x_1, x_2 \in \mathbb{I}^\circ$ with $x_1 < x_2$. If $|\Pi'|^q$ is quasi-geometrically convex on*

$[x_1, x_2]$, *for some* $q \geq 1$, $x \in [x_1, x_2]$, $\lambda \in [0,1]$ *and* $\alpha > 0$, *then the fractional integral inequalities are given as:*

$$\begin{aligned}
&\left| x_1\left(\ln\frac{x}{x_1}\right)^{\alpha+1}\int_0^1 (t^\alpha-\lambda)\left(\frac{x}{x_1}\right)^t \Pi'\left(x^t x_1^{1-t}\right)dt \right.\\
&\left. -x_2\left(\ln\frac{x_2}{x}\right)^{\alpha+1}\int_0^1 (t^\alpha-\lambda)\left(\frac{x}{x_2}\right)^t \Pi'\left(x^t x_2^{1-t}\right)dt \right|\\
\leq\ & A_1^{1-\frac{1}{q}}(\alpha,\lambda)\left\{ x_1\left(\ln\frac{x}{x_1}\right)^{\alpha+1}\left(\sup\left\{|\Pi'(x)|^q, |\Pi'(x_1)|^q\right\}\right)^{\frac{1}{q}} B_1^{\frac{1}{q}}(x,\alpha,\lambda,q)\right.\\
&\left. +x_2\left(\ln\frac{x_2}{x}\right)^{\alpha+1}\left(\sup\left\{|\Pi'(x)|^q, |\Pi'(x_2)|^q\right\}\right)^{\frac{1}{q}} B_2^{\frac{1}{q}}(x,\alpha,\lambda,q)\right\},
\end{aligned}$$

*where*

$$\begin{aligned}
A_1(\alpha,\lambda) &= \frac{2\alpha\lambda^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\lambda,\\
B_1(x,\alpha,\lambda,q) &= \int_0^1 |t^\alpha-\lambda|\left(\frac{x}{x_1}\right)^{qt} ds,\\
B_2(x,\alpha,\lambda,q) &= \int_0^1 |t^\alpha-\lambda|\left(\frac{x}{x_2}\right)^{qt} ds.
\end{aligned}$$

**Theorem 300** ([187]). *Assume that* $\Pi : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ *is a GA-convex function such that* $\Pi \in L_1[x_1, x_2]$, *where* $x_1, x_2 \in \mathbb{I}$ *with* $x_1 < x_2$ *and* $\alpha > 0$. *If* $g : [x_1, x_2] \to \mathbb{R}$ *is non-negative, integrable and geometrically symmetric with respect to* $\sqrt{x_1x_2}$, *then the fractional integral inequalities are given as:*

$$\begin{aligned}
&\Pi(\sqrt{x_1x_2})\left[{}_HJ^\alpha_{x_1+}g(x_2) + {}_HJ^\alpha_{x_2-}g(x_1)\right] \leq \left[{}_HJ^\alpha_{x_1+}(\Pi g)(x_2) + {}_HJ^\alpha_{x_2-}(\Pi g)(x_1)\right]\\
\leq\ & \frac{\Pi(x_1)+\Pi(x_2)}{2}\left[{}_HJ^\alpha_{x_1+}g(x_2) + {}_HJ^\alpha_{x_2-}g(x_1)\right].
\end{aligned}$$

**Theorem 301** ([187]). *Assume that* $\Pi$ *and* $g$ *are defined as in the Theorem 300. If* $|\Pi'|$ *is GA-convex on* $[x_1, x_2]$, *then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|\left(\frac{\Pi(x_1)+\Pi(x_2)}{2}\right)\left[{}_HJ^\alpha_{x_1+}g(x_2) + {}_HJ^\alpha_{x_2-}g(x_1)\right] - \left[{}_HJ^\alpha_{x_1+}(\Pi g)(x_2) + {}_HJ^\alpha_{x_2-}(\Pi g)(x_1)\right]\right|\\
\leq\ & \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{x_2}{x_1}\right)}{\Gamma(\alpha+1)}\left[|\Pi'(x_1)|\int_0^{1/2}[(1-u)^\alpha-u^\alpha][(1-u)a^{1-u}x_2^u+ua^u x_2^{1-u}]du\right.\\
&\left.+|\Pi'(x_2)|\int_0^{1/2}[(1-u)^\alpha-u^\alpha][ua^{1-u}x_2^u+(1-u)a^u x_2^{1-u}]du.\right.
\end{aligned}$$

**Theorem 302** ([187]). *Assume that* $\Pi$ *and* $g$ *are defined as in the Theorem 300. If* $|\Pi'|^q, q > 1$ *is GA-convex on* $[x_1, x_2]$, *then fractional integral inequality is given as:*

$$\begin{aligned}
&\left|\left(\frac{\Pi(x_1)+\Pi(x_2)}{2}\right)\left[{}_HJ^\alpha_{x_1+}g(x_2) + {}_HJ^\alpha_{x_2-}g(x_1)\right] - \left[{}_HJ^\alpha_{x_1+}(\Pi g)(x_2) + {}_HJ^\alpha_{x_2-}(\Pi g)(x_1)\right]\right|\\
\leq\ & \frac{\|g\|_\infty \ln^{\alpha+1-\frac{2}{q}}\left(\frac{x_2}{x_1}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\\
&\times\left[(x_2^q - qx_1^q\ln\frac{x_2}{x_1} - x_1^q)|\Pi'(x_1)|^q + \left(x_1^q + qx_2^q\ln\frac{x_2}{x_1} - x_2^q\right)|\Pi'(x_2)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

*if* $\alpha > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *and*

$$\left|\left(\frac{\Pi(x_1) + \Pi(x_2)}{2}\right)\left[{}_H J^{\alpha}_{x_1+} g(x_2) + {}_H J^{\alpha}_{x_2-} g(x_1)\right] - \left[{}_H J^{\alpha}_{x_1+}(\Pi g)(x_2) + {}_H J^{\alpha}_{x_2-}(\Pi g)(x_1)\right]\right|$$

$$\leq \frac{\|g\|_\infty \ln^{\alpha+1-\frac{2}{q}}\left(\frac{x_2}{x_1}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p + 1}\right]^{\frac{1}{p}}$$

$$\times\left[(x_2^q - q x_1^q \ln\frac{x_2}{x_1} - x_1^q)|\Pi'(x_1)|^q + \left(x_1^q + q x_2^q \ln\frac{x_2}{x_1} - x_2^q\right)|\Pi'(x_2)|^q\right]^{\frac{1}{q}},$$

*if* $0 < \alpha \leq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

## 11. H-H-Type Inequalities via Proportional Fractional Integral

In this subsection we present H-H inequalities for convex functions via proportional fractional integral.

**Definition 62** ([188])**.** *The left and right generalized proportional fractional integral operators are respectively defined by*

$$(\mathcal{F}^{\alpha,\rho}_{x_1}\Pi)(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_{x_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\Pi(s)ds$$

*and*

$$(\mathcal{F}^{\alpha,\rho}_{b}\Pi)(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_{t}^{b} e^{\frac{\rho-1}{\rho}(s-t)}(s-t)^{\alpha-1}\Pi(s)ds,$$

*where* $\rho \in (0,1]$, *and* $\alpha \in \mathbb{C}$ *and* $\Re(\alpha) > 0$.

**Theorem 303** ([189])**.** *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *with* $0 \leq x_1 < x_2$ *and* $\Pi \in L_1[x_1, x_2]$. *If* $\Pi$ *is a convex mapping on* $[x_1, x_2]$, *then, the following inequality is valid:*

$$\Pi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{\alpha\rho^\alpha\Gamma(\alpha)}{2(x_2 - x_1)^\alpha\, {}_1F_1\left(\alpha, \alpha+1, \frac{\rho-1}{\rho}(x_2 - x_1)\right)}\left[(\mathcal{F}^{\alpha,\rho}_{x_2}\Pi)(x_1) + (\mathcal{F}^{\alpha,\rho}_{x_1}\Pi)(x_2)\right],$$

*where* ${}_1F_1$ *is the hypergeometric function.*

**Definition 63** ([190])**.** *The left and right generalized proportional fractional integral operators of a function* $\Pi$ *with respect to* $\Pi$, *where* $\Pi$ *is a continuous and strictly increasing function on* $[x_1, x_2]$, *are respectively defined by*

$$(\mathcal{F}^{\alpha,\rho,\psi}_{x_1}\Pi)(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_{x_1}^{t} e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(s))}(\psi(t) - \psi(s))^{\alpha-1}\psi'(s)\Pi(s)ds$$

*and*

$$(\mathcal{F}^{\alpha,\rho,\psi}_{x_2}\Pi)(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_{t}^{x_2} e^{\frac{\rho-1}{\rho}(\psi(s)-\psi(t))}(\psi(s) - \psi(t))^{\alpha-1}\psi'(s)\Pi(s)ds,$$

*where* $\rho \in (0,1]$, *and* $\alpha \in \mathbb{C}$ *and* $\Re(\alpha) > 0$.

**Theorem 304** ([190])**.** *Let* $\psi : \mathbb{I} \to [x_1, x_2] \subseteq \mathbb{R}$ *with* $0 \leq x_1 < x_2$, *be a continuous and strictly increasing function and* $\Pi : [x_1, x_2] \to \mathbb{R}$ *be a convex differentiable function on* $(x_1, x_2)$ *satisfying that* $\Pi \circ \psi : \mathbb{I} \to \mathbb{R}$ *is an integrable mapping on* $\mathbb{I}$. *Then we have:*

$$\Pi\left(x_1 + x_2 - \frac{x+y}{2}\right)$$

$$\leq \Pi(x_1)+\Pi(x_2)-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\Big[\mathcal{F}^{\alpha,\rho,\psi}_{(\psi^{-1}(x)+}(\Pi\circ\psi)(\psi^{-1}(y))+\mathcal{F}^{\alpha,\rho,\psi}_{(\psi^{-1}(y))-}(\Pi\circ\psi)(\psi^{-1}(x))\Big]$$
$$\leq \Pi(x_1)+\Pi(x_2)-\Pi\Big(\frac{x_1+x_2}{2}\Big),$$

*and*

$$\begin{aligned}&\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\\ \leq\;& \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\Big[\mathcal{F}^{\alpha,\rho,\psi}_{(\psi^{-1}(x_1+x_2-y)+}(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-x))\\ &+\mathcal{F}^{\alpha,\rho,\psi}_{(\psi^{-1}(x_1+x_2-x)-}(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-y))\Big]\\ \leq\;& \frac{1}{2}[\Pi(x_1+x_2-y)+\Pi(x_1+x_2-x)]\leq \Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2}.\end{aligned}$$

**Theorem 305** ([190])**.** *Assume that $\Pi$ is as in Theorem 304. Then fractional integral inequality is given as:*

$$\begin{aligned}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\;\leq\;& \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\Big[\mathcal{F}^{\alpha,\rho,\psi}_{\left(\psi^{-1}\left(x_1+x_2-\frac{x+y}{2}\right)\right)+}(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-y))\\ &+\mathcal{F}^{\alpha,\rho,\psi}_{\left(\psi^{-1}\left(x_1+x_2-\frac{x+y}{2}\right)\right)-}(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-x))\Big]\\ \leq\;& \Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2}.\end{aligned}$$

**Definition 64** ([191])**.** *For an integrable function $\Pi$ and a strictly continuous function $\Pi$ on $[x_1,x_2]$, $\sigma\in(0,1]$, $\alpha\in\mathbb{C}$ and $\Re(\alpha)>0$, the left and right proportional k-fractional integral operators are respectively defined by*

$$({}^{k,\psi}\mathcal{J}^{\alpha,\sigma}_{x_1}\Pi)(t)=\frac{1}{\sigma^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_{x_1}^{t}e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\frac{\alpha}{k}-1}\psi'(s)\Pi(s)ds$$

*and*

$$({}^{k,\psi}\mathcal{J}^{\alpha,\sigma}_{x_1}\Pi)(t)=\frac{1}{\sigma^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_{t}^{x_2}e^{\frac{\sigma-1}{\sigma}(\psi(s)-\psi(t))}(\psi(s)-\psi(t))^{\frac{\alpha}{k}-1}\psi'(s)\Pi(s)ds.$$

**Theorem 306** ([192])**.** *Assume that $\Pi:[x_1,x_2]\to\mathbb{R}$ is a convex differentiable function defined on $(x_1,x_2)$ and $\psi:[x_1,x_2]\to[x_1,x_2]$ be a continuous strictly increasing function with $0\leq x_1<x_2$ and $(\Pi\circ\psi):[x_1,x_2]\to\mathbb{R}$ is an integrable function on $[x_1,x_2]$. Then, for all $x,y\in[x_1,x_2]$ the fractional integral inequalities are given as:*

$$\begin{aligned}&\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\\ \leq\;& \Pi(x_1)+\Pi(x_2)-\frac{\sigma^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}^{k,\psi}\mathcal{J}^{\alpha,\sigma}_{\psi^{-1}(x)+}(\Pi\circ\psi)(\psi^{-1}(y))\\ &+{}^{k,\psi}\mathcal{J}^{\alpha,\sigma}_{\psi^{-1}(y)-}(\Pi\circ\psi)(\psi^{-1}(x))\Big]\\ \leq\;& \Pi(x_1)+\Pi(x_2)-\Pi\Big(\frac{x+y}{2}\Big),\end{aligned}$$

*and*

$$\begin{aligned}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\;\leq\;& \frac{\sigma^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}^{k,\psi}\mathcal{J}^{\alpha,\sigma}_{\psi^{-1}(x_1+x_2-y)+}(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-x))\\ &+{}^{k,\psi}\mathcal{J}^{\alpha,\sigma}_{\psi^{-1}(x_1+x_2-x)-}(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-y))\Big]\end{aligned}$$

$$\leq \frac{\Pi(x_1+x_2-y)+\Pi(x_1+x_2-x)}{2}$$
$$\leq \Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2}.$$

**Theorem 307** ([192]). *Assume that $\Pi$ is as in Theorem 306. Then, for all $x, y \in [x_1, x_2]$ the fractional integral inequalities are given as:*

$$\begin{aligned}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big) &\leq \frac{\sigma^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}^{k,\psi}\mathcal{J}^{\alpha,\sigma}_{\psi^{-1}\left(x_1+x_2-\frac{x+y}{2}\right)+}(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-x)) \\ &\quad +{}^{k,\psi}\mathcal{J}^{\alpha,\sigma}_{\psi^{-1}\left(x_1+x_2-\frac{x+y}{2}\right)-}(\Pi\circ\psi)(\psi^{-1}(x_1+x_2-y))\Big] \\ &\leq \Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2}.\end{aligned}$$

## 12. H-H-Type Inequalities via Raina Integral Operator

In [193], Raina introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x)=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k,\ \ \rho,\lambda>0,\ \ |x|<\infty,$$

where the coefficients $\sigma(k)$ ($k \in \mathbb{N} = \mathbb{N} \cup \{0\}$ ) are a bounded sequence of positive real numbers. In [194] defined the following left-sided and right-sided fractional integral operators, respectively:

$$(J^{\sigma}_{\rho,\lambda,a+,w}\phi)(x)=\int_{x_1}^{x}(x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[w(x-t)^{\rho}\big]\phi(t)dt,\ \ x>x_1>0,$$

$$(J^{\sigma}_{\rho,\lambda,b-,w}\phi)(x)=\int_{x}^{x_2}(t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[w(t-x)^{\rho}\big]\phi(t)dt,\ \ 0<x<x_2.$$

In the following theorems we present H-H inequalities for convex functions via Raina integral operator.

**Theorem 308** ([195]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is a differentiable function on $[x_1, x_2]$ with $x_1 < x_2$. If $|\Pi'|$ is a convex function on $(x_1, x_2)$, then fractional integral inequality is given as:*

$$\begin{aligned}&\Bigg|\frac{1}{2(x_2-x_1)^{\lambda}}\Big[(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)+(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)\Big] \\ &-\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[w(x_2-x_1)^{\rho}\big]\Pi\Big(\frac{x_1+x_2}{2}\Big)\Bigg| \\ \leq\ &\frac{x_2-x_1}{4}\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\big[|w|(x_2-x_1)^{\rho}\big]\big[|\Pi'(x_1)|+|\Pi'(x_2)|\big],\end{aligned}$$

*where* $\sigma_1(k)=\sigma(k)\Big(\lambda+\rho k+3-\frac{1}{2^{\lambda+\rho k-1}}\Big),\ \ \rho,\lambda>0, w\in\mathbb{R}.$

**Theorem 309** ([195]). *Assume that $\Pi$ is as in Theorem 308. If $|\Pi'|^q$ is a convex function and $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then fractional integral inequality is given as:*

$$\Bigg|\frac{1}{2(x_2-x_1)^{\lambda}}\Big[(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)+(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)\Big]$$

$$
\begin{aligned}
& -\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[w(x_2-x_1)^{\rho}\big]\Pi\Big(\frac{x_1+x_2}{2}\Big)\Bigg| \\
\leq\ & \frac{x_2-x_1}{2}\Big[\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]\Big(\frac{|\Pi'(x_1)|^q+|\Pi'(x_2)|^q}{2}\Big)^{\frac{1}{q}} \\
& +\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]\Big(\Big[\frac{1}{8}|\Pi'(x_1)|^q+\frac{3}{8}|\Pi'(x_2)|^q\Big]^{\frac{1}{q}} \\
& +\Big[\frac{3}{8}|\Pi'(x_1)|^q+\frac{1}{8}|\Pi'(x_2)|^q\Big]^{\frac{1}{q}}\Big)\Big] \\
\leq\ & \mathcal{F}^{\sigma_3}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]\big[|\Pi'(x_1)|+|\Pi'(x_2)|\big],
\end{aligned}
$$

*where* $\rho,\lambda>0$, $w\in\mathbb{R}$ *and*

$$
\begin{aligned}
\sigma_2(k) &= \sigma(k)\Big(\frac{1}{p(\lambda+\rho k)+1}\Big(1-\frac{1}{2^{p(\lambda+\rho k)}}\Big)\Big]^{\frac{1}{p}}, \\
\sigma_3(k) &= \sigma(k)\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\Big(1+\Big[\frac{4}{p(\lambda+\rho k)+1}\Big(1-\frac{1}{2^{p(\lambda+\rho k)}}\Big)\Big]^{\frac{1}{p}}\Big).
\end{aligned}
$$

**Theorem 310** ([196]). *Assume that* $\Pi:[x_1,x_2]\to\mathbb{R}$ *is a differentiable function on* $(x_1,x_2)$ *with* $x_1<x_2$. *If* $|\Pi'|$ *is a convex function on* $[x_1,x_2]$, *then fractional integral inequality is given as:*

$$
\begin{aligned}
& \Bigg|\frac{1}{2(x_2-x_1)^{\lambda}}\Big[(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)+(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)\Big] \\
& -\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[w(x_2-x_1)^{\rho}\big]\Pi\Big(\frac{x_1+x_2}{2}\Big)\Bigg| \\
\leq\ & \frac{x_2-x_1}{2}\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]\big[|\Pi'(x_1)|+|\Pi'(x_2)|\big],
\end{aligned}
$$

*where* $\rho,\lambda>0$, $w\in\mathbb{R}$ *and* $\sigma_4(k)=\sigma(k)\left(\frac{1}{2}+\frac{(\frac{1}{2})^{\lambda+\rho k}-1}{\lambda+\rho k+1}\right)$.

**Theorem 311** ([196]). *Assume that* $\Pi$ *is as in Theorem 310. If* $|\Pi'|^q$ *is a convex function and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then fractional integral inequality is given as:*

$$
\begin{aligned}
& \Bigg|\frac{1}{2(x_2-x_1)^{\lambda}}\Big[(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)+(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)\Big] \\
& -\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[w(x_2-x_1)^{\rho}\big]\Pi\Big(\frac{x_1+x_2}{2}\Big)\Bigg| \\
\leq\ & \frac{x_2-x_1}{2}\Big[\mathcal{F}^{\sigma_5}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big] \\
& \times\Big\{\Big(\frac{3|\Pi'(x_1)|^q+|\Pi'(x_2)|^q}{4}\Big)^{\frac{1}{q}}+\Big(\frac{|\Pi'(x_1)|^q+3|\Pi'(x_2)|^q}{4}\Big)^{\frac{1}{q}}\Big\},
\end{aligned}
$$

*where* $\rho,\lambda>0$, $w\in\mathbb{R}$ *and* $\sigma_5(k)=\sigma(k)\left[\Big(\frac{(\frac{1}{2})^{(\lambda+\rho k)p+1}}{(\lambda+\rho k)p+1}\Big)^{\frac{1}{p}}+\Big(\int_{1/2}^{1}(1-t^{\lambda+\rho k})^p dt\Big)^{\frac{1}{p}}\right]$.

**Theorem 312** ([197]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is a convex function on $[x_1, x_2]$ with $x_1 < x_2$. Then fractional integral inequalities are given as:*

$$\begin{aligned} \Pi\Big(\frac{x_1+x_2}{2}\Big) &\leq \frac{(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)+(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)}{2(x_2-x_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]} \\ &\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}. \end{aligned}$$

**Theorem 313** ([197]). *Assume that the function $\Pi$ is as in Theorem 312. If $|\Pi'|$ is a convex function on $[x_1, x_2]$, then fractional integral inequality is given as:*

$$\begin{aligned} &\left|\frac{\Pi(x_1)+\Pi(x_2)}{2} - \frac{(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)+(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)}{2(x_2-x_1)^{\lambda}\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]}\right| \\ \leq\ & \frac{(x_2-x_1)\mathcal{F}^{\sigma_7}_{\rho,\lambda+2}\big[|w|(x_2-x_1)^{\rho}\big]}{2\mathcal{F}^{\sigma_7}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]}\big(|\Pi'(x_1)|+|\Pi'(x_2)|\big), \end{aligned}$$

*where* $\sigma_7 = \sigma(k)\Big(1-\Big(\frac{1}{2}\Big)^{k\rho+\lambda}\Big)$.

**Theorem 314** ([198]). *Assume that $\Pi : [x_1, x_2] \to \mathbb{R}$ is a positive function with $x_1 < x_2$ and $\Pi \in L_1[x_1, x_2]$. If $\Pi$ is a convex function on $[x_1, x_2]$, then $BS$ is a convex function and monotonically non-decreasing on $[0, 1]$ and for all $\rho, \lambda > 0$ and $w \geq 0$ we have the fractional integral inequalities are given as:*

$$\begin{aligned} \Pi\Big(\frac{x_1+x_2}{2}\Big) &= BS(0) \leq BS(\xi) \leq BS(1) \\ &= \frac{(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)+(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)}{2(x_2-x_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]} \end{aligned}$$

*where*

$$\begin{aligned} BS(\xi) &= \frac{1}{2(x_2-x_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]}\int_{x_1}^{x_2}\Pi\Big(\xi x+(1-\xi)\frac{x_1+x_2}{2}\Big) \\ &\quad\times\Big[(x_2-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[|w|(x_2-x)^{\rho}\big]+(x-x_1)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[|w|(x-x_1)^{\rho}\big]\Big]dx. \end{aligned}$$

**Theorem 315** ([198]). *Assume that $\Pi$ is as in Theorem 314. Then $BY$ is a convex function and monotonically non-decreasing on $[0, 1]$ and for all $\rho, \lambda > 0$ and $w \geq 0$ we have that the fractional integral inequalities are given as:*

$$\begin{aligned} \frac{(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)+(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)}{2(x_2-x_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]} &= BY(0) \leq BY(\xi) \leq BY(1) \\ &= \frac{\Pi(x_1)+\Pi(x_2)}{2}, \end{aligned}$$

*where*

$$\begin{aligned} BY(\xi) &= \frac{1}{4(x_2-x_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[|w|(x_2-x_1)^{\rho}\big]}\int_{x_1}^{x_2}\Big[\Pi\Big(\Big(\frac{1+\xi}{2}\Big)x_1+\Big(\frac{1-\xi}{2}\Big)x\Big) \\ &\quad\times\Big(\Big(\frac{2x_2-x_1-x}{2}\Big)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\Big[w\Big(\frac{2x_2-x_1-x}{2}\Big)^{\rho}\Big] \\ &\quad+\Big(\frac{x-x_1}{2}\Big)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\Big[w\Big(\frac{x-x_1}{2}\Big)^{\rho}\Big]\Big) \end{aligned}$$

$$+\Pi\Big(\Big(\frac{1+\xi}{2}\Big)x_1+\Big(\frac{1-\xi}{2}\Big)x\Big)\Big(\Big(\frac{x_2-x}{2}\Big)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\Big[w\Big(\frac{x_2-x}{2}\Big)^{\rho}\Big]$$
$$+\Big(\frac{x+x_2-2x_1}{2}\Big)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\Big[w\Big(\frac{x+x_2-2x_1}{2}\Big)^{\rho}\Big]\Big)\Big]dx.$$

**Theorem 316** ([199])**.** *Suppose that $\Pi:[x_1,x_2]\to\mathbb{R}$ is a convex function. Then:*

$$\begin{aligned}\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big) &\leq \Pi(x_1)+\Pi(x_2)-\frac{(J^{\sigma}_{\rho,\lambda,x+,w}\Pi)(y)+(J^{\sigma}_{\rho,\lambda,y-,w}\Pi)(x)}{2(y-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(y-x)^{\rho}]}\\ &\leq \Pi(x_1)+\Pi(x_2)-\Pi\Big(\frac{x+y}{2}\Big),\end{aligned}$$

*and*

$$\begin{aligned}&\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\\ \leq\ &\frac{(J^{\sigma}_{\rho,\lambda,(x_1+x_2-y)+,w}\Pi)(x_1+x_2-x)+(J^{\sigma}_{\rho,\lambda,(x_1+x_2-x)-,w}\Pi)(x_1+x_2-y)}{2(y-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(y-x)^{\rho}]}\\ \leq\ &\Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2},\end{aligned}$$

*for all $x,y\in[x_1,x_2]$, $x<y$ and $\lambda,\rho,w>0$.*

**Theorem 317** ([199])**.** *Suppose that $\Pi:[x_1,x_2]\to\mathbb{R}$ is a convex function. Then:*

$$\begin{aligned}&\Pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\\ \leq\ &\frac{2^{\lambda-1}}{(y-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(y-x)^{\rho}\frac{1}{2^{\rho}}]}\\ &\times\Big[(J^{\sigma}_{\rho,\lambda,\left(x_1+x_2-\frac{x+y}{2}\right)+,w}\Pi)(x_1+x_2-y)+(J^{\sigma}_{\rho,\lambda,\left(x_1+x_2-\frac{x+y}{2}\right)-,w}\Pi)(x_1+x_2-x)\Big]\\ \leq\ &\Pi(x_1)+\Pi(x_2)-\frac{\Pi(x)+\Pi(y)}{2},\end{aligned}$$

*for all $x,y\in[x_1,x_2]$, $x<y$ and $\lambda,\rho,w>0$.*

**Theorem 318** ([199])**.** *Assume that $\Pi:[x_1,x_2]\to\mathbb{R}$ is a differentiable function on $(x_1,x_2)$ with $x_1<x_2$. If $|\Pi'|$ is a convex function on $[x_1,x_2]$, then fractional integral inequalities are given as:*

$$\begin{aligned}&\Bigg|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(y-x)^{\rho}]\frac{\Pi(x_1+x_2-x)+\Pi(x_1+x_2-y)}{2}-\frac{1}{2(y-x)^{\lambda}}\\ &\times\Big[(J^{\sigma}_{\rho,\lambda,(x_1+x_2-x)-,w}\Pi)(x_1+x_2-y)+(J^{\sigma}_{\rho,\lambda,(x_1+x_2-y)+,w}\Pi)(x_1+x_2-x)\Big]\Bigg|\\ \leq\ &(y-x)\mathcal{F}^{\sigma_7}_{\rho,\lambda+2}[|w|(y-x)^{\rho}]\Big[|\Pi'(x_1)|+\Pi'(x_2)|-\frac{|\Pi'(x)|+|\Pi'(y)|}{2}\Big],\end{aligned}$$

*for all $x,y\in[x_1,x_2]$, $x<y$ and $\lambda,\rho,w>0$.*

We now give H-H inequalities for functions which have bounded second derivative via Raina integral operator.

**Theorem 319** ([200])**.** *Let $\alpha,w\in\mathbb{R}^+$, $\Pi:[x_1,x_2]\to\mathbb{R}$ $(x_1<x_2)$ be a positive, twice differentiable function, and $\Pi\in L_1(x_1,x_2)$. In addition, let $\Pi''$ be bounded on $[x_1,x_2]$. Then:*

$$\frac{m}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}\left(\frac{x_1+x_2}{2}-x\right)^2\{(x-x_1)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(x-x_1)^\rho]$$
$$+(x_2-x)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(x-x_1)^\rho]\}dx$$
$$\leq \frac{1}{2(x_2-x_1)^\alpha}\left[\left(J^{\sigma}_{\rho,\alpha,x_1+,w}\Pi\right)(x_2)+\left(J^{\sigma}_{\rho,\alpha,x_2-,w}\Pi\right)(x_1)-\Pi\left(\frac{x_1+x_2}{2}\right)\mathcal{F}^{\sigma}_{\rho,\alpha+1}[w(x_2-x_1)^\rho]\right]$$
$$\leq \frac{M}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}\left(\frac{x_1+x_2}{2}-x\right)^2\{(x-x_1)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(x-x_1)^\rho]$$
$$+(x_2-x)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(x-x_1)^\rho]\}dx$$

*and*

$$\frac{-M}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}(x-x_1)(x_2-x)\{(x-x_1)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(x-x_1)^\rho]$$
$$+(x_2-x)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(x-x_1)^\rho]\}dx$$
$$\leq \frac{1}{2(x_2-x_1)^\alpha}\left[\left(J^{\sigma}_{\rho,\alpha,x_1+,w}\Pi\right)(x_2)+\left(J^{\sigma}_{\rho,\alpha,x_2-,w}\Pi\right)(x_1)-\frac{\Pi(x_1)+\Pi(x_2)}{2}\mathcal{F}^{\sigma}_{\rho,\alpha+1}[w(x_2-x_1)^\rho]\right]$$
$$\leq \frac{-m}{2(x_2-x_1)^\alpha}\int_{x_1}^{\frac{x_1+x_2}{2}}(x-x_1)(x_2-x)\{(x-x_1)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(x-x_1)^\rho]$$
$$+(x_2-x)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(x-x_1)^\rho]\}dx,$$

*where* $m=\inf_{t\in[x_1,x_2]}\Pi''(t)$ *and* $M=\sup_{t\in[x_1,x_2]}\Pi''(t)$.

**Theorem 320** ([200]). *Assume that* $\Pi$ *is as in Theorem 319. If* $\Pi'(x_1+x_2-t)\geq\Pi'(t)$ *for all* $t\in\left[x_1,\frac{x_1+x_2}{2}\right]$*, then fractional integral inequalities are given as:*

$$\Pi\left(\frac{x_1+x_2}{2}\right)\mathcal{F}^{\sigma}_{\rho,\alpha+1}[w(x_2-x_1)^\rho] \leq \frac{1}{2(x_2-x_1)^\alpha}\left[\left(J^{\sigma}_{\rho,\alpha,x_1+,w}\Pi\right)(x_2)+\left(J^{\sigma}_{\rho,\alpha,x_2-,w}\Pi\right)(x_1)\right]$$
$$\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}\mathcal{F}^{\sigma}_{\rho,\alpha+1}[w(x_2-x_1)^\rho].$$

H-H inequalities for $\eta$-convex functions via Raina integral operator are given in the next.

**Definition 65** ([201]). *A function* $\Pi:[x_1,x_2]\to\mathbb{R}$ *is said to be* $\eta$*-convex function on* $[x_1,x_2]$ *if the inequality*

$$\Pi(tx+(1-t)y)\leq\Pi(y)+t\eta(\Pi(x),\Pi(y)),$$

*holds for any* $x,y\in[x_1,x_2]$ *and* $t\in[0,1]$*, and* $\eta$ *is defined by* $\eta:\Pi([x_1,x_2])\times\Pi([x_1,x_2])\to\mathbb{R}$.

**Theorem 321** ([202]). *Assume that* $\Pi:[x_1,x_2]\to\mathbb{R}$ *is a* $\eta$*-convex function such that* $\eta:\Pi([x_1,x_2])\times\Pi([x_1,x_2])\to\mathbb{R}$ *is upper bounded by* $M_\eta$*, then fractional integral inequalities are given as:*

$$\Pi\left(\frac{x_1+x_2}{2}\right)-M_\eta\leq\frac{(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2)+(J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)}{2(x_2-x_1)^\lambda\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[|w|(x_2-x_1)^\rho]}$$
$$\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}+\frac{\eta(\Pi(x_1),\Pi(x_2))+\eta(\Pi(x_2),\Pi(x_1))}{2}\frac{\mathcal{F}^{\sigma_6}_{\rho,\lambda+2}[|w|(x_2-x_1)^\rho]}{\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[|w|(x_2-x_1)^\rho]}$$
$$\leq \frac{\Pi(x_1)+\Pi(x_2)}{2}+M_\eta\frac{\mathcal{F}^{\sigma_6}_{\rho,\lambda+2}[|w|(x_2-x_1)^\rho]}{\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[|w|(x_2-x_1)^\rho]}.$$

*where* $\rho, \lambda > 0$, $w \in \mathbb{R}$ *and* $\sigma_6 = \sigma(k)(k\rho + \lambda)$.

**Theorem 322** ([202]). *Assume that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is a differentiable function on* $(x_1, x_2)$ *with* $x_1 < x_2$. *If* $|\Pi'|$ *is an* $\eta$*-convex function on* $[x_1, x_2]$, *then fractional integral inequality is given as:*

$$\begin{aligned} &\left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{(J^{\sigma}_{\rho,\lambda,x_1+,w}\Pi)(x_2) + (J^{\sigma}_{\rho,\lambda,x_2-,w}\Pi)(x_1)}{2(x_2 - x_1)^{\lambda}\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}\left[|w|(x_2 - x_1)^{\rho}\right]} \right| \\ \leq\ & \frac{(x_2 - x_1)\mathcal{F}^{\sigma_7}_{\rho,\lambda+2}\left[|w|(x_2 - x_1)^{\rho}\right]}{2\mathcal{F}^{\sigma_7}_{\rho,\lambda+1}\left[|w|(x_2 - x_1)^{\rho}\right]}(2|\Pi'(x_2)| + \eta(|\Pi'(x_1)|, |\Pi'(x_2)|), \end{aligned}$$

*where* $\sigma_7 = \sigma(k)\left(1 - \left(\frac{1}{2}\right)^{k\rho+\lambda}\right)$.

**Theorem 323** ([203]). *Suppose that* $\Pi : [x_1, x_2] \to \mathbb{R}$ *is an* $\eta$*-convex function such that* $\eta$ *is bounded above by* $M_\eta$, *then for* $\alpha > 0$, *the fractional integral inequality is given as:*

$$\begin{aligned} \Pi\left(\frac{x_1 + x_2}{2}\right) - M_\eta &\leq \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2) + J^{\alpha}_{x_2-}\Pi(x_1)\right] \\ &\leq \frac{\Pi(x_1) + \Pi(x_2)}{2} + \frac{\alpha(\eta(\Pi(x_1), \Pi(x_2)) + \eta(\Pi(x_2), \Pi(x_1)))}{2(\alpha+1)} \\ &\leq \frac{\Pi(x_1) + \Pi(x_2)}{2} + \frac{\alpha M_\eta}{\alpha+1}. \end{aligned}$$

**Theorem 324** ([203]). *Assume that the function* $\Pi$ *is as in Theorem 323. If* $|\Pi'|$ *is an* $\eta$*-convex function on* $[x_1, x_2]$, *then fractional integral inequality is given as:*

$$\begin{aligned} &\left| \frac{\Pi(x_1) + \Pi(x_2)}{2} - \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2) + J^{\alpha}_{x_2-}\Pi(x_1)\right] \right| \\ \leq\ & \frac{x_2 - x_1}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[2|\Pi'(x_2)| + \eta(|\Pi'(x_1)|, |\Pi'(x_2)|)\right]. \end{aligned}$$

**Theorem 325** ([203]). *Assume that the function* $\Pi$ *is as in Theorem 323. If* $|\Pi'|$ *is an* $\eta$*-convex function on* $[x_1, x_2]$ *and* $0 < \alpha \leq 1$, *then fractional integral inequality is given as:*

$$\begin{aligned} &\left| \Pi\left(\frac{x_1 + x_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2) + J^{\alpha}_{x_2-}\Pi(x_1)\right] \right| \\ \leq\ & \frac{x_2 - x_1}{2^{\alpha+1}(\alpha+1)}\left[|\Pi'(x_1)| + |\Pi'(x_2)| + \eta(|\Pi'(x_1)|, |\Pi'(x_2)|) + \eta(|\Pi'(x_2)|, |\Pi'(x_1)|)\right]. \end{aligned}$$

**Theorem 326** ([203]). *Assume that the function* $\Pi$ *is as in Theorem 323. If* $|\Pi'|^q$ *is* $\eta$*-convex function on* $[x_1, x_2]$ *for some* $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $0 < \alpha \leq 1$, *then fractional integral inequality is given as:*

$$\begin{aligned} &\left| \Pi\left(\frac{x_1 + x_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2(x_2 - x_1)^{\alpha}}\left[J^{\alpha}_{x_1+}\Pi(x_2) + J^{\alpha}_{x_2-}\Pi(x_1)\right] \right| \\ \leq\ & \frac{x_2 - x_1}{2^{\alpha+1}(\alpha p+1)^{\frac{1}{p}}}\left[\left(\frac{4|\Pi'(x_1)|^q + \eta(|\Pi'(x_2)|^q, |\Pi'(x_1)|^q)}{4}\right)^{\frac{1}{q}} \right. \\ & \left. + \left(\frac{4|\Pi'(x_1)|^q + \eta(|\Pi'(x_2)|^q, |\Pi'(x_1)|^q)}{4}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

**Definition 66** ([204])**.** *For $k > 0$, let $\psi : [x_1, x_2] \to \mathbb{R}$ be an increasing and positive function on $(x_1, x_2]$, having a continuous derivative $\psi'$ on $(x_1, x_2)$. The left and right-sided generalized k-fractional integrals of $\Pi$ with respect to the function $\psi$ on $[x_1, x_2]$ are defined, respectively, as follows:*

$$J^{\sigma,k,\psi}_{\rho,\lambda,x_1+,w}\Pi(x) = \int_{x_1}^{x} \psi'(t)(\psi(x) - \psi(t))^{\frac{\lambda}{k}-1}\mathcal{F}^{\sigma,k}_{\rho,\lambda}[w(\psi(x) - \psi(t))^{\rho}]\Pi(t)dt, \ \ x > x_1,$$

*and*

$$J^{\sigma,k,\psi}_{\rho,\lambda,x_2-,w}\Pi(x) = \int_{x}^{x_2} \psi'(t)(\psi(t) - \psi(x))^{\frac{\lambda}{k}-1}\mathcal{F}^{\sigma,k}_{\rho,\lambda}[w(\psi(t) - \psi(t))^{\rho}]\Pi(t)dt, \ \ x < x_2,$$

*where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and $\mathcal{F}^{\sigma,k}_{\rho,\lambda}(x) = \sum_{m=1}^{\infty} \frac{\sigma(m)}{k\Gamma_k(\rho km + \lambda)} x^m$ is the k-Raina function.*

**Theorem 327** ([204])**.** *Let $\psi : [x_1, x_2] \to \mathbb{R}$ be an increasing and positive function on $(x_1, x_2]$, having a continuous derivative $\psi'$ on $(x_1, x_2)$. If $\Pi$ is a convex function on $[x_1, x_2]$, then k-fractional integral inequality is given as:*

$$\begin{aligned} \Pi\Big(\frac{x_1 + x_2}{2}\Big) &\leq \frac{J^{\sigma,k,\psi}_{\rho,\lambda,x_2-,w}[\Pi(x_1) + \Pi(x_2)] + J^{\sigma,k,\Pi}_{\rho,\lambda,x_1+,w}[\Pi(x_1) + \Pi(x_2)]}{4k(\psi(x_2) - \psi(x_1))^{\frac{\lambda}{k}}\mathcal{F}^{\sigma,k}_{\rho,\lambda}[w(\psi(x_2) - \psi(x_1))^{\rho}]} \\ &\leq \frac{\Pi(x_1) + \Pi(x_2)}{2}. \end{aligned}$$

**Theorem 328** ([205])**.** *Assume that $\psi$ is as in Theorem 327. If $\Pi$ is convex on $[x_1, x_2]$, then k-fractional integral inequality is given as:*

$$\begin{aligned} \Pi\Big(\frac{x_1 + x_2}{2}\Big) &\leq \frac{J^{\sigma,k,\psi}_{\rho,\lambda,\frac{x_1+x_2}{2}-,w}[\Pi(x_1) + \Pi(x_2)] + J^{\sigma,k,\psi}_{\rho,\lambda,\frac{x_1+x_2}{2}+,w}[\Pi(x_1) + \Pi(x_2)]}{4k\Big[\psi(x_2) - \psi\Big(\frac{x_1+x_2}{2}\Big)\Big]^{\frac{\lambda}{k}}\mathcal{F}^{\sigma,k}_{\rho,\lambda+k}\Big[w\Big(\psi(x_2) - \psi\Big(\frac{x_1+x_2}{2}\Big)\Big)^{\rho}\Big]} \\ &\quad \times \frac{1}{4k\Big[\psi\Big(\frac{x_1+x_2}{2}\Big) - \psi(x_1)\Big]^{\frac{\lambda}{k}}\mathcal{F}^{\sigma,k}_{\rho,\lambda+k}\Big[w\Big(\psi\Big(\frac{x_1+x_2}{2}\Big) - \psi(x_1)\Big)^{\rho}\Big]} \\ &\leq \frac{\Pi(x_1) + \Pi(x_2)}{2}. \end{aligned}$$

**Theorem 329** ([205])**.** *Assume that $\Pi$ is as in Theorem 327. If $\Pi$ is a differentiable mapping on $(x_1, x_2)$, such that $\Pi' \in L_1[x_1, x_2]$ and $|\Pi'|$ is a convex function on $[x_1, x_2]$, then:*

$$\begin{aligned} &\Bigg|\frac{1}{2k}\Big[J^{\sigma,k,\psi}_{\rho,\lambda,\frac{x_1+x_2}{2}-,w}[\Pi(x_1) + \Pi(x_2)] + J^{\sigma,k,\psi}_{\rho,\lambda,\frac{x_1+x_2}{2}+,w}[\Pi(x_1) + \Pi(x_2)]\Big] \\ &-\Big(\Big[\psi(x_2) - \psi\Big(\frac{x_1 + x_2}{2}\Big)\Big]^{\frac{\lambda}{k}}\mathcal{F}^{\sigma,k}_{\rho,\lambda+k}\Big[w\Big(\psi(x_2) - \psi\Big(\frac{x_1 + x_2}{2}\Big)\Big)^{\rho}\Big] \\ &+\Big[\psi\Big(\frac{x_1 + x_2}{2}\Big) - \psi(x_1)\Big]^{\frac{\lambda}{k}}\mathcal{F}^{\sigma,k}_{\rho,\lambda+k}\Big[w\Big(\psi\Big(\frac{x_1 + x_2}{2}\Big) - \psi(x_1)\Big)^{\rho}\Big]\Big)\Pi\Big(\frac{x_1 + x_2}{2}\Big)\Bigg| \\ \leq\ &\frac{x_2 - x_1}{4}(\mathcal{F}^{\sigma^*,k}_{\rho,\lambda+k}[w])(|\Pi'(x_1)| + |\Pi'(x_2)|), \end{aligned}$$

*where*

$$\sigma^*(m) = \sigma(m)\int_0^1 \Big\{\Big[\psi(x_2) - \psi\Big(\frac{s}{2}x_1 + \frac{2 - s}{2}x_2\Big)\Big]^{\frac{\lambda}{k}+\rho m}$$

$$+\left[\psi\left(\frac{s}{2}x_1+\frac{2-s}{2}x_2\right)-\psi(x_1)\right]^{\frac{\lambda}{k}+\rho m}\Bigg\}ds.$$

## 13. Conclusions

Our objective in this review paper was to present and provide a comprehensive and up-to-date review on H-H-type inequalities pertaining to fractional integral operators. We presented results including integral inequalities of the H-H-type through various classes of convexity. H-H inequalities via Green's functions are also included, and H-H-type inequalities via preinvex functions are also presented. In the fractional integral operators, it includes R-L fractional integral, Katugampola fractional integral, $k$-R-L fractional integral, $(k,s)$-R-L fractional integral, C-F fractional integral, R-L fractional integrals of a function with respect to another function, Hadamard fractional integral and Raina fractional integral operator.

This review was prepared to keep in mind the theoretical and practical importance of the H-H-type inequalities. We believe that the present review will motivate and provide a platform for the researchers working on H-H-type inequalities to learn about the available work on the topic before developing new results. Future research regarding this review paper is fascinating. Our review paper might inspire a good number of additional studies.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| $AG(\log)$-convex | arithmetic–geometric convex |
| C-F | Caputo–Fabrizio |
| GA-convex | geometric–arithmetically convex |
| GA-$s$-convex | geometric–arithmetically $s$-convex |
| GG-convex | multiplicatively convex function |
| G-L | Godunova–Levin |
| H-H | Hermite–Hadamard |
| H-H-M | Hermite–Hadamard–Mercer |
| $m$-HH convex | $m$-harmonic harmonically convex |
| R-L | Riemann–Liouville |

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