

## Article

# Ergodicity and Related Bounds for One Particular Class of Markovian Time—Varying Queues with Heterogeneous Servers and Customer’s Impatience

Yacov Satin <sup>1</sup>, Rostislav Razumchik <sup>2,3</sup>, Ivan Kovalev <sup>1</sup> and Alexander Zeifman <sup>1,2,4,5,\*</sup>

<sup>1</sup> Department of Applied Mathematics, Vologda State University, 15 Lenina Str., 160000 Vologda, Russia; yacovi@mail.ru (Y.S.); kovalev.iv96@yandex.ru (I.K.)

<sup>2</sup> Federal Research Center “Computer Science and Control” of the Russian Academy of Sciences, 44-2 Vavilov Str., 119133 Moscow, Russia; rrazumchik@ipiran.ru

<sup>3</sup> Department of Applied Probability and Informatics, Peoples’ Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya Str., 117198 Moscow, Russia

<sup>4</sup> Vologda Research Center of the Russian Academy of Sciences, 556A Gorky Str., 160014 Vologda, Russia

<sup>5</sup> Moscow Center for Fundamental and Applied Mathematics, Moscow State University, 119991 Moscow, Russia

\* Correspondence: a\_zeifman@mail.ru

**Abstract:** We consider a non-standard class of Markovian time: varying infinite capacity queues with possibly heterogeneous servers and impatience. We assume that during service time, a customer may switch to the faster server (with no delay), when such a server becomes available and no other customers are waiting. As a result, customers in the queue may become impatient and leave it. Under this setting and with certain restrictions on the intensity functions, the quantity of interest, the total number of customers in the system, is the level-dependent birth-and-death process (BDP). In this paper, for the first time in the literature, explicit upper bounds for the distance between two probability distributions of this BDP are obtained. Using the obtained ergodicity bounds in combination with the sensitivity bounds, we assess the stability of BDP under perturbations. Truncation bounds are also given, which allow for numerical solutions with guaranteed truncation errors. Finally, we provide numerical results to support the findings.

**Keywords:** nonstationary queuing system; impatience; birth-death process; ergodicity; bounds; limiting characteristics

**MSC:** 60J27; 60J28; 60K25



**Citation:** Satin, Y.; Razumchik, R.; Kovalev, I.; Zeifman, A. Ergodicity and Related Bounds for One Particular Class of Markovian Time—Varying Queues with Heterogeneous Servers and Customer’s Impatience. *Mathematics* **2023**, *11*, 1979. <https://doi.org/10.3390/math11091979>

Academic Editor: Manuel Alberto M. Ferreira

Received: 29 March 2023

Revised: 18 April 2023

Accepted: 20 April 2023

Published: 22 April 2023



**Copyright:** © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

In this paper, consideration is given to certain aspects of the performance evaluation of one particular class of Markovian time: varying queues with heterogeneous servers and customer impatience. Both single- and multiple-class multiserver queues, either with or without impatience, have been the subjects of extensive research for many decades and remain so. Therefore, substantial literature on the topic exists. For the current state of art from various points of view, both theoretical and practical, one can refer, for example, to the introduction sections of [1–13].

The class of queues, investigated in this paper, consists solely of  $M_t/M_t/S/\infty$  queues with customer impatience. All service times are exponential, and the service intensity of server  $i$  is  $\mu_i(t)$ . Therefore, servers are allowed to be heterogeneous. Moreover, they are assumed to be numbered as  $1, 2, \dots, S$  without repetition by decreasing service speed. To make the latter possible we assume that  $\mu_i(t)$  is of the form  $\mu_i(t) = f(t)\mu_i$ , where  $f(t)$  is a locally integrable (non necessarily continuous) bounded positive function and  $0 < \mu_i < \infty$ . Customers arrive one by one with the intensity  $\lambda(t)$  and either occupy one server, if there is one idle, or one place in the queue. The basic quantities under consideration will be  $X(t)$

: the total number of customer in the system at the instant  $t$ , and its distributions. Thus, the queuing discipline is irrelevant as long as it remains work-conserving and non-preemptive (for example, FIFO, LIFO, or RANDOM).

The state space of  $X(t)$  is  $\{0, 1, 2, \dots\}$  and, from the description given above, it is clear that  $X(t)$  is the level-dependent BDP. Let us denote its time-dependent intensity matrix (generator) by  $A(t) = (a_{ij}(t))_{i,j=0}^{\infty}$  (its form is given in Section 2). The distribution of  $X(t)$  can be represented as a probability vector  $\mathbf{p}(t) = (p_0(t), p_1(t), p_2(t), \dots)^T$ .

There exist various techniques to calculate performance measures for time-varying queues (see, for example, Introduction in [14–17]). Probably the most widely applicable (or at least most popular nowadays, due to increasing computer power) set of techniques consists of numerical methods for systems of ordinary differential equations (ODEs). Whenever the queues' capacities are infinite, the following two questions usually arise, when one attempts to compute stationary (or limiting) performance characteristics of queues using an ODE technique: (i) how to solve the Kolmogorov system  $\frac{d}{dt}\mathbf{p}(t) = A(t)\mathbf{p}(t)$ ,  $\mathbf{p}(t)^T\mathbf{1} = 1$ , having infinitely many equations? (ii) how to understand that the limiting regime is reached? Traditionally, (i) is circumvented using truncation. Yet, there is no “rule of thumb” for choosing the truncation threshold, except for “the larger the better”. With respect to (ii), stationary (or limiting) characteristics are usually considered to be identical to the solution of the ODEs at some distant time interval. However, usually it cannot be specified in advance how to choose the position and the length of the “distant time interval”, on which the solution of the system must be found. It can happen that the steady-state is detected prematurely (see [18]).

There are methods in the literature that can solve (i) and (ii) for certain continuous-time Markov chains (see [19–23]). In those cases where the stationary distribution of the Markov chain can be efficiently calculated in advance, solution techniques equipped with the steady-state detection exist (see, for example, [24]). However, in any case, an ODE technique, used to compute the distribution of the queue size or its moments, benefits from the answer to question (ii). This answer follows from the convergence (ergodic) properties of Markov chains. In addition, even though those properties have been the subject of many research papers, it remains difficult to obtain practically useful general ergodicity bounds. In this paper, we show that one well-known method, which is based on the notion of the logarithmic norm of a linear operator and utilizes the properties of linear systems of differential equations, may yield explicit ergodicity bounds for certain time-varying multi-server queues with impatience. Using these bounds, both questions (i) and (ii) can be answered. It must be noticed that the ergodicity results available in the literature for level-(in)dependent nonhomogeneous BDP (and  $X(t)$  is such a Markov chain) can do the job as well (see, for example, [25,26]). However, being usually quite general, and requiring guessing or estimation of certain quantities, it can prove unclear how to apply them in specific use cases. The novel explicit ergodicity bounds provided in this paper constitute its main contribution. As an additional utility of these bounds, we show that, in combination with the sensitivity bounds, they allow assessment of the stability of  $X(t)$  under perturbations.

In the description of the class of queues under investigation given above, two ingredients are missed: a server selection rule for the newly arriving customers and the impatience mechanism. When the servers are not identical and an arriving customer sees more than one idle server, they need a rule to select a server. Various selection rules can be found in the literature. Probably the most popular (since [27]) is random selection, according to which arrivals choose randomly any of the idle servers. When arriving customers choose the first idle server, this is the ordered entry rule (see [28–31]). Assuming the servers are numbered, if a customer occupies the idle server with the lowest number, then this is an ordered hunt rule (see [32]). General selection rules, where arrivals select one of the idle servers with a certain probability, have also been considered (see [33,34]). In this paper, we assume that the ordered hunt rule is adopted. Moreover, in order to avoid the state space collapse, being the main obstacle in the analysis of multi-server heterogeneous queues,

we additionally assume that the following service mechanism (known in the literature; see, for example, [10]) is implemented in the system: during the service time, a customer switches to the fastest server (without any delay) when such a server becomes available and there are no other customers waiting. The choice of the impatience mechanism is dictated by the limitations of the method used to obtain the ergodicity bounds. As will be seen, the method heavily relies on the notion of the logarithmic norm of a linear operator, which in the considered case is equal to  $\sup_i \{a_{ii}(t) + \sum_{j \neq i} |a_{ji}(t)|\}$ . For it to be finite, the generator  $A(t)$  has to be bounded (element-wise) for any  $t$ . This condition conflicts with the classic impatience mechanism, according to which the impatience intensity grows as the queue-size increases. Therefore, in this paper we assume that the probability that exactly one customer leaves the queue during the interval  $(t, t + \Delta)$ , given that there are  $i$  customers in the system, is  $\zeta_i(t)\Delta + o(\Delta)$ ,  $i > S$ , and  $\zeta_{S+1}(t), \zeta_{S+2}(t), \dots$  with a single finite upper bound.

In what follows, by  $\|\cdot\|$  we denote the  $l_1$ -norm, i.e., if  $\mathbf{x}$  is an  $(l + 1)$ -dimensional column vector then  $\|\mathbf{x}\| = \sum_{k=0}^l |x_k|$ . If  $\mathbf{x}$  is a probability vector, and then  $\|\mathbf{x}\| = 1$ . The choice of operator norms will be the one induced by the  $l_1$ -norm in column vectors, i.e.,  $\|A\| = \sup_{0 \leq j \leq l} \sum_{i=0}^l |a_{ij}|$  for a linear operator  $A$ .

### 2. Model Description

Let  $X(t)$  denote the total number of customers in the model at time  $t \geq 0$ . Then  $X(t)$  is the continuous-time BDP with the state space  $\{0, 1, 2, \dots\}$ . Denote the birth intensity by  $\lambda(t)$  and the death intensity by  $\nu_k(t)$  if  $k$  customers are in the model at the instant  $t$ . We assume that  $\nu_k(t)$  can be represented in the form

$$\nu_k(t) = \begin{cases} f(t) \sum_{i=1}^k \mu_k, & \text{if } 1 \leq k \leq S, \\ f(t) \sum_{i=1}^S \mu_k + \zeta(t)\theta_k, & \text{if } k > S, \end{cases} \tag{1}$$

where  $\mu_1 \geq \dots \geq \mu_S > 0$  and  $\{\theta_k, k \geq S + 1\}$  is a bounded monotonically non-decreasing sequence of positive numbers i.e.,  $\theta_k \leq \theta_{k+1}$  and there exists some  $M \in (0, \infty)$  such that  $\theta_k \leq M$  for all  $k$ . Another restriction imposed on the intensities  $\lambda(t)$  and  $\{\nu_k(t), k \geq 1\}$  is that they are locally integrable on  $[0, \infty)$  (not necessarily continuous) functions and bounded in the following sense: there exists some  $L \in (0, \infty)$  such that  $\sup_{k \geq 1} |\lambda(t) + \nu_k(t)| \leq L$  for almost all  $t \geq 0$ .

Since  $X(t)$  is the BDP, then its transposed generator, further denoted by  $A(t) = (a_{ij}(t))_{i,j=0}^\infty$ , has the form:

$$A(t) = \begin{pmatrix} -\lambda(t) & \nu_1(t) & 0 & 0 & \dots \\ \lambda(t) & -(\lambda(t) + \nu_1(t)) & \nu_2(t) & 0 & \dots \\ 0 & \lambda(t) & -(\lambda(t) + \nu_2(t)) & \nu_3(t) & \dots \\ 0 & 0 & \lambda(t) & -(\lambda(t) + \nu_3(t)) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Given an appropriate initial condition, the Kolmogorov forward equations for the distribution  $\mathbf{p}(t) = (p_0(t), p_1(t), p_2(t), \dots)^T$  of  $X(t)$  can be written as

$$\frac{d}{dt} \mathbf{p}(t) = A(t) \mathbf{p}(t). \tag{2}$$

Since  $\|A(t)\| = 2 \sup_k (\lambda(t) + \nu_k(t)) \leq 2L < \infty$  for almost all  $t \geq 0$ , the linear operator  $A(t)$  is bounded and locally integrable for  $t \in [0, \infty)$ . Therefore (2) is the system of differential equations in the space  $l_1$  with the bounded linear operator and thus it has a unique solution for arbitrary initial conditions.

### 3. Ergodicity Bounds

All the ergodicity bounds, as shown in this section, are obtained through the application of the same method, known as the “logarithmic norm” method. It based on the

notion of the logarithmic norm (see, for example, [14,35]) and utilizes the properties of linear systems of differential equations and thus is directly applicable to (2).

### 3.1. Null Ergodicity

Fix any  $\varepsilon \in (0, 1)$  and define the decreasing sequence of positive numbers  $\{d_k, k \geq 0\}$  by  $d_k = (1 - \varepsilon)^k$ . Let  $D = \text{diag}(d_0, d_1, d_2, \dots)$ . Since the inverse  $D^{-1}$  always exists, the system of Equation (2) can be rewritten in the following equivalent form:

$$\frac{d}{dt}(D\mathbf{p}(t)) = (DA(t)D^{-1})D\mathbf{p}(t), \tag{3}$$

where

$$DA(t)D^{-1} = \begin{pmatrix} -\lambda(t) & \frac{v_1(t)}{1-\varepsilon} & 0 & 0 & \dots \\ (1-\varepsilon)\lambda(t) & -(\lambda(t) + v_1(t)) & \frac{v_2(t)}{1-\varepsilon} & 0 & \dots \\ 0 & (1-\varepsilon)\lambda(t) & -(\lambda(t) + v_2(t)) & \frac{v_3(t)}{1-\varepsilon} & \dots \\ 0 & 0 & (1-\varepsilon)\lambda(t) & -(\lambda(t) + v_3(t)) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Denote by  $-\alpha_k(t)$  the sum of all elements in the  $k$ th column of this matrix. By direct inspection, it can be seen that  $\alpha_0(t) = \lambda(t)\varepsilon$  and  $\alpha_k(t) = \varepsilon\left(\lambda(t) - \frac{v_k(t)}{1-\varepsilon}\right)$  for  $k \geq 1$ . Since for a fixed  $t \geq 0$  the sequence of transition rates  $v_k(t)$  is non-decreasing in  $k$  and bounded, then  $v_\infty(t) = \lim_{k \rightarrow \infty} v_k(t)$  exists. Therefore,  $\alpha_k(t) \geq \alpha_{k+1}(t)$  and

$$\alpha_*(t) = \inf_{k \geq 0} \alpha_k(t) = \varepsilon\left(\lambda(t) - \frac{v_\infty(t)}{1-\varepsilon}\right), \quad t \geq 0.$$

**Theorem 1.** *If  $\int_0^\infty \alpha_*(t)dt = +\infty$  for some  $\varepsilon \in (0, 1)$ , then the process  $\{X(t), t \geq 0\}$  is null ergodic, and for any non-negative integer  $N$  it holds that*

$$P(X(t) \leq N) \leq \min\left(1, e^{-\int_0^t \alpha_*(u)du} \sum_{k=0}^\infty (1-\varepsilon)^{k-N} p_k(0)\right), \quad t \geq 0. \tag{4}$$

**Proof.** Recall, that  $D = \text{diag}(d_1, d_2, \dots)$ , where  $d_k = (1 - \varepsilon)^k$ . From the logarithmic norm method, it follows that  $\|D\mathbf{p}(t)\| \leq e^{-\int_0^t \alpha_*(u)du} \|D\mathbf{p}(0)\|$  for any initial condition  $\mathbf{p}(0)$ . Fix any non-negative integer  $N$ . Since  $d_{k+1} < d_k < 1$  for  $k \geq 1$ , then

$$d_N \sum_{k=0}^N p_k(t) \leq \sum_{k=0}^N d_k p_k(t) \leq \|D\mathbf{p}(t)\|.$$

□

In the homogeneous case, i.e., when  $\lambda(t) = \lambda$  and  $v_k(t) = v_k$  for all  $k \geq 1$ , the null ergodicity condition of the Theorem 1 is reduced to  $\lambda/v_\infty > 1$ , i.e., to the natural condition where the offered load exceeds 1. The right part of (4) takes a simpler form, once the initial probability distribution  $\{p_k(0), k \geq 0\}$  has finite support. Under the conditions of the Theorem 1 it is clearly seen that the probability that the process  $X(t)$  is concentrated in a finite set of states  $[a, b] \in (0, \infty)$  is equal to

$$P(a \leq X(t) \leq b) = e^{-\int_0^t \alpha_*(u)du} \frac{(1-\varepsilon)^{a-1} - (1-\varepsilon)^b}{(1-\varepsilon)^{a+b-1}} \sum_{k=0}^\infty (1-\varepsilon)^k p_k(0),$$

and gradually diffuses, i.e., approaches 0 as the time increases indefinitely. Concentration inequalities can be used to construct bounds on the moments of  $X(t)$ ; for example,

the Markov inequality gives the following lower bound for the expected number  $E(X(t))$  of customers in the model at a time  $t \geq 0$ :

$$E(X(t)) \geq N \max \left( 0, 1 - e^{-\int_0^t \alpha_*(u) du} \sum_{k=0}^{\infty} (1 - \varepsilon)^{k-N+1} p_k(0) \right).$$

### 3.2. Weak Ergodicity

Put  $p_0(t) = 1 - \sum_{i \geq 1} p_i(t)$ . Consider now the reduced forward Kolmogorov system (2) in the form

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \quad t \geq 0, \tag{5}$$

where  $\mathbf{f}(t) = (\lambda(t), 0, 0, \dots)^T$ ,  $\mathbf{z}(t) = (p_1(t), p_2(t), \dots)^T$  and

$$B(t) = \begin{pmatrix} -(2\lambda(t) + v_1(t)) & v_2(t) - \lambda(t) & -\lambda(t) & -\lambda(t) & \dots \\ \lambda(t) & -(\lambda(t) + v_2(t)) & v_3(t) & 0 & \dots \\ 0 & \lambda(t) & -(\lambda(t) + v_3(t)) & v_4(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If instead of the sequence  $\{d_k = (1 - \varepsilon)^k, k \geq 0\}$  used in the previous section, one considers an arbitrary sequence of positive numbers  $\{d_k, k \geq 0\}$  and the triangular (instead of the diagonal) matrix

$$D = \begin{pmatrix} d_0 & d_0 & d_0 & \dots \\ 0 & d_1 & d_1 & \dots \\ 0 & 0 & d_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then for  $d_0 = 1$  (as shown in ([25], Section 3)) the sum  $-\alpha_k(t)$  of all elements in the  $k$ th column of the matrix  $DA(t)D^{-1}$  is equal to

$$\alpha_k(t) = \lambda(t) + v_{k+1}(t) - \frac{d_{k+1}}{d_k} \lambda(t) - \frac{d_{k-1}}{d_k} v_k(t), \quad k \geq 0, \quad d_{-1} = 1. \tag{6}$$

According to the definition of weak ergodicity (see, for example, [14]), the process  $\{X(t), t \geq 0\}$  will be weakly ergodic, if  $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  for any initial conditions  $\mathbf{p}^*(0)$  and  $\mathbf{p}^{**}(0)$ , where  $\mathbf{p}^*(t)$  and  $\mathbf{p}^{**}(t)$  are the corresponding solutions of (2). As shown in ([25] Theorem 3.1), the quantity  $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|$  for any regular birth and death process (and thus for  $X(t)$  as well) can be upper bounded by applying the logarithmic norm method; specifically, the relation (3.11) gives the explicit form of the bound:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq e^{-\int_0^t \alpha_*(u) du} \sum_{i=1}^{\infty} \frac{4 \sum_{k=0}^{i-1} d_k}{\inf_{k \geq 1} d_k} |p_i^*(0) - p_i^{**}(0)|, \tag{7}$$

where  $\alpha_*(t) = \inf_{k \geq 0} \alpha_k(t)$ . The next theorem gives the sequence  $\{d_k, k \geq 0\}$  and the sufficient condition, under which the right part of the inequality (7) tends to 0 as time increases indefinitely, and therefore  $X(t)$  is weakly ergodic.

**Theorem 2.** *If  $\int_0^\infty (v_S(t) - d\lambda(t)) dt = +\infty$  for a  $d \in (1, 1 + 1 + \mu_S / \sum_{i=1}^{S-1} \mu_i]$ , then the process  $\{X(t), t \geq 0\}$  is weakly ergodic and for any initial conditions  $\mathbf{p}^*(0)$  and  $\mathbf{p}^{**}(0)$  it holds that*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 4e^{-\frac{d-1}{d} \int_0^t (v_S(u) - d\lambda(u)) du} \sum_{i=1}^{\infty} \left( 1 + \frac{d^{i-1} - 1}{d - 1} \right) |p_i^*(0) - p_i^{**}(0)|, \quad t \geq 0. \tag{8}$$

**Proof.** Assume that  $\int_0^\infty (v_S(t) - d\lambda(t)) dt = +\infty$  and  $d \in (1, 1 + 1 + \mu_S / \sum_{i=1}^{S-1} \mu_i]$ . Define the increasing sequence of positive numbers  $\{d_k, k \geq 0\}$  by  $d_0 = 1, d_k = d^{k-1}$  for  $k \geq 1$ .

Then the expression for  $\alpha_k(t)$  given by (6) is reduced to  $\alpha_k(t) = v_{k+1}(t) - d^{-1}v_k(t) - (d-1)\lambda(t)$ . Now, depending on whether  $k < S$  or  $k \geq S$ , the values of  $\alpha_k(t)$  can be lower bounded. Indeed, from the exact form of  $v_k(t)$ , given by (1), and the assumption  $\theta_k \leq \theta_{k+1}$ , it follows that

$$\alpha_k(t) \geq \begin{cases} \left[ (1 - d^{-1}) \sum_{i=1}^k \mu_i + \mu_{k+1} \right] f(t) - (d - 1)\lambda(t), & \text{if } 0 \leq k \leq S - 1, \\ (1 - d^{-1}) \sum_{i=1}^S \mu_i f(t) - (d - 1)\lambda(t), & \text{if } k \geq S. \end{cases}$$

Consider the following two functions on  $(1, \infty)$ :

$$\zeta_1(x) = \min \left( \mu_1, \mu_2 + (1 - x^{-1}) \sum_{i=1}^1 \mu_i, \dots, \mu_S + (1 - x^{-1}) \sum_{i=1}^{S-1} \mu_i \right),$$

$$\zeta_2(x) = (1 - x^{-1}) \sum_{i=1}^S \mu_i.$$

Both are increasing and bounded, that is  $\mu_S < \zeta_1(x) < \mu_1$  and  $0 < \zeta_2(x) < \sum_{i=1}^S \mu_i$ . Moreover  $\zeta_2(x) < \mu_S$  as long as  $x \in (1, 1 + \mu_S / \sum_{i=1}^{S-1} \mu_i]$ . Therefore, as long as  $x$  varies in this region, we have  $\zeta_1(x) > \zeta_2(x)$  and, therefore,  $\inf_{k \geq 0} \alpha_k(t) \geq \alpha_S(t)$ . Now the statement of the theorem follows from (7). □

**Corollary 1.** Under the conditions of Theorem 2 the process  $X(t)$  has the limiting mean and if  $\inf_{i \geq 1} \frac{d_i}{i} > 0$  then

$$\left| \sum_{i=0}^{\infty} i p_i(t) - m^*(t) \right| \leq \frac{1}{\inf_{i \geq 1} \frac{d_i}{i}} \|D\mathbf{p}(t) - \mathbf{p}^*(t)\|. \tag{9}$$

where  $\mathbf{p}^*(t) = (p_0^*(t), p_1^*(t), p_2^*(t), \dots)^T$  is the limiting distribution of  $X(t)$ .

**Proof.** Since in Theorem 2  $d_i = d^{i-1}$ ,  $i \geq 1$  and thus  $\|D\mathbf{p}^*(t)\| < \infty$  for any  $t \geq 0$  we can see that the limiting mean

$$\sum_{i=1}^{\infty} i p_i^*(t) = \sum_{i=1}^{\infty} d_i \frac{1}{d_i} p_i^*(t) \leq \frac{1}{\inf_{i \geq 1} \frac{d_i}{i}} \|D\mathbf{p}^*(t)\|$$

exists. Now for any fixed  $X(0)$  we find that

$$\left| \sum_{i=0}^{\infty} i(p_i(t) - m(t)) \right| \leq \frac{1}{\inf_{i \geq 1} \frac{d_i}{i}} \sum_{i=1}^{\infty} d_i |p_i(t) - p_i^*(t)|$$

which is identical to the right part of (9). □

Note that if all the intensity functions are periodic and the smallest common multiple of the periods is  $T$ , then the sufficient weak ergodicity condition of Theorem 2 is reduced to  $\int_0^T (v_S(t) - d\lambda(t))dt > 0$ , and from the Corollary 1 it follows that the limiting mean will have a period equal to  $T$ .

**Corollary 2.** Assume  $\lambda(t) = \lambda$  and  $v_k(t) = v_k$  for all  $k \geq 1$ . If  $\lambda < v_\infty$ , then  $X(t)$  is exponentially ergodic. If  $\pi = (\pi_0, \pi_1, \dots)^T$  is its stationary distribution, then

$$\|\mathbf{p}(t) - \pi\| \leq 4e^{-\alpha t} \sum_{i=1}^{\infty} \sum_{k=0}^{i-1} d_k |p_i(0) - \pi_i|, \quad t \geq 0, \tag{10}$$

where the sequence  $\{d_k, k \geq 0\}$  and the parameter  $\alpha > 0$  can be given explicitly.

**Proof.** Put  $d_0 = 1$  and  $d_k = \prod_{i=1}^k (1 + \varepsilon_{i-1})$ ,  $k \geq 1$ , for some positive sequence of real numbers  $\varepsilon_0, \varepsilon_1, \dots$ . Then from (6) it follows that

$$\alpha_k \geq v_k \frac{\varepsilon_{k-1}}{1 + \varepsilon_{k-1}} - \lambda \varepsilon_k.$$

Let  $\lambda < v_\infty$ . Then, there must exist a positive  $\varepsilon$  such that  $(1 + \varepsilon)\lambda < v_\infty$  or, equivalently,  $v_\infty \frac{\varepsilon}{1 + \varepsilon} - \lambda \varepsilon > 0$ . Since  $v_k$  is a non-decreasing sequence, then one can find the smallest  $k^*$  for which simultaneously

$$v_{k^*+1} \frac{\varepsilon}{1 + \varepsilon} - \lambda \varepsilon > 0 \text{ and } v_{k^*} \frac{\varepsilon}{1 + \varepsilon} - \lambda \varepsilon \leq 0.$$

Put  $\varepsilon_k = \varepsilon$  for  $k \geq k^*$ . Then

$$\inf_{k \geq k^*+1} \alpha_k \geq \underbrace{v_{k^*+1} \frac{\varepsilon}{1 + \varepsilon} - \lambda \varepsilon}_{=\alpha}.$$

It can be shown that if  $\varepsilon$  and  $\varepsilon_{k^*-1}, \dots, \varepsilon_1$  are chosen consecutively from the following intervals:

$$\varepsilon \in \left( 0, \frac{\prod_{i=1}^{k^*} v_i}{\sum_{i=1}^{k^*} \lambda^i \prod_{j=1}^{k^*-i} v_j} \right), \varepsilon_k \in \left( \frac{\lambda \varepsilon_{k+1}}{v_{k+1} - \lambda \varepsilon_{k+1}}, \frac{\prod_{i=1}^k v_i}{\sum_{i=1}^k \lambda^i \prod_{j=1}^{k-i} v_j} \right),$$

then it is guaranteed that  $\alpha_k \geq \alpha$  for all  $1 \leq k \leq k^* - 1$ . Hence,  $\inf_{k \geq 1} \alpha_k \geq \alpha$ , and thus (10) follows from (7).  $\square$

#### 4. Perturbation and Truncation Bounds

Since  $X(t)$  is the BDP, the results regarding perturbation and truncation bounds, obtained for quite a general setting in [36–38], are applicable.

We start with the perturbation bounds. Consider a Markov chain, say  $\bar{X}(t)$ , with a state space identical to  $X(t)$  and with such a generator, say  $\bar{A}(t) = (\bar{a}_{ij}(t))_{i,j=0}^\infty$ , that  $\bar{A}(t) = A(t) + \hat{A}(t)$  holds to all  $t \geq 0$ . Here  $\hat{A}(t) = (\hat{a}_{ij}(t))_{i,j=0}^\infty$  is such a matrix that is small or  $\hat{A}(t) \rightarrow 0$  as  $t$  increases. Denote the state probabilities of the “perturbed” chain by  $\bar{p}_i(t)$ . Using the normalization condition  $\bar{p}_0(t) = 1 - \sum_{i \geq 1} \bar{p}_i(t)$ , the system corresponding  $\bar{X}(t)$  can be written as follows:

$$\frac{d}{dt} \bar{\mathbf{z}}(t) = \bar{B}(t) \bar{\mathbf{z}}(t) + \bar{\mathbf{f}}(t). \tag{11}$$

Here the matrix  $\bar{B}(t)$  with the elements  $\bar{b}_{ij}(t) = \bar{a}_{ij}(t) - \bar{a}_{i0}(t)$ . Let the perturbed intensities  $\bar{\lambda}(t)$  and  $\bar{v}_k(t)$  be such that  $|\lambda(t) - \bar{\lambda}(t)| \leq \hat{\varepsilon}$ ,  $|v_k(t) - \bar{v}_k(t)| \leq \hat{\varepsilon}$  for  $t \geq 0$ . Then for all  $t \geq 0$  we have the upper bounds:

$$\|B(t)\|_{1D} \leq (1 + d)|\lambda(t)| + \frac{1 + d}{d} |v_k(t)| \leq (1 + d)L,$$

$$\|f(t)\|_{1D} = |\lambda(t)| \leq \Lambda,$$

$$\|B(t) - \bar{B}(t)\|_{1D} \leq (1 + d)|\lambda(t) - \bar{\lambda}(t)| + \frac{1 + d}{d} |v_k(t) - \bar{v}_k(t)| \leq \frac{(1 + d)^2}{d} \hat{\varepsilon},$$

$$\|f(t) - \bar{f}(t)\|_{1D} \leq |\lambda(t) - \bar{\lambda}(t)| \leq \hat{\varepsilon}.$$

In the article in [37], the following theorem was proven.



**Theorem 3.** *If a Markov chain  $X(t)$  is 1D-exponentially weakly ergodic*

$$e^{-\frac{d-1}{d} \int_0^t (v_S(u) - d\lambda(u)) du} \leq Me^{-a(t-s)} \tag{12}$$

for all  $0 \leq s \leq t$ , then  $\bar{X}(t)$  is also 1D-exponentially weakly ergodic and the following perturbation estimate in the 1D-norm holds:

$$\limsup_{t \rightarrow \infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_{1D} \leq \frac{M\hat{\epsilon}(M(1+d)^2\Lambda + ad)}{a(ad - M(1+d)^2\hat{\epsilon})}. \tag{13}$$

If  $W = \inf_{i \geq 1} \frac{d_i}{i} > 0$ , then both chains  $X(t)$  and  $\bar{X}(t)$  have limiting means and

$$\limsup_{t \rightarrow \infty} |\phi(t) - \bar{\phi}(t)| \leq \frac{M\hat{\epsilon}(M(1+d)^2\Lambda + ad)}{Wa(ad - M(1+d)^2\hat{\epsilon})}. \tag{14}$$

Assume that the conditions of the theorem are fulfilled, i.e.,

$$\|B(t) - \bar{B}(t)\|_{1D} \leq \chi(t), \quad \|\mathbf{f}(t) - \bar{\mathbf{f}}(t)\|_{1D} \leq \chi(t), \tag{15}$$

where  $\chi(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Then the bound from [36]

$$\|\hat{\mathbf{z}}(t)\| \leq Me^{-at}\|\hat{\mathbf{z}}(0)\| + M\left(e^{-a(t-t_*)}\frac{\epsilon_0}{a} + \frac{\epsilon}{a}\right)\left(1 + M\|\bar{\mathbf{z}}(0)\| + \frac{M(F + \epsilon_0)}{a - \epsilon_0}\right), \tag{16}$$

where

$$\|\mathbf{f}(t)\|_{1D} \leq F, \chi(0) = \epsilon_0, \epsilon \in (0, \epsilon_0), \chi(t_*) = \epsilon, \tag{17}$$

yields the perturbation bound for the considered  $X(t)$ .

Let us consider the truncation bound. Consider the copy of the process  $X(t)$  but with the finite state space  $\{0, 1, 2, \dots, N\}$  (denote it by  $X^*(t)$ ). Let its birth intensity  $\lambda^*(t) = \lambda(t)$  and death intensity by  $v_k^*(t) = v_k(t)$ , when there are, in total,  $k$  customers in the model. Denote its transposed generator by  $A^*(t)$  and its probability vector by  $\mathbf{p}^*(t)$ . For the Markov chain  $X^*(t)$ , instead of (2), we have

$$\frac{d}{dt}\mathbf{p}^*(t) = A^*(t)\mathbf{p}^*(t). \tag{18}$$

Assume that for  $t \geq 0$

$$\lambda(t) \leq \Lambda \leq L < \infty, \quad v_n(t) \leq \Delta_n \leq L < \infty. \tag{19}$$

In order to obtain a truncation bound, one needs to find proper sequences  $\{d_k\}$  and  $\{d_k^*\}$ . Assume that for the sequence  $\{d_k\}$  there exist positive numbers  $M$  and  $\alpha$ , such that

$$e^{-\int_s^t \alpha(\tau) d\tau} \leq Me^{-\alpha(t-s)}, \tag{20}$$

for all  $0 \leq s \leq t$ , where  $\alpha(t) = \inf_{k \geq 1} \alpha_k(t)$  and

$$\alpha_k(t) = \begin{cases} \lambda(t) + v_1(t) - \frac{d_2}{d_1}\lambda(t), & k = 1, \\ \lambda(t) + v_{k+1}(t) - \frac{d_{k+2}}{d_{k+1}}\lambda(t) - \frac{d_k}{d_{k+1}}v_k(t), & k > 1. \end{cases}$$

Assume that for the sequence  $\{d_k^*\}$  there exist positive numbers  $M^*$  and  $\alpha^*$ , such that

$$e^{-\int_s^t \alpha^*(\tau) d\tau} \leq M^*e^{-\alpha^*(t-s)} \tag{21}$$



for all  $0 \leq s \leq t$ , where  $\alpha^*(t) = \min \alpha_k^*(t)$  and

$$\alpha_k^*(t) = \begin{cases} \lambda(t) + v_1(t) - \frac{d_2^*}{d_1^*} \lambda(t), & k = 1, \\ \lambda(t) + v_{k+1}(t) - \frac{d_{k+2}^*}{d_{k+1}^*} \lambda(t) - \frac{d_k^*}{d_{k+1}^*} v_k(t), & k > 1. \end{cases}$$

Denote

$$G_k = \sum_{j=1}^k d_j, G_k^* = \sum_{j=1}^k d_j^*, d = d_1, \quad W = \inf_{k>0} \left( \frac{G_k}{k} \right).$$

The following theorem holds.

**Theorem 4.** Assume the Markov chains  $X(t)$  and  $X^*(t)$  satisfy (20) and (21) and  $X(0) = X^*(0) = 0$ . Then the following bounds of truncations hold:

$$\| \mathbf{p}(t) - \mathbf{p}^*(t) \| \leq \frac{4M M^* \Lambda d_{i+1}^* G_{N+1} \Lambda}{d \alpha \alpha^* G_N^*}, \tag{22}$$

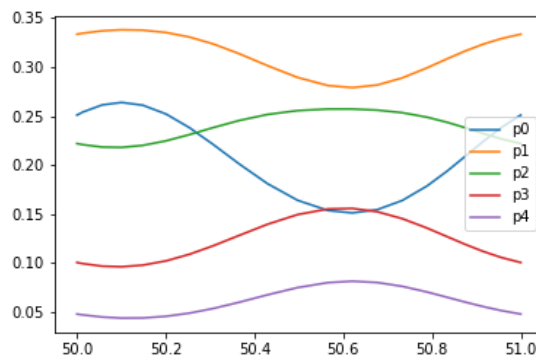
$$\| \mathbf{p}(t) - \mathbf{p}^*(t) \|_{1E} \leq \frac{4M M^* \Lambda d_{i+1}^* G_{N+1} \Lambda}{W \alpha \alpha^* G_N^*}. \tag{23}$$

### 5. Examples

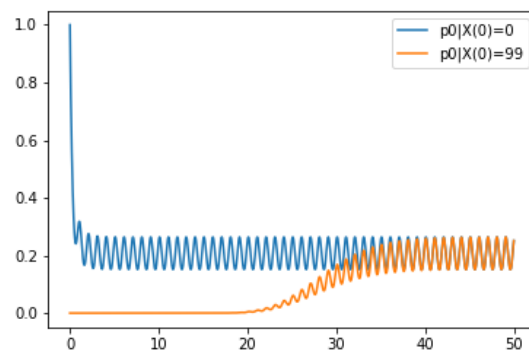
In this section, we will consider three different examples, demonstrating the applicability of the obtained results. The values of the parameters used do not follow any real data but were chosen for illustration purposes only. As Example 1, let us consider the homogeneous system with three servers (i.e.,  $S = 3$ ), periodic service intensity  $\mu_1(t) = \mu_2(t) = \mu_3(t) = \mu(t) = 2 + \cos 2\pi t$ , and periodic arrival intensity  $\lambda(t) = 3 + \sin 2\pi t$ . Assume that the impatience intensity is time-independent and has the form  $\zeta_k(t) = 0.4(1 - k^{-1}), k > S$ . Thus,  $\zeta = 0.1$  and  $\theta_k = 4(1 - k^{-1})$ . Then from (6) it follows that  $\alpha_k(t) = \mu(t) + (1 - d^{-1})k\mu(t) - (d - 1)\lambda(t)$ . The conditions of Theorem 2 are satisfied if  $d = 3/2$  and the function  $\alpha_*(t)$  in (7) is equal to

$$\alpha_*(t) = \alpha_0(t) = \alpha_3(t) = \mu(t) - 0.5\lambda(t) = 0.5 + \cos 2\pi t - 0.5 \sin 2\pi t.$$

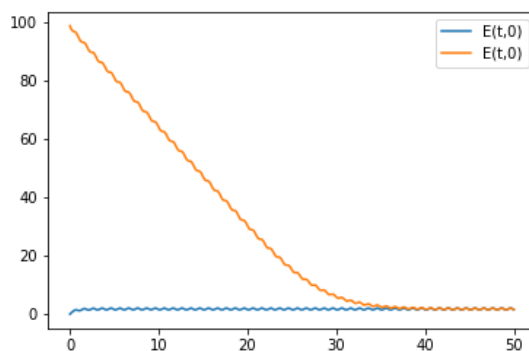
In Figure 1, it is shown how the probabilities  $p_0(t), p_1(t), p_2(t), p_3(t)$ , and  $p_4(t)$  behave as  $t$  increases, if initially the system was empty. In this case, the functions quite quickly approach their limiting values. Convergence to the limiting value of the empty system probability  $p_0(t)$  is shown in Figure 2. It can be seen that starting from approximately  $t = 40 = t^*$  the system “forgets” its initial state and the distribution of  $X(t)$  for  $t > t^*$  can be regarded as limiting. Moreover, since the limiting distribution of  $X(t)$  is periodic, it is sufficient to solve (numerically) the system of ODEs only in the interval  $[0, t^* + T]$ , where  $T$  is the smallest common multiple of the periods of  $\lambda(t)$  and  $\mu(t)$  i.e.,  $T = 1$ . The probability distribution of  $X(t)$  in the interval  $[t^*, t^* + T]$  is the estimate of the limiting probability distribution of  $X(t)$ . Convergence of the conditional mean number of customers in the system is shown in Figure 3. Starting from the instant  $t = t^*$ , the system “forgets” its initial state and the value of  $\sum_{i=0}^{\infty} i p_i(t)$  can be regarded as the limiting value of the mean number of customers.



**Figure 1.** Example 1. Behavior of the probabilities  $p_0(t)$ ,  $p_1(t)$ ,  $p_2(t)$ ,  $p_3(t)$ , and  $p_4(t)$  given that initially the system was empty (i.e.,  $p_0(0) = 1$ ).



**Figure 2.** Example 1. Convergence and the limiting value of the empty system probability  $p_0(t)$  given two different initial conditions:  $p_0(0) = 1$  (blue line),  $p_{99}(0) = 1$  (orange line).



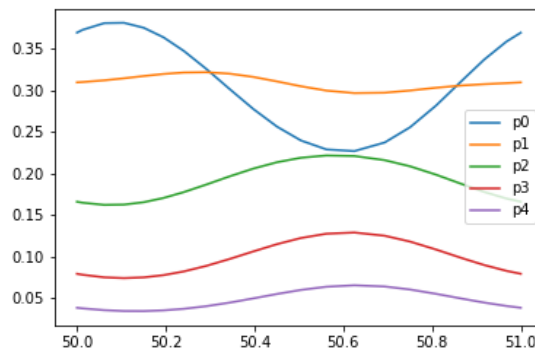
**Figure 3.** Example 1. Convergence of the conditional mean  $\sum_{i=0}^{\infty} ip_i(t)$  number of customers in the system given two different initial conditions:  $p_0(0) = 1$  (blue line),  $p_{99}(0) = 1$  (orange line).

If one increases (some of) the service intensities, then the rate of convergence is not expected to slow. This is illustrated by the Example 2 (see Figures 4–6), in which the service intensities are changed to

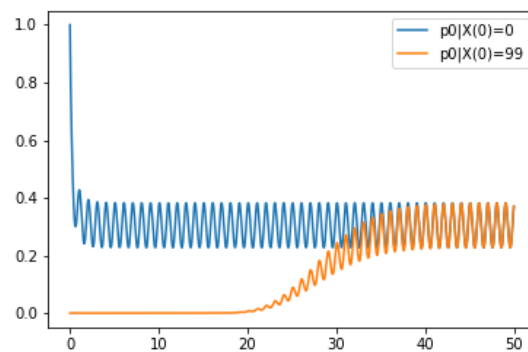
$$\mu_1(t) = 1.5(2 + \cos 2\pi t), \mu_2(t) = 2 + \cos 2\pi t, \mu_3(t) = 0.5(2 + \cos 2\pi t). \quad (24)$$

The conditions of the Theorem 2 are still satisfied if  $d = 3/2$ , but the function  $\alpha_*(t)$  in (7) is now equal to

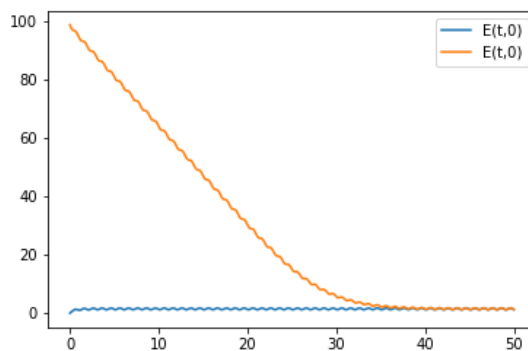
$$\alpha_*(t) = 0.5 + \cos 2\pi t - 0.5 \sin 2\pi t.$$



**Figure 4.** Example 2. Behavior of the probabilities  $p_0(t)$ ,  $p_1(t)$ ,  $p_2(t)$ ,  $p_3(t)$  and  $p_4(t)$ , given that initially the system was empty (i.e.,  $p_0(0) = 1$ ).



**Figure 5.** Example 2. Convergence and the limiting value of the empty system probability  $p_0(t)$ , given two different initial conditions:  $p_0(0) = 1$  (blue line),  $p_{99}(0) = 1$  (orange line).



**Figure 6.** Example 2. Convergence of the conditional mean  $\sum_{i=0}^{\infty} ip_i(t)$  number of customers in the system, given two different initial conditions:  $p_0(0) = 1$  (blue line),  $p_{99}(0) = 1$  (orange line).

As the last example, let us consider a heterogeneous system with three servers, periodic arrival intensity  $\lambda(t) = 3 + \sin 2\pi t$ , and periodic service intensities (24). Let  $k_{max}$  be a positive integer,  $\eta > 0$  and  $q \in (0, 1]$ . If we put  $\zeta = \eta q k_{max}$  and

$$\theta_k = \begin{cases} \frac{k-S}{k_{max}}, & k = S + 1, S + 2, \dots, S + k_{max}, \\ 1, & k > S + k_{max} \end{cases}$$

then, as long as  $k < S + k_{max}$ , the time-independent impatience intensity  $\zeta_k(t) = \zeta \theta_k$  is equal to  $(k\eta)q$ . In other words, the adopted impatience mechanism mimics the classic

one (with impatience intensity  $k\eta$  and probability of successful impatience  $q$ ), as long as the queue size stays below  $S + k_{max}$ . Figures 6 and 7 show the convergence and the limiting value of the empty system probability  $p_0(t)$  for  $q = 0.2$  and various values of  $k_{max} = 2, 3, 10, 200$ . The limiting values of the conditional mean are shown in the Figures 8 and 9. For  $k_{max} > 10$  the results are almost indistinguishable from the case  $k_{max} = 10$ . The rate of convergence here is identical to in Example 2, and for  $d = 1.9$  we have

$$\alpha_*(t) = \frac{9}{19}(0.3 + 3 \cos 2\pi t - 1.9 \sin 2\pi t).$$

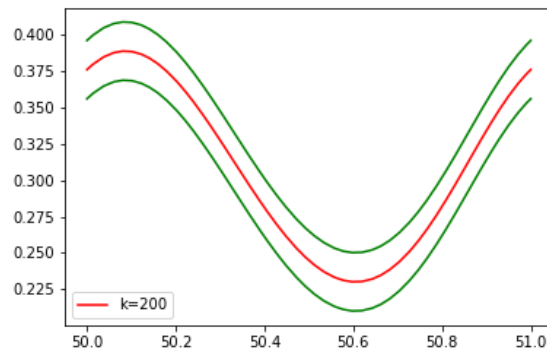


Figure 7. Example 3. The perturbation bounds (green lines) for the limiting values of  $p_0(t)$  for  $k = 200$ .

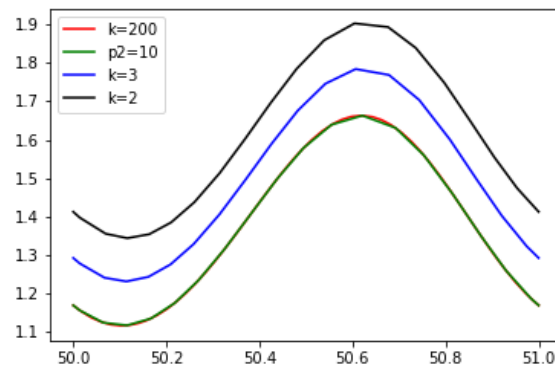


Figure 8. Example 3. Behavior of the limiting values of  $p_0(t)$ , given that initially the system was empty (i.e.,  $p_0(0) = 1$ ) for various values of  $k_{max}$ .

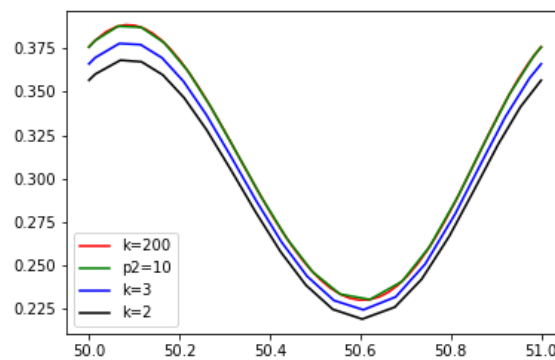


Figure 9. Example 3. Behavior of the limiting conditional mean  $\sum_{i=0}^{\infty} ip_i(t)$  number of customers in the system for various values of  $k_{max}$ .

By fixing  $\alpha = 0.142$  and  $M = 3$ , we obtain the following bound on the rate of convergence:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 12e^{-0.142t} \sum_{i=1}^{\infty} \left(1 + \frac{1.9^{i-1} - 1}{0.9}\right) |p_i^*(0) - p_i^{**}(0)|, t \geq 0. \tag{25}$$

The perturbation bound

$$\limsup_{t \rightarrow \infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_{1D} \leq \frac{1126\hat{\epsilon}}{0.142 - 13.3\hat{\epsilon}}. \tag{26}$$

and thus, if  $\hat{\epsilon} = 0.000001$ , we have that  $\limsup_{t \rightarrow \infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_{1D} \leq 0.02$ .

In order to obtain a truncation bound, let us consider the sequence  $\{d^*\}$  with  $d = \frac{100}{99}$ . Then

$$\alpha^*(t) = \frac{49}{1650} + \frac{3}{100} \cos(2\pi t) - \frac{1}{99} \sin(2\pi t)$$

By putting  $\alpha^* = \frac{49}{1650}$ ,  $M^* = \exp(\frac{3/100+1/99}{\pi})$ ,  $\Lambda = 4$ , we have that  $G_{200}^* \geq 6 \times 10^{55}$  and  $G_{200} \approx 639$ . Therefore

$$\|\mathbf{p}(t) - \mathbf{p}^*(t)\| \leq \frac{4 \times 3 \times 2 \times 4}{0.14(49/1650)} \frac{639 \times 4}{6 \times 10^{55}} \leq 10^{-50}.$$

### 6. Conclusions

It is well known that numerical solution techniques for the computation of time-dependent probability distributions of Markovian systems can benefit from methods providing ergodicity bounds, because the latter can indicate how to choose the position and the length of the “distant time interval” (in the periodic case) on which the solution has to be computed. They are also helpful whenever state space truncation is required, since they allow for guaranteed truncation error. In this paper, explicit ergodicity, perturbation, and truncation bounds for a particular class of Markovian time-varying queue with heterogeneous servers and customer impatience were obtained. Further research in this area could concentrate on the relaxation of the assumptions made about the transition intensities. For example, for the presented solution, the monotonicity of  $\theta_k$  is crucial (it guarantees the existence of the  $\lim_{k \rightarrow \infty} \nu_k(t)$ ). If, in addition, the transition intensities are time-independent, the condition  $\lambda \neq \lim_{k \rightarrow \infty} \nu_k$  gives exponential/weak/null ergodicity. If the monotonicity assumption is dropped, other justifications need to be found.

**Author Contributions:** Conceptualization, supervision, A.Z.; methodology, A.Z. and Y.S.; investigation, software, validation, visualization, Y.S. and I.K.; writing, R.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** The research was supported by the Ministry of Science and Higher Education of the Russian Federation, project No. 075-15-2020-799.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

### References

1. Haight, F.A. Queueing with reneging. *Metrika* **1959**, *2*, 186–197. [CrossRef]
2. Hasenbein, J.; Perry, D. Introduction: Queueing systems special issue on queueing systems with abandonments. *Queueing Syst.* **2013**, *75*, 111–113. [CrossRef]
3. Van Ommeren, J.K.; Baer, N.; Mishra, N.; Roy, D. Batch service systems with heterogeneous servers. *Queueing Syst.* **2020**, *95*, 251–269. [CrossRef]
4. Klimenok, V.; Dudin, A.; Vishnevsky, V. Priority Multi-Server Queueing System with Heterogeneous Customers. *Mathematics* **2020**, *8*, 1501. [CrossRef]
5. Harchol-Balter, M.; Osogami, T.; Scheller-Wolf, A.; Wierman, A. Multi-Server Queueing Systems with Multiple Priority Classes. *Queueing Syst.* **2005**, *51*, 331–360. [CrossRef]

6. Jouini, O.; Roubos, A. On multiple priority multi-server queues with impatience. *J. Oper. Res. Soc.* **2014**, *65*, 616–632. [[CrossRef](#)]
7. Puha, A.L.; Ward, A.R. Scheduling an Overloaded Multiclass Many-Server Queue with Impatient Customers. In *Operations Research & Management Science in the Age of Analytics*; INFORMS TutORials: New York, NY, USA, 2019; pp. 189–217. [[CrossRef](#)]
8. Dong, J.; Ibrahim, R. SRPT Scheduling Discipline in Many-Server Queues with Impatient Customers. *Manag. Sci.* **2021**, *67*, 7708–7718. [[CrossRef](#)]
9. Efrosinin, D.; Stepanova, N.; Sztrik, J. Algorithmic Analysis of Finite-Source Multi-Server Heterogeneous Queueing Systems. *Mathematics* **2021**, *9*, 2624. [[CrossRef](#)]
10. Kumar, R.; Sharma, S. Transient analysis of a Markovian queueing model with multiple-heterogeneous servers, and customers' impatience. *Opsearch* **2021**, *58*, 540–556. [[CrossRef](#)]
11. Melikov, A.Z.; Ponomarenko, L.A.; Mekhbaliyeva, E.V. Analyzing the Models of Systems with Heterogeneous Servers. *Cybern. Syst. Anal.* **2020**, *56*, 89–99. [[CrossRef](#)]
12. Dudin, A.; Dudina, O.; Dudin, S.; Samouylov, K. Analysis of Multi-Server Queue with Self-Sustained Servers. *Mathematics* **2021**, *9*, 2134. [[CrossRef](#)]
13. Bhati, A.; Pillai, S.R.B.; Vaze, R. On the Age of Information of a Queueing System with Heterogeneous Servers. In Proceedings of the National Conference on Communications (NCC), Kanpur, India, 27–30 July 2021; pp. 1–6. [[CrossRef](#)]
14. Zeifman, A.; Satin, Y.; Kovalev, I.; Razumchik, R.; Korolev, V. Facilitating Numerical Solutions of Inhomogeneous Continuous Time Markov Chains Using Ergodicity Bounds Obtained with Logarithmic Norm Method. *Mathematics* **2021**, *9*, 42. [[CrossRef](#)]
15. Arns, M.; Buchholz, P.; Panchenko, A. On the numerical analysis of inhomogeneous continuous-time Markov chains. *Inform. J. Comput.*, **2010**, *22*, 416–432. [[CrossRef](#)]
16. Schwarz, J.A.; Selinka, G.; Stollitz, R. Performance analysis of time-dependent queueing systems: Survey and classification. *Omega* **2016**, *63*, 170–189. [[CrossRef](#)]
17. Sidje, R.B.; Burrage, K.; Macnamara, S. Inexact uniformization method for computing transient distributions of Markov chains. *Siam J. Sci. Comput.* **2007**, *29*, 2562–2580. [[CrossRef](#)]
18. Zapreev, I.S.; Katoen, J.-P. Safe on-the-fly steady-state detection for time-bounded reachability. In Proceedings of the 3rd International Conference on the Quantitative Evaluation of Systems, Riverside, CA, USA, 11–14 September 2006. [[CrossRef](#)]
19. Down, D.; Meyn, S.P.; Tweedie, R.L. Exponential and Uniform Ergodicity of Markov Processes. *Ann. Probab.* **1995**, *23*, 1671–1691. [[CrossRef](#)]
20. Meyn, S.P.; Tweedie, R.L. *Markov Chains and Stochastic Stability*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2012.
21. Zeifman, A.; Satin, Y.; Kiseleva, K.; Korolev, V. On the Rate of Convergence for a Characteristic of Multidimensional Birth-Death Process. *Mathematics* **2019**, *7*, 477. [[CrossRef](#)]
22. Masuyama, H. Error bounds for augmented truncations of discrete-time block-monotone Markov chains under geometric drift conditions. *Adv. Appl. Probab.* **2015**, *47*, 83–105. [[CrossRef](#)]
23. Tweedie, R. L. Truncation approximations of invariant measures for Markov chains, *J. Appl. Probab.* **1998**, *35*, 517–536. [[CrossRef](#)]
24. Burak, M. R.; Korytkowski, P. Inhomogeneous CTMC Birth-and-Death Models Solved by Uniformization with Steady-State Detection. *Acm Trans. Model. Comput. Simul. (Tomacs)* **2020**, *30*, 1–18. [[CrossRef](#)]
25. Zeifman, A.I. Upper and lower bounds on the rate of convergence for nonhomogeneous birth and death processes. *Stoch. Process. Their Appl.* **1995**, *59*, 157–173. [[CrossRef](#)]
26. Granovsky B.L.; Zeifman A. Nonstationary Queues: Estimation of the Rate of Convergence. *Queueing Syst.* **2004**, *46*, 363–388. .:QUES.0000027991.19758.b4 [[CrossRef](#)]
27. Gumbel, H. Waiting Lines with Heterogeneous Servers. *Oper. Res.* **1960**, *8*, 504–511. [[CrossRef](#)]
28. Cooper, R. B. Queues with ordered servers that work at different rates. *Opsearch* **1976**, *13*, 69–78
29. Nawijn, W.M. A note on many-server queueing systems with ordered entry with applications to conveyor theory. *J. Appl. Probab.* **1983**, *20*, 144–152. [[CrossRef](#)]
30. Razumchik, R., Zaryadov, I. Stationary Blocking Probability in Multi-server Finite Queueing System with Ordered Entry and Poisson Arrivals. In *Distributed Computer and Communication Networks, Proceedings of the DCCN 2015—Communications in Computer and Information Science, Moscow, Russia, 19–22 October 2016*; Vishnevsky, V., Kozyrev, D., Eds.; Springer: Cham, Switzerland, 2016; Volume 601, pp. 344–357. .:36 [[CrossRef](#)]
31. Meykhanadzhyan, L.; Matyushenko, S.; Pyatkina, D.; Razumchik, R. Revisiting joint stationary distribution in two finite capacity queues operating in parallel. *Inform. Primen.* **2017**, *11*, 106–112. [[CrossRef](#)]
32. Baxley, R.V.N. The Multiple-Server Queue with Heterogeneous Service Times. Ph.D. Thesis, Georgia Institute of Technology, Atlanta, GA, USA, 1973.
33. Yu, O. S., Neuts, M. F. The steady state solution of a heterogeneous-server queue with Erlang service times. *Tims Stud. Manag. Sci.*, **1977**, *7*, 199–213.
34. Grassmann, W.K.; Zhao, Y.Q. Heterogeneous Multiserver Queues with General Input. *Infor. Inf. Syst. Oper. Res.* **1997**, *35*, 208–224. [[CrossRef](#)]
35. Zeifman, A.; Razumchik, R.; Satin, Y.; Kovalev, I. Ergodicity Bounds for the Markovian Queue with Time-Varying Transition Intensities, Batch Arrivals and One Queue Skipping Policy. *Appl. Math. Comput.* **2021**, *395*, 125846. [[CrossRef](#)]

36. Zeifman, A.I.; Korolev, V.Y.; Razumchik, R.V.; Satin, Y.A.; Kovalev, I.A. Limiting Characteristics of Queueing Systems with Vanishing Perturbations. *Dokl. Math.* **2022**, *106*, 375–379. [[CrossRef](#)]
37. Zeifman, A.; Korolev, V.; Satin, Y. Two approaches to the construction of perturbation bounds for continuous-time Markov chains *Mathematics* **2020**, *8*, 253. [[CrossRef](#)]
38. Satin, Y.A.; Razumchik, R.V.; Zeifman, A.I.; Kovalev, I.A. Upper bound on the rate of convergence and truncation bound for non-homogeneous birth and death processes on *Z. Appl. Math. Comput.* **2022**, *423*, 127009. [[CrossRef](#)]

**Disclaimer/Publisher's Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.