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On ν-Level Interval of Fuzzy Set for Fractional Order Neutral Impulsive Stochastic Differential System

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Abstract: The main concern of this paper is to investigate the existence and uniqueness of a fuzzy neutral impulsive stochastic differential system with Caputo fractional order driven by fuzzy Brownian motion using fuzzy numbers with bounded ν-level intervals that are convex, normal and upper-semicontinuous. Fuzzy Itô process, Grönwall's inequality and the Banach fixed-point theorem are employed to probe the local and global existence. An analytical example is provided to examine the theoretical results.

Keywords: fuzzy stochastic differential equation; fuzzy Itô process; neutral impulsive system; Caputo fractional derivative; Banach fixed-point theorem

MSC: 60H10; 34F05; 34A07; 34A12

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1. Introduction

Fractional order derivatives have gained prominence and appeal among researchers in recent decades. The primary advantage of fractional calculus is that fractional derivatives can be used to describe the memory and hereditary qualities of diverse materials and processes. Fractional calculus is important in many disciplines of practical research, including blood flow, control theory, economics, etc. [1,2]. Several articles [3–5] have reported the existence results for neutral differential equations with fractional order in the Caputo sense. Various proposed fuzzy differential systems furnish better models in the frame of ambiguity. Noise frequently causes fluctuations in deterministic systems so it is inevitable to replace deterministic models with stochastic models. The behaviour of uncertainty is concerned with the stochastic mutability of an entire attainable consequence of states though ambiguity connected with the blurred confine of models. Stochastic differential equations (SDEs) and fuzzy SDEs are effective tools for modelling fuzzy random phenomena. Dynamical systems with fuzziness are designed by fuzzy stochastic differential systems affected by fuzzy stochastic noise. The basic statistical properties of fuzzy stochastic processes are considered in [6].

Many experts are investigating fuzzy postulates [7–12], which currently have increasing applications in a wide range of engineering disciplines and also in finance [13–16]. Many interpretations of facts in economics, smart fluid technology, bioinformatics, and neural networks have successfully used perturbation terms in crisp stochastic differential systems [17]. A number of studies [12,18–23] have employed fuzzy stochastic differential and integral equations for applications, including the presentation of a model for application in population dynamics. Zhu et al. [24] examined the uniqueness and existence of stochastic set-valued differential equations with fractional Brownian motion, and Jafari et al. [15] studied fuzzy stochastic differential equations driven by fractional Brownian motion. Arhrrabi et al. [25] inspected the existence, uniqueness and stability for fuzzy fractional stochastic differential systems driven by fractional Brownian motion using an approximation method.

The reason for investigators to concentrate on fuzzy impulsive differential equations is the sudden switch over of conditions in numerous processes. Benchohra et al. [26] investigated fuzzy solutions for impulsive differential equations. Priyadharshini et al. [27] proposed the existence and uniqueness of fuzzy fractional stochastic differential systems with impulses with granular derivatives using a contraction principle. Bao et al. [28] and Chadha et al. [29] undertook work on impulsive neutral stochastic differential equations with infinite delay in the frame of the Caputo fractional order. Narayanamoorthy et al. [30] studied the approximate controllability for impulsive linear fuzzy stochastic differential equations under non-local conditions. Maheswari et al. [31] and Anguraj et al. [32] examined the existence and stability behavior of neutral impulsive stochastic differential equations. The results of [33] showed that impulses can facilitate the stability of stochastic differential equations when the original system is not stable.

Examples of manifesting the existence and uniqueness of various systems have been reported in numerous articles. Balasubramainiam et al. [34] studied the existence and uniqueness of fuzzy solutions for semilinear fuzzy integro-differential equations using fuzzy numbers whose values were normal, convex upper-semicontinuous, and with compactly supported intervals. Abuasbeh et al. [16] investigated the existence and uniqueness of fuzzy fractional stochastic differential systems driven by fractional Brownian motion with non-local conditions using an approximation method. Arhrrabi et al. [35] explored the existence and uniqueness of solutions for fuzzy fractional stochastic differential equations under generalized Hukuhara differentiability, using the principle of contraction mappings. Luo et al. [36] explored a new kind of Caputo fractional fuzzy stochastic differential equation with delay and established the existence using a monotone iterative technique. Using a fuzzy controller function, Chaharpashlou et al. [37] stabilized the random operator for a type of fractional stochastic Voltera integral equation. The authors of [38] addressed the exact controllability for Caputo fuzzy fractional evolution equations in the credibility space from the perspective of the Liu process. The concepts of global and local existence and uniqueness were presented in [39] for the fuzzy fractional functional evolution equation by employing the contraction principle and successive approximation.

The determined efforts which have inspired this article include the work of Anil kumar et al. [40] on fuzzy fractional differential systems with non-instantaneous impulses, which inspect the local and global existence using contraction mapping and Grönwall's inequality. The weak uniqueness of fuzzy stochastic differential equations driven by fuzzy Brownian motion was explored by Didier et al. [41]. The fuzzy stochastic integral driven by fuzzy Brownian motion with metric between fuzzy numbers and the limit of sequences of fuzzy numbers was taken into account in [42]. The primary purpose of this paper is to investigate the existence and uniqueness using a ν-cut method of the fuzzy fractional order neutral impulsive stochastic differential system operated by fuzzy Brownian motion.

We explore below the solution for the considered fractional order fuzzy neutral impulsive stochastic differential system.

$$\begin{aligned} {}^{c}D^{\mu}_{\zeta}[z(\zeta)+q(\zeta,z(\zeta))] &= h(\zeta,z(\zeta))d\zeta+\varrho(\zeta,z(\zeta))d\tilde{B}_{\zeta}, \qquad \zeta\in[0,T], \qquad \zeta\neq\zeta_m \\ \Delta z(\zeta_m) &= b_m(z(\zeta_m)), \qquad \zeta=\zeta_m, \qquad m=1,2,3,4....k \\ z(\zeta) &= \eta(\zeta), \qquad \zeta\in[-\tau,0] \end{aligned} \tag{1}$$

where ${}^{c}D^{\mu}_{\zeta}$ is a $\mu\in(0,1)$ order Caputo derivative, and $z(\zeta)$ is the fuzzy function structure. Here, $0=\zeta_0\leq\zeta_1\leq ....\zeta_i\leq\zeta_{i+1}=T<\infty$. $\tilde{B}_{\zeta}$ is the fuzzy Brownian motion. The pertinent functions are $q,h,\varrho : V\times\Omega\times F^d_R\to F^d_R$; where F^d_R signify the group of fuzzy numbers that are convex, normal, upper-semi continuous with bounded ν-level intervals. $z(\zeta_m^+)$ and $z(\zeta_m^-)$ depict the right and left limit of the fuzzy function at ζ_m, $\Delta z(\zeta_m) = z(\zeta_m^+) - z(\zeta_m^-)$ where $z(\zeta_m^+) = \lim_{\epsilon \in 0^+} z(\zeta_m + \epsilon)$ and $z(\zeta_m^-) = \lim_{\epsilon \in 0^+} z(\zeta_m - \epsilon)$, $\eta = \{\eta(\zeta), \zeta \in [-\tau, 0]\}, \eta : [-\tau, 0] \to F_R^d$ is a continuous function.

The contribution of this article are:

(i) This study explores the existence of fuzzy neutral impulsive fractional stochastic systems with fuzzy metrices and fuzzy Brownian motion for the first time in the literature.
(ii) An example is provided to illustrate the theory.

The article is structured as follows: In Section 2, the core definitions that are fundamental to the paper are presented. In Section 3, the local existence and uniqueness of the considered system is established. In Section 4, the global existence and uniqueness of the considered system is established. In Section 5, an example with ν-cut is provided for the proposed system. Finally, the conclusions are drawn in Section 6.

2. Preliminaries

This part of the article sets out some notations, rudimentary definitions, and major lemmas that are utilized for the leading proof. Let us indicate C^F as the collection of all fuzzy valued continuous functions on V: = [0, T] and L^F as the collection of all fuzzy valued Labesgue integrable functions on V. In addition, we specify $PC^F(U, F_R^d) = \{z : U \to F_R^d\}$ as the space of fuzzy functions that are piecewise continuous functions where $U = [-\tau, 0) \bigcup [0, T]$.

Definition 1. ***Hausdorff metric*** [9]: *The distance of two sets that are nonempty bounded subsets of R^d as*

$$\mathbf{d}_{\mathcal{H}}(y, c) := \max\left\{ \sup_{\hat{y} \in y} \inf_{\hat{c} \in c} ||\hat{y} - \hat{c}||_{R^d}, \sup_{\hat{c} \in c} \inf_{\hat{y} \in y} ||\hat{y} - \hat{c}||_{R^d} \right\} \qquad y, c \in K_R^d$$

We elucidate the structure (Ω, A^F, P) to be the complete probability space with filtration $\{A_\zeta^F \in V := [0, T]\}$ contented by regular conditions. The proceeding values from K_R^d, i.e., (the collection of all non-empty subsets of R^d that are convex and compact). The fuzzy Brownian motion $\{\tilde{B}_\zeta, \zeta \in V\}$ is defined on $(\Omega, A^F, \{A_\zeta^F\}_{\zeta \in V}, P)$.

Signifying $M(\Omega, A^F, K_R^d)$ as the family of A^F-measurable multi-valued random variables and also $L^p(\Omega, A^F, P, F_R^d)$ as the set of all L^p-integrably bounded. The function $G : \Omega \to K_R^d$ with state $\{\omega \in \Omega : G(\omega) \cup O \neq \Phi\} \in A^F$ is satisfied for every open set $O \in R^d$. The function $F : \Omega \to K_R^d$ is a multi-valued random variable if, and only if, F is a $A^F \backslash B_{d_H}$ measurable function (B_{d_H} denotes the Borel σ-algebra generated by the metric d_H in K_R^d).

Definition 2. ***Fuzzy random variable*** [9]: *A mapping $z : \Omega \to F_R^d$ is claimed to be a fuzzy random variable if $[z]^\nu : \Omega \to K_R^d$ is an A^F-measurable set valued random variable $\forall\ \nu \in (0, 1]$ with $[z]^\nu(\omega) := [z(\omega)]^\nu$.*

Considering the σ-algebra B_{d_∞} generated by the topology induced by the metric d_∞ in K_R^d, the interpretation is analogous to the $A^F \backslash B_{d_\infty}$ measurability for $z : \Omega \to F_R^d$.

Definition 3. ***L^p-integrably bounded*** [23]: *Fuzzy random variable $z : \Omega \to F_R^d$ to L^p-integrable that are bounded , $p \geq 1$ if $\omega \mapsto [z(\omega)]^\nu \in L^p(\Omega, A^F, P, F_R^d, \hat{\delta}_p)$ is complete.*

Definition 4. ***Fuzzy stochastic process*** [41]: *We term the mapping $z : V \times \Omega \to F_R^d$ a fuzzy stochastic process if $\forall\ \zeta \in V$ the function $z(\zeta) : \Omega \to F_R^d$ is a fuzzy random variable. We affirm that a fuzzy stochastic process z is $\mathbf{d}_\infty$-continuous if a stochastic process h is continuous and it is $\{A_\zeta^F\}_{\zeta \in V}$ adapted and measurable.*

Definition 5. ***Fuzzy Brownian motion*** *[41]: A fuzzy stochastic process* $\{z(\zeta), \zeta \in [0,T], 0 < T < \infty\}$ *is a fuzzy Brownian motion on the space* (Ω, A^F, P) *if, and only if,* $\forall \nu \in (0,1]$*, the process*

$$[\tilde{B}_\zeta]^\nu = [(\tilde{B}_\zeta)^\nu_q, (\tilde{B}_\zeta)^\nu_r]$$

is an interval-Brownian Motion on (Ω, A^F, P) *and* $\tilde{B}_\zeta = \cup_{\nu\in(0,1]}[\tilde{B}^\nu_\zeta]$

Definition 6. ***Fuzzy Membership function*** *[7]: The mapping* $\mathcal{M}_{\mathfrak{fn}} : R^d \to [0,1]$ *that satisfies*

(1) *If* $\mathcal{M}_{\mathfrak{fn}}(\rho) = 1, \rho \in R^d$*, then* $\mathfrak{fn}$ *is interpreted as complete membership.*
(2) *If* $0 < \mathcal{M}_{\mathfrak{fn}}(\rho) < 1$*, then* $\mathfrak{fn}$ *is interpreted as partial membership.*
(3) *If* $\mathcal{M}_{\mathfrak{fn}}(\rho) = 0$*, then* $\mathfrak{fn}$ *is interpreted as non-membership.*

Definition 7. ***Fuzzy number*** *[7]: A fuzzy set* $\mathfrak{fn}$ *is claimed as a fuzzy number if it assures*

(1) $\mathfrak{fn}$ *is normal, i.e., for* $\rho \in R^d$ $\mathcal{M}_{\mathfrak{fn}}(\rho) = 1$.
(2) $\mathfrak{fn}$ *is fuzzy convex, i.e.,* $\mathfrak{fn}(\delta\rho + (1-\delta)(\hat{\rho})) \geq min\{\mathfrak{fn}(\rho) + \mathfrak{fn}(\hat{\rho})\}, \forall\delta \in [0,1], \rho, \hat{\rho} \in R^d$
(3) $\mathfrak{fn}$ *is upper semi-continuous on* R^d
(4) $\mathfrak{fn}$ *is compactly supported, i.e.,* $cl\{\rho \in R^d; \mathcal{M}_{\mathfrak{fn}}(\rho) > 0\}$ *is compact.*

Definition 8. ***ν-level set*** *[11]: The ν-level set of the fuzzy set* $\mathfrak{fn}$ *is defined as*

$$[\mathfrak{fn}]^\nu = \{\rho\backslash\rho \in R^d; \mathfrak{fn}(\rho) > 0\}, \qquad \nu \in (0,1]$$

and $[\mathfrak{fn}]^0 = cl\{\rho \in R^d; \mathfrak{fn}(\rho) \geq 0\}$ *, cl denotes closure and* $[\mathfrak{fn}]^0$ *is compact.*

Define the ν-level set of $\mathfrak{fn}$ *as* $[\mathfrak{fn}]^\nu = [\mathfrak{fn}^\nu_l, \mathfrak{fn}^\nu_r], \mathfrak{fn}_l, \mathfrak{fn}_r$ *are left and right branch. In consequence, for any two fuzzy numbers,*

$$\begin{aligned} \hat{y} + \hat{c} &= [\hat{y}]^\nu + [\hat{c}]^\nu = \{\rho + \hat{\rho} : \rho \in [\hat{y}]^\nu, \hat{\rho} \in [\hat{c}]^\nu\}, \nu \in (0,1] \\ \sigma\hat{c} &= \{\sigma\rho : \hat{\rho} \in [\hat{c}]^\nu\}, \nu \in (0,1] \end{aligned}$$

Definition 9 ([9])**.** *The distance between fuzzy numbers in the Hausdorff space is defined as*

$$\mathbf{d}_\mathcal{H}(y,c) := \sup_{\nu\in(0,1]} \max\{\mathbf{d}(|y^\nu_l - c^\nu_l|, |y^\nu_r - c^\nu_r|)\} = \sup_{\nu\in(0,1]} \max\{\mathbf{d}([y]^\nu, [c]^\nu)\}$$

Clearly $(F^d_R, \mathbf{d}_\mathcal{H})$ *is a complete metric space and the metric sustains the properties*

(1) $\mathbf{d}_\mathcal{H}(y + c, x + c) = \mathbf{d}(y,x), \ \forall y, x \in F^d_R$
(2) $\mathbf{d}_\mathcal{H}(y + c, x + a) = \mathbf{d}_\mathcal{H}(y,x) + \mathbf{d}_\mathcal{H}(c,a), \ \forall y, x, c, a \in F^d_R$
(3) $\mathbf{d}_\mathcal{H}(\lambda y, \lambda x) = |\lambda|\mathbf{d}_\mathcal{H}(y,x), \ \forall y, x \in F^d_R$

Definition 10 ([10])**.** *We define*

$$\mathbf{d}_\mathcal{H}([y]^\nu, [x]^\nu) = \max\{\mathbf{d}([y]^\nu, [x]^\nu), \mathbf{d}([x]^\nu, [y]^\nu); \nu \in (0,1]\} \ y, x \in F^d_R$$

Hence, $(F^d_R, \mathbf{d}_\mathcal{H})$ *forms a complete metric space.*

Definition 11 ([10])**.** *The supremum metric* $\mathbf{d}_\infty$ *on* F^d_R *is defined by*

$$\mathbf{d}_\infty(y,x) = \sup\{\mathbf{d}_\mathcal{H}([y]^\nu, [x]^\nu) : \ \nu \in (0,1] \ \forall y, x \in F^d_R\}$$

Now, $\mathbf{d}_\infty$ *is a metric in* F^d_R *and* $(F^d_R, \mathbf{d}_\infty)$ *forms a complete metric space.*

Definition 12 ([10])**.** *We define the metric*

$$\tilde{\mathcal{H}}_1(y,x) = \sup\{\mathbf{d}_\infty(y(\zeta), x(\zeta)) : \zeta \in U, y, x \in PC^F(U, F^d_R)\}$$

It is direct that $(PC^F(U, F^d_R), \tilde{\mathcal{H}}_1)$ *is a complete metric space.*

Lemma 1 ([10]). *For $p, q \in F_R^d, \nu \in (0,1]$, we have*

$$\begin{aligned} [p+q]^\nu &= [p_u^\nu + q_u^\nu,\ p_v^\nu + q_v^\nu] \\ [p \times q]^\nu &= [min\{h_i^\nu, h_j^\nu\},\ max\{h_i^\nu,\ h_j^\nu\}],\ i,j = u,v \\ [p-q]^\nu &= [p_u^\nu - q_u^\nu,\ p_v^\nu - q_v^\nu] \end{aligned}$$

Definition 13. ***Fuzzy integral*** *[8]: The integral of a function $z(\zeta) : V \to F_R^d$, which is measurable and integrably bounded is in the configuration*

$$\begin{aligned} \left[\int_0^\zeta z(\zeta)d\zeta\right]^\nu &:= \int_0^\zeta [z(\zeta)]^\nu d\zeta \\ &= \left\{\int_0^\zeta \hat{z}(\zeta)d\zeta \backslash \hat{z} : V \to F_R^d is\ a\ measurable\ selection\ for [z(.)]^\nu\ \ \nu \in (0,1]\right\} \end{aligned}$$

Theorem 1. *(Fuzzy Itô process) [41] For $(\Upsilon(\zeta))_{\zeta \geq 0},\ (\hat{\Upsilon}(\zeta))_{\zeta \geq 0} \in \mathcal{L}^2(F_R^d)$, we have*

$$\mathbf{E}\left[\mathbf{d}_\infty^2\left(\int_0^\zeta \Upsilon(\kappa)d\tilde{B}_\kappa, \int_0^\zeta \hat{\Upsilon} d\tilde{B}_\kappa\right)\right] \leq \mathbf{E}\left[\int_0^\zeta \mathbf{d}_\infty^2(\Upsilon(\kappa), \hat{\Upsilon}(\kappa))d\kappa\right]$$

Proof. From the definition of $\mathbf{d}_\infty$ for $\zeta \geq 0$

Now, consider $\mathbf{E}\left[\mathbf{d}_\infty^2\left(\int_0^\zeta \Upsilon(\kappa)d\tilde{B}_\kappa, \int_0^\zeta \hat{\Upsilon}(\kappa)d\tilde{B}_\kappa\right)\right]$

$$\begin{aligned} &= \mathbf{E}[\sup_{\nu \in (0,1]} \mathbf{d}_{\mathcal{H}}^2\left(\int_0^\zeta [(\Upsilon)(\kappa)d\tilde{B}_\kappa]^\nu, \int_0^\zeta [(\hat{\Upsilon})(\kappa)d\tilde{B}_\kappa]^\nu\right)] \\ &\leq \mathbf{E}[\sup_{\nu \in (0,1]} \mathbf{d}_{\mathcal{H}}^2\left(\int_0^\zeta [(\Upsilon)]^\nu(\kappa)(d\tilde{B}_\kappa)_q^\nu, \int_0^\zeta [(\hat{\Upsilon})]^\nu(\kappa)(d\tilde{B}_\kappa)_q^\nu)\right)] \\ &\leq \mathbf{E}[\sup_{\nu \in (0,1]} \int_0^\zeta \mathbf{d}_{\mathcal{H}}^2([\Upsilon]^\nu(\kappa), [\hat{\Upsilon}]^\nu(\kappa))d\kappa] \\ &\leq \mathbf{E}[\int_0^\zeta \sup_{\nu \in (0,1]} \mathbf{d}_{\mathcal{H}}^2([\Upsilon]^\nu(\kappa), [\hat{\Upsilon}]^\nu(\kappa))d\kappa] \\ &\leq \mathbf{E}[\int_0^\zeta \mathbf{d}_\infty^2(\Upsilon(\kappa), \hat{\Upsilon}(\kappa))d\kappa] \end{aligned}$$

We have $d\tilde{B}_\kappa = [(d\tilde{B}_\kappa^{\nu_q}), (d\tilde{B}_\kappa)^{\nu_r}]$

Hence, the proof. □

Definition 14 ([2]). *Let $z : [u,v] \to F_R^d$, the fuzzy Riemann–Liouville integral of fuzzy valued function z is*

$$(\mathfrak{I}_{u^+}^\mu z)(\zeta) = \frac{1}{\Gamma\mu}\int_u^\zeta (\zeta-\kappa)^{\mu-1} z(\kappa)d\kappa,\ u \leq \zeta,\ 0 < \mu \leq 1$$

Definition 15 ([2]). *The fuzzy Caputo differentiability of z is*

$${}^c\mathfrak{D}_{u^+}^\mu z(\zeta) = \frac{1}{\Gamma n-\mu}\int_u^\zeta (\zeta-\kappa)^{n-\mu-1}(\mathfrak{D}z^n)(\kappa)d\kappa = \mathfrak{I}_{u^+}^{1-\mu}(\mathfrak{D}z^{(n)})(\zeta),\ \zeta > u\ , n-1 < \mu < n$$

In particular, for $\mu \in (0,1)$

$${}^c\mathfrak{D}^\mu z(\zeta) = \frac{1}{\Gamma\mu}\int_u^\zeta (\zeta-\kappa)^{\mu-1} z(\kappa)d\kappa$$

Lemma 2. *If $z(\zeta) \in \mathfrak{C}^{\mathfrak{F}} \bigcap \mathfrak{L}^{\mathfrak{F}}, 0 < \mu < 1$, then the unique solution of*

$${}^c\mathfrak{D}_\zeta^\mu z(\zeta) = u(\zeta), \zeta \in [0,T]$$

is given by

$$z(\zeta) = \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} u(\kappa)d\kappa$$

Definition 16. *An A_ζ^F -adapted fuzzy stochastic process $z : V \times \Omega \to F_R^d$ is reckoned to be the solution of the proposed system(1) if $z(0) = \eta(0)$ where z is $\mathbf{d}_\infty$ continuous , $z \in \mathbf{L}^p(V \times \Omega, \hat{N}; F_R^d)$ is disposed to be as*

$$z(\zeta) = \begin{cases} \eta(\zeta), \quad \zeta \in [-\tau, 0] \\ \eta(0) + q(0,\eta(0)) - q(\zeta, z(\zeta)) + \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, z(\kappa))d\kappa \\ \quad + \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} \varrho(\kappa, z(\kappa))d\tilde{B}_\kappa, \quad \forall\zeta \in [0,\zeta_1] \\ \eta(0) + q(0,\eta(0)) - q(\zeta, z(\zeta)) + \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, z(\kappa))d\kappa \\ \quad + \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} \varrho(\kappa, z(\kappa))d\tilde{B}_\kappa + \sum_{m=1}^k b_m(z(\zeta_m)), \\ \quad \forall\zeta \in (\zeta_m, \zeta_{m+1}] \end{cases} \tag{2}$$

3. Existence of Local Solutions via Contraction Principle

Theorem 2. *If the hypothesis*

$\mathfrak{H}(1)$ *For η the A^F is measurable, we retain*

$$\mathbf{E}\mathbf{d}_\infty^2(\eta(\zeta), 0) < \infty$$

$\mathfrak{H}(2)$ *For all q,h,ϱ and $\Upsilon, \hat{\Upsilon}$, we retain*

$$\begin{aligned} &(i)\ \mathbf{d}_{\mathcal{H}}^2([q(\zeta,\Upsilon(\zeta))]^\nu, [q(\zeta,\hat{\Upsilon}(\zeta))]^\nu) \le \hat{q}\mathbf{d}_{\mathcal{H}}^2([\Upsilon(\zeta)]^\nu, [\hat{\Upsilon}(\zeta)]^\nu) \\ &(ii)\ \mathbf{d}_{\mathcal{H}}^2([h(\zeta,\Upsilon(\zeta))]^\nu, [h(\zeta,\hat{\Upsilon}(\zeta))]^\nu) \le \hat{h}\mathbf{d}_{\mathcal{H}}^2([\Upsilon(\zeta)]^\nu, [\hat{\Upsilon}(\zeta)]^\nu) \\ &(iii)\ \mathbf{d}_{\mathcal{H}}^2([\varrho(\zeta,\Upsilon(\zeta))]^\nu, [\varrho(\zeta,\hat{\Upsilon}(\zeta))]^\nu) \le \hat{\varrho}\mathbf{d}_{\mathcal{H}}^2([\Upsilon(\zeta)]^\nu, [\hat{\Upsilon}(\zeta)]^\nu) \end{aligned}$$

$\mathfrak{H}(3)$ *For b_m, we retain*

$$\mathbf{d}_{\mathcal{H}}^2([b_m(\Upsilon(\zeta))]^\nu, [b_m(\hat{\Upsilon}(\zeta))]^\nu) \le \hat{\mathfrak{b}}_{\mathrm{m}}\mathbf{d}_{\mathcal{H}}^2([\Upsilon(\zeta)]^\nu, [\hat{\Upsilon}(\zeta)]^\nu)$$

is satisfied, then system (1) possibly has a local unique solution on U.

Proof. Defining an operator $\Theta : PC^F(U, F_R^d) \to: PC^F(U, F_R^d)$

The solution of the system (1) is

$$(\Theta\Upsilon)(\zeta) = \begin{cases} \eta(\zeta), \quad \zeta \in [-\tau, 0] \\ \eta(0) + q(0,\eta(0)) - q(\zeta, \Upsilon(\zeta)) + \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, \Upsilon(\kappa))d\kappa \\ \quad + \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} \varrho(\kappa, \Upsilon(\kappa))d\tilde{B}_\kappa, \quad \forall\zeta \in [0,\zeta_1] \\ \eta(0) + q(0,\eta(0)) - q(\zeta, \Upsilon(\zeta)) + \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, \Upsilon(\kappa))d\kappa \\ \quad + \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} \varrho(\kappa, \Upsilon(\kappa))d\tilde{B}_\kappa + \sum_{m=1}^k b_m(\Upsilon(\zeta_m)), \\ \quad \forall\zeta \in (\zeta_m, \zeta_{m+1}] \end{cases} \tag{3}$$

Now, we show that the operator Θ owns a fixed point, that provides the solution of the proposed system (1). We crack the proof over three segments.

Case 1: If $\zeta \in [-\tau, 0]$ and $\Upsilon, \hat{\Upsilon} \in PC^F(U, F_R^d)$, we know that

$$(\Theta\Upsilon)\zeta = \eta(\zeta) \;\; and \;\; (\Theta\hat{\Upsilon})\zeta = \eta(\zeta)$$

Therefore

$$\tilde{\mathcal{H}}_1^2(\Theta\Upsilon(\zeta), \Theta\hat{\Upsilon}(\zeta)) = 0$$

Hence, Θ is a contraction in $[-\tau, 0]$
Case 2: When $\zeta \in [0, \zeta_1]$ and $\Upsilon, \hat{\Upsilon} \in PC^F(U, F_R^d)$, we could explore

$$\begin{aligned}(\Theta\Upsilon)(\zeta) = \eta(0) + q(0, \eta(0)) - q(\zeta, \Upsilon(\zeta)) &+ \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, \Upsilon(\kappa))d\kappa \\ &+ \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} \varrho(\kappa, \Upsilon(\kappa))d\tilde{B}_\kappa \\ (\Theta\hat{\Upsilon})(\zeta) = \eta(0) + q(0, \eta(0)) - q(\zeta, \hat{\Upsilon}(\zeta)) &+ \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, \hat{\Upsilon}(\kappa))d\kappa \\ &+ \frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} \varrho(\kappa, \hat{\Upsilon}(\kappa))d\tilde{B}_\kappa\end{aligned}$$

Now, $\mathbf{E}\big[\mathbf{d}_{\mathcal{H}}^2\big([(\Theta\Upsilon)(\zeta)]^\nu, [(\Theta\hat{\Upsilon})(\zeta)]^\nu\big)\big]$

$$\begin{aligned}= \;\; &\mathbf{E}\Bigg[\mathbf{d}_{\mathcal{H}}^2\Bigg([\eta(0)]^\nu + [q(0,\eta(0))]^\nu - [q(\zeta, \Upsilon(\zeta))]^\nu + \left[\frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, \Upsilon(\kappa))d\kappa\right]^\nu + \\ &\left[\frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\varrho(\kappa, \Upsilon(\kappa))d\tilde{B}_\kappa\right]^\nu, \; [\eta(0)]^\nu + [q(0,\eta(0))]^\nu - [q(\zeta, \hat{\Upsilon}(\zeta))]^\nu + \\ &\left[\frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, \hat{\Upsilon}(\kappa))d\kappa\right]^\nu + \left[\frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\varrho(\kappa, \hat{\Upsilon}(\kappa))d\tilde{B}_\kappa\right]^\nu\Bigg)\Bigg]\end{aligned}$$

$$\begin{aligned}\leq \;\; &3\mathbf{E}\Big[\mathbf{d}_{\mathcal{H}}^2\Big(-[q(\zeta, \Upsilon(\zeta))]^\nu, -[q(\zeta, \hat{\Upsilon}(\zeta))]^\nu\Big)\Big] \\ &+3\mathbf{E}\Bigg[\mathbf{d}_{\mathcal{H}}^2\Bigg(\left[\frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, \Upsilon(\kappa))d\kappa\right]^\nu, \left[\frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1} h(\kappa, \hat{\Upsilon}(\kappa))d\kappa\right]^\nu\Bigg)\Bigg] \\ &+3\mathbf{E}\Bigg[\mathbf{d}_{\mathcal{H}}^2\Bigg(\left[\frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\varrho(\kappa, \Upsilon(\kappa))d\tilde{B}_\kappa\right]^\nu, \left[\frac{1}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\varrho(\kappa, \hat{\Upsilon}(\kappa))d\tilde{B}_\kappa\right]^\nu\Bigg)\Bigg]\end{aligned}$$

By using the hypothesis and Theorem 1, we have

$$\begin{aligned}\mathbf{E}[\mathbf{d}_{\mathcal{H}}^2\big([(\Theta\Upsilon)(\zeta)]^\nu, [(\Theta\hat{\Upsilon})(\zeta)]^\nu\big)] \;\; \leq \;\; &-3\hat{\mathfrak{q}}\mathbf{E}[\mathbf{d}_{\mathcal{H}}^2\big([\Upsilon(\zeta)]^\nu, [\hat{\Upsilon}(\zeta)]^\nu\big)] \\ &+\frac{3\hat{h}}{\Gamma\mu}\mathbf{E}[\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{d}_{\mathcal{H}}^2\big([\Upsilon(\kappa)]^\nu, [\hat{\Upsilon}(\kappa)]^\nu\big)d\kappa] \\ &+\frac{3\hat{\varrho}}{\Gamma\mu}\mathbf{E}[\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{d}_{\mathcal{H}}^2\big([\Upsilon(\kappa)]^\nu, [\hat{\Upsilon}(\kappa)]^\nu\big)d\kappa]\end{aligned}$$

Now, by Definition 11, we have

$$
\begin{aligned}
&\mathbf{E}[\mathbf{d}_\infty^2(\Theta\Upsilon(\zeta),\Theta\hat{\Upsilon}(\zeta))] = \mathbf{E}[\sup_{\nu\in(0,1]} \mathbf{d}_{\mathcal{H}}^2([\Theta\Upsilon(\zeta)]^\nu,[\Theta\hat{\Upsilon}(\zeta)]^\nu)] \\
\leq\;& -3\hat{\mathfrak{q}}\mathbf{E}[\sup_{\nu\in(0,1]} \mathbf{d}_{\mathcal{H}}^2([\Upsilon(\zeta)]^\nu,[\hat{\Upsilon}(\zeta)]^\nu)] + \frac{3\hat{h}}{\Gamma\mu}\mathbf{E}[\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{d}_{\mathcal{H}}^2 \sup_{\nu\in(0,1]}([\Upsilon(\kappa)]^\nu,[\hat{\Upsilon}(\kappa)]^\nu)d\kappa] \\
&+\frac{3\hat{\varrho}}{\Gamma\mu}\mathbf{E}[\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{d}_{\mathcal{H}}^2 \sup_{\nu\in(0,1]}([\Upsilon(\kappa)]^\nu,[\hat{\Upsilon}(\kappa)]^\nu)d\kappa] \\
\leq\;& -3\hat{\mathfrak{q}}\mathbf{E}[\mathbf{d}_\infty^2(\Upsilon(\zeta),\hat{\Upsilon}(\zeta))] + \frac{3\hat{h}}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{E}[\mathbf{d}_\infty^2(\Upsilon(\kappa),\hat{\Upsilon}(\kappa))]d\kappa \\
&+\frac{3\hat{\varrho}}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{E}[\mathbf{d}_\infty^2(\Upsilon(\kappa),\hat{\Upsilon}(\kappa))]d\kappa
\end{aligned}
$$

According to Definition 12, we have

$$
\begin{aligned}
&\mathbf{E}[\tilde{\mathcal{H}}_1^2(\Theta\Upsilon,\Theta\hat{\Upsilon})] = \mathbf{E}[\sup_{\zeta\in[0,\zeta_1]} \mathbf{d}_\infty^2(\Theta\Upsilon,\Theta\hat{\Upsilon})] \\
\leq\;& -3\hat{\mathfrak{q}}\mathbf{E}[\sup_{\zeta\in[0,\zeta_1]} \mathbf{d}_\infty^2(\Upsilon(\zeta),\hat{\Upsilon}(\zeta))] + \frac{3\hat{h}}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{E}[\sup_{\zeta\in[0,\zeta_1]} \mathbf{d}_\infty^2(\Upsilon(\kappa),\hat{\Upsilon}(\kappa))]d\kappa \\
&+\frac{3\hat{\varrho}}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{E}[\sup_{\zeta\in[0,\zeta_1]} \mathbf{d}_\infty^2(\Upsilon(\kappa),\hat{\Upsilon}(\kappa))]d\kappa \\
\leq\;& -3\hat{\mathfrak{q}}\mathbf{E}[\tilde{\mathcal{H}}_1^2(\Upsilon,\hat{\Upsilon})] + \frac{3\hat{h}}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{E}[\tilde{\mathcal{H}}_1^2(\Upsilon,\hat{\Upsilon})]d\kappa \\
&+\frac{3\hat{\varrho}}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\mathbf{E}[\tilde{\mathcal{H}}_1^2(\Upsilon,\hat{\Upsilon})]d\kappa
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\tilde{\mathcal{H}}_1^2(\Theta\Upsilon,\Theta\hat{\Upsilon}) &\leq -3\hat{\mathfrak{q}}\tilde{\mathcal{H}}_1^2(\Upsilon,\hat{\Upsilon}) + \frac{3\hat{h}}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\tilde{\mathcal{H}}_1^2(\Upsilon,\hat{\Upsilon})d\kappa \\
&\quad+\frac{3\hat{\varrho}}{\Gamma\mu}\int_0^\zeta (\zeta-\kappa)^{\mu-1}\tilde{\mathcal{H}}_1^2(\Upsilon,\hat{\Upsilon})d\kappa \\
\tilde{\mathcal{H}}_1^2(\Theta\Upsilon,\Theta\hat{\Upsilon}) &\leq \left[-3\hat{\mathfrak{q}} + 3\frac{\hat{h}+\hat{\varrho}}{\Gamma\mu}T^\mu\right]\tilde{\mathcal{H}}_1^2(\Upsilon,\hat{\Upsilon}) \\
\tilde{\mathcal{H}}_1^2(\Theta\Upsilon,\Theta\hat{\Upsilon}) &\leq \mathfrak{L}_1\tilde{\mathcal{H}}_1^2(\Upsilon,\hat{\Upsilon}),\ \ \textit{where}\ \ \mathfrak{L}_1 = \left[-3\hat{\mathfrak{q}} + 3\frac{\hat{h}+\hat{\varrho}}{\Gamma\mu}T^\mu\right]
\end{aligned}
$$

Hence, Θ is contraction in $[0,\zeta_1]$

Case 3: When $\zeta\in(\zeta_m,\zeta_{m+1}]$ and $\Upsilon,\hat{\Upsilon}\in PC^F(U,F_R^d)$, we could explore

$$
\begin{aligned}
(\Theta\Upsilon)(\zeta) = \eta(0)+q(0,\eta(0))-q(\zeta,\Upsilon(\zeta)) &+ \frac{1}{\Gamma\mu}\int_{\zeta_m}^\zeta (\zeta-\kappa)^{\mu-1}h(\kappa,\Upsilon(\kappa))d\kappa \\
&+ \frac{1}{\Gamma\mu}\int_{\zeta_m}^\zeta (\zeta-\kappa)^{\mu-1}\varrho(\kappa,\Upsilon(\kappa))d\tilde{B}_\kappa + \sum_{m=1}^k b_m(\Upsilon(\zeta_m)) \\
(\Theta\hat{\Upsilon})(\zeta) = \eta(0)+q(0,\eta(0))-q(\zeta,\hat{\Upsilon}(\zeta)) &+ \frac{1}{\Gamma\mu}\int_{\zeta_m}^\zeta (\zeta-\kappa)^{\mu-1}h(\kappa,\hat{\Upsilon}(\kappa))d\kappa \\
&+ \frac{1}{\Gamma\mu}\int_{\zeta_m}^\zeta (\zeta-\kappa)^{\mu-1}\varrho(\kappa,\hat{\Upsilon}(\kappa))d\tilde{B}_\kappa + \sum_{m=1}^k b_m(\hat{\Upsilon}(\zeta_m))
\end{aligned}
$$

Now, $\mathbf{E}\big[\mathbf{d}^2_{\mathcal{H}}\big([(\Theta\Upsilon)(\zeta)]^\nu,[(\Theta\hat{\Upsilon})(\zeta)]^\nu\big)\big]$

$$= \mathbf{E}\Bigg[\mathbf{d}^2_{\mathcal{H}}\Bigg([\eta(0)]^\nu + [q(0,\eta(0))]^\nu - [q(\zeta,\Upsilon(\zeta))]^\nu + \left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa,\Upsilon(\kappa))d\kappa\right]^\nu + \left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa,\Upsilon(\kappa))d\tilde{B}_\kappa\right]^\nu + \left[\sum_{m=1}^{k} b_m(\Upsilon(\zeta_m))\right]^\nu,\ [\eta(0)]^\nu + [q(0,\eta(0))]^\nu - [q(\zeta,\hat{\Upsilon}(\zeta))]^\nu + \left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa,\hat{\Upsilon}(\kappa))d\kappa\right]^\nu + \left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa,\hat{\Upsilon}(\kappa))d\tilde{B}_\kappa\right]^\nu + \left[\sum_{m=1}^{k} b_m\big(\hat{\Upsilon}(\zeta_m)\big)\right]^\nu\Bigg)\Bigg]$$

$$\leq 4\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\Big(-[q(\zeta,\Upsilon(\zeta))]^\nu, -\big[q(\zeta,\hat{\Upsilon}(\zeta))\big]^\nu\Big)\Big] + 4\mathbf{E}\Bigg[\mathbf{d}^2_{\mathcal{H}}\Bigg(\left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa,\Upsilon(\kappa))d\kappa\right]^\nu, \left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa,\hat{\Upsilon}(\kappa))d\kappa\right]^\nu\Bigg)\Bigg] + 4\mathbf{E}\Bigg[\mathbf{d}^2_{\mathcal{H}}\Bigg(\left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa,\Upsilon(\kappa))d\tilde{B}_\kappa\right]^\nu, \left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa,\hat{\Upsilon}(\kappa))d\tilde{B}_\kappa\right]^\nu\Bigg)\Bigg] + 4\mathbf{E}\Bigg[\mathbf{d}^2_{\mathcal{H}}\Bigg(\left[\sum_{m=1}^{k} b_m(\Upsilon(\zeta_m))\right]^\nu, \left[\sum_{m=1}^{k} b_m\big(\hat{\Upsilon}(\zeta_m)\big)\right]^\nu\Bigg)\Bigg]$$

By using the hypothesis and Theorem 1

$$\leq -4\hat{\mathfrak{q}}\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([\Upsilon(\zeta)]^\nu,[\hat{\Upsilon}(\zeta)]^\nu\big)\Big] + \frac{4\hat{h}}{\Gamma\mu}\mathbf{E}\left[\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{d}^2_{\mathcal{H}}\big([\Upsilon(\kappa)]^\nu,[\hat{\Upsilon}(\kappa)]^\nu\big)d\kappa\right] + \frac{4\hat{\varrho}}{\Gamma\mu}\mathbf{E}\left[\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{d}^2_{\mathcal{H}}\Big([\Upsilon(\kappa)]^\nu,\big[\hat{\Upsilon}(\kappa)\big]^\nu\Big)d\kappa\right] + 4\sum_{m=1}^{k}\hat{\mathfrak{b}}_{\mathfrak{m}}\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\Big([\Upsilon(\zeta_m)]^\nu,\big[\hat{\Upsilon}(\zeta_m)\big]^\nu\Big)\Big]$$

Now, by using Definition 11, we have

$$\mathbf{E}[\mathbf{d}^2_\infty(\Theta\Upsilon(\zeta),\Theta\hat{\Upsilon}(\zeta))] = \mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([\Theta\Upsilon(\zeta)]^\nu,[\Theta\hat{\Upsilon}(\zeta)]^\nu)]$$

$$\leq -4\hat{\mathfrak{q}}\mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([\Upsilon(\zeta)]^\nu,[\hat{\Upsilon}(\zeta)]^\nu)] + \frac{4\hat{h}}{\Gamma\mu}\mathbf{E}[\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([\Upsilon(\kappa)]^\nu,[\hat{\Upsilon}(\kappa)]^\nu)d\kappa] + \frac{4\hat{\varrho}}{\Gamma\mu}\mathbf{E}[\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([\Upsilon(\kappa)]^\nu,[\hat{\Upsilon}(\kappa)]^\nu)d\kappa] + 4\sum_{m=1}^{k}\hat{\mathfrak{b}}_{\mathfrak{m}}\mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([\Upsilon(\zeta_m)]^\nu,[\hat{\Upsilon}(\zeta_m)]^\nu)]$$

$$\leq -4\hat{\mathfrak{q}}\mathbf{E}[\mathbf{d}^2_\infty(\Upsilon(\zeta),\hat{\Upsilon}(\zeta))] + \frac{4\hat{h}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\mathbf{d}^2_\infty(\Upsilon(\kappa),\hat{\Upsilon}(\kappa))]d\kappa + \frac{4\hat{\varrho}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\mathbf{d}^2_\infty(\Upsilon(\kappa),\hat{\Upsilon}(\kappa))]d\kappa + 4\sum_{m=1}^{k}\hat{\mathfrak{b}}_{\mathfrak{m}}\mathbf{E}[\mathbf{d}^2_\infty(\Upsilon(\zeta_m),\hat{\Upsilon}(\zeta_m))]$$

According to the Definition 12, $\mathbf{E}\big[\tilde{\mathcal{H}}^2_1(\Theta\Upsilon,\Theta\hat{\Upsilon})\big] = \mathbf{E}\left[\sup_{\zeta\in[0,\zeta_1]}\mathbf{d}^2_\infty(\Theta\Upsilon,\Theta\hat{\Upsilon})\right]$

$$
\begin{aligned}
\le\ & -4\hat{\mathfrak{q}}\mathbf{E}\left[\sup_{\zeta\in[\zeta_m,\zeta_{m+1}]}\mathbf{d}^2_\infty\big(\Upsilon(\zeta),\hat{\Upsilon}(\zeta)\big)\right]+\frac{4\hat{h}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}\left[\sup_{\zeta\in[\zeta_m,\zeta_{m+1}]}\mathbf{d}^2_\infty\big(\Upsilon(\kappa),\hat{\Upsilon}(\kappa)\big)\right]d\kappa \\
& +\frac{4\hat{\varrho}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}\left[\sup_{\zeta\in[\zeta_m,\zeta_{m+1}]}\mathbf{d}^2_\infty\big(\Upsilon(\kappa),\hat{\Upsilon}(\kappa)\big)\right]d\kappa+4\sum_{m=1}^{k}\hat{\mathfrak{b}}_{\mathfrak{m}}\mathbf{E}\left[\sup_{\zeta\in[\zeta_m,\zeta_{m+1}]}\mathbf{d}^2_\infty\big(\Upsilon(\zeta_m),\hat{\Upsilon}(\zeta_m)\big)\right] \\
\le\ & -4\hat{\mathfrak{q}}\mathbf{E}\Big[\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon})\Big]+\frac{4\hat{h}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}\Big[\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon})\Big]d\kappa+\frac{4\hat{\varrho}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}\Big[\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon})\Big]d\kappa \\
& +4\sum_{m=1}^{k}\hat{\mathfrak{b}}_{\mathfrak{m}}\mathbf{E}\Big[\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon})\Big]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\tilde{\mathcal{H}}^2_1(\Theta\Upsilon,\Theta\hat{\Upsilon}) &\le -4\hat{\mathfrak{q}}\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon})+\frac{4\hat{h}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon})d\kappa \\
&\quad +\frac{4\hat{\varrho}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon})d\kappa+4\sum_{m=1}^{k}\hat{\mathfrak{b}}_m\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon}) \\
\tilde{\mathcal{H}}^2_1(\Theta\Upsilon,\Theta\hat{\Upsilon}) &\le \left[-4\hat{\mathfrak{q}}+4\frac{\hat{h}+\hat{\varrho}}{\Gamma\mu}T^\mu+4\sum_{m=1}^{k}\hat{\mathfrak{b}}_m\right]\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon}) \\
\tilde{\mathcal{H}}^2_1(\Theta\Upsilon,\Theta\hat{\Upsilon}) &\le \tilde{\mathfrak{L}}_2\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon}),\ \ \textit{where}\ \ \tilde{\mathfrak{L}}_2=\left[-4\hat{\mathfrak{q}}+4\frac{\hat{h}+\hat{\varrho}}{\Gamma\mu}T^\mu+4\sum_{m=1}^{k}\hat{\mathfrak{b}}_{\mathfrak{m}}\right]
\end{aligned}
$$

Hence, Θ is the contraction in $\zeta\in(\zeta_m,\zeta_{m+1}]$.
Consequently, we complete the proof by concluding as

$$
\tilde{\mathcal{H}}^2_1(\Theta\Upsilon,\Theta\hat{\Upsilon})=\sup_{\zeta\in V}\mathbf{d}^2_\infty(\Theta\Upsilon,\Theta\hat{\Upsilon})\le\tilde{\mathfrak{L}}\tilde{\mathcal{H}}^2_1(\Upsilon,\hat{\Upsilon})
$$

Therefore, Θ is a contraction in a strict manner on $PC^F(U,F^d_R)$ and, thus, by the Banach fixed-point theorem shows that Θ has a unique fixed point for the proposed fuzzy system. □

4. Existence of Global Solutions via Gronwall Inequality

Lemma 3. *Ref. [40] Let $\mathcal{Y}(\zeta,,\kappa)\ge 0$ be a continuous function on $0\le\zeta\le T$. If there are positive constants $\mathfrak{s},\mathfrak{r},\mu$ such that*

$$
\mathcal{Y}(\zeta,\kappa)\le\mathfrak{s}+\mathfrak{r}\int_0^\zeta(\zeta-\kappa)^{\mu-1}\mathcal{Y}(\zeta,\kappa)d\kappa,0\le\zeta\le T
$$

then there exists a constant $\mathfrak{m}$ such that $\mathcal{Y}(\zeta,\kappa)\le\mathfrak{m}$ for $0\le\zeta\le T$

Theorem 3. *Let the functions $q,h,\varrho: V\times\Omega\times F^d_R\to F^d_R$ retain the claimed assumptions and provided that*

$$
\begin{aligned}
\mathbf{d}^2_{\mathcal{H}}([q(\zeta,z(\zeta))]^\nu,[0]^\nu)\le\hat{\mathfrak{Q}}\mathbf{d}^2_{\mathcal{H}}([z(\zeta)]^\nu,[0]^\nu)\\
\mathbf{d}^2_{\mathcal{H}}([h(\zeta,z(\zeta))]^\nu,[0]^\nu)\le\hat{H}\mathbf{d}^2_{\mathcal{H}}([z(\zeta)]^\nu,[0]^\nu)\\
\mathbf{d}^2_{\mathcal{H}}([\varrho(\zeta,z(\zeta))]^\nu,[0]^\nu)\le\hat{\rho}\mathbf{d}^2_{\mathcal{H}}([z(\zeta)]^\nu,[0]^\nu)
\end{aligned}
$$

Then the system (1) has a unique solution z on $\zeta\in[-\tau,T]$.

Proof. We address the solution of the system (1), using the theorem (2) up to $||z||$ staying bounded. Thus, we need to claim z exists on $[-\tau,T]$; then it is bounded as ζ increases to T.

Here, the proof is divided into three segments

case (i) When ζ in the interval $[-\tau, 0)$, we explore

$$\tilde{\mathcal{H}}_1^2(z,0) = 0$$

case (ii) When $\zeta \in [0, \zeta_1]$, we have

$$z(\zeta) = \eta(0) + q(0,\eta(0)) - q(\zeta, z(\zeta)) + \frac{1}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa, z(\kappa))d\kappa + \frac{1}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa, z(\kappa))d\tilde{B}_\kappa$$

Now, $\mathbf{E}\big[\mathbf{d}^2_{\mathcal{H}}\big([z(\zeta)]^\nu, [0]^\nu\big)\big]$

$$= \mathbf{E}\Bigg[\mathbf{d}^2_{\mathcal{H}}\Bigg([\eta(0)]^\nu + [q(0,\eta(0))]^\nu - [q(\zeta,z(\zeta))]^\nu + \Bigg[\frac{1}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa,z(\kappa))d\kappa\Bigg]^\nu + \Bigg[\frac{1}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa,z(\kappa))d\tilde{B}_\kappa\Bigg]^\nu, [0]^\nu)\Bigg]$$

$$\begin{aligned}\leq\ & 5\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}([\eta(0)]^\nu, [0]^\nu) + 5\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([q(0,\eta(0))]^\nu, [0]^\nu\big)\Big] - 5\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([q(\zeta,z(\zeta))]^\nu, [0]^\nu\big)\Big] \\ & +5\mathbf{E}\Bigg[\mathbf{d}^2_{\mathcal{H}}\Bigg(\Bigg[\frac{1}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa,z(\kappa))d\kappa\Bigg]^\nu, [0]^\nu\Bigg)\Bigg] \\ & +5\mathbf{E}\Bigg[\mathbf{d}^2_{\mathcal{H}}\Bigg(\Bigg[\frac{1}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa,z(\kappa))d\tilde{B}_\kappa\Bigg]^\nu, [0]^\nu\Bigg)\Bigg]\end{aligned}$$

By using assumptions and Theorem 1, we have

$$\begin{aligned}\leq\ & 5\mathbf{E}[\mathbf{d}^2_{\mathcal{H}}([\eta(0)]^\nu, [0]^\nu)] + 5\mathbf{E}[\mathbf{d}^2_{\mathcal{H}}([q(0,\eta(0))]^\nu, [0]^\nu)] - 5\hat{\mathfrak{Q}}\mathbf{E}[\mathbf{d}^2_{\mathcal{H}}([z(\zeta)]^\nu, [0]^\nu)] \\ & +\frac{5\hat{H}}{\Gamma\mu}\mathbf{E}[\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{d}^2_{\mathcal{H}}([z(\kappa)]^\nu, [0]^\nu)d\kappa] + \frac{5\hat{\rho}}{\Gamma\mu}\mathbf{E}[\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{d}^2_{\mathcal{H}}([\Upsilon(\kappa)]^\nu, [0]^\nu)d\kappa]\end{aligned}$$

By Definition 11, $\mathbf{E}[\mathbf{d}^2_\infty(z(\zeta), 0)] = \mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([z(\zeta)]^\nu, [0]^\nu)]$

$$\begin{aligned}\leq\ & 5\mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([\eta(0)]^\nu, [0]^\nu)] + 5\mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([q(0,\eta(0))]^\nu, [0]^\nu)] - 5\hat{\mathfrak{Q}}\mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([z(\zeta)]^\nu, [0]^\nu)] \\ & +\frac{5\hat{H}}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([z(\zeta)]^\nu, [0]^\nu)]d\kappa + \frac{5\hat{\rho}}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}^2_{\mathcal{H}}([z(\zeta)]^\nu, [0]^\nu)]d\kappa\end{aligned}$$

$$\begin{aligned}\leq\ & 5\mathbf{E}[\mathbf{d}^2_\infty([\eta(0)]^\nu, [0]^\nu)] + 5\mathbf{E}[\mathbf{d}^2_\infty([q(0,\eta(0))]^\nu, [0]^\nu)] - 5\hat{\mathfrak{Q}}\mathbf{E}[\mathbf{d}^2_\infty(z(\zeta), 0)] \\ & +\frac{5\hat{H}}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\mathbf{d}^2_\infty(z(\zeta), 0)]d\kappa + \frac{5\hat{\rho}}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\mathbf{d}^2_\infty(z(\zeta), 0)]d\kappa\end{aligned}$$

According to Definition 12, $\mathbf{E}[\tilde{\mathcal{H}}_1^2(z,0)] = \mathbf{E}[\sup_{\zeta\in[0,\zeta_1]}\mathbf{d}^2_\infty(z(\zeta), 0)]$

$$\begin{aligned}\leq\ & 5\mathfrak{C}_1 + 5\mathfrak{C}_2 - 5\hat{\mathfrak{Q}}\mathbf{E}[\sup_{\zeta\in[0,\zeta_1]}\mathbf{d}^2_\infty(z(\zeta), 0)] + \frac{5\hat{H}}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\sup_{\zeta\in[0,\zeta_1]}\mathbf{d}^2_\infty(z(\zeta), 0)]d\kappa \\ & +\frac{5\hat{\rho}}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\sup_{\zeta\in[0,\zeta_1]}\mathbf{d}^2_\infty(z(\zeta), 0)]d\kappa\end{aligned}$$

$$\leq \quad 5\mathfrak{C}_1 + 5\mathfrak{C}_2 - 5\hat{\mathfrak{Q}}\mathbf{E}[\tilde{\mathcal{H}}_1^2(z,0)] + \frac{5\hat{H}}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\tilde{\mathcal{H}}_1^2(z,0)]d\kappa$$
$$+\frac{5\hat{\rho}}{\Gamma\mu}\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}[\tilde{\mathcal{H}}_1^2(z,0)]d\kappa$$

where $\mathfrak{C}_1 = \mathbf{d}_\infty^2(\eta(0),0),\ \mathfrak{C}_2 = \mathbf{d}_\infty^2(q(\eta(0),0))$

Thus, we have

$$\tilde{\mathcal{H}}_1^2(z,0) \leq k_1 + k_2\int_0^{\zeta}(\zeta-\kappa)^{\mu-1}\tilde{\mathcal{H}}_1^2(z,0)d\kappa,\ \ where\ \ k_1 = \frac{5(\mathfrak{C}_1+\mathfrak{C}_2)}{(1+5\mathfrak{Q})},\ k_2 = \frac{5(\hat{H}+\hat{\rho})}{(1+5\hat{\mathfrak{Q}})\Gamma\mu}$$

case (iii) When $\zeta \in (\zeta_m, \zeta_{m+1}]$, we explore

$$z(\zeta) = \eta(0) + q(0,\eta(0)) - q(\zeta,z(\zeta)) + \frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa,z(\kappa))d\kappa + \frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa,z(\kappa))d\tilde{B}_\kappa + \sum_{m=1}^{k} b_m(z(\zeta_m))$$

Now, $\mathbf{E}\big[\mathbf{d}_{\mathcal{H}}^2\big([z(\zeta)]^\nu,[0]^\nu\big)\big]$

$$= \mathbf{E}\Bigg[\mathbf{d}_{\mathcal{H}}^2\Bigg([\eta(0)]^\nu + [q(0,\eta(0))]^\nu - [q(\zeta,z(\zeta))]^\nu + \left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}h(\kappa,z(\kappa))d\kappa\right]^\nu + \left[\frac{1}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\varrho(\kappa,z(\kappa))d\tilde{B}_\kappa\right]^\nu + \left[\sum_{m=1}^{k} b_m(z(\zeta_m))\right]^\nu, [0]^\nu\Bigg)\Bigg]$$

Thus, by the assumptions and Theorem 1, we have

$$\leq \quad 6\mathbf{E}\Big[\mathbf{d}_{\mathcal{H}}^2\big([\eta(0)]^\nu,[0]^\nu\big)\Big] + 6\mathbf{E}\Big[\mathbf{d}_{\mathcal{H}}^2\big([q(0,\eta(0))]^\nu,[0]^\nu\big)\Big] - 6\mathbf{E}\Big[\mathbf{d}_{\mathcal{H}}^2\big([q(\zeta,z(\zeta))]^\nu,[0]^\nu\big)\Big]$$
$$+\frac{6}{\Gamma\mu}\mathbf{E}\left[\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{d}_{\mathcal{H}}^2\big([h(\kappa,z(\kappa))]^\nu,[0]^\nu\big)d\kappa\right] + \frac{6}{\Gamma\mu}\mathbf{E}\left[\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{d}_{\mathcal{H}}^2\big([\varrho(\kappa,z(\kappa))]^\nu,[0]^\nu\big)d\kappa\right]$$
$$+6\mathbf{E}\left[\sum_{m=1}^{k}\mathbf{d}_{\mathcal{H}}^2\big([b_m(z(\zeta_m))]^\nu,[0]^\nu\big)\right]$$
$$\leq \quad 6\mathbf{E}\Big[\mathbf{d}_{\mathcal{H}}^2([\eta(0)]^\nu,[0]^\nu)\Big] + 6\mathbf{E}\Big[\mathbf{d}_{\mathcal{H}}^2\big([q(0,\eta(0))]^\nu,[0]^\nu\big)\Big] - 6\hat{\mathfrak{Q}}\mathbf{E}\Big[\mathbf{d}_{\mathcal{H}}^2\big([z(\zeta)]^\nu,[0]^\nu\big)\Big]$$
$$+\frac{6\hat{H}}{\Gamma\mu}\mathbf{E}\left[\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{d}_{\mathcal{H}}^2\big([z(\kappa)]^\nu,[0]^\nu\big)d\kappa\right] + \frac{6\hat{\rho}}{\Gamma\mu}\mathbf{E}\left[\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{d}_{\mathcal{H}}^2\big([z(\kappa)]^\nu,[0]^\nu\big)d\kappa\right]$$
$$+6\sum_{m=1}^{k}\hat{b}_m\mathbf{E}\Big[\mathbf{d}_{\mathcal{H}}^2\big([z(\zeta_m)]^\nu,[0]^\nu\big)\Big]$$

By Definition 11, we have $\mathbf{E}[\mathbf{d}_\infty^2(z(\zeta),0)] = \mathbf{E}[\sup_{\nu\in(0,1]}\mathbf{d}_{\mathcal{H}}^2([z(\zeta)]^\nu,[0]^\nu)]$

$$\leq \quad 6\mathfrak{C}_1 + 6\mathfrak{C}_2 - 6\hat{\mathfrak{Q}}\mathbf{E}\Big[\mathbf{d}_\infty^2(z(\zeta),0)\Big] + \frac{6\hat{H}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}\Big[\mathbf{d}_\infty^2(z(\zeta),0)\Big]d\kappa$$
$$+\frac{6\hat{\rho}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}\Big[\mathbf{d}_\infty^2(z(\zeta),0)\Big]d\kappa + 6\sum_{m=1}^{k}\hat{b}_m\mathbf{E}\Big[\mathbf{d}_\infty^2(z(\zeta),0)\Big]$$

where $\mathfrak{C}_1 = \mathbf{d}_\infty^2(\eta(0),0), \mathfrak{C}_2 = \mathbf{d}_\infty^2(q(\eta(0),0))$

According to Definition 12, $\mathbf{E}\left[\tilde{\mathcal{H}}_1^2(z,0)\right] = \mathbf{E}\left[\sup_{\zeta\in[\zeta_m,\zeta_{m+1}]} \mathbf{d}_\infty^2(z(\zeta),0)\right]$

$$\leq 6\mathfrak{C}_1 + 6\mathfrak{C}_2 - 6\hat{\mathfrak{Q}}\mathbf{E}\left[\tilde{\mathcal{H}}_1^2(z,0)\right] + \frac{6\hat{H}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}\left[\tilde{\mathcal{H}}_1^2(z,0)\right]d\kappa$$
$$+\frac{6\hat{\rho}}{\Gamma\mu}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\mathbf{E}\left[\tilde{\mathcal{H}}_1^2(z,0)\right]d\kappa + 6\sum_{m=1}^{k}\hat{b}_m\mathbf{E}\left[\tilde{\mathcal{H}}_1^2(z,0)\right]$$

$$\tilde{\mathcal{H}}_1^2(z,0) \leq \hat{k}_{1_m} + \hat{k}_{2_m}\int_{\zeta_m}^{\zeta}(\zeta-\kappa)^{\mu-1}\tilde{\mathcal{H}}_1^2(z,0)d\kappa$$

where $\hat{k}_{1_m} = \frac{6(\mathfrak{C}_1+\mathfrak{C}_2)}{\Gamma\mu(1+6\mathfrak{Q}-6\sum_{m=1}^{k}b_m)}, m = 1,2,3,...k$, $\hat{k}_{2_m} = \frac{6(\hat{H}+\hat{\varrho})}{\Gamma\mu(1+6\mathfrak{Q}-6\sum_{m=1}^{k}b_m)}, m = 1,2,3,...k$

Thus, from case(i)–(iii), we have

$$\tilde{\mathcal{H}}_1^2(z,0) \leq M_1 + M_2\int_{-\tau}^{T}(\zeta-\kappa)^{\mu-1}\tilde{\mathcal{H}}_1^2(z,0)d\kappa$$

where $M_1 = \max_{1\leq m\leq k}\{k_1,\hat{k}_{1_m}\}$, $M_2 = \max_{1\leq m\leq k}\{k_2,\hat{k}_{2_m}\}$,

In order that

$$\tilde{\mathcal{H}}_1^2(z,0) \leq M_1 e^{M_2\int_{-\tau}^{T}(\zeta-\kappa)^{\mu-1}d\kappa}$$

Hence,

$$\tilde{\mathcal{H}}_1^2(z,0) \leq M_3, \text{ where } M_3 = \max_{1\leq m\leq k}\{M_1 e^{M_2\int_{-\tau}^{T}(\zeta-\kappa)^{\mu-1}d\kappa}\}$$

From Lemma (3), we conclude $\tilde{\mathcal{H}}_1^2(z,0) = ||z||^2 \leq M_3$ (i.e.,) z is bounded. Clearly our solution exists globally in the interval $[-\tau,T]$. □

5. Example

Considering the neutral impulsive Caputo-order fuzzy fractional stochastic differential system with fuzzy Brownian motion

$$\begin{aligned}
{}_0^cD_\zeta^{\frac{1}{2}}[z(\zeta)+\bar{4}\zeta] &= \bar{4}\zeta z^2(\zeta) + \bar{4}\zeta^2 z^2(\zeta)d\tilde{B}_\zeta, \ \ \zeta\in[0,T] \\
\Delta z(\zeta) &= \frac{cos(m\zeta)}{e^{m\zeta}}z(\zeta_m), \zeta = \zeta_m, m = 1,2,3,4,5 \\
z(\zeta) &= \eta(\zeta) = \zeta + \bar{2}, \ \zeta\in[-\tau,0]
\end{aligned}$$

It is noted that, $\zeta_0 = 0 < \zeta_1 < ....\zeta_m < \zeta_{m+1} = T$, and for the fuzzy number, the ν-cut is $[4]^\nu = [\nu+3, 5-\nu]$ with $\nu \in (0,1]$ $\mu \in (0,1)$. Here $q(\zeta,z(\zeta)) = \bar{4}\zeta$,$h(\zeta,z(\zeta)) = \bar{4}\zeta z^2(\zeta)$, $\varrho(\zeta,z(\zeta)) = \bar{4}\zeta^2 z^2(\zeta)$, $b_m(\zeta,z(\zeta_m)) = \frac{cos(m\zeta)}{e^{m\zeta}}z(\zeta_m)$, m = 1, 2, 3, 4, 5.

Now, the solution for the system with the ν-cut method is furnished below

$$z(\zeta) = \begin{cases}
\zeta + \bar{2}, & \zeta\in[-\tau,0] \\
-\bar{4}\zeta - 2 + \frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\bar{4}\kappa z^2(\kappa)d\kappa + \frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\bar{4}\kappa^2 z^2(\kappa)d\tilde{B}_\kappa, & \forall\zeta\in[0,\zeta_1] \\
-\bar{4}\zeta - 2 + \frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\bar{4}\kappa z^2(\kappa)d\kappa + \frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\bar{4}\kappa^2 z^2(\kappa)d\tilde{B}_\kappa + \sum_{m=1}^{5}\frac{cos(m\zeta)}{e^{m\zeta}}z(\zeta_m), & \forall\zeta\in(\zeta_m,\zeta_{m+1}]
\end{cases}$$

Now, the ν-level set is noticeable for the example

$$\begin{aligned}[\bar{4}\zeta z^2(\zeta)]^\nu &= [(\nu+3)\zeta(z_q^\nu(\zeta))^2, (5-\nu)\zeta(z_r^\nu(\zeta))^2] \\ [\bar{4}\zeta^2 z^2(\zeta)d\tilde{B}_\zeta]^\nu &= [(\nu+3)\zeta^2(z_q^\nu(\zeta))^2(d\tilde{B}_\zeta)_q^\nu, (5-\nu)\zeta^2(z_r^\nu(\zeta))^2(d\tilde{B}_\zeta)_r^\nu]\end{aligned}$$

Therefore, we now deduce the uniqueness for the instance

$$\begin{aligned}
&\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([h(\zeta,\Upsilon(\zeta))]^\nu,[h(\zeta,\hat{\Upsilon}(\zeta))]^\nu\big)\Big] \\
=\ &\mathbf{E}\bigg[\mathbf{d}^2_{\mathcal{H}}\bigg([\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}h(\zeta,\Upsilon(\zeta))d\kappa]^\nu,[\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}h(\zeta,\hat{\Upsilon}(\zeta))d\kappa]^\nu\bigg)\bigg] \\
=\ &\mathbf{E}\bigg[\mathbf{d}^2_{\mathcal{H}}\bigg(\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}[\bar{4}\kappa\Upsilon^2(\kappa)]^\nu d\kappa,\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}[\bar{4}\kappa\hat{\Upsilon}^2(\kappa)]^\nu d\kappa\bigg)\bigg] \\
=\ &\mathbf{E}\bigg[\mathbf{d}^2_{\mathcal{H}}\bigg(\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}[(\nu+3)\kappa(\Upsilon_q^\nu(\kappa))^2,(5-\nu)\kappa(\Upsilon_r^\nu(\kappa))^2]d\kappa, \\
&\quad\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}[(\nu+3)\kappa(\hat{\Upsilon}_q^\nu(\kappa))^2,(5-\nu)\kappa(\hat{\Upsilon}_r^\nu(\kappa))^2]d\kappa\bigg)\bigg] \\
\leq\ &\mathbf{E}\bigg[max\{\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}|(\nu+3)\kappa(\Upsilon_q^\nu(\kappa))^2-(\nu+3)\kappa(\hat{\Upsilon}_q^\nu(\kappa))^2|^2d\kappa, \\
&\quad\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}|(5-\nu)\kappa(\Upsilon_r^\nu(\kappa))^2-(5-\nu)\kappa(\hat{\Upsilon}_r^\nu(\kappa))^2|^2\}d\kappa\bigg] \\
\leq\ &\mathbf{E}\bigg[max\{\frac{(\nu+3)}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\kappa|(\Upsilon_q^\nu(\kappa))^2-(\hat{\Upsilon}_q^\nu(\kappa))^2|^2d\kappa, \\
&\quad\frac{(5-\nu)}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}|\kappa(\Upsilon_r^\nu(\kappa))^2-\kappa(\hat{\Upsilon}_r^\nu(\kappa))^2|^2\}d\kappa\bigg] \\
\leq\ &T\frac{(5-\nu)}{\sqrt{\pi}}\mathbf{E}\bigg[\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}max\{|\Upsilon_q^\nu(\kappa)-\hat{\Upsilon}_q^\nu(\kappa)|^2|\Upsilon_q^\nu(\kappa)+\hat{\Upsilon}_q^\nu(\kappa)|^2, \\
&\quad|\Upsilon_r^\nu(\kappa)-\hat{\Upsilon}_r^\nu(\kappa)|^2|\Upsilon_r^\nu(\kappa)+\hat{\Upsilon}_r^\nu(\kappa)|^2\}d\kappa\bigg]
\end{aligned}$$

$$\begin{aligned}
\leq\ &T\frac{(5-\nu)}{\sqrt{\pi}}\mathbf{E}\bigg[\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\max_{\frac{-1}{3}\leq\zeta\leq 2}\{|\Upsilon_q^\nu(\kappa)-\hat{\Upsilon}_q^\nu(\kappa)|^2|\Upsilon_q^\nu(\kappa)+\hat{\Upsilon}_q^\nu(\kappa)|^2, \\
&\quad|\Upsilon_r^\nu(\kappa)-\hat{\Upsilon}_r^\nu(\kappa)|^2|\Upsilon_r^\nu(\kappa)+\hat{\Upsilon}_r^\nu(\kappa)|^2\}d\kappa\bigg] \\
\leq\ &T\frac{(5-\nu)}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\max_{\frac{-1}{3}\leq\zeta\leq 2}\{|\Upsilon_q^\nu(\kappa)+\hat{\Upsilon}_q^\nu(\kappa)|^2\}\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([\Upsilon(\zeta)]^\nu,[\hat{\Upsilon}(\zeta)]^\nu\big)d\kappa\Big] \\
\leq\ &\hat{h}\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([\Upsilon(\zeta)]^\nu,[\hat{\Upsilon}(\zeta)]^\nu\big)\Big]
\end{aligned}$$

where $\hat{h} = T\frac{(5-\nu)}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\max_{\frac{-1}{3}\leq\zeta\leq 2}\{|\Upsilon_q^\nu(\kappa)+\hat{\Upsilon}_q^\nu(\kappa)|^2\}d\kappa$

Hence, it satisfies the hypothesis.

In addition,

$$\begin{aligned}
&\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([\varrho(\zeta,\Upsilon(\zeta))]^\nu,[\varrho(\zeta,\hat{\Upsilon}(\zeta))]^\nu\big)\Big]\\
=\;&\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\Big([\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\varrho(\zeta,\Upsilon(\zeta))d\tilde{B}_\kappa]^\nu,[\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\varrho(\zeta,\hat{\Upsilon}(\zeta))d\tilde{B}_\kappa]^\nu\Big)\Big]\\
=\;&\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\Big(\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}[\bar{4}\kappa^2\Upsilon^2(\kappa)d\tilde{B}_\kappa]^\nu,\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}[\bar{4}\kappa^2\hat{\Upsilon}^2(\kappa)d\tilde{B}_\kappa]^\nu\Big)\Big]\\
=\;&\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\Big(\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}[(\nu+3)\kappa^2(\Upsilon^\nu_q(\kappa))^2(d\tilde{B}^\nu_\kappa)_q,(5-\nu)\kappa^2(\Upsilon^\nu_r(\kappa))^2(d\tilde{B}^\nu_\kappa)_r],\\
&\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}[(\nu+3)\kappa^2(\hat{\Upsilon}^\nu_q(\kappa))^2(d\tilde{B}^\nu_\kappa)_q,(5-\nu)\kappa^2(\hat{\Upsilon}^\nu_r(\kappa))^2(d\tilde{B}^\nu_\kappa)_r]\Big)\Big]\\
\leq\;&\mathbf{E}\Big[max\{\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}|(\nu+3)\kappa^2(\Upsilon^\nu_q(\kappa))^2-(\nu+3)\kappa^2(\hat{\Upsilon}^\nu_q(\kappa))^2|^2(dB^\nu_\kappa)_q,\\
&\frac{1}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}|(5-\nu)\kappa^2(\Upsilon^\nu_r(\kappa))^2-(5-\nu)\kappa^2(\hat{\Upsilon}^\nu_r(\kappa))^2|^2\}(d\tilde{B}^\nu_\kappa)_r\Big]\\
\leq\;&\mathbf{E}\Big[max\{\frac{(\nu+3)}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\kappa|(\Upsilon^\nu_q(\kappa))^2-(\hat{\Upsilon}^\nu_q(\kappa))^2|^2(d\tilde{B}^\nu_\kappa)_q,\\
&\frac{(5-\nu)}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}|\kappa(\Upsilon^\nu_r(\kappa))^2-\kappa(\hat{\Upsilon}^\nu_r(\kappa))^2|^2\}(d\tilde{B}^\nu_\kappa)_r\Big]\\
\leq\;&T^2\frac{(5-\nu)}{\sqrt{\pi}}\mathbf{E}\Big[\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}max\{|\Upsilon^\nu_q(\kappa)-\hat{\Upsilon}^\nu_q(\kappa)|^2|\Upsilon^\nu_q(\kappa)+\hat{\Upsilon}^\nu_q(\kappa)|^2,\\
&|\Upsilon^\nu_r(\kappa)-\hat{\Upsilon}^\nu_r(\kappa)|^2|\Upsilon^\nu_r(\kappa)+\hat{\Upsilon}^\nu_r(\kappa)|^2\}d\kappa\Big]\\
\leq\;&T^2\frac{(5-\nu)}{\sqrt{\pi}}\mathbf{E}\Big[\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\max_{\frac{-1}{3}\leq\zeta\leq2}\{|\Upsilon^\nu_q(\kappa)-\hat{\Upsilon}^\nu_q(\kappa)|^2|\Upsilon^\nu_q(\kappa)+\hat{\Upsilon}^\nu_q(\kappa)|^2,\\
&|\Upsilon^\nu_r(\kappa)-\hat{\Upsilon}^\nu_r(\kappa)|^2|\Upsilon^\nu_r(\kappa)+\hat{\Upsilon}^\nu_r(\kappa)|^2\}d\kappa\Big]\\
\leq\;&T^2\frac{(5-\nu)}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\max_{\frac{-1}{3}\leq\zeta\leq2}\{|\Upsilon^\nu_q(\kappa)+\hat{\Upsilon}^\nu_q(\kappa)|^2\}\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([\Upsilon(\zeta)]^\nu,[\hat{\Upsilon}(\zeta)]^\nu\big)d\kappa\Big]\\
\leq\;&\hat{\varrho}\mathbf{E}\Big[\mathbf{d}^2_{\mathcal{H}}\big([\Upsilon(\zeta)]^\nu,[\hat{\Upsilon}(\zeta)]^\nu\big)\Big]
\end{aligned}$$

where $\hat{\varrho}=T^2\frac{(5-\nu)}{\sqrt{\pi}}\int_0^\zeta(\zeta-\kappa)^{\frac{-1}{2}}\max_{\frac{-1}{3}\leq\zeta\leq2}\{|\Upsilon^\nu_q(\kappa)+\hat{\Upsilon}^\nu_q(\kappa)|^2\}d\kappa$

Hence, it satisfies the hypothesis.

Now, the ν-level set for fuzzy number $\bar{1}=[\nu,2-\nu]\ \forall\ \nu\in(0,1]$ and the ν set for the impulse are as follows:

$$\begin{aligned}
[\sum_{m=1}^{5}b_m(z(\zeta_m))]^\nu &= \sum_{m=1}^{5}[\frac{cosm\zeta}{e^{m\zeta}}z(\zeta_m)]^\nu\\
&= \sum_{m=1}^{5}\frac{cosm\zeta}{e^{m\zeta}}[(\nu,2-\nu)[z(\zeta_m)]^\nu]\\
&= \sum_{m=1}^{5}\frac{cosm\zeta}{e^{m\zeta}}[\nu z^\nu_q(\zeta_m),(2-\nu)z^\nu_r(\zeta_m)]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{d}^2_{\mathcal{H}}\left([\sum_{m=1}^{5} b_m(\zeta, \Upsilon(\zeta))]^{\nu}, [\sum_{m=1}^{5} b_m(\zeta, \hat{\Upsilon}(\zeta))]^{\nu}\right)\right] \\
= \; & \mathbf{E}\left[\mathbf{d}^2_{\mathcal{H}}\left(\sum_{m=1}^{5} \frac{cosm\zeta}{e^{m\zeta}}[\nu\Upsilon^{\nu}_{q}(\zeta_m), (2-\nu)\Upsilon^{\nu}_{r}(\zeta_m)], \sum_{m=1}^{5} \frac{cosm\zeta}{e^{m\zeta}}[\nu\hat{\Upsilon}^{\nu}_{q}(\zeta_m), (2-\nu)\hat{\Upsilon}^{\nu}_{r}(\zeta_m)]\right)\right] \\
= \; & \mathbf{E}\left[max\{\nu \sum_{m=1}^{5} \frac{cosm\zeta}{e^{m\zeta}}|\Upsilon^{\nu}_{q}(\zeta_m) - \hat{\Upsilon}^{\nu}_{q}(\zeta_m)|^2, (2-\nu) \sum_{m=1}^{5} \frac{cosm\zeta}{e^{m\zeta}}|\Upsilon^{\nu}_{r}(\zeta_m) - \hat{\Upsilon}^{\nu}_{r}(\zeta_m)|^2\}\right] \\
\leq \; & (2-\nu)\mathbf{E}\left[\sum_{m=1}^{5} \frac{cosmT}{e^{mT}} max\{|\Upsilon^{\nu}_{q}(\zeta_m) - \hat{\Upsilon}^{\nu}_{q}(\zeta_m)|^2, |\Upsilon^{\nu}_{r}(\zeta_m) - \hat{\Upsilon}^{\nu}_{r}(\zeta_m)|^2|\}\right] \\
\leq \; & (2-\nu) \sum_{m=1}^{5} \frac{cosmT}{e^{mT}} \mathbf{E}\left[\mathbf{d}^2_{\mathcal{H}}([\Upsilon(\zeta)]^{\nu}, [\hat{\Upsilon}(\zeta)]^{\nu})\right] \\
\leq \; & \hat{\mathfrak{b}}\mathbf{E}\left[\mathbf{d}^2_{\mathcal{H}}([\Upsilon(\zeta)]^{\nu}, [\hat{\Upsilon}(\zeta)]^{\nu})\right]
\end{aligned}$$

where $\hat{\mathfrak{b}} = (2-\nu) \sum_{m=1}^{5} \frac{cosmT}{e^{mT}}$

Hence, it satisfies the hypothesis.

Thus, all the hypotheses are addressed. Hence, the system has a unique fuzzy solution.

6. Conclusions

The solution of the fuzzy fractional neutral impulsive stochastic differential equation possessing global and local existence is demonstrated in this paper. We put forward a ν-cut method to obtain the uniqueness and existence results. Future research on fuzzy time-delay fractional stochastic differential equations driven by fuzzy Brownian motion with non-instantaneous impulses could benefit from the findings of this study.

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References

1. Dhayal, R.; Malik, M.; Abbas, S. Solvability and optimal controls of noninstantaneous impulsive stochastic fractional differential equation of order q∈(1, 2). *Stochastics* **2020**, *62*, 1275–1283. [CrossRef]
2. Podlubny, I. *Fractioanl Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applicatons*; Academic Press: San Diego, CA, USA, 1999.
3. Alqudah, M.A.; Ravichandran, C.; Abdeljawad, T.; Valliammal, N. New results on Caputo fractional-order neutral differential inclusions without compactness. *Adv. Differ. Equ.* **2019**, *1*, 528. [CrossRef]
4. Dos Santos, J.P.C.; Arjunan, M.M.; Cuevas, C. Existence results for fractional neutral integro-differential equations with state-dependent delay. *Comput. Math. Appl.* **2011**, *62*, 1275–1283. [CrossRef]

5. Valliammal, N.; Ravichandran, C.; Nisar, K.S. Solutions to fractional neutral delay differential nonlocal systems. *Chaos Solitons Fractals* **2020**, *138*, 109912. [CrossRef]
6. Fukuda, T. Basic statistical properties of fuzzy stochastic processes. *Otemon Econ. Stud.* **1991**, *24*, 89–120.
7. Zadeh, L.A. Fuzzy sts. *Inf. Control.* **1965**, *8*, 338–353. [CrossRef]
8. Kaleva, O. Fuzzy differential equations. *Fuzzy Sets Syst.* **1987**, *24*, 301–317. [CrossRef]
9. Puri, M.L.; Ralescu, D.A.; Zadeh, L. Fuzzy random variables. *Read. Fuzzy Sets Intell. Syst.* **1993**, 265–271. [CrossRef]
10. Diamond, P.; Kloeden, P.E. *Metric Spaces of Fuzzy Sets: Theory and Applications*; World Scientific: Singapore, 1994.
11. Huibert, K. Fuzzy random variables-I.definitions and theorems. *Inf. Sci.* **1978**, *15*, 1–29.
12. Malinowski, M.T. On random fuzzy differential equations. *Fuzzy Sets Syst.* **2009**, *160*, 3152–3165. [CrossRef]
13. Abuasbeh, K.; Shafqat, R.; Niazi, A.U.; Awadalla, M. Nonlocal fuzzy fractional stochastic evolution equations with fractional Brownian motion of order (1, 2). *AIMS Math.* **2022**, *7*, 19344–19358. [CrossRef]
14. Jafari, H.; Farahani, H.; Paripour, M. Fuzzy Malliavin derivative and linear Skorod fuzzy stochastic differential equation. *J. Intell. Fuzzy Syst.* **2018**, *35*, 2447–2458. [CrossRef]
15. Jafari, H.; Malinowski, M.T.; Ebadi, M.J. Fuzzy stochastic differential equations driven by fractional Brownian motion. *Adv. Differ. Equ.* **2021**, *16*. [CrossRef]
16. Abuasbeh, K.; Shafqat, R. Fractional Brownian motion for a system of fuzzy fractional stochastic differential equation. *J. Math.* **2022**, *2022*, 3559035. [CrossRef]
17. Oksendal. *Stochastic Differential Equations*; Springer: Berlin/Heidelberg, Germany, 1998.
18. Malinowski, M.T.; Michta, M. Fuzzy stochastic integral equations. *Dyn. Syst. Appl.* **2010**, *19*, 473–494.
19. Malinowski, M.T. Bipartite fuzzy stochastic differential equations with global Lipschitz condition. *Math. Probl. Eng.* **2016**, 2016, 3830529. [CrossRef]
20. Feng, Y. Fuzzy stochastic differential systems. *Fuzzy Sets Syst.* **2000**, *115*, 351–363. [CrossRef]
21. Kim, J.H. On fuzzy stochastic differential equations. *J. Korean Math. Soc.* **2005**, *42*, 153–169. [CrossRef]
22. Malinowski, M.T. Itô type stochastic fuzzy differential equations with delay. *Syst. Control. Lett.* **2012**, *61*, 692–701. [CrossRef]
23. Malinowski, M.T.; Michta, M. Stochastic fuzzy differential equations with an application. *Kybernetika* **2011**, *61*, 123–143.
24. Zhu, J.; Liang, Y.; Fei, W. On uniqueness and existence of solutions to stochastic set-valued differential equations with fractional Brownian motions. *Syst. Sci. Control. Eng.* **2020**, *8*, 618–627. [CrossRef]
25. Arhrrabi, E.; Melliani, S.; Chadli, L.S. Existence and Stability of solutoins of fuzzy fractional stochastic differential equations with fractional Brownian Motions. *Adv. Fuzzy Syst.* **2021**, *2021*, 3948493.
26. Benchohra, M.; Nieto, J.J.; Ouahab, A. Fuzzy solutions for impulsive differential equations. *Commun. Appl. Anal.* **2007**, *11*, 379–394.
27. Priyadharsini, J.; Balasubramaniam, P. Existence of fuzzy fractional stochastic differential system with impulses. *Comput. Appl. Math.* **2020**, *39*, 1–21. [CrossRef]
28. Bao, H.; Cao, J. Existence of the solutions for fractional stochastic impulsive neutral functional differential equations with infinite delay. *Adv. Differ. Equ.* **2017**, *66*. [CrossRef]
29. Chadha, A.; Pandey, D.N. Existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay. *Nonlinear Anal.* **2015**, *128*, 149–175. [CrossRef]
30. Narayanamoorthy, S.; Sowmiya, S. Approximate controllability results for impulsive linear fuzzy stochastic differential equations under nonlocal conditions. *Int. J. Fuzzy Log. Syst.* **2015**, *5*, 27–36.
31. Maheswari, R.; Karunanithi, S. Asymptotic stability of stochastic impulsive neutral partial functional differential equations. *Int. J. Comput. Appl.* **2014**, *85*, 23–26. [CrossRef]
32. Anguraj, A.; Banupriya, K.; Baleanu, D.; Vinodkumar, A. On neutral impulsive stochastic differential equations with Poisson jumps. *Adv. Differ. Equ.* **2018**, *290*. [CrossRef]
33. Xia, M.; Liu, L.; Fang, J.; Zhang, Y. Stability analysis for a class of stochastic differential equations with impulses. *Mathematics* **2023**, *11*, 1541. [CrossRef]
34. Balasubramainiam, P.; Muralishankar, S. Existence and uniqueness of fuzzy solution for semilinear fuzzy integrodifferential equations with nonlocal conditions. *Comput. Math. Appl.* **2004**, *47*, 1115–1122. [CrossRef]
35. Arhrrabi, E.; Elomari, M.; Melliani, S.; Chadli, L.S. Existence and Uniqueness Results of Fuzzy Fractional Stochastic Differential Equations with Impulsive. In *Recent Advances in Fuzzy Sets Theory, Fractional Calculus, Dynamic Systems and Optimization*; Springer International Publishing: Cham, Switzerland, 2022; pp. 147–163.
36. Luo, D.; Wang, X.; Caraballo, T.; Zhu, Q. Ulam-Hyers stability of Caputo-type fractional fuzzy stochastic differential equations with delay. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *15*, 107229. [CrossRef]
37. Chaharpashlou, R.; Atangana, A.; Saadati, R. On the fuzzy stability results for fractional stochastic Volterra integral equation. *Discret. Contin. Dyn. Syst.* **2021**, *14*, 3529–3539. [CrossRef]
38. Niazi, A.U.; Iqbal, N.; Shah, R.; Wannalookkhee, F.; Nonlaopon, K. Controllability for fuzzy fractional evolution equations in credibility space. *Fractal Fract.* **2021**, *5*, 112. [CrossRef]
39. Abuasbeh, K.; Shafqat, R.; Niazi, A.U.; Awadalla, M. Local and global existence and uniqueness of solution for class of fuzzy fractional functional evolution equation. *J. Funct. Spaces* **2022**, *2022*, 7512754. [CrossRef]
40. Kumar, A.; Malik, M.; Sajid, M.; Baleanu, D. Existence of local and global solutions to fractional order fuzzy delay differential equation with non-instantaneuos impulses. *AIMS Math.* **2021**, *7*, 2348–2369. [CrossRef]

41. Didier, K.S.; Rebecca, W.O.; Rosten, M.M.; Christopher, B.O.; Patient, K.K.; Remon, M. Fuzzy stochastic differential equations driven by a fuzzy Brownian Motion. *J. Appl. Math. Phys.* **2022**, *10*, 641–655. [CrossRef]
42. Seya, D.K.; Makengo, RM.; Remon, M.; Rebecca, W.O. Fuzzy itô integral driven by a fuzzy brownian motion. *J. Fuzzy Set Valued Anal.* **2015**, *3*, 232–244. [CrossRef]