mathematics

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Towards a Proof of Bahri–Coron's Type Theorem for Mixed Boundary Value Problems

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Abstract: We consider a nonlinear variational elliptic problem with critical nonlinearity on a bounded domain of $\mathbb{R}^n$, $n \geq 3$ and mixed Dirichlet–Neumann boundary conditions. We study the effect of the domain's topology on the existence of solutions as Bahri–Coron did in their famous work on the homogeneous Dirichlet problem. However, due to the influence of the part of the boundary where the Neumann condition is prescribed, the blow-up picture in the present setting is more complicated and makes the mixed boundary problems different with respect to the homogeneous ones. Such complexity imposes modification of the argument of Bahri–Coron and demands new constructions and extra ideas.

Keywords: nonlinear elliptic problems; critical nonlinearities; variational structures; mixed boundary conditions; topological methods

MSC: 35J20

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Citation: Alghanemi, A.; Chaabane, S.; Chtioui, H.; Soumaré, A. Towards a Proof of Bahri–Coron's Type Theorem for Mixed Boundary Value Problems. *Mathematics* **2023**, *11*, 1955. https://doi.org/10.3390/math11081955

Academic Editor: Ravi P. Agarwal

Received: 15 March 2023
Revised: 17 April 2023
Accepted: 18 April 2023
Published: 20 April 2023

1. Introduction

In this paper, we are concerned with a mixed boundary problem of the form

$$\begin{cases} -\Delta u &= u^{\frac{n+2}{n-2}}, \ u > 0 & \text{in} & \Omega, \\ u &= 0 & \text{on} & \Gamma_0, \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on} & \Gamma_1, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\frac{\partial}{\partial \nu}$ denotes the derivation with respect to the outward unit normal ν on Γ_1. We suppose that Γ_0 and Γ_1 are (n-1)-dimensional submanifolds of $\partial\Omega$ having positive Hausdorff measures. We are looking for solutions to problem (1) in the Sobolev space

$$V(\Omega) = \{u \in H^1(\Omega), \ \ u = 0 \text{ on } \Gamma_0\}.$$

Let

$$\Sigma = \{u \in V(\Omega), \ \ \|u\|^2 := \int_\Omega |\nabla u|^2 dx = 1\} \text{ and } \Sigma^+ = \{u \in \Sigma, \ \ u > 0\}.$$

For any $u \in V(\Omega)$, $u \neq 0$, we define

$$J(u) = \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega u^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}}.$$

It is straightforward to see that the solutions of the problem (1) correspond to the critical points of the variational functional $J(u)$ subject to the constraint $u \in \Sigma^+$. However J involves the exponent $2^* = \frac{2n}{n-2}$, which is the critical exponent of the Sobolev embeddings $V(\Omega) \hookrightarrow L^q(\Omega), \quad q \leq 2^*$. In contrast with the subcritical cases, $q < 2^*$, the Sobolev embedding is not compact for $q = 2^*$. It results in the corresponding variational structure presenting a lack of compactness, which can be seen in the fact that J does not satisfy the Palais–Smale condition. This makes the problem of finding positive critical points of J particulary difficult.

Elliptic equations involving Laplacian operator on bounded domains with mixed boundary conditions arise in real applications, for example, in hydrodynamics; see [1,2]. Generally, nonlinear problems subject to various boundary conditions appear in many different branches of the applied sciences, including physics (e.g., steady-state heat flux modeling), chemistry (e.g., Keller–Segel model for parabolic equations in chemotaxis), biology (e.g., Gierer–Meinhardt system in the formation of biological models), and engineering. See for example [3–5] and references therein.

By continuity of the Sobolev embedding for $q = 2^*$, the functional J is lower bounded on Σ. Let

$$S(\Omega) := inf_{u \in \Sigma} J(u).$$

A first attempt to find solutions of (1) could be to see if $S(\Omega)$ may be achieved. In [6], Lions–Pacella and Tricarico studied the minimizing sequences of J following the concentration compactness principle of Lions [7]. As product, under some geometrical conditions on $\overline{\Omega}$, it is shown that it is possible to prove that $S(\Omega)$ is achieved. See corollaries 2.1 and 2.2 of [6]. The existence of a bounded domain for which $S(\Omega)$ is achieved is strange and makes the study of mixed boundary problems of type (1) different compared with the classical homogeneous Dirichlet problem. For more conditions that ensure $S(\Omega)$ is achieved for problems such as (1), we refer to [8–12].

Problem (1) does not always have a solution. A necessary condition to obtain a solution has been established in [6]. It is the following Pohozaev-type identity:

$$\frac{1}{2}\int_{\Gamma_0}(x.\nu)\left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma = \frac{1}{2}\int_{\Gamma_1}(x.\nu)|\nabla u|^2 d\sigma - \frac{n-2}{2n}\int_{\Gamma_1}(x.\nu)u^{\frac{2n}{n-2}}d\sigma.$$

Particulary, (1) has no solution provided:

$$x.\nu = 0 \text{ on } \Gamma_1 \text{ and } x.\nu > 0 \text{ on } \Gamma_0.$$

For examples of domains satisfying such a condition, we refer to [6]. However, there are other conditions on $\overline{\Omega}$ obtained by some Sobolev inequalities (see [13,14]) that only guarantee $S(\Omega)$ is not achieved. An example of such domains is given by Pacella and Tricarico in [15] by considering domains bounded by two concentric spheres with Γ_1 denoting the interior sphere. Using the so-called "isoperimetric constant of Ω relative to Γ_1", see [15], it is proved that $S(\Omega)$ is not achieved whatever the radius of the two spheres. When $S(\Omega)$ is not achieved, a natural question arises: Could one find positive critical points of J of energy levels larger than $S(\Omega)$?

An analysis of Palais–Smale sequences of the function J has been performed by Grossi and Pacella [16]. As a consequence of it, a positive answer to the above question has been derived for bounded domains Ω with two holes, one of which is very small; Γ_1 denotes the boundaries of the interior two holes and $\Gamma_0 = \partial\Omega \setminus \Gamma_1$.

Observe that the case of domains bounded by two concentric spheres with Γ_1 denotes the boundary of the interior ball is not included in the existence results of [16]. Motivated by the work of Bahri–Coron [17] and aiming to include a larger class of domains Ω in the existence results of problem (1), we develop in the present article and the subsequent one [18] an approach that allows us to include all possible cases of bounded domains with an arbitrary number of holes of arbitrary sizes.

$$(H): \text{ Let } D \subset \mathbb{R}^n, \ \ n \geq 3, \text{ be a bounded domain with smooth boundary.}$$

$$\text{Let } k \geq 1, \ \ x_1, ..., x_k \in D \text{ and } R_1, ..., R_k > 0 \text{ such that}$$

$$\overline{B(x_i, R_i)} \subset D, \ \ \forall\, i = 1, ..., \ k \text{ and } \overline{B(x_i, R_i)} \cap \overline{B(x_j, R_j)} = \emptyset, \ \ \forall\, 1 \leq i \neq j \leq k.$$

Here, $\overline{B(x, R)}$ denotes the closure of the ball of center x and radius R. Let

$$\Omega = D \setminus \cup_{i=1}^{k} \overline{B(x_i, R_i)} \ \ , \ \ \Gamma_1 = \cup_{i=1}^{k} \partial B(x_i, R_i) \text{ and } \Gamma_0 = \partial\Omega \setminus \Gamma_1 = \partial D.$$

In the following, we state the main theorem that we shall prove in this paper and the subsequent one [18].

Theorem 1. *If* Ω, Γ_0 *and* Γ_1 *satisfy description* (H), *then* (1) *has a solution.*

The aim of this paper is to prepare the field to prove Theorem 1. We first establish under the assumption that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ a strong maximum principle for the Laplacian operator with mixed Dirichlet–Neumann boundary conditions. After that we extend the analysis of [17] (see also [19]) to the present setting and prove useful estimates involving asymptotic expansions of the variational functional J. Then, we develop an algebraic topological method and prove Theorem 1 under an additional topological condition. See Theorem 3 below.

Notice that, although the general scheme of our proof falls within the analysis and topological techniques of Bahri–Coron [17], the same techniques cannot be extended to the present framework. Indeed, with respect to the homogeneous Dirichlet problem traited by Bahri–Coron, the case of mixed boundary problem presents new phenomena. Namely, due to the influence of the boundary part where the Neumann condition is assumed, the blow-up configuration is completely different and more complicated. It is described by interior and boundary blow-up points as well as mixed configurations. See [6,16]. This leads to additional difficulties and obstacles to apply Bahri–Coron's approach and requires novelties in the proof.

2. A Maximum Principle Theorem

In this section, we prove an $L^{\infty}-$ estimate for solutions of mixed boundary value problems with Laplace operator. Let Ω be a bounded connected domain of $\mathbb{R}^n$, $n \geq 1$ with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ (Figure 1).

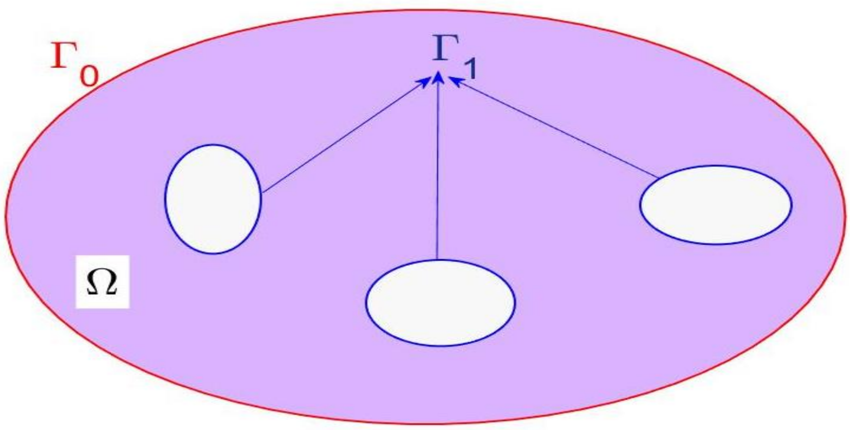

Figure 1. A simplified figure of Ω.

Let $k > \frac{n}{2}$, $f_0 \in H^k(\Omega)$, $f_1 \in H^{k+\frac{3}{2}}(\Gamma_0)$ and $f_2 \in H^{k+\frac{1}{2}}(\Gamma_1)$. Denote $u = u(f_0, f_1, f_2)$ be the solution of the following problem:

$$(\mathcal{N})\begin{cases} -\Delta u & = & f_0 & \text{in} & \Omega, \\ u & = & f_1 & \text{on} & \Gamma_0, \\ \frac{\partial u}{\partial \nu} & = & f_2 & \text{on} & \Gamma_1. \end{cases}$$

We then have

Theorem 2. *for every $x \in \overline{\Omega}$, it holds:*

$$m_0 w_0(x) + m_1 w_1(x) + m_2 w_2(x) \leq u(x) \leq M_0 w_0(x) + M_1 w_1(x) + M_2 w_2(x).$$

where w_0, w_1 and w_2 are the solutions of the following boundary value problems:

$$\begin{cases} -\Delta w_0 & = & 1 & in & \Omega, \\ w_0 & = & 0 & on & \Gamma_0, \\ \frac{\partial w_0}{\partial \nu} & = & 0 & on & \Gamma_1. \end{cases}, \begin{cases} -\Delta w_1 & = & 0 & in & \Omega, \\ w_1 & = & 1 & on & \Gamma_0, \\ \frac{\partial w_1}{\partial \nu} & = & 0 & on & \Gamma_1. \end{cases}, \begin{cases} -\Delta w_2 & = & 0 & in & \Omega, \\ w_2 & = & 0 & on & \Gamma_0, \\ \frac{\partial w_2}{\partial \nu} & = & 1 & on & \Gamma_1. \end{cases}$$

and

$$\begin{cases} m_0 = \inf_{x\in\overline{\Omega}} f_0(x), \\ m_1 = \inf_{x\in\Gamma_0} f_1(x), \\ m_2 = \inf_{x\in\Gamma_1} f_2(x). \end{cases}, \quad \begin{cases} M_0 = \sup_{x\in\overline{\Omega}} f_0(x), \\ M_1 = \sup_{x\in\Gamma_0} f_1(x), \\ M_2 = \sup_{x\in\Gamma_1} f_2(x). \end{cases}$$

The proof of Theorem 2 requires the following three Lemmas.

Lemma 1. *Let $u_0 = u_0(f_0)$ be the solution of*

$$\begin{cases} -\Delta u_0 & = & f_0 & in & \Omega, \\ u_0 & = & 0 & on & \Gamma_0, \\ \frac{\partial u_0}{\partial \nu} & = & 0 & on & \Gamma_1. \end{cases}$$

Then for any $x \in \overline{\Omega}$, we have

$$m_0 w_0(x) \leq u_0(x) \leq M_0 w_0(x).$$

Proof. Let $z_0 = u_0 - m_0 w_0$ and $Z_0 = u_0 - M_0 w_0$. We then have

$$\begin{cases} \Delta z_0 & = & m_0 - f_0 \leq 0 & \text{in} & \Omega, \\ z_0 & = & 0 & \text{on} & \Gamma_0, \\ \frac{\partial z_0}{\partial \nu} & = & 0 & \text{on} & \Gamma_1. \end{cases}, \quad \begin{cases} \Delta Z_0 & = & M_0 - f_0 \geq 0 & \text{in} & \Omega, \\ Z_0 & = & 0 & \text{on} & \Gamma_0, \\ \frac{\partial Z_0}{\partial \nu} & = & 0 & \text{on} & \Gamma_1. \end{cases}$$

By regularity Theorems, see [20,21], we have $(z_0, Z_0) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. Indeed, under assumption $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, the solutions z_0 and Z_0 lie in $H^{k+2}(\Omega) \hookrightarrow C^2(\overline{\Omega})$ since $k > \frac{n}{2}$.

Let $(x_0, y_0, X_0, Y_0) \in \overline{\Omega}^4$, such that

$$z_0(x_0) = \inf_{x\in\overline{\Omega}} z_0(x) \quad ; \quad z_0(y_0) = \sup_{x\in\overline{\Omega}} z_0(x) \quad ; \quad Z_0(X_0) = \inf_{x\in\overline{\Omega}} Z_0(x) \quad ; \quad Z_0(Y_0) = \sup_{x\in\overline{\Omega}} Z_0(x).$$

To proof the left side inequality of the Lemma, we distinguish two cases.

If z_0 is a constant function on Ω, then by condition $z_0 = 0$ in Γ_0, we obtain $z_0 \equiv 0$ on Ω.
If z_0 not constant, then x_0 have to be in $\partial\Omega$ and satisfies $\frac{\partial z_0}{\partial \nu}(x_0) < 0$, (see [22], Lemma 3.4). It follows that $x_0 \in \Gamma_0$ and therefore $z_0(x_0) = 0$. Consequently,

$$u_0(x) \geq m_0 w_0(x), \ \forall\, x \in \overline{\Omega}.$$

Similarly, we proof the right side inequality of the Lemma. Indeed,

- If Z_0 is a constant function on Ω, we then have $Z_0 \equiv 0$ on Ω.
- If not, $Y_0 \in \partial\Omega$ and satisfies $\frac{\partial Z_0}{\partial \nu}(Y_0) > 0$. It follows that $Y_0 \in \Gamma_0$. Therefore,
$$u_0(x) \leq M_0 w_0(x), \quad \text{for every } x \in \overline{\Omega}.$$

□

Lemma 2. *Let $u_1 = u_1(f_1)$ be the solution of*

$$\begin{cases} -\Delta u_1 &= & 0 & in & \Omega, \\ u_1 &= & f_1 & on & \Gamma_0, \\ \frac{\partial u_1}{\partial \nu} &= & 0 & on & \Gamma_1. \end{cases}$$

Then for any $x \in \overline{\Omega}$, we have

$$m_1 w_1(x) \leq u_1(x) \leq M_1 w_1(x).$$

Proof. Let $z_1 = u_1 - m_1 w_1$ and $Z_1 = u_1 - M_1 w_1$. We then have

$$\begin{cases} \Delta z_1 &= & 0 & \text{in} & \Omega, \\ z_1 &= & f_1 - m_1 & \text{on} & \Gamma_0, \\ \frac{\partial z_1}{\partial \nu} &= & 0 & \text{on} & \Gamma_1. \end{cases}, \quad \begin{cases} \Delta Z_1 &= & 0 & \text{in} & \Omega, \\ Z_1 &= & f_1 - M_1 & \text{on} & \Gamma_0, \\ \frac{\partial Z_1}{\partial \nu} &= & 0 & \text{on} & \Gamma_1. \end{cases}$$

Let $(x_1, y_1, X_1, Y_1) \in \overline{\Omega}^4$ such that

$$z_1(x_1) = \inf_{x \in \overline{\Omega}} z_1(x) \ ; \quad z_1(y_1) = \sup_{x \in \overline{\Omega}} z_1(x) \ ; \quad Z_1(X_1) = \inf_{x \in \overline{\Omega}} Z_1(x) \quad ; \quad Z_1(Y_1) = \sup_{x \in \overline{\Omega}} Z_1(x).$$

- If z_1 is a constant function on Ω, from the fact that $z_1 = f_1 - m_1 \geq 0$ on Γ_0, we obtain
$$u_1(x) - m_1 w_1(x) \geq 0, \ \forall \ x \in \overline{\Omega}.$$
- If z_1 is not constant, then x_1 have to be in $\partial\Omega$ and satisfies $\frac{\partial z_1}{\partial \nu}(x_1) < 0$. Therefore, $x_1 \in \Gamma_0$ where $z_1(x_1) = f_1(x_1) - m_1 \geq 0$. Consequently,
$$z_1(x) = u_1(x) - m_1 w_1(x) \geq 0, \ \ \forall \ x \in \overline{\Omega}.$$

Moreover,

- If Z_1 is a constant function on Ω, then by condition $Z_1 = f_1 - M_1 \leq 0$ on Γ_0, we get
$$Z_1(x) = u_1(x) - M_1 w_1(x) \leq 0, \ \ \forall \ x \in \overline{\Omega}.$$
- If Z_1 is not constant, then Y_1 have to be in $\partial\Omega$ and satisfies $\frac{\partial Z_1}{\partial \nu}(Y_1) > 0$, this implies that $Y_1 \in \Gamma_0$, where $Z_1(Y_1) = f_1(Y_1) - M_1 \leq 0$. Therefore,
$$Z_1(x) = u_1(x) - M_1 w_1(x) \leq 0, \ \forall \ x \in \overline{\Omega}.$$

□

Lemma 3. *Let $u_2 = u_2(f_2)$ be the solution of*

$$\begin{cases} -\Delta u_2 &= & 0 & in & \Omega, \\ u_2 &= & 0 & on & \Gamma_0, \\ \frac{\partial u_2}{\partial \nu} &= & f_2 & on & \Gamma_1. \end{cases}$$

Then for every any $x \in \overline{\Omega}$, we have

$$m_2 w_2(x) \leq u_2(x) \leq M_2 w_2(x).$$

Proof. Let $z_2 = u_2 - m_2 w_2$ and $Z_2 = u_2 - M_2 w_2$. We then have:

$$\left\{ \begin{array}{rclcl} \Delta z_2 & = & 0 & \text{in} & \Omega, \\ z_2 & = & 0 & \text{on} & \Gamma_0, \\ \frac{\partial z_2}{\partial \nu} & = & f_2 - m_2 & \text{on} & \Gamma_1. \end{array} \right. , \quad \left\{ \begin{array}{rclcl} \Delta Z_2 & = & 0 & \text{in} & \Omega, \\ Z_2 & = & 0 & \text{on} & \Gamma_0, \\ \frac{\partial Z_2}{\partial \nu} & = & f_2 - M_2 & \text{on} & \Gamma_1. \end{array} \right.$$

Let $(x_2, y_2, X_2, Y_2) \in \overline{\Omega}^4$ such that:

$$z_2(x_2) = \inf_{x \in \overline{\Omega}} z_2(x)\ , \quad z_2(y_2) = \sup_{x \in \overline{\Omega}} z_2(x)\ , \quad Z_2(X_2) = \inf_{x \in \overline{\Omega}} Z_2(x) \quad , \quad Z_2(Y_2) = \sup_{x \in \overline{\Omega}} Z_2(x).$$

As the proof of previous Lemmas, we distinguish the following cases:

- If z_2 is constant on Ω, then $z_2 \equiv 0$ on Ω, since $z_2 = 0$ on Γ_0.
- If not, x_2 have to be in $\partial\Omega$ and satisfies $\frac{\partial z_2}{\partial \nu}(x_2) < 0$. Therefore, $x_2 \in \Gamma_0$ where $z_2(x_2) = 0$. It follows that,

$$z_2(x) = u_2(x) - m_2 w_2(x) \geq 0, \quad \forall x \in \overline{\Omega}.$$

Concerning Z_2,

- If Z_2 is constant on Ω, then by condition $Z_2 = 0$ on Γ_0, we get $Z_2 \equiv 0$ on Ω.
- If not, Y_2 have to be in $\partial\Omega$ and satisfies $\frac{\partial Z_2}{\partial \nu}(Y_2) > 0$. Consequently $Y_2 \in \Gamma_0$, where $Z_2(Y_2) = 0$. Therefore,

$$Z_2(x) = u_2(x) - M_2 w_2(x) \leq 0, \quad \forall x \in \overline{\Omega}.$$

□

Proof of Theorem 2. Let $u = u(f_0, f_1, f_2)$ be the solution of problem $(\mathcal{N})$. We decompose u as follows:

$$u = u_0(f_0) + u_1(f_1) + u_2(f_2).$$

where $u_i(f_i),\ i = 0,\ 1,\ 2$, are the solutions of Lemmas 1–3. The estimate of $u(x),\ x \in \overline{\Omega}$, follows from the estimates of $u_i(f_i),\ i = 0,\ 1,\ 2$.

We end this section by stating an $L^\infty -$ estimate of the solution of problem $(\mathcal{N})$. The estimate is a direct consequence of Theorem 2. □

Corollary 1. *Let $k > \frac{n}{2}$. There exists a positive constant $c > 0$, such that for every $f_0 \in H^k(\Omega)$, $f_1 \in H^{k+\frac{3}{2}}(\Gamma_0)$ and $f_2 \in H^{k+\frac{1}{2}}(\Gamma_1)$, the solution $u = u(f_0, f_1, f_2)$ of problem $(\mathcal{N})$ satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq c\Big[\|f_0\|_{L^\infty(\Omega)} + \|f_1\|_{L^\infty(\Gamma_0)} + \|f_2\|_{L^\infty(\Gamma_1)}\Big].$$

3. Asymptotic Analysis

Problem (1) has a variational structure. Indeed, if u is a critical point of J in Σ^+, then $J(u)^{\frac{n-2}{4}} u$ is a solution of (1). Due to the compactness defect of the Sobolev embedding $V(\Omega) \hookrightarrow L^{2^*}(\Omega)$, the functional J fails to satisfy the Palais–Smale condition on Σ^+. In order to describe the sequences failing the Palais–Smale condition, we introduce in the following a family of "almost solutions" of problem (1).

For any $a \in \Omega \cup \Gamma_1$, and $\lambda > 0$, we define

$$\delta_{(a,\lambda)}(x) = \beta_n \left(\frac{\lambda}{1 + \lambda^2 |x - a|^2} \right)^{\frac{n-2}{2}}, x \in \mathbb{R}^n,$$

where β_n is a fixed positive constant which depends only on n and chosen so that

$$-\Delta\delta_{(a,\lambda)} = \delta_{(a,\lambda)}^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n.$$

In the case of $a \in \Gamma_1$, we define an almost solution $P\delta_{(a,\lambda)}$ by

$$P\delta_{(a,\lambda)}(x) = \Psi_a(x)\delta_{(a,\lambda)}(x),\ x \in \Omega,$$

where Ψ_a is a smooth cut-off function defined by

$$\Psi_a(x) = 1 \text{ if } x \in B(a, \frac{\rho}{2}) \text{ and } \Psi_a(x) = 0 \text{ if } x \in B(a,\rho)^c.$$

Here, ρ is a positive constant depending on a and chosen so that $P\delta_{(a,\lambda)} = 0$ on Γ_0. In the case of $a \in \Omega$, we define $P\delta_{(a,\lambda)}$ as the unique solution in $V(\Omega)$ of

$$\begin{cases} -\Delta u &= \delta_{(a,\lambda)}^{\frac{n+2}{n-2}} & \text{in} & \Omega, \\ u &= 0 & \text{on} & \Gamma_0, \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on} & \Gamma_1. \end{cases}$$

Denote $d_1 = d_1(a) = min\big(d(a,\Gamma_0)^n, d(a,\Gamma_1)^{n+1}\big)$. Setting

$$\varphi_{(a,\lambda)} = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}.$$

In the following proposition, we estimate $\varphi_{(a,\lambda)}$. It involves the regular part of the Green function for the Laplacian operator under mixed Dirichlet–Neumann boundary conditions.

Proposition 1. *Let $(a,\lambda) \in \Omega \times \mathbb{R}_+^*$ and let $H(a,.)$ be the regular part of the Green's function associated with problem (1). We then have*

$$\varphi_{(a,\lambda)} = \frac{H(a,.)}{\lambda^{\frac{n-2}{2}}} + O\Big(\frac{1}{\lambda^{\frac{n+2}{2}} d_1}\Big).$$

Proof. Let

$$\Phi_{(a,\lambda)}(x) = \varphi_{(a,\lambda)}(x) - \frac{H(a,x)}{\lambda^{\frac{n-2}{2}}},\ \ x \in \Omega.$$

Using the fact that $H(a,.)$ satisfies

$$\begin{cases} -\Delta H(a,x) &= 0 & \text{in} & \Omega, \\ H(a,x) &= \frac{1}{|a-x|^{n-2}} & \text{on} & \Gamma_0, \\ \frac{\partial H}{\partial \nu}(a,x) &= \frac{\partial}{\partial \nu}\Big(\frac{1}{|a-x|^{n-2}}\Big) & \text{on} & \Gamma_1, \end{cases}$$

it holds

$$\begin{cases} -\Delta\Phi_{(a,\lambda)} &= 0 & \text{in} & \Omega\,, \\ \Phi_{(a,\lambda)} &= \delta_{(a,\lambda)} - \frac{1}{\lambda^{\frac{n-2}{2}}|x-a|^{n-2}} & \text{on} & \Gamma_0, \\ \frac{\partial \Phi_{(a,\lambda)}}{\partial \nu} &= \frac{\partial}{\partial \nu}(\delta_{(a,\lambda)} - \frac{1}{\lambda^{\frac{n-2}{2}}|x-a|^{n-2}}) & \text{on} & \Gamma_1. \end{cases}$$

Thus, by Corollary 1 we obtain

$$\|\Phi_{(a,\lambda)}\|_{L^\infty(\Omega)} \le C \max\Big(\Big\|\delta_{(a,\lambda)} - \frac{1}{\lambda^{\frac{n-2}{2}}|x-a|^{n-2}}\Big\|_{L^\infty(\Gamma_0)}, \Big\|\frac{\partial}{\partial \nu}(\delta_{(a,\lambda)} - \frac{1}{\lambda^{\frac{n-2}{2}}|x-a|^{n-2}})\Big\|_{L^\infty(\Gamma_1)}\Big). \tag{2}$$

Observe that

$$\begin{aligned}\delta_{(a,\lambda)}(x) &= \int_{\mathbb{R}^n} \frac{1}{|\, x-y\,|^{n-2}} \delta_{(a,\lambda)}^{\frac{n+2}{n-2}}(y)dy \\ &= \beta_n^{\frac{n+2}{n-2}} \int_{\mathbb{R}^n} \frac{1}{|\, x-y\,|^{n-2}} \frac{\lambda^{\frac{n+2}{2}}}{(1+\lambda^2 \,|\, y-a\,|^2)^{\frac{n+2}{2}}} dy.\end{aligned}$$

A change of variables, $z = \lambda(y-a)$ yields

$$\delta_{(a,\lambda)}(x) = \frac{\beta_n^{\frac{n+2}{n-2}}}{\lambda^{\frac{n-2}{2}}} \int_{\mathbb{R}^n} \frac{1}{|\, x-a-\frac{z}{\lambda}\,|^{n-2}} \frac{dz}{(1+|\, z\,|^2)^{\frac{n+2}{2}}}.$$

Let $\rho > 0$,

$$\begin{aligned}\delta_{(a,\lambda)}(x) &= \frac{\beta_n^{\frac{n+2}{n-2}}}{\lambda^{\frac{n-2}{2}}} \Big[\int_{|z|<\lambda\rho} \frac{1}{|\, x-a\,|^{n-2}} \frac{dz}{(1+|\, z\,|^2)^{\frac{n+2}{2}}} \\ &\quad +(n-2) \int_{|z|<\lambda\rho} \frac{1}{|\, x-a\,|^{n}} \frac{< x-a, z/\lambda > dz}{(1+|\, z\,|^2)^{\frac{n+2}{2}}} \\ &\quad +O\Big(\int_{|z|<\lambda\rho} \frac{1}{|\, x-a\,|^{n}} \frac{|\, z/\lambda\,|^2\, dz}{(1+|\, z\,|^2)^{\frac{n+2}{2}}} \Big) \Big] \\ &\quad +O\Big(\frac{1}{\lambda^{\frac{n-2}{2}}} \int_{|z|\geq\lambda\rho} \frac{1}{|\, x-a-\frac{z}{\lambda}\,|^{n-2}} \frac{dz}{(1+|\, z\,|^2)^{\frac{n+2}{2}}} \Big).\end{aligned}$$

Now, using the fact that

$$\int_{|z|<\lambda\rho} \frac{< x-a, z/\lambda > dz}{(1+|\, z\,|^2)^{\frac{n+2}{2}}} = 0,$$

$$\int_{|z|<\lambda\rho} \frac{|\, z\,|^2\, dz}{(1+|\, z\,|^2)^{\frac{n+2}{2}}} = O(1),$$

and

$$\int_{|z|\geq\lambda\rho} \frac{1}{|\, x-a-\frac{z}{\lambda}\,|^{n-2}} \frac{dz}{(1+|\, z\,|^2)^{\frac{n+2}{2}}} = O\Big(\frac{1}{\lambda^2}\Big),$$

we obtain that

$$\delta_{(a,\lambda)}(x) = \frac{1}{\lambda^{\frac{n-2}{2}}\,|\, x-a\,|^{n-2}} + O\Big(\frac{1}{\lambda^{\frac{n+2}{2}}\,|\, x-a\,|^{n}}\Big) + O\Big(\frac{1}{\lambda^{\frac{n+2}{2}}}\Big).$$

Observe that for any $x \in \Gamma_0$, we have $|\, x-a\,| \geq d(a,\Gamma_0)$. Therefore,

$$\delta_{(a,\lambda)}(x) - \frac{1}{\lambda^{\frac{n-2}{2}}\,|\, x-a\,|^{n-2}} = O\Big(\frac{1}{\lambda^{\frac{n+2}{2}} d(a,\Gamma_0)^n}\Big). \tag{3}$$

From another part,

$$\begin{aligned}\frac{\partial\delta_{(a,\lambda)}}{\partial\nu}(x) &= \frac{\beta_n^{\frac{n+2}{n-2}}}{\lambda^{\frac{n-2}{2}}}\int_{\mathbb{R}^n}\frac{\partial}{\partial\nu}\Big(\frac{1}{\mid x-a-\frac{z}{\lambda}\mid^{n-2}}\Big)\frac{dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\\ &= \beta_n^{\frac{n+2}{n-2}}\frac{\partial}{\partial\nu}\Big(\frac{1}{\lambda^{\frac{n-2}{2}}\mid x-a\mid^{n-2}}\Big)\int_{|z|<\lambda\rho}\frac{dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\\ &\quad+(n-2)\beta_n^{\frac{n+2}{n-2}}\frac{\partial}{\partial\nu}\int_{|z|<\lambda\rho}\frac{1}{\mid x-a\mid^n}\frac{<x-a,z/\lambda>dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\\ &\quad+O\Big(\frac{1}{\lambda^{\frac{n-2}{2}}}\int_{|z|<\lambda\rho}\frac{\partial}{\partial\nu}\Big(\frac{1}{\mid x-a\mid^n}\Big)\frac{\mid z/\lambda\mid^2 dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\Big)\\ &\quad+O\Big(\frac{1}{\lambda^{\frac{n-2}{2}}}\int_{|z|\geq\lambda\rho}\frac{\partial}{\partial\nu}\Big(\frac{1}{\mid x-a-\frac{z}{\lambda}\mid^{n-2}}\Big)\frac{dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\Big).\end{aligned}$$

We have

$$\Big|\int_{|z|<\lambda\rho}\frac{\partial}{\partial\nu}\big(\frac{1}{\mid x-a\mid^n}\big)\frac{\mid z/\lambda\mid^2 dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\Big|\leq n\int_{|z|<\lambda\rho}\big(\frac{1}{\mid x-a\mid^{n+1}}\big)\frac{\mid z/\lambda\mid^2 dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}},$$

and

$$\Big|\int_{|z|\geq\lambda\rho}\frac{\partial}{\partial\nu}\big(\frac{1}{\mid x-a-\frac{z}{\lambda}\mid^n}\big)\frac{\mid z/\lambda\mid^2 dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\Big|\leq(n-2)\int_{|z|\geq\lambda\rho}\big(\frac{1}{\mid x-a-\frac{z}{\lambda}\mid^{n-1}}\big)\frac{\mid z/\lambda\mid^2 dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}.$$

Therefore,

$$\begin{aligned}\frac{\partial\delta_{(a,\lambda)}}{\partial\nu}(x) &= \frac{\partial}{\partial\nu}\Big(\frac{1}{\lambda^{\frac{n-2}{2}}\mid x-a\mid^{n-2}}\Big)\\ &\quad+O\Big(\frac{1}{\lambda^{\frac{n-2}{2}}}\int_{|z|<\lambda\rho}\frac{1}{\mid x-a\mid^{n+1}}\frac{\mid z/\lambda\mid^2 dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\Big)\\ &\quad+O\Big(\frac{1}{\lambda^{\frac{n-2}{2}}}\int_{|z|\geq\lambda\rho}\frac{1}{\mid x-a-\frac{z}{\lambda}\mid^{n-1}}\frac{dz}{(1+\mid z\mid^2)^{\frac{n+2}{2}}}\Big).\end{aligned}$$

Using the same reasoning as before, we obtain

$$\frac{\partial\delta_{(a,\lambda)}}{\partial\nu}(x)-\frac{\partial}{\partial\nu}\Big(\frac{1}{\lambda^{\frac{n-2}{2}}\mid x-a\mid^{n-2}}\Big)=O\Big(\frac{1}{\lambda^{\frac{n+2}{2}}d(a,\Gamma_1)^{n+1}}\Big). \tag{4}$$

The proof follows from (2)–(4). □

Let h and q be positive integers such that $0\leq q\leq h$ and let $\varepsilon>0$. Define

$$\begin{aligned}V(h,q,\varepsilon) &= \Big\{u\in\Sigma,\ \text{s.t.},\ \exists\, a_1,\dots,a_q\in\Gamma_1,\ \exists\, a_{q+1},\dots,a_h\in\Omega,\ \exists\,\lambda_1,\ \dots,\lambda_h>\varepsilon^{-1}\text{ and}\\ &\quad\alpha_1,\dots,\alpha_h>0\text{ satisfiying }\Big\|u-\textstyle\sum_{i=1}^h\alpha_i P\delta_{(a_i,\lambda_i)}\Big\|<\varepsilon,\ \text{ with }\lambda_i d(a_i,\partial\Omega)>\varepsilon^{-1},\\ &\quad\forall\, i=q+1,\dots,h\text{ and }\varepsilon_{i,j}<\varepsilon\ \forall\, 1\leq i\neq j\leq h\Big\}.\end{aligned}$$

Here, $\varepsilon_{i,j}=\Big(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2\Big)^{\frac{-n-2}{2}}$.

As in [17], we parameterize the sets $V(h,q,\varepsilon)$ as follows. For $u\in V(h,q,\varepsilon)$, we consider the minimization problem

$$\begin{aligned}min\Big\{\ &\Big\|u-\textstyle\sum_{i=1}^q\alpha_i P\delta_{(a_i,\lambda_i)}-\sum_{i=q+1}^h\alpha_i P\delta_{(a_i,\lambda_i)}\Big\|,\ \text{ s.t.},\ a_1,\dots,a_q\in\Gamma_1,\ a_{q+1},\dots,a_h\in\Omega,\\ &\alpha_i>0,\ \lambda_i>\varepsilon^{-1},\ \ \forall\, i=1,\dots h\Big\}.\end{aligned}$$

Arguing as in ([17], Proposition 7), we have

Proposition 2. *Let $h \in \mathbb{N}$. There exists $\varepsilon_h > 0$ such that for any $\varepsilon \in (0, \varepsilon_h)$ and for any $u \in V(h, q, \varepsilon)$, $0 \le q \le h$, the above minimization problem admits a unique solution $(\overline{\alpha}, \overline{a}, \overline{\lambda})$ (up to permutation). Denoting*

$$v = u - \sum_{i=1}^{h} \overline{\alpha_i} P\delta_{(\overline{a_i}, \overline{\lambda_i})},$$

then v satisfies the following orthogonality condition,

$$(V_0):\ < v, \Psi >= \int_\Omega \nabla v \nabla \Psi = 0,\ \forall \Psi \in \left\{ P\delta_{(a_i,\lambda)},\ \frac{\partial P\delta_{(a_i,\lambda)}}{\partial \lambda},\ \frac{\partial P\delta_{(a_i,\lambda)}}{\partial a_i}\ \forall\ i = 1, ..., h \right\}$$

Following the concentration compactness principle of [6,16,17], we have the following result.

Proposition 3. *Assume that (1) has no solution. Let $(u_k)_k$ be a sequence of Σ^+ such that $J(u_k) \longrightarrow c$ and $\partial J(u_k) \longrightarrow 0$. Then there exist integers h and q, $h \ge 1$, $0 \le q \le h$ and a subsequence $(u_{k_\ell})_\ell$ of $(u_k)_k$ such that $u_{k_\ell} \in V(h, q, \varepsilon_\ell)$, $\forall\ \ell \ge 1$.*
Here, the sequence $(\varepsilon_\ell)_\ell$ is positive and tends to zero. Moreover,

$$J(u_{k_\ell}) \longrightarrow (2h - q)^{\frac{2}{n}} S, \ \text{as } \ell \longrightarrow \infty,$$

where S is a fixed constant given by

$$S = \left(\frac{\beta_n^{2^*}}{2} \int_{\mathbb{R}^n} \frac{dy}{(1 + |y|^2)^n} \right)^{\frac{2}{n}}. \tag{5}$$

Let us note that under the assumption that (1) has no solution,

$$\inf_{u \in \Sigma} J(u) = S;$$

see ([16], Lemma 3.5).

Fix $\tilde{\Sigma}$ be a compact set in Ω. For $h \in \mathbb{N}$ and $\lambda > 0$, we denote

$$B(h, \lambda, \tilde{\Sigma}) = \left\{ u = \frac{\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda_i)}}{\left\| \sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda_i)} \right\|},\ a_i \in \tilde{\Sigma},\ \ \alpha_i \in [0, 1],\ i = 1,\ ...,\ h \text{ and } \sum_{i=1}^{h} \alpha_i = 1 \right\}.$$

For $u = \frac{\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda_i)}}{\left\| \sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda_i)} \right\|} \in B(h, \lambda, \tilde{\Sigma})$, we may assume that $a_i \ne a_j$, $\forall\ 1 \le i \ne j \le h$, (if not, $u \in B(h-1, \lambda, \tilde{\Sigma})$). Denote $da = \min_{1 \le i \ne j \le h} |a_i - a_j|$. Then it holds

Proposition 4. *For any $u \in B(h, \lambda, \tilde{\Sigma})$, we have*

$$J(u) = 2^{\frac{2}{n}} S \frac{\sum_{i=1}^h \alpha_i^2}{\left(\sum_{i=1}^h \alpha_i^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}} \Big\{ 1 - \left(2S^{\frac{n}{2}} \right)^{-1} \frac{c_0}{\lambda^{n-2}} \Big[\sum_{i=1}^{h} \Big(\frac{\alpha_i^2}{\sum_{k=1}^h \alpha_k^2} - 2 \frac{\alpha_i^{\frac{2n}{n-2}}}{\sum_{k=1}^h \alpha_k^{\frac{2n}{n-2}}} \Big) H(a_i, a_i) -$$
$$\sum_{i \ne j} \Big(\frac{\alpha_i \alpha_j}{\sum_{k=1}^h \alpha_k^2} - 2 \frac{\alpha_i^{\frac{n+2}{n-2}} \alpha_j}{\sum_{k=1}^h \alpha_k^{\frac{2n}{n-2}}} \Big) \Big(\frac{1}{|a_i - a_j|^{n-2}} - H(a_i, a_j) \Big) \Big] \Big\} + O\Big(\frac{1}{(\lambda d_a)^{n-1}} \Big),$$

where S is defined in (5) and $c_0 = \beta_n^{\frac{n+2}{n-2}} \int_{\mathbb{R}^n} \frac{dz}{(1+|z|^2)^{\frac{n+2}{2}}}$.

Proof. Let us denote
$$J(u) = \frac{\|u\|^2}{\left(\int_\Omega u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}} = \frac{N}{D}.$$

We claim that
$$\begin{aligned} N \quad = \quad & \left(2S^{\frac{n}{2}} \sum_{i=1}^h \alpha_i^2\right)\left\{1 - \left(2S^{\frac{n}{2}} \sum_{i=1}^h \alpha_i^2\right)^{-1} \frac{c_0}{\lambda^{n-2}}\left[\sum_{i=1}^h \alpha_i^2 H(a_i, a_i) \right.\right. \\ & \left.\left. - \sum_{i \neq j} \alpha_i \alpha_j \left(\frac{1}{|a_i - a_j|^{n-2}} - H(a_i, a_j)\right)\right]\right\} + O(\frac{1}{(\lambda d_a)^{n-1}}). \end{aligned} \tag{6}$$

Indeed,
$$N = \sum_{i=1}^h \alpha_i^2 \int_\Omega |\nabla P\delta_{(a_i,\lambda)}|^2 dx + \sum_{i \neq j} \alpha_i \alpha_j \int_\Omega \nabla P\delta_{(a_i,\lambda)} \nabla P\delta_{(a_j,\lambda)} dx.$$

□

For any $i = 1, \ ..., \ h$, we have
$$\int_\Omega \left|\nabla P\delta_{(a_i,\lambda)}\right|^2 = \int_\Omega \nabla\delta_{(a_i,\lambda)} \nabla P\delta_{(a_i,\lambda)} = \int_\Omega \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}}(\delta_{(a_i,\lambda)} - \varphi_{(a_i,\lambda)}).$$

Observe that
$$\int_\Omega \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} = \int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} - \int_{\mathbb{R}^n\setminus\Omega} \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} = 2S^{\frac{n}{2}} + O\left(\frac{1}{\lambda^n}\right).$$

Moreover, by Proposition 1, we have
$$\begin{aligned} \int_\Omega \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \varphi_{(a_i,\lambda)} \quad &= \quad \int_\Omega \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}}\left(\frac{H(x,a_i)}{\lambda^{\frac{n-2}{2}}} + O\left(\frac{1}{\lambda^{\frac{n+2}{2}} d_1}\right)\right) \\ &= \quad \frac{1}{\lambda^{\frac{n-2}{2}}}\left[\int_{B(a_i,\rho/2)} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} H(x,a_i) + \int_{\Omega\setminus B(a_i,\rho/2)} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} H(x,a_i)\right] + O\left(\frac{1}{\lambda^n}\right). \end{aligned}$$

Expanding $H(x, a_i)$ around a_i, we obtain
$$\int_\Omega \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \varphi_{(a_i,\lambda)} \quad = \quad c_0 \frac{H(a_i,a_i)}{\lambda^{n-2}} + O\left(\frac{1}{\lambda^n}\right).$$

Therefore,
$$\int_\Omega \left|\nabla P\delta_{(a_i,\lambda)}\right|^2 = 2S^{\frac{n}{2}} - c_0 \frac{H(a_i,a_i)}{\lambda^{n-2}} + O\left(\frac{1}{\lambda^n}\right). \tag{7}$$

Moreover, for any $i \neq j$ we have
$$\begin{aligned} \int_\Omega \nabla P\delta_{(a_i,\lambda)} \nabla P\delta_{(a_j,\lambda)} \quad &= \quad \int_\Omega \delta_{(a_j,\lambda)}^{\frac{n+2}{n-2}}(\delta_{(a_i,\lambda)} - \varphi_{(a_i,\lambda)}) \\ &= \quad \int_{\mathbb{R}^n} \delta_{(a_j,\lambda)}^{\frac{n+2}{n-2}} \delta_{(a_i,\lambda)} - \int_{\mathbb{R}^n\setminus\Omega} \delta_{(a_j,\lambda)}^{\frac{n+2}{n-2}} \delta_{(a_i,\lambda)} - \int_\Omega \delta_{(a_j,\lambda)}^{\frac{n+2}{n-2}} \varphi_{(a_i,\lambda)}. \end{aligned}$$

Observe that
$$\int_{\mathbb{R}^n} \delta_{(a_j,\lambda)}^{\frac{n+2}{n-2}} \delta_{(a_i,\lambda)} \quad = \quad \frac{c_0}{\lambda^{n-2}|a_i - a_j|^{n-2}} + O\left(\frac{1}{(\lambda d_a)^{n-1}}\right),$$

$$\int_{\mathbb{R}^n\setminus\Omega} \delta_{(a_j,\lambda)}^{\frac{n+2}{n-2}} \delta_{(a_i,\lambda)} \leq \int_{\mathbb{R}^n\setminus\Omega} (\delta_{(a_j,\lambda)}^{\frac{2n}{n-2}} + \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}}) = O\left(\frac{1}{\lambda^n}\right),$$

and

$$\int_\Omega \delta_{(a_j,\lambda)}^{\frac{n+2}{n-2}} \varphi_{(a_i,\lambda)} = c_0 \frac{H(a_i,a_j)}{\lambda^{n-2}} + O\Big(\frac{1}{\lambda^n}\Big).$$

Hence,

$$\int_\Omega \nabla P\delta_{(a_i,\lambda)} \nabla P\delta_{(a_j,\lambda)} = \frac{c_0}{\lambda^{n-2}} \left(\frac{1}{|a_i - a_j|^{n-2}} - H(a_i,a_j) \right) + O\Big(\frac{1}{(\lambda d_a)^{n-1}}\Big). \tag{8}$$

Using (7) and (8), Claim (6) follows.
We now estimate the denomerator of $J(u)$. We claim that

$$\begin{aligned} D &= \Big(2S^{\frac{n}{2}} \textstyle\sum_{i=1}^h \alpha_i^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \Big\{ 1 - 2\Big(2S^{\frac{n}{2}} \textstyle\sum_{i=1}^h \alpha_i^{\frac{2n}{n-2}}\Big)^{-1} \frac{c_0}{\lambda^{n-2}} \Big[\textstyle\sum_{i=1}^h \alpha_i^{\frac{2n}{n-2}} H(a_i,a_i) - \\ &\quad \textstyle\sum_{i\neq j} \alpha_i^{\frac{n+2}{n-2}} \alpha_j \Big(\frac{1}{|a_i-a_j|^{n-2}} - H(a_i,a_j) \Big) \Big] \Big\} + O\Big(\frac{1}{(\lambda d_a)^{n-1}}\Big). \end{aligned} \tag{9}$$

Indeed, let $\rho_a > 0$ small enough such that for any $i = 1 \ldots,\ h$, $B_i := B(a_i,\rho_a) \subset \Omega$ and $B_i \cap B_j = \emptyset$, $\forall\ 1 \le i \ne j \le h$. We may assume $\rho_a = d_a$. We then have

$$\begin{aligned} D^{\frac{n}{n-2}} &= \int_\Omega \Big(\sum_{k=1}^h \alpha_k P\delta_{(a_k,\lambda)} \Big)^{\frac{2n}{n-2}} \\ &= \int_{\cup B_i} \Big(\sum_{k=1}^h \alpha_k P\delta_{(a_k,\lambda)} \Big)^{\frac{2n}{n-2}} + \int_{\Omega \setminus \cup B_i} \Big(\sum_{k=1}^h \alpha_k P\delta_{(a_k,\lambda)} \Big)^{\frac{2n}{n-2}}. \end{aligned}$$

Let $1 \le i \le h$. To estimate $I_i := \int_{B_i} \big(\sum_{k=1}^h \alpha_k P\delta_{(a_k,\lambda)} \big)^{\frac{2n}{n-2}} dx$, we write

$$\sum_{k=1}^h \alpha_k P\delta_{(a_k,\lambda)} = \alpha_i \delta_{(a_i,\lambda)} + \sum_{k\neq i} \alpha_k P\delta_{(a_k,\lambda)} - \alpha_i \varphi_{(a_i,\lambda)}$$

$$\begin{aligned} \Big(\sum_{k=1}^h \alpha_k P\delta_{(a_k,\lambda)} \Big)^{\frac{2n}{n-2}} &= \alpha_i^{\frac{2n}{n-2}} \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} + \frac{2n}{n-2} \alpha_i^{\frac{n+2}{n-2}} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \Big(\sum_{k\neq i} \alpha_k P\delta_{(a,\lambda)} - \alpha_i \varphi_{(a_i,\lambda)} \Big) \\ &\quad + O\Big(\frac{\lambda^{n-2}}{(\lambda d_a)^{2(n-2)}} \delta_{(a_i,\lambda)}^{\frac{4}{n-2}} \Big). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{B_i} \Big(\sum_{k=1}^h \alpha_k P\delta_{(a_k,\lambda)} \Big)^{\frac{2n}{n-2}} &= \alpha_i^{\frac{2n}{n-2}} \int_{B_i} \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} \\ &\quad + \frac{2n}{n-2} \alpha_i^{\frac{n+2}{n-2}} \int_{B_i} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \Big(\sum_{k\neq i} \alpha_k P\delta_{(a,\lambda)} - \alpha_i \varphi_{(a_i,\lambda)} \Big) + O\Big(\frac{1}{(\lambda d_a)^{n-1}} \Big). \end{aligned}$$

We now compute

$$\int_{B_i} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} P\delta_{(a_k,\lambda)} = \int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} P\delta_{(a_k,\lambda)} - \int_{B_i^c} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} P\delta_{(a_k,\lambda)}.$$

We have

$$\int_{B_i^c} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} P\delta_{(a_k,\lambda)} = O\Big(\frac{1}{(\lambda d_a)^{n-1}} \Big).$$

Using Proposition 1, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} P\delta_{(a_k,\lambda)} &= \int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \Big(\delta_{(a_k,\lambda)} - \frac{H(x,a_k)}{\lambda^{\frac{n-2}{2}}} + O\Big(\frac{1}{\lambda^{\frac{n+2}{2}}}\Big)\Big) \\ &= \int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \delta_{(a_k,\lambda)} - \frac{1}{\lambda^{\frac{n-2}{2}}} \int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} H(x,a_k) + O\Big(\frac{1}{\lambda^{\frac{n+2}{2}}} \int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}}\Big) \\ &= \frac{c_0}{\lambda^{n-2}}\left(\frac{1}{|a_i-a_j|^{n-2}} - H(a_i,a_j)\right) + O\Big(\frac{1}{\lambda^n}\Big).\end{aligned}$$

Therefore,

$$\int_{B_i} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} P\delta_{(a_k,\lambda)} = \frac{c_0}{\lambda^{n-2}}\left(\frac{1}{|a_i-a_j|^{n-2}} - H(a_i,a_j)\right) + O\Big(\frac{1}{(\lambda d_a)^{n-1}}\Big).$$

In addition,

$$\begin{aligned}\int_{B_i} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \varphi_{(a_i,\lambda)} &= \int_{B_i} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \Big(\frac{H(x,a_i)}{\lambda^{\frac{n-2}{2}}} + O\Big(\frac{1}{\lambda^{\frac{n+2}{2}}}\Big)\Big) \\ &= \frac{1}{\lambda^{\frac{n-2}{2}}} \int_{B_i} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} H(x,a_i) + O\Big(\frac{1}{\lambda^{\frac{n+2}{2}}} \int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}}\Big) \\ &= \frac{1}{\lambda^{\frac{n-2}{2}}}\Big[H(a_i,a_i)\Big(\int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} - \int_{B_i^c} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}}\Big) + O\Big(|x-a_i| \int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}}\Big)\Big] \\ &\quad + O\Big(\frac{1}{(\lambda d_a)^{n-1}}\Big) \\ &= \frac{1}{\lambda^{\frac{n-2}{2}}}\Big[c_0 \frac{H(a_i,a_i)}{\lambda^{\frac{n-2}{2}}} + O\Big(\frac{1}{\lambda^{\frac{n+2}{2}}}\Big) + O\Big(\frac{1}{\lambda^{\frac{n}{2}}}\Big)\Big] + O\Big(\frac{1}{(\lambda d_a)^{n-1}}\Big) \\ &= c_0 \frac{H(a_i,a_i)}{\lambda^{n-2}} + O\Big(\frac{1}{(\lambda d_a)^{n-1}}\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\int_{B_i} \Big(\sum_{k=1}^{h} \alpha_k P\delta_{(a_k,\lambda)}\Big)^{\frac{2n}{n-2}} &= \alpha_i^{\frac{2n}{n-2}} \int_{B_i} \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} - \frac{2n}{n-2} \alpha_i^{\frac{2n}{n-2}} c_0 \frac{H(a_i,a_i)}{\lambda^{n-2}} \\ &\quad + \frac{2n}{n-2} \frac{c_0}{\lambda^{n-2}} \sum_{k \neq i} \alpha_i^{\frac{n+2}{n-2}} \alpha_k \left(\frac{1}{|a_i-a_j|^{n-2}} - H(a_i,a_j)\right) \\ &\quad + O\Big(\frac{1}{(\lambda d_a)^{n-1}}\Big),\end{aligned}$$

and therefore,

$$\begin{aligned}D^{\frac{n}{n-2}} &= 2S^{\frac{n}{2}} \textstyle\sum_{i=1}^h \alpha_i^{\frac{2n}{n-2}} + \frac{2n}{n-2} \frac{c_0}{\lambda^{n-2}} \Big[\sum_{i=1}^h \alpha_i^{\frac{2n}{n-2}} H(a_i,a_i) \\ &\quad + \textstyle\sum_{i \neq j} \alpha_i^{\frac{n+2}{n-2}} \alpha_j \left(\frac{1}{|a_i-a_j|^{n-2}} - H(a_i,a_j)\right)\Big] + O\Big(\frac{1}{(\lambda d_a)^{n-1}}\Big).\end{aligned}$$

This concludes the proof of Claim (9). The expansion of Proposition 4 follows from (6) and (9).

We now prove the following Lemma

Lemma 4. *For any* $u = \frac{\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda_i)}}{\left\|\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda_i)}\right\|} \in B(h,\lambda,\tilde{\Sigma})$, *we have*

$$J(u) \leq \left\{ \frac{\int_\Omega (\sum_{i=1}^h \alpha_i \delta_{(a_i,\lambda)})^{\frac{2n}{n-2}}}{\int_\Omega (\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda)})^{\frac{2n}{n-2}}} \right\}^{\frac{n-2}{2n}} \left(\sum_{i=1}^h \int_\Omega \gamma_i \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}},$$

where

$$\gamma_i = \frac{\alpha_i \delta_{(a_i,\lambda)}}{\sum_{k=1}^h \alpha_k \delta_{(a_k,\lambda)}}.$$

Proof. Let $u = \frac{\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda)}}{\left\| \sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda)} \right\|} \in B(h,\lambda,\tilde{\Sigma})$. We have

$$J(u) = \frac{\int_\Omega |\nabla \sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda)}|^2 dx}{\left(\int_\Omega \left(\nabla \sum_{i=1}^h \alpha_i^{\frac{2n}{n-2}} P\delta_{(a_i,\lambda)} \right)^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}}.$$

Using Holder's inequality,

$$\int_\Omega |\nabla (\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda)})|^2 \leq \Big[\int_\Omega \Big(\sum_{i=1}^h \alpha_i \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \Big)^{\frac{2n}{n+2}} \Big]^{\frac{n+2}{2n}} \Big[\int_\Omega \Big(\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2}} \Big]^{\frac{n-2}{2n}}.$$

Observe that

$$\begin{aligned} \int_\Omega \Big(\sum_{i=1}^h \alpha_i \delta_{(a_i,\lambda)}^{\frac{n+2}{n-2}} \Big)^{\frac{2n}{n+2}} &\leq \Big[\int_\Omega \Big(\sum_{i=1}^h \alpha_i \delta_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2}} \Big]^{\frac{n-2}{n+2}} \Big[\int_\Omega \Big(\sum_{i=1}^h \gamma_i \delta_{(a_i,\lambda)}^{\frac{8n}{n^2-4}} \Big)^{\frac{n+2}{4}} \Big]^{\frac{4}{n+2}} \\ &\leq \Big[\int_\Omega \Big(\sum_{i=1}^h \alpha_i \delta_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2}} \Big]^{\frac{n-2}{n+2}} \Big[\sum_{i=1}^h \int_\Omega \gamma_i \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} \Big]^{\frac{4}{n+2}}. \end{aligned}$$

It follows that

$$\int_\Omega |\nabla (\sum_{i=1}^h \alpha_i P_{(a_i,\lambda)}^{\delta})|^2 \leq \Big[\int_\Omega \Big(\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2}} \Big]^{\frac{n-2}{2n}} \Big[\int_\Omega \Big(\sum_{i=1}^h \alpha_i \delta_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2}} \Big]^{\frac{n-2}{2n}} \Big[\sum_{i=1}^h \int_\Omega \gamma_i \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} \Big]^{\frac{2}{n}}.$$

This concludes the proof. □

Proposition 5. *Let $h \in \mathbb{N}$. For any $\varepsilon > 0$, there exists $\lambda(h,\varepsilon) > 0$ such that for every $\lambda > \lambda(h,\varepsilon)$, we have*

$$J(B(h,\lambda,\tilde{\Sigma})) \subset (S, (2h+\varepsilon)^{\frac{2}{n}} S).$$

Proof. By Lemma 4, we know that for any $u \in B(h,\lambda,\tilde{\Sigma})$, we have

$$J(u) \leq \left\{ \frac{\int_\Omega (\sum_{i=1}^h \alpha_i \delta_{(a_i,\lambda)})^{\frac{2n}{n-2}}}{\int_\Omega (\sum_{i=1}^h \alpha_i P\delta_{(a_i,\lambda)})^{\frac{2n}{n-2}}} \right\}^{\frac{n-2}{2n}} \left(\sum_{i=1}^h \int_\Omega \gamma_i \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}}.$$

Observe that

$$\begin{aligned} \int_\Omega \Big(\sum_{i=1}^h \int_\Omega \gamma_i \delta_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2}} dx &\leq \sum_{i=1}^h \int_\Omega \gamma_i \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} dx \\ &\leq 2hS^{\frac{n}{2}} + \sum_{i=1}^h \int_{\Omega^c} \gamma_i \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} dx, \end{aligned}$$

since

$$\int_{\mathbb{R}^n} \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} dx = 2S^{\frac{n}{2}}.$$

Using the fact that

$$\int_{\Omega^c} \delta_{(a_i,\lambda)}^{\frac{2n}{n-2}} dx = O(\frac{1}{\lambda^n}),\ \forall\, i = 1,\ ..., h,$$

we obtain

$$\int_\Omega \Big(\sum_{i=1}^{h} \int_\Omega \gamma_i \delta_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2}} dx \leq 2hS^{\frac{n}{2}} \Big(1 + O(\frac{1}{\lambda^n}) \Big).$$

Therefore,

$$J(u) \leq (2h)^{\frac{2}{n}} S \left\{ \frac{\int_\Omega (\sum_{i=1}^{h} \alpha_i \delta_{(a_i,\lambda)})^{\frac{2n}{n-2}}}{\int_\Omega (\sum_{i=1}^{h} \alpha_i P\delta_{(a_i,\lambda)})^{\frac{2n}{n-2}}} \right\}^{\frac{n-2}{2n}} (1 + O(\frac{1}{\lambda^n})). \tag{10}$$

Using the estimate of Proposition 1, we obtain

$$J(u) \leq (2h)^{\frac{2}{n}} S(1 + O(\frac{1}{\lambda^{\frac{n-2}{2}}})). \tag{11}$$

Let $\varepsilon > 0$. for λ large enough, the inclusion of Proposition 5 is valid. □

The above expansion can be improved for h large enough. Namely,

Proposition 6. *There exists $h_0 \in \mathbb{N}$ and $\lambda(h_0) > 0$ such that for any $h \geq h_0$ and $\lambda > \lambda(h_0)$,*

$$J(B(h, \lambda, \tilde{\Sigma})) \subset (S, (2h)^{\frac{2}{n}} S).$$

Proof. Let $u = \frac{\sum_{i=1}^{h} \alpha_i P\delta_{(a_i,\lambda_i)}}{\left\| \sum_{i=1}^{h} \alpha_i P\delta_{(a_i,\lambda_i)} \right\|} \in B(h, \lambda, \tilde{\Sigma})$. We distinguish two cases.
Case 1. Assume that there exists $i,\ 1 \leq i \leq h$ such that $\alpha_i << 1$. By elementary computation, we obtain

$$J(u) \leq J(\sum_{j=1; j\neq i}^{h} \alpha_j (1-\alpha_i)^{-1} P\delta_{(a_j,\lambda)}) + O(\alpha_i).$$

We know from Proposition 5 that

$$J(\sum_{j=1; j\neq i}^{h} \alpha_j (1-\alpha_i)^{-1} P\delta_{(a_j,\lambda_j)}) \leq (2(h-1) + \varepsilon)^{\frac{2}{n}} S.$$

Thus, for α_i small enough, we have

$$J(u) \leq (2h)^{\frac{2}{n}} S.$$

Case 2. Assume that $\alpha_i,\ i = 1,\ ...,\ h$ is lower bounded by a fixed positive constant. If d_a is small enough, using expansion of Proposition 4, we obtain

$$J(u) \leq (2h)^{\frac{2}{n}} S.$$

If d_a is lower bounded by a fixed positive constant, we deduce from Proposition 4 the existence of three positive constants c, $\tilde{c_1}$ and $\tilde{c_2}$ such that

$$J(u) \leq (2h)^{\frac{2}{n}} S\Big(1 - \frac{c}{\lambda^{n-2}}(\tilde{c_1}h - \tilde{c_2})\Big) + O(\frac{1}{\lambda^{n-1}}).$$

Thus, for h_0 such that $\tilde{c_1}h_0 - \tilde{c_2} > 0$ and for λ large enough, we have

$$J(u) \leq (2h)^{\frac{2}{n}} S.$$

This finishes the proof. □

4. Topological Arguments

In this section, we extend the topological approach of Bahri–Coron [17] to the framework of mixed boundary problems. Due to the effect of boundary blow-up points, the same techniques cannot be applied in the present setting, and therefore new constructions and extra ideas will be required. That is what we will do in this section. We think that such an approach will be helpful to prove Theorem 1. This is subject of the forthcoming paper [18]. Nevertheless, it leads to prove Theorem 1 under an additional topological condition; see Theorem 3 below.

Assume that (1) has no solution. Under the assumption of Theorem 3, we construct a sequence of maps of topological pairs in Σ^+ which induces a sequences of non trivial homomorphisms of relative homological groups. However, by using the asymptotic expansions of Section 3, we prove that the induced homomorphisms sequence is trivial from a certain rank. This leads to a contradiction.
First, let us introduce the gradient vector field of the functional J, and it follows that will be used to deform the level sets of J. Let

$$-\partial J : \Sigma \longrightarrow T\Sigma$$

be the gradient field of J, and let

$$\eta(.,.) : [0, \infty) \times \Sigma \longrightarrow \Sigma$$

be the associated flow. For any $u \in \Sigma$, $t \longmapsto \eta(t, u)$ is the unique solution of

$$\begin{cases} \dot{\eta}(t,u) = -\partial J(\eta(t,u)), \\ \eta(0,u) = u. \end{cases}$$

A direct computation shows that J decreases a long $\eta(t, u)$, $\|\partial J(\eta(t, u))\| \longrightarrow 0$, as $t \longrightarrow \infty$ and if $u \in \Sigma^+$, then $\eta(t, u) \in \Sigma^+, \forall t \geq 0$.

Let $u \in \Sigma^+$. It follows from Proposition 3 that under the assumption that J has no critical point in Σ^+, there exists a unique positive integer $h = h(u)$ and a unique integer $q = q(u)$, such that $0 \leq q \leq h$, so that the following holds: For any $\varepsilon > 0$, and there exists $t_\varepsilon > 0$ such that for any $t > t_\varepsilon$, $\eta(t, u) \in V(h, q, \varepsilon)$. Consequently,

$$J(\eta(t,u)) \longrightarrow (2h - q)^{\frac{2}{n}} S, \text{ as } t \longrightarrow \infty.$$

Here, q represents the number of concentation points of $\eta(t, u)$ that lie in the boundary part Γ_1 and S is defined in (5). The levels $(2h - q)^{\frac{2}{n}} S$, $h \geq 1$, $0 \leq q \leq h$, are called critical values at infinity.

Let $c > 0$. We define

$$J_c = \{u \in \Sigma^+, J(u) \leq c\}.$$

Using the classical deformation lemma, we have

Proposition 7. *Assume that J has no critical point in Σ^+. Let h be a positive integer and let q be an integer such that $0 \le q \le h$. For any two real numbers c_1 and c_2 such that*

$$(2h-q-1)^{\frac{2}{n}} S < c_1 \le c_2 \le (2h-q)^{\frac{2}{n}} S,$$

we have

$$J_{c_2} \simeq J_{c_1},$$

where $\simeq$ denotes retract by deformation.

Proof. We use the gradient flow $\eta(.,.)$ to deform J_{c_2} onto J_{c_1}. Since J decreases along $\eta(.,.)$ and J has no critical values nor critical values at infinity in (c_1, c_2), then $J_{c_2} \simeq J_{c_1}$. □

For any $h \ge 1$, let ε_h be a fixed positive constant subjected to Proposition 2.

Proposition 8. *Assume that J has no critical points in Σ^+. For any $h \ge 1$, there exists a fixed constant $\delta_h > 0$ such that if a flow line $\eta(t,u)$ moves from $V(h,q,\frac{\varepsilon_h}{2})$ to $V(h,q,\varepsilon_h)^c$, then $J(\eta(t,u))$ decreases by at least δ_h. Here $0 \le q \le h$.*

Proof. Assume that there exists $t_1 < t_2$ such that $\eta(t_1,u) \in V(h,q,\frac{\varepsilon_h}{2})$, $\eta(t_2,u) \in V(h,q,\varepsilon_h)^c$, and $\eta(t,u) \in V(h,q,\varepsilon_h) \backslash V(h,q,\frac{\varepsilon_h}{2})$, $\forall t \in (t_1,t_2)$. It follows from Propostion 3 that there exists $\alpha_{\varepsilon_h} > 0$ such that

$$\|\partial J(\eta(t,u))\| \ge \alpha_{\varepsilon_h}, \quad \forall t \in (t_1,t_2).$$

Moreover, by estimate (119) of [23], we know that there exists $\beta_{\varepsilon_h} > 0$ such that

$$d\Big(V(h,q,\varepsilon_h)^c, V(h,q,\frac{\varepsilon_h}{2})\Big) \ge \beta_{\varepsilon_h}.$$

Thus,

$$\begin{aligned} J\big(\eta(t_2,u)\big) - J\big(\eta(t_1,u)\big) &= -\int_{t_1}^{t_2} \|\partial J(\eta(t,u))\|^2 dt \\ &\le -\alpha_{\varepsilon_h} \int_{t_1}^{t_2} \|\partial J(\eta(t,u))\| dt \\ &\le -\alpha_{\varepsilon_h} \beta_{\varepsilon_h}, \end{aligned}$$

since

$$\beta_{\varepsilon_h} \le d\big(\eta(t_2,u), \eta(t_1,u)\big) \le \int_{t_1}^{t_2} \|\partial J(\eta(t,u))\| dt.$$

The result follows for $\delta_h = \alpha_{\varepsilon_h} \beta_{\varepsilon_h}$. □

Next, we shall use the following notations. Let

$$V_\Omega(h,\varepsilon_h) = V(h,0,\varepsilon_h)$$

and

$$V_{\Gamma_1}(h,\varepsilon_h) = V(h,q,\varepsilon_h), \text{ with } q \ne 0.$$

Proposition 9. *Assume that J has no critical point in Σ^+. Let $u \in V_\Omega(h,\frac{\varepsilon_h}{2})$. If there exixts a positive time t_1 such that $\eta(t_1,u) \in V_{\Gamma_1}(h',\varepsilon_{h'})$, for some positive integer h', then $J\big(\eta(t,u)\big)$ decreases by at least δ_h, where δ_h is the given constant of Propostion 8.*

Proof. Let $u = \sum_{i=1}^h P\delta_{(a_i,\lambda)} + v \in V_\Omega(h,\frac{\varepsilon_h}{2})$. Before the time t_1 at which $\eta(t_1,u) \in V_{\Gamma_1}(h',\varepsilon_{h'})$, the flow line $\eta(t,u)$ has to leave $V_\Omega(h,\varepsilon_h)$, since at $t = 0$, all the indices $i, 1 \le i \le h$, satisfy $\lambda(0) d\big(a_i(0), \partial\Omega\big) > \left(\frac{\varepsilon_h}{2}\right)^{-1}$, and at $t = 1$, there exists at least an index i, $1 \leq i \leq h'$, satisfying $\lambda(t_1)d\big(a_i(t_1), \partial\Omega\big) < \varepsilon_{h'}$. Therefore, the flow line $\eta(t,u)$ moves from $V(h,0,\frac{\varepsilon_h}{2})$ to $V(h,0,\varepsilon_h)^c$. The result follows from Proposition 8. □

For any $h \geq 1$, we fix

$$\gamma_h = \frac{1}{4}\min\Big(\delta_h,\ S\big((2h+1)^{\frac{2}{n}} - (2h)^{\frac{2}{n}}\big)\Big).$$

From Proposition 3, we know that for any $h \geq 1$, there exists N_h large enough such that

$$J(u) \leq (2h)^{\frac{2}{n}} S + \gamma_h,\ \forall u \in V_\Omega(h, \frac{\varepsilon_h}{N_h}).$$

For simplicity, we shall denote $W_u\Big(V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big)$ the unstable manifold of $\underset{1\leq p\leq h}{\cup} V_\Omega(p,\frac{\varepsilon_p}{N_p})$ with respect to the gradient flow. More precisely,

$$W_u\Big(V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big) = \Big\{\eta(t,u),\ \text{ s.t., } t \geq 0 \text{ and } u \in \underset{1\leq p\leq h}{\cup} V_\Omega(p,\frac{\varepsilon_p}{N_p})\Big\}.$$

Define

$$W_h = J_{(2h)^{\frac{2}{n}} S + \gamma_h} \cap W_u\Big(V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big),$$

and

$$A_h = J_{(2h)^{\frac{2}{n}} S} \cap W_u\Big(V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big).$$

In this way it is easy to verify that

$$...A_{h-1} \subset W_{h-1} \subset A_h \subset W_h... ,$$

with $A_0 = W_0 = \emptyset$. We now prove the following result.
Let $M_2 \subset M_1$ be two topological spaces. We denote $H_*(M_1, M_2)$, $* \geq 0$, the homology group of order $*$ of the pair (M_1, M_2).

Proposition 10. *Assume that (1) has no solution. For any $h \geq 1$, there exists $\varepsilon_1(h) > 0$ so that the following holds. For any $\bar{\varepsilon}_0 \in (0, \varepsilon_1(h))$ there exists a continious map*

$$r : (W_h, A_h) \longrightarrow \Big(J_{(2h)^{\frac{2}{n}} S + \bar{\varepsilon}_0} \cap V_\Omega(h,\frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}} S} \cap V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big),$$

which induces an isomorphism

$$r^* : H_*(W_h, A_h) \longrightarrow H_*\Big(J_{(2h)^{\frac{2}{n}} S + \bar{\varepsilon}_0} \cap V_\Omega(h,\frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}} S} \cap V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big),\ * \geq 0.$$

Proof. If we assume that J has no critical point in Σ^+, it follows from Proposition 7 that for any $\bar{\varepsilon} \in (0, \gamma_h)$,

$$(W_h, A_h) \simeq (J_{(2h)^{\frac{2}{n}} S + \bar{\varepsilon}} \cap W_u\Big(V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big), A_h).$$

Let

$$r_1 : (W_h, A_h) \longrightarrow (J_{(2h)^{\frac{2}{n}} S + \bar{\varepsilon}} \cap W_u\Big(V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big), A_h)$$

be the associated deformation retract. Using Proposition 3, the following holds.
For any $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ such that for any $\bar{\varepsilon} \in (0, \varepsilon_1)$,

$$J_{(2h)^{\frac{2}{n}} S + \bar{\varepsilon}} \cap W_u\Big(V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big) \backslash J_{(2h)^{\frac{2}{n}} S} \cap W_u\Big(V_\Omega(h,\frac{\varepsilon_h}{N_h})\Big) \subset V_\Omega(h, \varepsilon).$$

Particularly, for $\varepsilon = \frac{\varepsilon_h}{N_h}$, there exists $\varepsilon_1(h) > 0$ such that for any $\bar{\varepsilon}_0 \in (0, \varepsilon_1(h))$, we have

$$J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap W_u\Big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big) \backslash J_{(2h)^{\frac{2}{n}}S} \cap W_u\Big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big) \subset V_\Omega(h, \frac{\varepsilon_h}{N_h}).$$

Note that for given sets A_1, A_2, B and A such that

$$A_1 \cap B \backslash A_2 \cap B \subset A, \text{with } A \subset B,$$

then,

$$A_1 \cap B \backslash A_2 \cap B = A_1 \cap A \backslash A_2 \cap A.$$

We apply this in our statement. It results that the pairs

$$\Bigg(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap W_u\Big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big), J_{(2h)^{\frac{2}{n}}S} \cap W_u\Big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big)\Bigg)$$

and

$$\Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}}S} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big).$$

are homotopical equivalent, since $V_\Omega(h, \frac{\varepsilon_h}{N_h}) \subset W_u\Big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big)$. Let

$$r_2 : \Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap W_u\Big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big), J_{(2h)^{\frac{2}{n}}S} \cap W_u\Big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big)\Big) \longrightarrow$$
$$\Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}}S} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big)$$

be the associated homotopy equivalence. We define

$$r = r_2 \circ r_1.$$

Using the fact that r_1 and r_2 two homotopy equivalences, r^* is then an isomorphism. □

Let Ω be a bounded domain of $\mathbb{R}^n, n \geq 3$, satisfaying the condition of Theorem 1. Then, there exists at least an $(n-1)-$dimensional sphere $\tilde{\Sigma}$ included in Ω such that, if we denote $i : \tilde{\Sigma} \hookrightarrow \Omega$ the natural injection, then the induced homorphism of groups $i^* : H_*(\tilde{\Sigma}) \hookrightarrow H_*(\Omega)$ is not null for $* = n-1$. Here, $H_*(M), * \geq 0$, denotes the homology groups of a topological space M.

Let us introduce the following notations. For $h \geq 1$ we denote

$$\Delta_{h-1} = \Big\{(\alpha_1, ..., \alpha_h), s.t., \alpha_i \in [0,1], \forall i = 1, ..., h \text{ and } \sum_{i=1}^{n} \alpha_i = 1\Big\}$$

and

$$B_h(\tilde{\Sigma}) = \Big\{\sum_{i=1}^{n} \alpha_i \delta_i, s.t., (\alpha_1, ..., \alpha_h) \in \triangle_{h-1} \text{ and } (a_1, ..., a_h) \in \tilde{\Sigma}^h\Big\}.$$

Here, δ_{a_i} is the Dirac distribution at a_i.

In the following two Propositions, we construct a sequence of non trivial homomorphisms ϕ_h^*,

$h \geq 1$, between the relative homological groups $H_*\big(B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big)$ and $H_*(W_h, A_h)$.

Proposition 11. *Assume that (1) has no solution. For any positive integer h, the homology group $H_*(W_h, A_h), * \geq 1$, has a structure of a module over the cohomology group $H^*(\Omega^h/_{\mathfrak{S}_h})$, where $\mathfrak{S}_h$ is the permutation group of order h. Moreover, there exists a sequence of homomorphisms*

$$\phi_h^* : H_*\big(B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big) \longrightarrow H_*(W_h, A_h),$$

such that for any $h \geq 1$, ϕ_h^* is $H^*(\Omega^h/_{\mathfrak{S}_h})-$linear.

Proof. Let $h \geq 1$. Define the projection

$$\chi : V_\Omega(h, \varepsilon_h) \longrightarrow \Omega^h/_{\mathfrak{S}_h}$$

$$u = \sum_{i=1}^{n} \alpha_i P\delta_{(a_i,\lambda)} + v \longmapsto (a_1, ..., a_h)$$

Let

$$\chi_1 := \chi_{|_{J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0}} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})}$$

be the restriction of χ on $J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})$. The mapping χ_1 induces a homomorphism

$$(\chi_1)_* : H^*(\Omega^h/_{\mathfrak{S}_h}) \longrightarrow H^*\Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big).$$

The cap product, see [24],

$$H^*\Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big) \otimes H_*\Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}}S} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big) \longrightarrow$$

$$H_*\Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}}S} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big)$$

provides $H_*\Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}}S} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big)$ a structure of a module over $H^*\Big(J_{(2h)^{\frac{2}{n}}S+\bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big)$ and therefore over $H^*(\Omega^h/_{\mathfrak{S}_h})$ through the homomorphism $(\chi_1)_*$. Using now the isomorphism r^* given in Proposition 10, $H_*(W_h, A_h)$ obtains the structure of an $H^*(\Omega^h/_{\mathfrak{S}_h})$ module.

We now construct the required sequence of homomorphisms ϕ_h^*.

For $h \geq 1$, we define an equivalence relation $\sim$ on $\tilde{\Sigma}^h \times \Delta_{h-1}$ by

$$\big((a_1, ..., a_h), (\alpha_1, ..., \alpha_h)\big) \sim \big((a_{\sigma(1)}, ..., a_{\sigma(h)}), (\alpha_{\sigma(1)}, ..., \alpha_{\sigma(h)})\big),\ \sigma \in \mathfrak{S}_h.$$

$B_h(\tilde{\Sigma})$ can be viewed as the quotient space $\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}$ of $\tilde{\Sigma}^h \times \Delta_{h-1}$ with respect to $\sim$.

Define

$$\pi_h : B_h(\tilde{\Sigma}) \longrightarrow \tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1},\ \sum_{i=1}^{n} \alpha_i \delta_{a_i} \mapsto \big((a_1, ..., a_h), (\alpha_1, ..., \alpha_h)\big).$$

Let

$$S_h = \Big\{(a_1, ..., a_h) \in \tilde{\Sigma}^h,\ \text{ s.t., } a_i = a_j, \text{ for some } i \neq j\Big\}.$$

Following [24], there exits a $\mathfrak{S}_h-$equivariant tubular neighborhood T_h of S_h in $\tilde{\Sigma}^h$ such that $\overline{T}_h$ retracts by deformation on S_h. The above projection π_h induces a map of pairs denoted again π_h,

$$\pi_h : \big(B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big) \longrightarrow \big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, S_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times \partial\Delta_{h-1}\big).$$

It is easy to see that π_h is an homeomorphism and therefore induces an isomorphism

$$\pi_h^* : H_*\big(B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big) \longrightarrow H_*\big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, S_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times \partial\Delta_{h-1}\big).$$

Let

$$i_h : \big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, S_h \times \triangle_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times \partial\Delta_{h-1}\big) \longrightarrow \big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \overline{T}_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times \partial\Delta_{h-1}\big)$$

be the natural injection. Using the fact that S_h is a strong deformation retract of $\overline{T}_h$, i_h defines an homotopy equivalence and hence induces an isomorphism

$$i_h^* : H_*\big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, S_h \times \triangle_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times \partial\Delta_{h-1}\big) \longrightarrow H_*\big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \overline{T}_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times \partial\Delta_{h-1}\big).$$

Let us denote

$$\tilde{\Sigma}_0^h = \tilde{\Sigma}^h \setminus T_h,$$

and for $\theta \in (0,1)$, we also denote

$$\Delta_{h-1}^\theta = \Big\{(\alpha_1, ..., \alpha_h) \in \Delta_{h-1}, s.t., \frac{\alpha_i}{\alpha_j} \in [1-\theta, 1+\theta], \forall i \neq j\Big\}.$$

We note that $(\Delta_{h-1}^\theta)^c$ retracts by deformation on $\partial\Delta_{h-1}$. As a consequence there exists an isomorphism

$$j_h^* : H_*\big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \overline{T}_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times \partial\Delta_{h-1}\big) \longrightarrow H_*\big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \overline{T}_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times (\Delta_{h-1}^\theta)^c\big).$$

By excision of $T_h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} T_h \times (\Delta_{h-1}^\theta)^c$, we get the existence of an isomorphism

$$\varphi_h^* : H_*(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \overline{T}_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times (\Delta_{h-1}^\theta)^c) \longrightarrow H_*(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \partial\tilde{\Sigma}_0^h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times (\Delta_{h-1}^\theta)^c),$$

since $T_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} T_h \times (\Delta_{h-1}^\theta)^c$ is open in $\overline{T}_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times (\Delta_{h-1}^\theta)^c$.

Let λ be a large positive constant. We define

$$f_{\lambda,h} : \tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1} \longrightarrow \Sigma^+$$

$$\Big((a_1, ..., a_h), (\alpha_1, ..., \alpha_h)\Big) \mapsto \frac{\sum_{i=1}^n \alpha_i P\delta_{(a_i,\lambda)}}{\| \sum_{i=1}^n \alpha_i P\delta_{(a_i,\lambda)} \|}.$$

Using expansions of Propositions 5 and 6, we can select $\theta \in (0,1)$ and a diameter $d(T_h)$ of T_h small enough so that for λ large enough, we have

$$f_{\lambda,h}(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}) \subset J_{(2h)^{\frac{2}{n}} S + \overline{\varepsilon_0}},$$

and

$$f_{\lambda,h}\big(\partial\tilde{\Sigma}_0^h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}_0^h \times (\Delta_{h-1}^\theta)^c\big) \subset J_{(2h)^{\frac{2}{n}} S}.$$

In addition, using the fact that $| a_i - a_j | \geq d(T_h), i \neq j$, for any $(a_1, ..., a_h) \in \tilde{\Sigma}_0^h \cup \partial\tilde{\Sigma}_0^h$, we obtain for λ large,

$$f_{\lambda,h}(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}) \subset \underset{1 \leq p \leq h}{\cup} V_\Omega(p, \frac{\varepsilon_p}{N_p}) \subset W_u\big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\big),$$

and

$$f_{\lambda,h}\big(\partial\tilde{\Sigma}_0^h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}_0^h \times (\Delta_{h-1}^\theta)^c\big) \subset \underset{1 \leq p \leq h}{\cup} V_\Omega(p, \frac{\varepsilon_p}{N_p}) \subset W_u\big(V_\Omega(h, \frac{\varepsilon_h}{N_h})\big).$$

Therefore, for $\bar{\varepsilon}_0 \in (0, \gamma_h)$, $f_{\lambda,h}$ defines a map denoted again $f_{\lambda,h}$,

$$f_{\lambda,h} : \Big(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \partial\tilde{\Sigma}_0^h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}_0^h \times (\Delta_{h-1}^\theta)^c\Big) \longrightarrow \big(W_h, A_h\big).$$

Thus, it induces an homomorphism

$$f_{\lambda,h}^* : H_*\Big(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \partial\tilde{\Sigma}_0^h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}_0^h \times (\Delta_{h-1}^\theta)^c\Big) \longrightarrow H_*\big(W_h, A_h\big).$$

The required sequence of homomorphism is

$$\Phi_h^* = f_{\lambda,h}^* \circ \varphi_h^* \circ j_h^* \circ i_h^* \circ \pi_h^*, \; h \geq 1.$$

To prove that Φ_h^* defines a $H^*(\Omega^h/_{\mathfrak{S}_h})-$linear map. We consider the following commutative diagram, analogue to the one of ([17], diagram (17)). Let

$$a_h : \tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1} \longrightarrow \Omega^h/_{\mathfrak{S}_h}$$

be the first projection. Using expansions of Propositions 5 and 6, it is easy to check that $f_{\lambda,h}$ maps

$$\Big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \overline{T}_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times (\Delta_{h-1}^\theta)^c\Big) \text{ onto } (W_h, A_h).$$

Moreover, using the fact that $\mid a_i - a_j \mid \geq d(T_h), i \neq j$, for any $(a_1, ..., a_h) \in \tilde{\Sigma}_0^h \cup \partial\tilde{\Sigma}_0^h$, we obtain for λ large,

$$f_{\lambda,h}(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}^\theta) \subset V_\Omega(h, \frac{\varepsilon_h}{N_h})$$

and

$$f_{\lambda,h}(\partial\tilde{\Sigma}_0^h \times \Delta_{h-1}^\theta \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}_0^h \times \partial\Delta_{h-1}^\theta) \subset V_\Omega(h, \frac{\varepsilon_h}{N_h}).$$

Thus, $f_{\lambda,h}$ defines a map denoted again $f_{\lambda,h}$,

$$f_{\lambda,h} : \Big(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}^\theta, \partial\tilde{\Sigma}_0^h \times \Delta_{h-1}^\theta \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}_0^h \times \partial\Delta_{h-1}^\theta\Big) \longrightarrow \Big(J_{(2h)^{\frac{2}{n}} S + \bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}} S} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big). \tag{12}$$

Consider the following diagram:

$$\begin{array}{ccc}
\Big(\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}\overline{T}_h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}^h \times (\Delta_{h-1}^\theta)^c\Big) & \xrightarrow{f_{\lambda,h}} & (W_h, A_h) \\
\Big\downarrow i_1 & & \Big\downarrow i_2 \\
\Big(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}^\theta, \partial\tilde{\Sigma}_0^h \times \Delta_{h-1}^\theta \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}_0^h \times \partial\Delta_{h-1}^\theta\Big) & \xrightarrow{f_{\lambda,h}} & \Big(J_{(2h)^{\frac{2}{n-2}} \frac{S}{2n} + \bar{\varepsilon}_0} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h}), J_{(2h)^{\frac{2}{n}} S} \cap V_\Omega(h, \frac{\varepsilon_h}{N_h})\Big) \\
\Big\downarrow i_3 & & \Big\downarrow \chi \\
\tilde{\Sigma}^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1} & \xrightarrow{a_h} & \Omega^h/_{\mathfrak{S}_h},
\end{array} \tag{13}$$

where i_1, i_2, and i_3 are the natural injection. It is easy to verify that the above diagram is commutative. Moreover, i_1^* and i_2^* are two isomorphisms. Thus we are in the same position of diagram (17) of [17]. The $H^*(\Omega^h|_{\mathfrak{S}_h})-$linearity of Φ_h^* follows from the same argument of ([17], Proposition 9). $\square$

We now prove the following result:

Proposition 12. *Assume that (1) has no solution and assume condition (A) below. Then,*

$$\Phi_h^*\Big(\big[B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big]\Big) \neq 0,\ \forall\ h \geq 1 \text{ and } * = nh-1.$$

Proof. From the construction of the proof of Proposition 11,

$$k_h^* := \varphi_h^* \circ j_h^* \circ i_h^* \circ \pi_h^*$$

defines an isomorphism:

$$k_h^* : H_*\big(B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big) \longrightarrow H_*\Big(\tilde{\Sigma}_0^h \times \Delta_{h-1}, \partial\tilde{\Sigma}_0^h \times \Delta_{h-1} \underset{\mathfrak{S}_h}{\cup} \tilde{\Sigma}_0^h \times (\Delta_{h-1}^\theta)^c\Big).$$

Using the fact that $(\Delta_{h-1}^\theta)^c$ retracts by deformation on $\partial\Delta_{h-1}$, k_h^* induces an isomorphism denoted again

$$k_h^* : H_*\big(B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big) \longrightarrow H_*\Big(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \partial(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1})\Big).$$

Observe that $\tilde{\Sigma}_0^h \times \Delta_{h-1}$ is a manifold of dimension $nh-1$ with boundary. Therefore,

$$[\tilde{\Sigma}_0^h \times \Delta_{h-1}, \partial(\tilde{\Sigma}_0^h \times \Delta_{h-1})],$$

defines a non-zero class in $H_*\Big(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \partial(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1})\Big)$ for $* = nh-1$.

Denote

$$\big[B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big] = k_h^{*-1}\Big(\big[\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1}, \partial(\tilde{\Sigma}_0^h \underset{\mathfrak{S}_h}{\times} \Delta_{h-1})\big]\Big)$$

and

$$\partial_1 : H_*\big(B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big) \longrightarrow H_{*-1}\big(B_{h-1}(\tilde{\Sigma}), B_{h-2}(\tilde{\Sigma})\big)$$

are the usual connecting homomorphism. We introduce the following topological condition.
(A): Assume that there exists a connecting homomorphism

$$\partial_2 : H_*(W_h, A_h) \longrightarrow H_{*-1}(W_{h-1}, A_{h-1})$$

such that the following diagram is commutative

$$\begin{array}{ccc}
H_*\big(B_h(\tilde{\Sigma}),B_{h-1}(\tilde{\Sigma})\big) & \xrightarrow{\Phi_h^*} & H_*(W_h,A_h) \\
\Big\downarrow \partial_1 & & \Big\downarrow \partial_2 \\
H_*\big(B_{h-1}(\tilde{\Sigma}), B_{h-2}(\tilde{\Sigma})\big) & \xrightarrow{\Phi_{h-1}^*} & H_{*-1}(W_{h-1}, A_{h-1}).
\end{array}$$

Under assumption (A), the topological argument displayed in ([17] estimates (25) and (26)) shows that

$$\Phi_1^*\Big(\big[B_1(\tilde{\Sigma}), B_0(\tilde{\Sigma})\big]\Big) \neq 0 \Rightarrow \Phi_h^*\Big(\big[B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})\big]\Big) \neq 0,\ \forall\ h \geq 1.$$

We agree to suppose that $B_0(\tilde{\Sigma}) = \emptyset$. If $\Phi_1^*\Big(\big[B_1(\tilde{\Sigma}), B_0(\tilde{\Sigma})\big]\Big) = 0$, then

$$\Big(\chi^* \circ (i_2^*)^{-1} \circ \Phi_1^*\Big)\big(\big[B_1(\tilde{\Sigma}), B_0(\tilde{\Sigma})\big]\big) = 0 \text{ in } H_*(\Omega), \tag{14}$$

where χ and i_2 are defined in diagram (13). Observe that

$$\Big(\chi^* \circ (i_2^*)^{-1} \circ \Phi_1^*\Big)\big([B_1(\tilde{\Sigma}), B_0(\tilde{\Sigma})]\big) = \Big(\chi^* \circ (i_2^*)^{-1} \circ f_{\lambda,1}^*\Big)\big([\tilde{\Sigma}, \partial\tilde{\Sigma}]\big).$$

Using mapping (12), we have

$$f_{\lambda,1}(\tilde{\Sigma}, \partial\tilde{\Sigma}) \subset \Big(J_{2^{\frac{2}{n-2}}\frac{S}{2n}+\bar{\varepsilon}_0} \cap V_\Omega(1, \frac{\varepsilon_1}{N_1}), J_{2^{\frac{2}{n-2}}\frac{S}{2n}} \cap V_\Omega(1, \frac{\varepsilon_1}{N_1})\Big).$$

Thus,

$$(i_2^*)^{-1}\big(f_{\lambda,1}^*([\tilde{\Sigma}, \partial\tilde{\Sigma}])\big) = f_{\lambda,1}^*([\tilde{\Sigma}, \partial\tilde{\Sigma}]),$$

since i_2 is an equivalence of homotopy. It follows that

$$\Big(\chi^* \circ (i_2^*)^{-1} \circ \Phi_1^*\Big)\big([B_1(\tilde{\Sigma}), B_0(\tilde{\Sigma})]\big) = \chi^*\big(f_{\lambda,1}^*([\tilde{\Sigma}, \partial\tilde{\Sigma}])\big) = [\tilde{\Sigma}],$$

since $\partial\tilde{\Sigma} = \emptyset$. We therefore have from (14), $[\tilde{\Sigma}] = 0$ in $H_*(\Omega)$. This is absurd. Thus, $\Phi_1^*\Big([B_1(\tilde{\Sigma}), B_0(\tilde{\Sigma})]\Big) \neq 0$ and the proof of Proposition 12 follows. □

We prove now the following existence result.

Theorem 3. *Under assumption* (A), *Theorem 1 holds.*

Proof. Assume that (1) has no solution. It follows from proposition 12 that under assumption (A),

$$\Phi_h^*\Big([B_h(\tilde{\Sigma}), B_{h-1}(\tilde{\Sigma})]\Big) \neq 0, \ \forall\, h \geq 1.$$

However, the expansion of Proposition 6 shows that for h large enough, ϕ_h^* is a null homomorphism. Such a contradiction implies the existence of solutions of problem (1). □

5. Conclusions

This paper conjectured the existence of positive solutions of problem (1.1) on bounded domains with holes. We proved preliminary results and useful estimates for mixed Dirichlet–Neumann boundary value problems involving the standard Laplacian. Particularly, a strong maximum principle theorem has been established. We were able to evaluate the topological differences between the level sets of the associated energy functional. Precisely, the level sets corresponding to critical points at infinity of the associated variational problem. An additional difficulty compared to homogeneous Dirichlet problems lies in the complexity of the configuration of the critical points at infinity. In the present setting we have two types of critical points at infinity. Type1, containing only interior concentration points, and type2, containing at least a concentration point in the boundary part where the Neumann condition is prescribed. Although both types of critical points at infinity may have the same energy levels, our method was able to exclude the topological effect of critical points at infinity of type2 and prove the above conjecture under an algebraic topological condition. We believe that the used method and the obtained results constitute interesting steps to fully prove the aforementioned conjecture. We also believe that the used approach helps in studying related nonlinear problems such as scalar-curvature-type problems on bounded domains with mixed Dirichlet– Neumann boundary conditions and mixed elliptic problems driven by the fractional Laplacian. It is well known that the latter problems are motivated by previous works on mixed Dirichlet–Neumann boundary problems driven by the standard Laplacian. For recent progress in these directions, we may refer the reader to [25–28] and references therein.

Author Contributions: Conceptualization, H.C.; Formal analysis, A.A., S.C., H.C. and A.S.; Supervision, H.C.; Funding acquisition, A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: Part of this work was conducted when the third author enjoyed the hospitality of the "Institut des Hautes Etudes Scientifiques (IHES, Paris)". He takes the opportunity to acknowledge the excellent working conditions.

Conflicts of Interest: The authors declare no conflict of interest.

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