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Study of Log Convex Mappings in Fuzzy Aunnam Calculus via Fuzzy Inclusion Relation over Fuzzy-Number Space

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Abstract: In this paper, with the use of newly defined class up and down log–convex fuzzy-number valued mappings, we offer a few new and original mappings defined by applying some mild restrictions over the definition of up and down log–convex fuzzy-number valued mapping. With the use of these mappings, we are able to develop partners of Fejér-type inequalities for up and down log–convexity, which improve upon certain previously established findings. The discussion also includes these mappings' characteristics. Moreover, some nontrivial examples are also provided to prove the validation of our main results.

Keywords: up and down log–convex fuzzy-number valued mapping; fuzzy Aunnam integral operator; Hermite–Hadamard type inequalities; Jensen's type inequality; Schur's type inequality

MSC: 26A33; 26A51; 26D10

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1. Introduction

Convex sets and convex mappings have contributed significantly and fundamentally to the growth of numerous domains in the pure and practical sciences. Convexity theory describes a wide range of extremely intriguing breakthroughs, including a connection between many areas of mathematics, physics, economics, and engineering sciences. Convex sets, and their numerous extensions and generalizations have been thought about and investigated recently utilizing novel concepts and methodologies. The concept of invex mappings was first introduced to mathematical programming by Hanson [1], and it sparked a lot of interest. Ben-Israel and Mond [2] introduced invex sets and preinvex mappings. They demonstrated that the differentiable preinvex mappings are invex mappings and that, under some circumstances, the opposite is also true. Noor [3] showed that variational-like inequalities describe the minimum of the differentiable preinvex mappings. See [4,5] and the references therein for further information on preinvex mappings' applications, numerical techniques, variational-like inequalities, and other features. The log–convex mappings are known to yield inequalities more precisely than the convex mappings do. We also have the idea of exponentially convex (concave) mappings, which is closely related to log–convex mappings and has its roots in Bernstein [6]. Exponentially preinvex mappings and their variant forms were introduced, and many aspects of them were covered by Noor and Noor [7,8]. Big data analysis, machine learning, statistics, and information theory all heavily rely on exponentially convex mappings. See, for instance, the references in [9–13].

Recent research by Noor et al. [14] investigated the comparable formulation of log–convex mappings and demonstrated that they have many of the same characteristics as convex mappings. For instance, the mapping ex is not convex but is a log–convex mapping. Log–convex mappings, which include hypergeometric mappings such as Gamma and Beta, are crucial in a number of fields of pure and practical sciences. Strongly log-biconvex mappings were first discussed by Noor and Noor [15], who also looked at their characterization. It is demonstrated that the bivariational inequalities are a novel generalization of the variational inequalities that can be used to describe the optimality conditions of the biconvex mappings.

One of the most well-known inequalities in the theory of convex mappings, the Hermite–Hadamard inequality, was found by C. Hermite and J. Hadamard [16]. It has a geometrical meaning and several applications.

One of the most beneficial findings in mathematical analysis is the H-H inequality. It is also known as the classical equation of the H-H inequality.

The H-H inequality for convex mapping $\mathfrak{S} : K \to \mathbb{R}$ on an interval $K = [\varsigma, \mathfrak{v}]$

$$\mathfrak{S}\left(\frac{\mathfrak{v}+\varsigma}{2}\right) \leq \frac{1}{\mathfrak{v}-\varsigma}\int_{\varsigma}^{\mathfrak{v}} \mathfrak{S}(\mathfrak{x})d\mathfrak{x} \leq \frac{\mathfrak{S}(\mathfrak{v})+\mathfrak{S}(\varsigma)}{2}, \tag{1}$$

for $\mathfrak{v}, \varsigma \in K$.

We point out that the Hermite–Hadamard inequality is a straightforward extension of Jensen's inequality and may be thought of as a refinement of the idea of convexity. Recent years have seen a resurgence in interest in the Hermite–Hadamard inequality for convex mappings, and a stunning array of improvements and generalizations have been investigated.

Interval analysis is a subset of set-valued analysis, which is the study of sets in the context of mathematics and general topology. The Archimedean approach, which includes determining the circumference of a circle, is a well-known example of interval enclosure.

This theory addresses the interval uncertainty that exists in many computational and mathematical models of deterministic real-world systems. This method investigates interval variables as opposed to point variables and expresses computation results as intervals, eliminating mistakes that lead to incorrect findings. One of the initial goals of the interval-valued analysis was to account for the error estimates of finite-state machine numerical solutions. Interval analysis, which Moore first proposed in his well-known book [17], is one of the most important methods in numerical analysis. As a result, it has found applications in a wide range of industries, including computer graphics [18,19], differential equations for intervals [20], neural network output optimization [21], and many more.

On the other hand, a number of significant inequalities, including Hermite–Hadamard and Ostrowski, have recently been investigated for interval-valued mappings. Using the Hukuhara derivative for interval-valued mappings, Chalco-Cano et al. discovered Ostrowski-type inequalities for interval-valued mappings in [22,23]. Román-Flores et al. established the inequalities of Minkowski and Beckenbach for interval-valued mappings in [24]. Please refer to [25–28] for the others. However, for more generic set-valued maps, inequalities were investigated. Sadowska provided the Hermite–Hadamard inequality, for instance, in [29]. Results related to log–convex fuzzy-number valued mappings see [30–32]. Interested readers can view [33,34] for the other investigations. For more information, see [35–64] and the references therein.

The article is set up as follows: We discuss log fuzzy-number valued convex mappings with numerical estimates and related fuzzy Aunnam integral inequalities in Section 3 after examining the prerequisite material and important details on inequalities and interval-valued analysis in Section 2. Section 4 then derives Jensen and Schur's inequalities for log fuzzy-number valued convex mappings. To decide whether the predefined results are advantageous, examples and numerical estimations are also taken into consideration. Section 4 explores a quick conclusion and potential study directions connected to the findings in this work before we wrap things up.

2. Preliminaries

This section reloads key findings and terminology necessary for understanding the core outcomes. Let $\mathcal{X}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $Ʊ \in \mathcal{X}_C$ be defined by

$$Ʊ = [Ʊ_*, Ʊ^*] = \{\omega \in \mathbb{R} | Ʊ_* \leq \omega \leq Ʊ^*\}, (Ʊ_*, Ʊ^* \in \mathbb{R}). \tag{2}$$

If $Ʊ_* = Ʊ^*$, then $Ʊ$ is referred to be degenerate. In this article, all intervals will be non-degenerate intervals. If $Ʊ_* \geq 0$, then $[Ʊ_*, Ʊ^*]$ is referred to as a positive interval. The set of all positive intervals is denoted by $\mathcal{X}_C^+$ and defined as

$$\mathcal{X}_C^+ = \{[Ʊ_*, Ʊ^*] : [Ʊ_*, Ʊ^*] \in \mathcal{X}_C \text{ and } Ʊ_* \geq 0\}. \tag{3}$$

Let $i \in \mathbb{R}$ and $i \cdot Ʊ$ be defined by

$$i \cdot Ʊ = \begin{cases} [iƱ_*, iƱ^*] & \text{if } i > 0, \\ \{0\} & \text{if } i = 0, \\ [iƱ^*, iƱ_*] & \text{if } i < 0. \end{cases} \tag{4}$$

Then the Minkowski difference $Ʋ - Ʊ$, addition $Ʊ + Ʋ$ and $Ʊ \times Ʋ$ for $Ʊ, Ʋ \in \mathcal{X}_C$ are defined by

$$[Ʋ_*, Ʋ^*] + [Ʊ_*, Ʊ^*] = [Ʋ_* + Ʊ_*, Ʋ^* + Ʊ^*], \tag{5}$$

$$[Ʋ_*, Ʋ^*] \times [Ʊ_*, Ʊ^*] = [\min\{Ʋ_*Ʊ_*, Ʋ^*Ʊ_*, Ʋ_*Ʊ^*, Ʋ^*Ʊ^*\}, \max\{Ʋ_*Ʊ_*, Ʋ^*Ʊ_*, Ʋ_*Ʊ^*, Ʋ^*Ʊ^*\}] \tag{6}$$

$$[Ʋ_*, Ʋ^*] - [Ʊ_*, Ʊ^*] = [Ʋ_* - Ʊ^*, Ʋ^* - Ʊ_*], \tag{7}$$

Remark 1 ([48])**.** For given $\left[Ʋ_*, Ʋ^*\right], \left[Ʊ_*, Ʊ^*\right] \in \mathbb{R}_I$, we say that $\left[Ʋ_*, Ʋ^*\right] \leq_I \left[Ʊ_*, Ʊ^*\right]$ if and only if $Ʋ_* \leq Ʊ_*, Ʋ^* \leq Ʊ^*$, it is a partial interval or left and right order relation.

If $\left[Ʋ_*, Ʋ^*\right], \left[Ʊ_*, Ʊ^*\right] \in \mathbb{R}_I$, we say that $\left[Ʋ_*, Ʋ^*\right] \subseteq_I \left[Ʊ_*, Ʊ^*\right]$ if and only if $Ʊ_* \leq Ʋ_*, Ʋ^* \leq Ʊ^*$, it is an inclusion interval or up and down ($\mathcal{UD}$) order relation.

For $[Ʋ_*, Ʋ^*], [Ʊ_*, Ʊ^*] \in \mathcal{X}_C$, the Hausdorff–Pompeiu distance between intervals $[Ʋ_*, Ʋ^*]$, and $[Ʊ_*, Ʊ^*]$ is defined by

$$d_H([Ʋ_*, Ʋ^*], [Ʊ_*, Ʊ^*]) = max\{|Ʋ_* - Ʊ_*|, |Ʋ^* - Ʊ^*|\}. \tag{8}$$

It is a familiar fact that $(\mathcal{X}_C, d_H)$ is a complete metric space [41,44,45].

Definition 1 ([41,42])**.** A fuzzy subset L of $\mathbb{R}^+$ is distinguished by a mapping $\widetilde{\Lambda} : \mathbb{R}^+ \to [0,1]$ called the membership mapping of L. That is, a fuzzy subset L of $\mathbb{R}^+$ is a mapping $\widetilde{\Lambda} : \mathbb{R}^+ \to [0,1]$. So, for further study, we have chosen this notation. We appoint Ω to denote the set of all fuzzy subsets of $\mathbb{R}^+$.

Let $\widetilde{\Lambda} \in \Omega$. Then, $\widetilde{\Lambda}$ is referred to as a fuzzy number or fuzzy interval if the following properties are satisfied by $\widetilde{\Lambda}$:

(1) $\widetilde{\Lambda}$ should be normal if there exists $\omega \in \mathbb{R}^+$ and $\widetilde{\Lambda}(\omega) = 1$;

(2) $\widetilde{\Lambda}$ should be upper semi-continuous on $\mathbb{R}^+$ if for given $\omega \in \mathbb{R}^+$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\widetilde{\Lambda}(\omega) - \widetilde{\Lambda}(\varpi) < \varepsilon$ for all $\varpi \in \mathbb{R}^+$ with $|\omega - \varpi| < \delta$;

(3) $\widetilde{\Lambda}$ should be fuzzy convex that is

$$\widetilde{\Lambda}((1-\partial)\omega+\partial\varpi) \geq min\left(\widetilde{\Lambda}(\omega),\widetilde{\Lambda}(\varpi)\right), \text{ for all } \omega,\varpi \in \mathbb{R}^+, \text{ and } \partial \in [0,1]$$

(4) $\widetilde{\Lambda}$ should be compactly supported that is $cl\left\{\omega \in \mathbb{R}^+ \middle| \widetilde{\Lambda}(\omega) > 0\right\}$ is compact. We appoint Ω_C to denote the set of all fuzzy numbers of $\mathbb{R}^+$.

Definition 2 ([42])**.** Given $\widetilde{\Lambda} \in \Omega_C$, the level sets or cut sets are given by $\left[\widetilde{\Lambda}\right]^{\mathrm{i}} = \left\{\omega \in \mathbb{R}^+ \middle| \widetilde{\Lambda}(\omega) > \mathrm{i}\right\}$ for all $\mathrm{i} \in [0,1]$ and by $\left[\widetilde{\Lambda}\right]^0 = \left\{\omega \in \mathbb{R}^+ \middle| \widetilde{\Lambda}(\omega) > 0\right\}$. These sets are known as i-level sets or i-cut sets of $\widetilde{\Lambda}$.

Proposition 1 ([26])**.** Let $\widetilde{\Lambda},\widetilde{\varpi} \in \Omega_C$. Then relation " $\leq_{\mathbb{F}}$ " given on Ω_C by

$$\widetilde{\Lambda} \leq_{\mathbb{F}} \widetilde{\varpi} \text{ when and only when, } \left[\widetilde{\Lambda}\right]^i \leq_I \left[\widetilde{\varpi}\right]^i, \text{ for every } i \in [0,1],$$

it is a partial-order or left and right relation.

Proposition 2 ([35])**.** Let $\widetilde{\Lambda},\widetilde{\varpi} \in \Omega_C$. Then inclusion relation " $\supseteq_{\mathbb{F}}$ " given on Ω_C by

$$\widetilde{\Lambda} \supseteq_{\mathbb{F}} \widetilde{\varpi} \text{ when and only when, } \left[\widetilde{\Lambda}\right]^i \supseteq_I \left[\widetilde{\varpi}\right]^i, \text{ for every } i \in [0,1],$$

it is an up-and-down fuzzy inclusion relation.

Remember the approaching notions which are offered in the literature. If $\widetilde{\Lambda},\widetilde{\varpi} \in \Omega_C$ and $\partial \in \mathbb{R}$, then, for every $i \in [0,1]$, the arithmetic operations are defined by

$$\left[\widetilde{\Lambda} \oplus \widetilde{\varpi}\right]^i = \left[\widetilde{\Lambda}\right]^i + \left[\widetilde{\varpi}\right]^i, \tag{9}$$

$$\left[\widetilde{\Lambda} \otimes \widetilde{\varpi}\right]^i = \left[\widetilde{\Lambda}\right]^i \times \left[\widetilde{\varpi}\right]^i, \tag{10}$$

$$\left[\partial \odot \widetilde{\Lambda}\right]^i = \partial \cdot \left[\widetilde{\Lambda}\right]^i. \tag{11}$$

These operations follow directly from Equations (4), (6) and (7), respectively.

Theorem 1 ([41])**.** The space Ω_C dealing with a supremum metric i.e., for $\widetilde{\Lambda},\widetilde{\varpi} \in \Omega_C$

$$d_\infty\left(\widetilde{\Lambda},\widetilde{\varpi}\right) = \sup_{0 \leq i \leq 1} d_H\left(\left[\widetilde{\Lambda}\right]^i, \left[\widetilde{\varpi}\right]^i\right), \tag{12}$$

is a complete metric space, where H denotes the well-known Hausdorff metric on space of intervals.

Now we define and discuss some properties of Riemann integral operators of interval- and fuzzy-number valued mappings.

Theorem 2 ([41,43]). If $\mathfrak{S} : [\varsigma, \mathfrak{r}] \subset \mathbb{R} \to \mathcal{X}_C$ is an interval-valued mapping ($I\mathcal{VM}$) satisfying that $\mathfrak{S}(\omega) = [\mathfrak{S}_*(\omega), \mathfrak{S}^*(\omega)]$, where $\mathfrak{S}_*, \mathfrak{S}^* : [\varsigma, \mathfrak{r}] \to \mathbb{R}$ and $\mathfrak{S}_*(\omega) \leq \mathfrak{S}^*(\omega)$, $\forall \omega \in [\varsigma, \mathfrak{r}]$ then $\mathfrak{S}$ is Aumann integrable (IA-integrable) over $[\varsigma, \mathfrak{r}]$ when and only when; $\mathfrak{S}_*(\omega)$ and $\mathfrak{S}^*(\omega)$ both are integrable over $[\varsigma, \mathfrak{r}]$ such that

$$(IA)\int_{\varsigma}^{\mathfrak{r}} \mathfrak{S}(\omega)d\omega = \left[\int_{\varsigma}^{\mathfrak{r}} \mathfrak{S}_*(\omega)d\omega, \int_{\varsigma}^{\mathfrak{r}} \mathfrak{S}^*(\omega)d\omega\right] \tag{13}$$

Definition 3 ([43]). Let $\widetilde{\mathfrak{S}} : [\varsigma, \mathfrak{r}] \subset \mathbb{R} \to \Omega_C$ is fuzzy number valued mapping ($\mathcal{FNVM}$), whose parametrized form is given by $\mathfrak{S}_i : [c, d] \subset \mathbb{R} \to \mathcal{K}_C$ and defined as $\mathfrak{S}_i(\omega) = [\mathfrak{S}_*(\omega, i), \mathfrak{S}^*(\omega, i)]$ for every $\omega \in [\varsigma, \mathfrak{r}]$ and for every $i \in [0, 1]$. The fuzzy Aumann integral ($(\mathcal{FA})$-integral) of $\widetilde{\mathfrak{S}}$ over $[\varsigma, \mathfrak{r}]$, denoted by $(\mathcal{FA})\int_{\varsigma}^{\mathfrak{r}} \widetilde{\mathfrak{S}}(\omega)d\omega$, is defined level-wise by

$$\begin{aligned}\left[(\mathcal{FA})\int_{\varsigma}^{\mathfrak{r}} \widetilde{\mathfrak{S}}(\omega)d\omega\right]^{i} &= (IA)\int_{\varsigma}^{\mathfrak{r}} \mathfrak{S}_i(\omega)d\omega \\ &= \left\{\int_{\varsigma}^{\mathfrak{r}} \mathfrak{S}(\omega, i)d\omega : \mathfrak{S}(\omega, i) \in S(\mathfrak{S}_i)\right\},\end{aligned} \tag{14}$$

where $S(\mathfrak{S}_i) = \{\mathfrak{S}(., i) \to \mathbb{R} : \mathfrak{S}(., i)$is integrable and $\mathfrak{S}(\omega, i) \in \mathfrak{S}_i(\omega)\}$,for every $i \in [0, 1]$. $\widetilde{\mathfrak{S}}$ is $(\mathcal{FA})$-integrable over $[\varsigma, \mathfrak{r}]$ if $(\mathcal{FA})\int_{\varsigma}^{\mathfrak{r}} \widetilde{\mathfrak{S}}(\omega)d\omega \in \Omega_C$.

Theorem 3 ([26]). Let $\widetilde{\mathfrak{S}} : [\varsigma, \mathfrak{r}] \subset \mathbb{R} \to \Omega_C$ be a $\mathcal{FNVM}$. Then, $\widetilde{\mathfrak{S}}$ is $(\mathcal{FA})$-integrable over $[\varsigma, \mathfrak{r}]$ when and only when, $\mathfrak{S}_*(\omega, i)$ and $\mathfrak{S}^*(\omega, i)$ both are integrable over $[\varsigma, \mathfrak{r}]$. Moreover, if $\widetilde{\mathfrak{S}}$ is $(\mathcal{FA})$-integrable over $[\varsigma, \mathfrak{r}]$, then

$$\begin{aligned}\left[(\mathcal{FA})\int_{\varsigma}^{\mathfrak{r}} \widetilde{\mathfrak{S}}(\omega)d\omega\right]^{i} &= \left[\int_{\varsigma}^{\mathfrak{r}} \mathfrak{S}_*(\omega, i)d\omega, \int_{\varsigma}^{\mathfrak{r}} \mathfrak{S}^*(\omega, i)d\omega\right] \\ &= (I\mathcal{A})\int_{\varsigma}^{\mathfrak{r}} \mathfrak{S}_i(\omega)d\omega\end{aligned} \tag{15}$$

for every $i \in [0, 1]$.

Definition 4 ([51]). A mapping $\mathfrak{S} : \mathfrak{T} \to \mathbb{R}$ is referred to as log–convex mapping if

$$\mathfrak{S}(\mathfrak{v}\omega + (1 - \mathfrak{v})y) \leq \mathfrak{S}(\omega)^{\mathfrak{v}}\mathfrak{S}(y)^{1-\mathfrak{v}}, \forall \omega, y \in \mathfrak{T}, \mathfrak{v} \in [0, 1], \tag{16}$$

where $\mathfrak{S}(\omega) \geq 0$, where $\mathfrak{T}$ is a convex set. If (16) is inverted, then $\mathfrak{S}$ is referred to as log-concave.

Definition 5 ([49]). Let $\mathfrak{T}$ be a convex set. Then $\mathcal{FNVM}$ $\widetilde{\mathfrak{S}} : \mathfrak{T} \to \Omega_C$ is referred to as convex $\mathcal{FNVM}$ on $\mathfrak{T}$ if

$$\widetilde{\mathfrak{S}}(\mathfrak{v}\omega + (1 - \mathfrak{v})y) \leq_{\mathbb{F}} \mathfrak{v} \odot \widetilde{\mathfrak{S}}(\omega) \oplus (1 - \mathfrak{v}) \odot \widetilde{\mathfrak{S}}(y), \tag{17}$$

for all $\omega, y \in \mathfrak{T}, \mathfrak{v} \in [0, 1]$, where $\widetilde{\mathfrak{S}}(\omega) \geq_{\mathbb{F}} \widetilde{0}$. If (17) is inverted, then $\widetilde{\mathfrak{S}}$ is referred to as concave $\mathcal{FNVM}$ on $[\varsigma, \mathfrak{r}]$. $\widetilde{\mathfrak{S}}$ is affine if and only if it is both convex $\mathcal{FNVM}$ and concave $\mathcal{FNVM}$.

Definition 6 ([25]). Let $\mathfrak{T}$ be a convex set. Then $\mathcal{FNVM}$ $\widetilde{\mathfrak{S}} : \mathfrak{T} \to \Omega_C$ is referred to as log convex $\mathcal{FNVM}$ ($\mathcal{L}$-convex $\mathcal{FNVM}$) on $\mathfrak{T}$ if

$$\widetilde{\mathfrak{S}}(\mathfrak{v}\omega + (1 - \mathfrak{v})y) \leq_{\mathbb{F}} \widetilde{\mathfrak{S}}(\omega)^{\mathfrak{v}} \otimes \widetilde{\mathfrak{S}}(y)^{(1-\mathfrak{v})}, \tag{18}$$

for all $\omega, y \in \mathfrak{T}, \mathfrak{v} \in [0,1]$, where $\widetilde{\mathfrak{S}}(\omega) \geq_{\mathbb{F}} \widetilde{0}$. If (18) is inverted, then $\widetilde{\mathfrak{S}}$ is referred to as $\mathcal{L}$-concave $\mathcal{FNVM}$ on $[\varsigma, \mathfrak{v}]$. $\widetilde{\mathfrak{S}}$ is $\mathcal{L}$-affine if and only if it is both $\mathcal{L}$-convex $\mathcal{FNVM}$ and $\mathcal{L}$-concave $\mathcal{FNVM}$.

Definition 7. Let $\mathfrak{T}$ be a convex set. Then $\mathcal{FNVM}$ $\widetilde{\mathfrak{S}} : \mathfrak{T} \to \Omega_C$ is referred to as up and down log convex $\mathcal{FNVM}$ (U$\mathcal{DL}$-convex $\mathcal{FNVM}$) on $\mathfrak{T}$ if

$$\widetilde{\mathfrak{S}}(\mathfrak{v}\omega + (1-\mathfrak{v})y) \supseteq_{\mathbb{F}} \widetilde{\mathfrak{S}}(\omega)^{\mathfrak{v}} \otimes \widetilde{\mathfrak{S}}(y)^{(1-\mathfrak{v})}, \tag{19}$$

for all $\omega, y \in \mathfrak{T}, \mathfrak{v} \in [0,1]$, where $\widetilde{\mathfrak{S}}(\omega) \geq_{\mathbb{F}} \widetilde{0}$. If (19) is inverted, then $\widetilde{\mathfrak{S}}$ is referred to as U$\mathcal{DL}$-concave $\mathcal{FNVM}$ on $[\varsigma, \mathfrak{v}]$. $\widetilde{\mathfrak{S}}$ is U$\mathcal{DL}$-affine if and only if it is both U$\mathcal{DL}$-convex $\mathcal{FNVM}$ and U$\mathcal{DL}$-concave $\mathcal{FNVM}$.

Remark 2. If $\widetilde{\mathfrak{S}}$ is U$\mathcal{DL}$-convex $\mathcal{FNVM}$, then $g\widetilde{\mathfrak{S}}$ is also U$\mathcal{DL}$-convex $\mathcal{FNVM}$ for $g \geq 0$.

If $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathcal{J}}$ both are U$\mathcal{DL}$-convex $\mathcal{FNVM}$s, then $\max\left(\widetilde{\mathfrak{S}}(\omega), \widetilde{\mathcal{J}}(\omega)\right)$ is also U$\mathcal{DL}$-convex $\mathcal{FNVM}$.

Theorem 4. Let $\mathfrak{T}$ be a convex set and $\widetilde{\mathfrak{S}} : \mathfrak{T} \to \Omega_C$ be a $\mathcal{FNVM}$ with $\widetilde{\mathfrak{S}}(\omega) \geq_{\mathbb{F}} \widetilde{0}$, whose parametrized form is given by $\mathfrak{S}_i : \mathfrak{T} \subset \mathbb{R} \to \mathcal{K}_C^+ \subset \mathcal{K}_C$ and defined as

$$\mathfrak{S}_i(\omega) = \left[\mathfrak{S}_*(\omega, i), \mathfrak{S}^*(\omega, i)\right], \tag{20}$$

for all $\omega \in \mathfrak{T}$ and for all $i \in (0,1]$. Then $\widetilde{\mathfrak{S}}$ is U$\mathcal{DL}$-convex on $\mathfrak{T}$, if and only if, for all $i \in (0,1]$, $\mathfrak{S}_*(\omega, i)$ and $\mathfrak{S}^*(\omega, i)$ are $\mathcal{L}$-convex and $\mathcal{L}$-concave, respectively.

Proof. Let $\widetilde{\mathfrak{S}}$ be an $\mathcal{UDL}$-convex $\mathcal{FNVM}$ on $\mathfrak{T}$. Then, for all $\omega, y \in \mathfrak{T}$ and $\mathfrak{v} \in [0,1]$, we have

$$\widetilde{\mathfrak{S}}(\mathfrak{v}\omega + (1-\mathfrak{v})y) \supseteq_{\mathbb{F}} \widetilde{\mathfrak{S}}(\omega)^{\mathfrak{v}} \otimes \widetilde{\mathfrak{S}}(y)^{(1-\mathfrak{v})}. \tag{21}$$

Therefore, from (20) and Proposition 2, we have

$$\begin{aligned} &[\mathfrak{S}_*(\mathfrak{v}\omega + (1-\mathfrak{v})y, i), \mathfrak{S}^*(\mathfrak{v}\omega + (1-\mathfrak{v})y, i)] \\ &\qquad \supseteq_I \left[\mathfrak{S}_*(\omega, i)^{\mathfrak{v}}, \mathfrak{S}^*(\omega, i)^{\mathfrak{v}}\right] \times \left[\mathfrak{S}_*(y, i)^{(1-\mathfrak{v})}, \mathfrak{S}^*(y, i)^{(1-\mathfrak{v})}\right]. \end{aligned} \tag{22}$$

It follows that $\mathfrak{S}_*(\mathfrak{v}\omega + (1-\mathfrak{v})y, i) \leq \mathfrak{S}_*(\omega, i)^{\mathfrak{v}} \mathfrak{S}_*(y, i)^{(1-\mathfrak{v})}$ and $\mathfrak{S}^*(\mathfrak{v}\omega + (1-\mathfrak{v})y, i) \geq \mathfrak{S}^*(\omega, i)^{\mathfrak{v}} \mathfrak{S}^*(y, i)^{(1-\mathfrak{v})}$, for each $i \in (0,1]$. This shows that $\mathfrak{S}_*(\omega, i)$ and $\mathfrak{S}^*(\omega, i)$ both are $\mathcal{UDL}$-convex mappings.

Conversely, suppose that $\mathfrak{S}_*(\omega, i)$ and $\mathfrak{S}^*(\omega, i)$ both are $\mathcal{UDL}$-convex mappings. Then from the (19), it follows that $\widetilde{\mathfrak{S}}(\omega)$ is $\mathcal{UDL}$-convex $\mathcal{FNVM}$. □

Example 1. We consider the $\mathcal{FNVM}$ $\widetilde{\mathfrak{S}} : [1,8] \to \Omega_C$ established by,

$$\widetilde{\mathfrak{S}}(\omega)(s) = \begin{cases} \dfrac{s - e^{\frac{1}{\omega}}}{\frac{9}{2} - e^{\frac{1}{\omega}}} & s \in \left[e^{\frac{1}{\omega}}, \frac{9}{2}\right]; \\ \dfrac{10\omega - s}{10\omega - \frac{9}{2}} & s \in \left(\frac{9}{2}, 10\omega\right]; \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Then, for each $\mathrm{i} \in (0,1]$, we have $\mathfrak{S}_{\mathrm{i}}(\omega) = \left[(1-\mathrm{i})e^{\frac{1}{\omega}} + \frac{9}{2}\mathrm{i}, 10(1-\mathrm{i})\omega + \frac{9}{2}\mathrm{i}\right]$. Since end-point mappings $\mathfrak{S}_*(\omega,\mathrm{i})$, $\mathfrak{S}^*(\omega,\mathrm{i})$ are $\mathcal{L}$-convex and $\mathcal{L}$-concave mappings for each $\mathrm{i} \in (0,1]$, respectively, then by Theorem 4, $\widetilde{\mathfrak{S}}(\omega)$ is $\mathrm{U}\mathcal{DL}$-convex $\mathcal{F}\mathrm{N}\mathcal{VM}$.

Remark 3. If $\mathfrak{S}_*(\omega,\mathrm{i}) = \mathfrak{S}^*(\omega,\mathrm{i})$ with $\mathrm{i} = 1$, then $\mathrm{U}\mathcal{DL}$-convex $\mathcal{F}\mathrm{N}\mathcal{VM}$ becomes classical $\mathrm{U}\mathcal{DL}$-convex mapping [3].

3. Main Results

This section summarizes the study's principal findings. There are two subsections in this section. In the opening subsection, we present very fuzzy Aunnam integrals that are critical for estimating the Hermite–Hadamard (H-H) type inequality's inaccuracy for $U\mathcal{DL}$-convex $\mathcal{FNVM}$. In the second subsection, we find the results related to Jensen's and Schur's inequalities. Moreover, some exceptional cases are also acquired.

3.1. Hermite–Hadamard Type Inequalities

Theorem 5. $\widetilde{\mathfrak{S}} : [\varsigma, \vartheta] \to \Omega_C$ be a $\mathrm{U}\mathcal{DL}$-convex $\mathcal{F}\mathrm{N}\mathcal{VM}$, whose parametrized form is given by $\mathfrak{S}_{\mathrm{i}} : [\varsigma, \vartheta] \subset \mathbb{R} \to \mathcal{K}_C^+$ and provided as $\mathfrak{S}_{\mathrm{i}}(\omega) = [\mathfrak{S}_*(\omega,\mathrm{i}), \mathfrak{S}^*(\omega,\mathrm{i})]$ for all $\omega \in [\varsigma, \vartheta]$ and for all $\mathrm{i} \in (0,1]$. If $\widetilde{\mathfrak{S}} \in \mathcal{FA}_{([\varsigma,\vartheta],\mathrm{i})}$, then

$$\widetilde{\mathfrak{S}}\left(\frac{\varsigma+\vartheta}{2}\right) \supseteq_{\mathbb{F}} \exp\left[\frac{1}{\vartheta-\varsigma} \odot (\mathrm{FA})\int_{\varsigma}^{\vartheta} \ln\widetilde{\mathfrak{S}}(\omega)d\omega\right] \supseteq_{\mathbb{F}} \sqrt{\widetilde{\mathfrak{S}}(\varsigma) \otimes \widetilde{\mathfrak{S}}(\vartheta)}. \tag{24}$$

If $\widetilde{\mathfrak{S}}$ is $\mathrm{U}\mathcal{DL}$-concave, then (24) is inverted.

Proof. Let $\widetilde{\mathfrak{S}} : [\varsigma, \vartheta] \to \Omega_C$, $U\mathcal{DL}$-convex $\mathcal{FNVM}$. Then, by hypothesis, we have

$$\widetilde{\mathfrak{S}}\left(\frac{\varsigma+\vartheta}{2}\right) \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{S}}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta)\right]^{\frac{1}{2}} \otimes \left[\widetilde{\mathfrak{S}}((1-\mathfrak{v})\varsigma + \mathfrak{v}\vartheta)\right]^{\frac{1}{2}}.$$

Therefore, for every $i \in (0,1]$, we have

$$\begin{aligned} \mathfrak{S}_*\left(\tfrac{\varsigma+\vartheta}{2}, i\right) &\le [\mathfrak{S}_*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta, i)]^{\frac{1}{2}} \times [\mathfrak{S}_*((1-\mathfrak{v})\varsigma + \mathfrak{v}\vartheta, i)]^{\frac{1}{2}}, \\ \mathfrak{S}^*\left(\tfrac{\varsigma+\vartheta}{2}, i\right) &\ge [\mathfrak{S}^*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta, i)]^{\frac{1}{2}} \times [\mathfrak{S}^*((1-\mathfrak{v})\varsigma + \mathfrak{v}\vartheta, i)]^{\frac{1}{2}}. \end{aligned} \tag{25}$$

Taking logarithms on both sides of (25), then we obtain

$$\begin{aligned} 2ln\mathfrak{S}_*\left(\tfrac{\varsigma+\vartheta}{2}, i\right) &\le ln\mathfrak{S}_*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta, i) + ln\mathfrak{S}_*((1-\mathfrak{v})\varsigma + \mathfrak{v}\vartheta, i), \\ 2ln\mathfrak{S}^*\left(\tfrac{\varsigma+\vartheta}{2}, i\right) &\ge ln\mathfrak{S}^*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta, i) + ln\mathfrak{S}^*((1-\mathfrak{v})\varsigma + \mathfrak{v}\vartheta, i). \end{aligned}$$

Then,

$$\begin{aligned} 2\int_0^1 ln\mathfrak{S}_*\left(\tfrac{\varsigma+\vartheta}{2}, i\right)d\mathfrak{v} &\le \int_0^1 ln\mathfrak{S}_*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta, i)d\mathfrak{v} + \int_0^1 ln\mathfrak{S}_*((1-\mathfrak{v})\varsigma + \mathfrak{v}\vartheta, i)d\mathfrak{v}, \\ 2\int_0^1 ln\mathfrak{S}^*\left(\tfrac{\varsigma+\vartheta}{2}, i\right)d\mathfrak{v} &\ge \int_0^1 ln\mathfrak{S}^*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta, i)d\mathfrak{v} + \int_0^1 ln\mathfrak{S}^*((1-\mathfrak{v})\varsigma + \mathfrak{v}\vartheta, i)d\mathfrak{v}. \end{aligned}$$

It follows that

$$\begin{aligned} ln\mathfrak{S}_*\left(\tfrac{\varsigma+\vartheta}{2}, i\right) &\le \tfrac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta} ln\mathfrak{S}_*(\omega, i)d\omega, \\ ln\mathfrak{S}^*\left(\tfrac{\varsigma+\vartheta}{2}, i\right) &\ge \tfrac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta} ln\mathfrak{S}^*(\omega, i)d\omega, \end{aligned}$$

which implies that

$$\begin{gathered}\mathfrak{S}_*\left(\tfrac{\varsigma+\vartheta}{2}, i\right) \leq exp\left(\tfrac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta} ln\mathfrak{S}_*(\omega, i)d\omega\right),\\ \mathfrak{S}^*\left(\tfrac{\varsigma+\vartheta}{2}, i\right) \geq exp\left(\tfrac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta} ln\mathfrak{S}^*(\omega, i)d\omega\right).\end{gathered}$$

That is

$$\left[\mathfrak{S}_*\left(\frac{\varsigma+\vartheta}{2}, i\right), \mathfrak{S}^*\left(\frac{\varsigma+\vartheta}{2}, i\right)\right] \supseteq_I \left[exp\left(\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta} ln\mathfrak{S}_*(\omega, i)d\omega\right), exp\left(\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta} ln\mathfrak{S}^*(\omega, i)d\omega\right)\right].$$

Thus,

$$\widetilde{\mathfrak{S}}\left(\frac{\varsigma+\vartheta}{2}\right) \supseteq_{\mathbb{F}} exp\left[\frac{1}{\vartheta-\varsigma} \odot (\boldsymbol{FA})\int_{\varsigma}^{\vartheta} \boldsymbol{ln}\widetilde{\mathfrak{S}}(\omega)\boldsymbol{d}\omega\right]. \tag{26}$$

In a similar way as above, we have

$$exp\left[\frac{1}{\vartheta-\varsigma}(\boldsymbol{FA})\int_{\varsigma}^{\vartheta} \boldsymbol{ln}\widetilde{\mathfrak{S}}(\omega)\boldsymbol{d}\omega\right] \supseteq_{\mathbb{F}} \sqrt{\widetilde{\mathfrak{S}}(\varsigma) \otimes \widetilde{\mathfrak{S}}(\vartheta)}. \tag{27}$$

Combining (26) and (27), we have

$$\widetilde{\mathfrak{S}}\left(\frac{\varsigma+\vartheta}{2}\right) \supseteq_{\mathbb{F}} exp\left[\frac{1}{\vartheta-\varsigma}(\boldsymbol{FA})\int_{\varsigma}^{\vartheta} \boldsymbol{ln}\widetilde{\mathfrak{S}}(\omega)\boldsymbol{d}\omega\right] \supseteq_{\mathbb{F}} \sqrt{\widetilde{\mathfrak{S}}(\varsigma) \otimes \widetilde{\mathfrak{S}}(\vartheta)},$$

the required result. □

Remark 4. If $\mathfrak{S}_*(\omega, \mathrm{i}) = \mathfrak{S}^*(\omega, \mathrm{i})$ with $\mathrm{i} = 1$, then by (26), the following outcome can be obtained see [50]:

$$\mathfrak{S}\left(\frac{\varsigma+\vartheta}{2}\right) \leq exp\left[\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta} ln\mathfrak{S}(\omega)d\omega\right] \leq \sqrt{\mathfrak{S}(\varsigma) \times \mathfrak{S}(\vartheta)}.$$

Here, we achieve H-H Fejér type inequality for $\mathcal{UDL}$-convex $\mathcal{FNVM}$ To obtain H-H Fejér inequality for $\mathcal{UDL}$-convex $\mathcal{FNVM}$. Initially, we find the right part of H-H Fejér inequality. In the next Theorem 5, we will acquire the left part of H-H Fejér inequality.

Theorem 6. Let $\widetilde{\mathfrak{S}} : [\varsigma, \vartheta] \to \Omega_{\mathrm{C}}$ be a $\mathrm{U}\mathcal{DL}$-convex $\mathcal{FNVM}$ with $\varsigma < \vartheta$, whose parametrized form is given by $\mathfrak{S}_{\mathrm{i}} : [\varsigma, \vartheta] \subset \mathbb{R} \to \mathcal{K}_{\mathrm{C}}^{+}$ and provided as $\mathfrak{S}_{\mathrm{i}}(\omega) = [\mathfrak{S}_*(\omega, \mathrm{i}), \mathfrak{S}^*(\omega, \mathrm{i})]$ for all $\omega \in [\varsigma, \vartheta]$ and for all $\mathrm{i} \in (0, 1]$. If $\widetilde{\mathfrak{S}} \in \mathcal{FA}_{([\varsigma, \vartheta], \mathrm{i})}$ and $\mathfrak{Q} : [\varsigma, \vartheta] \to \mathbb{R}, \mathfrak{Q}(\omega) \geq 0,$ symmetric with respect to $\frac{\varsigma+\vartheta}{2}$, then

$$\frac{1}{\vartheta-\varsigma} \odot (\mathrm{FA})\int_{\varsigma}^{\vartheta} \left[\ln\widetilde{\mathfrak{S}}(\omega)\right]\mathfrak{Q}(\omega)\mathrm{d}\omega \supseteq_{\mathbb{F}} \ln\left[\widetilde{\mathfrak{S}}(\varsigma) \otimes \widetilde{\mathfrak{S}}(\vartheta)\right] \odot \int_0^1 \mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma + \mathfrak{v}\vartheta)\mathrm{d}\mathfrak{v}. \tag{28}$$

If $\widetilde{\mathfrak{S}}$ is $\mathrm{U}\mathcal{DL}$-concave, then (28) is inverted.

Proof. Let $\mathfrak{S}$ be a $\mathcal{UDL}$-convex $\mathcal{FNVM}$. Then, for each $i \in (0, 1]$, we have

$$\begin{aligned}&[ln\mathfrak{S}_*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta, i)]\mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta)\\ &\leq (\mathfrak{v}ln\mathfrak{S}_*(\varsigma, i) + (1-\mathfrak{v})ln\mathfrak{S}_*(\vartheta, i))\mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta),\\ &[ln\mathfrak{S}^*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta, i)]\mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta)\\ &\geq (\mathfrak{v}ln\mathfrak{S}^*(\varsigma, i) + (1-\mathfrak{v})ln\mathfrak{S}^*(\vartheta, i))\mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\vartheta),\end{aligned} \tag{29}$$

and

$$\begin{aligned}&[ln\,\mathfrak{S}_*((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r},i)]\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\\&\quad\le((1-\mathfrak{v})ln\,\mathfrak{S}_*(\varsigma,i)+\mathfrak{v}ln\,\mathfrak{S}_*(\mathfrak{r},i))\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r}),\\&[ln\,\mathfrak{S}^*((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r},i)]\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\\&\quad\ge((1-\mathfrak{v})ln\,\mathfrak{S}^*(\varsigma,i)+\mathfrak{v}ln\,\mathfrak{S}^*(\mathfrak{r},i))\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r}).\end{aligned}\tag{30}$$

After adding (29) and (30), and then integrating over $(0,1)$, we get

$$\begin{aligned}&\int_0^1[ln\,\mathfrak{S}_*(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r},i)]\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})\,d\mathfrak{v}\\&\quad+\int_0^1 ln\,\mathfrak{S}_*((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r},i)\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v}\\&\le\int_0^1\left[\begin{array}{l}ln\,\mathfrak{S}_*(\varsigma,i)\{\mathfrak{v}\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})+(1-\mathfrak{v})\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\}\\+ln\,\mathfrak{S}_*(\mathfrak{r},i)\{(1-\mathfrak{v})\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})+\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\}\end{array}\right]d\mathfrak{v},\\&\int_0^1[ln\,\mathfrak{S}^*((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r},i)]\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v}\\&\quad+\int_0^1 ln\,\mathfrak{S}^*(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r},i)\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})\,d\mathfrak{v}\\&\ge\int_0^1\left[\begin{array}{l}ln\,\mathfrak{S}^*(\varsigma,i)\{\mathfrak{v}\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})+(1-\mathfrak{v})\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\}\\+ln\,\mathfrak{S}^*(\mathfrak{r},i)\{(1-\mathfrak{v})\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})+\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\}\end{array}\right]d\mathfrak{v}\end{aligned}$$

$$\begin{aligned}&=2ln\,\mathfrak{S}_*(\varsigma,i)\int_0^1\mathfrak{v}\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})\,d\mathfrak{v}+2ln\,\mathfrak{S}_*(\mathfrak{r},i)\int_0^1\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v},\\&=2ln\,\mathfrak{S}^*(\varsigma,i)\int_0^1\mathfrak{v}\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})\,d\mathfrak{v}+2ln\,\mathfrak{S}^*(\mathfrak{r},i)\int_0^1\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v}.\end{aligned}$$

Since $\mathfrak{Q}$ is symmetric, then

$$\begin{aligned}&=2ln[\mathfrak{S}_*(\varsigma,i)\times\mathfrak{S}_*(\mathfrak{r},i)]\int_0^1\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v},\\&=2ln[\mathfrak{S}^*(\varsigma,i)\times\mathfrak{S}^*(\mathfrak{r},i)]\int_0^1\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v}.\end{aligned}\tag{31}$$

Since

$$\begin{aligned}&\int_0^1[ln\,\mathfrak{S}_*(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r},i)]\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})\,d\mathfrak{v}\\&\quad=\int_0^1[ln\,\mathfrak{S}_*((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r},i)]\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v}\\&\qquad=\frac{1}{\mathfrak{r}-\varsigma}\int_\varsigma^\mathfrak{r}[ln\,\mathfrak{S}_*(\omega,i)]\mathfrak{Q}(\omega)d\omega,\\&\int_0^1[ln\,\mathfrak{S}^*((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r},i)]\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v}\\&\quad=\int_0^1[ln\,\mathfrak{S}^*(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r},i)]\mathfrak{Q}(\mathfrak{v}\varsigma+(1-\mathfrak{v})\mathfrak{r})\,d\mathfrak{v}\\&\qquad=\frac{1}{\mathfrak{r}-\varsigma}\int_\varsigma^\mathfrak{r}[ln\,\mathfrak{S}^*(\omega,i)]\mathfrak{Q}(\omega)d\omega.\end{aligned}\tag{32}$$

From (31) and (32), we have

$$\begin{aligned}&\frac{1}{\mathfrak{r}-\varsigma}\int_\varsigma^\mathfrak{r}[ln\,\mathfrak{S}_*(\omega,i)]\mathfrak{Q}(\omega)d\omega\le ln[\mathfrak{S}_*(\varsigma,i)\times\mathfrak{S}_*(\mathfrak{r},i)]\int_0^1\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v},\\&\frac{1}{\mathfrak{r}-\varsigma}\int_\varsigma^\mathfrak{r}[ln\,\mathfrak{S}^*(\omega,i)]\mathfrak{Q}(\omega)d\omega\ge ln[\mathfrak{S}^*(\varsigma,i)\times\mathfrak{S}^*(\mathfrak{r},i)]\int_0^1\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v},\end{aligned}$$

that is

$$\begin{aligned}&\left[\frac{1}{\mathfrak{r}-\varsigma}\int_\varsigma^\mathfrak{r}[ln\,\mathfrak{S}_*(\omega,i)]\mathfrak{Q}(\omega)d\omega,\frac{1}{\mathfrak{r}-\varsigma}\int_\varsigma^\mathfrak{r}[ln\,\mathfrak{S}^*(\omega,i)]\mathfrak{Q}(\omega)d\omega\right]\\&\quad\supseteq_I\left[ln\,[\mathfrak{S}_*(\varsigma,i)\times\mathfrak{S}_*(\mathfrak{r},i)],\ ln\,[\mathfrak{S}^*(\varsigma,i)\times\mathfrak{S}^*(\mathfrak{r},i)]\right]\int_0^1\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})\,d\mathfrak{v},\end{aligned}$$

Hence

$$\frac{1}{\mathfrak{r}-\varsigma}\odot(FA)\int_\varsigma^\mathfrak{r}[ln\widetilde{\mathfrak{S}}(\omega)]\mathfrak{Q}(\omega)d\omega\supseteq_{\mathbb{F}}ln[\widetilde{\mathfrak{S}}(\varsigma)\otimes\widetilde{\mathfrak{S}}(\mathfrak{r})]\odot\int_0^1\mathfrak{v}\mathfrak{Q}((1-\mathfrak{v})\varsigma+\mathfrak{v}\mathfrak{r})d\mathfrak{v}.$$

This concludes the proof. □

Now, we present the following solution for $\mathcal{UDL}$-convex $\mathcal{FNVM}$ utilizing up and down fuzzy inclusion relation, which is associated with the left portion classical H-H Fejér type inequality.

Theorem 7. Let $\widetilde{\mathfrak{S}}: [\varsigma, \mathfrak{v}] \to \Omega_C$ be a $\mathcal{UDL}$-convex $\mathcal{FNVM}$ with $\varsigma < \mathfrak{v}$, whose parametrized form is given by $\mathfrak{S}_i: [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C^+$ and provided as $\mathfrak{S}_i(\omega) = [\mathfrak{S}_*(\omega, i), \mathfrak{S}^*(\omega, i)]$ for all $\omega \in [\varsigma, \mathfrak{v}]$ and for all $i \in (0, 1]$. If $\widetilde{\mathfrak{S}} \in \mathcal{FA}_{([\varsigma, \mathfrak{v}], i)}$ and $\mathfrak{Q}: [\varsigma, \mathfrak{v}] \to \mathbb{R}, \mathfrak{Q}(\omega) \geq 0$, symmetric with respect to $\frac{\varsigma+\mathfrak{v}}{2}$, and $\int_\varsigma^\mathfrak{v} \mathfrak{Q}(\omega)d\omega > 0$, then

$$ln\widetilde{\mathfrak{S}}\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\int_\varsigma^\mathfrak{v} \mathfrak{Q}(\omega)d\omega} \odot (FA) \int_\varsigma^\mathfrak{v} \left[ln\widetilde{\mathfrak{S}}(\omega)\right]\mathfrak{Q}(\omega)d\omega. \tag{33}$$

If $\widetilde{\mathfrak{S}}$ is a $\mathcal{UDL}$-concave, then (33) is inverted.

Proof. Since $\mathfrak{S}$ is a $\mathcal{UDL}$-convex then, for $i \in (0, 1]$ we have

$$\begin{aligned} 2ln\,\mathfrak{S}_*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right) &\leq ln\,\mathfrak{S}_*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v}, i) + ln\,\mathfrak{S}_*\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}, i\big), \\ 2ln\,\mathfrak{S}^*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right) &\geq ln\,\mathfrak{S}^*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v}, i) + ln\,\mathfrak{S}^*\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}, i\big). \end{aligned} \tag{34}$$

By multiplying (34) by $\mathfrak{Q}\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}\big) = \mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v})$ and integrate it by $\mathfrak{v}$ over $[0, 1]$, we obtain

$$\begin{aligned} &2\left[ln\,\mathfrak{S}_*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right)\right]\int_0^1 \mathfrak{Q}\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}\big)d\mathfrak{v} \\ &\quad\leq \int_0^1 [ln\,\mathfrak{S}_*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v}, i)]\mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v})d\mathfrak{v} \\ &\quad+ \int_0^1 \big[ln\,\mathfrak{S}_*\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}, i\big)\big]\mathfrak{Q}\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}\big)d\mathfrak{v}, \\ &2\left[ln\,\mathfrak{S}^*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right)\right]\int_0^1 \mathfrak{Q}\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}\big)d\mathfrak{v} \\ &\quad\geq \int_0^1 [ln\,\mathfrak{S}^*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v}, i)]\mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v})d\mathfrak{v} \\ &\quad+ \int_0^1 \big[ln\,\mathfrak{S}^*\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}, i\big)\big]\mathfrak{Q}\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}\big)d\mathfrak{v}. \end{aligned} \tag{35}$$

Since

$$\begin{aligned} &\int_0^1 [ln\,\mathfrak{S}_*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v}, i)]\mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v})\,d\mathfrak{v} \\ &\quad= \int_0^1 \big[ln\,\mathfrak{S}_*\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}, i\big)\big]\mathfrak{Q}\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}\big)\,d\mathfrak{v}, \\ &\quad= \frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^\mathfrak{v} [ln\,\mathfrak{S}_*(\omega, i)]\mathfrak{Q}(\omega)d\omega, \\ &\int_0^1 [ln\,\mathfrak{S}^*(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v}, i)]\mathfrak{Q}(\mathfrak{v}\varsigma + (1-\mathfrak{v})\mathfrak{v})\,d\mathfrak{v} \\ &\quad= \int_0^1 \big[ln\,\mathfrak{S}^*\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}, i\big)\big]\mathfrak{Q}\big((1-\mathfrak{v})\varsigma + \mathfrak{v}\mathfrak{v}\big)\,d\mathfrak{v}, \\ &\quad= \frac{1}{\mathfrak{v}-\varsigma}\int_\varsigma^\mathfrak{v} [ln\,\mathfrak{S}^*(\omega, i)]\mathfrak{Q}(\omega)d\omega. \end{aligned} \tag{36}$$

From (35) and (36), we have

$$\begin{aligned} ln\,\mathfrak{S}_*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right) &\leq \frac{1}{\int_\varsigma^\mathfrak{v} \mathfrak{Q}(\omega)d\omega}\int_\varsigma^\mathfrak{v} [ln\,\mathfrak{S}_*(\omega, i)]\mathfrak{Q}(\omega)d\omega, \\ ln\,\mathfrak{S}^*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right) &\geq \frac{1}{\int_\varsigma^\mathfrak{v} \mathfrak{Q}(\omega)d\omega}\int_\varsigma^\mathfrak{v} \big[ln\,\mathfrak{S}^*(\omega, i)\big]\mathfrak{Q}(\omega)d\omega. \end{aligned}$$

From which, we have

$$\begin{aligned} &\left[ln\,\mathfrak{S}_*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right), ln\,\mathfrak{S}^*\left(\frac{\varsigma+\mathfrak{v}}{2}, i\right)\right] \\ &\supseteq_I \frac{1}{\int_\varsigma^\mathfrak{v} \mathfrak{Q}(\omega)d\omega}\left[\int_\varsigma^\mathfrak{v} [ln\,\mathfrak{S}_*(\omega, i)]\mathfrak{Q}(\omega)d\omega,\ \int_\varsigma^\mathfrak{v} \big[ln\,\mathfrak{S}^*(\omega, i)\big]\mathfrak{Q}(\omega)d\omega\right], \end{aligned}$$

that is

$$ln\widetilde{\mathfrak{S}}\left(\frac{\varsigma+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\int_{\varsigma}^{\mathfrak{v}} \mathfrak{Q}(\omega)d\omega} \odot (FA)\int_{\varsigma}^{\mathfrak{v}} \left[ln\widetilde{\mathfrak{S}}(\omega)\right]\mathfrak{Q}(\omega)d\omega.$$

Then we complete the proof. □

Remark 5. If $\mathfrak{S}_*(\varsigma, \mathrm{i}) = \mathfrak{S}^*(\varsigma, \mathrm{i})$ with $\mathrm{i} = 1$, then from (30) and (35), the classical H-H Fejér inequality for $\mathcal{L}$-convex mapping can be acquired.

3.2. Jensen's and Schur's Inequalities for Log Convex Fuzzy-Number Valued Mappings

Here, we will prove Jensen's and Schur's inequality for $U\mathcal{DL}$-convex $\mathcal{FNVM}$s.

Theorem 8. Let $\mathrm{i}_i \in \mathbb{R}^+$, $\omega_i \in [\varsigma, \mathfrak{v}]$, $(i = 1, 2, 3, \ldots, k, k \geq 2)$ and $\widetilde{\mathfrak{S}}: [\varsigma, \mathfrak{v}] \to \Omega_C$ be a $\mathrm{U}\mathcal{DL}$-convex $\mathcal{F}\mathrm{N}\mathcal{VM}$, whose parametrized form is given by $\mathfrak{S}_\mathrm{i}: [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_C^+$ and provided as $\mathfrak{S}_\mathrm{i}(\omega) = [\mathfrak{S}_*(\omega, \mathrm{i}), \mathfrak{S}^*(\omega, \mathrm{i})]$ for all $\omega \in [\varsigma, \mathfrak{v}]$ and for all $\mathrm{i} \in (0, 1]$. Then

$$\widetilde{\mathfrak{S}}\left(\frac{1}{W_k}\sum_{i=1}^{k} i_i\,\omega_i\right) \supseteq_{\mathbb{F}} \prod_{i=1}^{k}\left[\widetilde{\mathfrak{S}}(\omega_i)\right]^{\frac{i_i}{W_k}}, \tag{37}$$

where $\mathrm{W}_k = \sum_{i=1}^{k} \mathrm{i}_i$. If $\widetilde{\mathfrak{S}}$ is $\mathrm{U}\mathcal{DL}$-concave, then (37) is inverted.

Proof. When $k = 2$, then (37) holds. Consider (37) also holds for $k = r - 1$, then

$$\widetilde{\mathfrak{S}}\left(\frac{1}{W_{r-1}}\sum_{i=1}^{r-1} i_i\,\omega_i\right) \supseteq_{\mathbb{F}} \prod_{i=1}^{r-1}\left[\widetilde{\mathfrak{S}}(\omega_i)\right]^{\frac{i_i}{W_{r-1}}}$$

Now, let us prove that (37) holds for $k = r$, we have

$$\widetilde{\mathfrak{S}}\left(\frac{1}{W_r}\sum_{i=1}^{r} i_i\,\omega_i\right) = \widetilde{\mathfrak{S}}\left(\frac{W_{r-2}}{W_r}\frac{1}{W_{r-2}}\sum_{i=1}^{r-2} i_i\,\omega_i \oplus \frac{i_{r-1}+i_r}{W_r}\left(\frac{i_{r-1}}{i_{r-1}+i_r}\omega_{r-1} + \frac{i_r}{i_{r-1}+i_r}\omega_r\right)\right).$$

Therefore, for every $i \in (0, 1]$, we have

$$\begin{aligned}
&\mathfrak{S}_*\left(\frac{1}{W_r}\sum_{i=1}^{r} i_i\,\omega_i, i\right)\\
&\leq \mathfrak{S}_*\left(\frac{W_{r-2}}{W_r}\frac{1}{W_{r-2}}\sum_{i=1}^{r-2} i_i\,\omega_i + \frac{i_{r-1}+i_r}{W_r}\left(\frac{i_{r-1}}{i_{r-1}+i_r}\omega_{r-1} + \frac{i_r}{i_{r-1}+i_r}\omega_r,\ i\right)\right),\\
&\leq \prod_{i=1}^{r-2}[\mathfrak{S}_*(\omega_i,\ i)]^{\frac{i_i}{W_r}}\left[\mathfrak{S}_*\left(\frac{i_{r-1}}{i_{r-1}+i_r}\omega_{r-1} + \frac{i_r}{i_{r-1}+i_r}\omega_r,\ i\right)\right]^{\frac{i_{r-1}+i_r}{W_r}},\\
&\leq \prod_{i=1}^{r-2}[\mathfrak{S}_*(\omega_i,\ i)]^{\frac{i_i}{W_r}}\left[\ [\mathfrak{S}_*(\omega_{r-1},\ i)]^{\frac{i_{r-1}}{i_{r-1}+i_r}}\,[\mathfrak{S}_*(\omega_r,\ i)]^{\frac{i_r}{i_{r-1}+i_r}}\right]^{\frac{i_{r-1}+i_r}{W_r}},\\
&\leq \prod_{i=1}^{r-2}[\mathfrak{S}_*(\omega_i,\ i)]^{\frac{i_i}{W_r}}\,[\mathfrak{S}_*(\omega_{r-1},\ i)]^{\frac{i_{r-1}}{W_r}}\,[\mathfrak{S}_*(\omega_r,\ i)]^{\frac{i_r}{W_r}},\\
&= \prod_{i=1}^{r}[\mathfrak{S}_*(\omega_i,\ i)]^{\frac{i_i}{W_r}}.
\end{aligned}$$

Similarly, for $\mathfrak{S}^*(\omega, i)$, we have

$$\mathfrak{S}^*\left(\frac{1}{W_r}\sum_{i=1}^{r} i_i\,\omega_i, i\right) \geq \prod_{i=1}^{r}[\mathfrak{S}^*(\omega_i,\ i)]^{\frac{i_i}{W_r}}.$$

From this, we have

$$\left[\mathfrak{S}_*\left(\frac{1}{W_r}\sum_{i=1}^{r} i_i\,\omega_i, i\right), \mathfrak{S}^*\left(\frac{1}{W_r}\sum_{i=1}^{r} i_i\,\omega_i, i\right)\right] \supseteq_I \left[\prod_{i=1}^{r}[\mathfrak{S}_*(\omega_i,\ i)]^{\frac{i_i}{W_r}}, \prod_{i=1}^{r}[\mathfrak{S}^*(\omega_i,\ i)]^{\frac{i_i}{W_r}}\right],$$

that is,

$$\widetilde{\mathfrak{S}}\left(\frac{1}{W_r}\sum_{i=1}^{r} i_i\,\omega_i\right) \supseteq_{\mathbb{F}} \prod_{i=1}^{r}\left[\widetilde{\mathfrak{S}}(\omega_i)\right]^{\frac{i_i}{W_r}},$$

and the result follows. □

If $i_1 = i_2 = i_3 = \cdots = i_k = 1$, then Theorem 8 reduces to the following result:

Corollary 1. Let $\omega_i \in [\varsigma, \nu]$, $(i = 1, 2, 3, \ldots, k, k \geq 2)$ and $\widetilde{\mathfrak{S}}: [\varsigma, \nu] \to \Omega_C$ be a $U\mathcal{DL}$-convex $\mathcal{FNVM}$, whose parametrized form is given by $\mathfrak{S}_i: [\varsigma, \nu] \subset \mathbb{R} \to \mathcal{K}_C^+$ and defined as $\mathfrak{S}_i(\omega) = [\mathfrak{S}_*(\omega, i), \mathfrak{S}^*(\omega, i)]$ for all $\omega \in [\varsigma, \nu]$ and for all $i \in (0, 1]$. Then,

$$\widetilde{\mathfrak{S}}\left(\frac{1}{k}\sum_{i=1}^{k} \omega_i\right) \supseteq_{\mathbb{F}} \prod_{i=1}^{k}\left[\widetilde{\mathfrak{S}}(\omega_i)\right]^{\frac{1}{k}} \tag{38}$$

If $\widetilde{\mathfrak{S}}$ is a $U\mathcal{DL}$-concave, then (38) is inverted.

Now in upcoming results, with the help of $U\mathcal{DL}$-convex $\mathcal{FNVM}$s, we will prove Schur's inequality and its generalized form.

Theorem 9. Let $\widetilde{\mathfrak{S}} : [\varsigma, \nu] \to \Omega_C$ be a $\mathcal{FNVM}$, whose parametrized form is given by $\mathfrak{S}_i : [\varsigma, \nu] \subset \mathbb{R} \to \mathcal{K}_C^+$ and provided as $\mathfrak{S}_i(\omega) = [\mathfrak{S}_*(\omega, i), \mathfrak{S}^*(\omega, i)]$ for all $\omega \in [\varsigma, \nu]$ and for all $i \in (0, 1]$. If $\widetilde{\mathfrak{S}}$ be a $U\mathcal{DL}$-convex $\mathcal{FNVM}$ then, for $\omega_1, \omega_2, \omega_3 \in [\varsigma, \nu]$, $\omega_1 < \omega_2 < \omega_3$ such that $\omega_3 - \omega_1, \omega_3 - \omega_2, \omega_2 - \omega_1 \in [0, 1]$, we have

$$\widetilde{\mathfrak{S}}(\omega_2)^{(\omega_3 - \omega_1)} \supseteq_{\mathbb{F}} \widetilde{\mathfrak{S}}(\omega_1)^{\omega_3 - \omega_2} \otimes \widetilde{\mathfrak{S}}(\omega_3)^{\omega_2 - \omega_1} \tag{39}$$

If $\widetilde{\mathfrak{S}}$ is a $U\mathcal{DL}$-concave, then (39) is inverted.

Proof. Let $\omega_1, \omega_2, \omega_3 \in [\varsigma, \nu]$ and $\omega_3 - \omega_1 > 0$. Taking $\lambda = \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1}$, then $\omega_2 = \lambda\omega_1 + (1 - \lambda)\omega_3$. Since $\widetilde{\mathfrak{S}}$ is a $U\mathcal{DL}$-convex $\mathcal{FNVM}$ then, by hypothesis, we have

$$\begin{aligned} \mathfrak{S}_*(\omega_2, i) &\leq [\mathfrak{S}_*(\omega_1, i)]^{\frac{\omega_3 - \omega_2}{\omega_3 - \omega_1}} [\mathfrak{S}_*(\omega_3, i)]^{\frac{\omega_2 - \omega_1}{\omega_3 - \omega_1}}, \\ \mathfrak{S}^*(\omega_2, i) &\geq [\mathfrak{S}^*(\omega_1, i)]^{\frac{\omega_3 - \omega_2}{\omega_3 - \omega_1}} [\mathfrak{S}^*(\omega_3, i)]^{\frac{\omega_2 - \omega_1}{\omega_3 - \omega_1}}. \end{aligned} \tag{40}$$

Taking "log" on both sides of (40), we have

$$\begin{aligned} (\omega_3 - \omega_1) ln\mathfrak{S}_*(\omega_2, i) &\leq (\omega_3 - \omega_2) ln\mathfrak{S}_*(\omega_1, i) + (\omega_2 - \omega_1) ln\mathfrak{S}_*(\omega_3, i), \\ (\omega_3 - \omega_1) ln\mathfrak{S}^*(\omega_2, i) &\geq (\omega_3 - \omega_2) ln\mathfrak{S}^*(\omega_1, i) + (\omega_2 - \omega_1) ln\mathfrak{S}^*(\omega_3, i). \end{aligned} \tag{41}$$

From (41), we have

$$\begin{aligned} \mathfrak{S}_*(\omega_2, i)^{(\omega_3 - \omega_1)} &\leq [\mathfrak{S}_*(\omega_1, i)]^{(\omega_3 - \omega_2)} [\mathfrak{S}_*(\omega_3, i)]^{(\omega_2 - \omega_1)}, \\ \mathfrak{S}^*(\omega_2, i)^{(\omega_3 - \omega_1)} &\geq [\mathfrak{S}^*(\omega_1, i)]^{(\omega_3 - \omega_2)} [\mathfrak{S}^*(\omega_3, i)]^{\omega_2 - \omega_1}. \end{aligned}$$

That is

$$\begin{aligned} &\left[\mathfrak{S}_*(\omega_2, \boldsymbol{i})^{(\omega_3 - \omega_1)}, \mathfrak{S}^*(\omega_2, \boldsymbol{i})^{(\omega_3 - \omega_1)}\right] \\ &\supseteq_I \left[[\mathfrak{S}_*(\omega_1, \boldsymbol{i})]^{(\omega_3 - \omega_2)} [\mathfrak{S}_*(\omega_3, \boldsymbol{i})]^{(\omega_2 - \omega_1)}, [\mathfrak{S}^*(\omega_1, \boldsymbol{i})]^{(\omega_3 - \omega_2)} [\mathfrak{S}^*(\omega_3, \boldsymbol{i})]^{(\omega_2 - \omega_1)}\right], \end{aligned}$$

Hence

$$\widetilde{\mathfrak{S}}(\omega_2)^{(\omega_3 - \omega_1)} \supseteq_{\mathbb{F}} \widetilde{\mathfrak{S}}(\omega_1)^{(\omega_3 - \omega_2)} \otimes \widetilde{\mathfrak{S}}(\omega_3)^{(\omega_2 - \omega_1)}.$$

□

Theorem 10. Let $\mathrm{i}_{i} \in \mathbb{R}^{+}$, $\omega_{i} \in [\varsigma, \mathfrak{v}]$, $(i = 1, 2, 3, \ldots, k, k \geq 2)$ and $\widetilde{\mathfrak{S}}: [\varsigma, \mathfrak{v}] \to \Omega_{C}$ be a $U\mathcal{DL}$-convex $\mathcal{FNVM}$, whose parametrized form is given by $\mathfrak{S}_{\mathrm{i}}: [\varsigma, \mathfrak{v}] \subset \mathbb{R} \to \mathcal{K}_{C}^{+}$ and provided as $\mathfrak{S}_{\mathrm{i}}(\omega) = [\mathfrak{S}_{*}(\omega, \mathrm{i}), \mathfrak{S}^{*}(\omega, \mathrm{i})]$ for all $\omega \in [\varsigma, \mathfrak{v}]$ and for all $\mathrm{i} \in (0,1]$. If $(\mathfrak{L}, \mathcal{U}) \subseteq [\varsigma, \mathfrak{v}]$ then,

$$\prod_{i=1}^{k} [\widetilde{\mathfrak{S}}(\omega_{i})]^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \supseteq_{\mathbb{F}} \prod_{i=1}^{k} \left([\widetilde{\mathfrak{S}}(\mathfrak{L})]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \otimes [\widetilde{\mathfrak{S}}(\mathcal{U})]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right), \tag{42}$$

where $W_{k} = \sum_{i=1}^{k} \mathrm{i}_{i}$. If $\widetilde{\mathfrak{S}}$ is $U\mathcal{DL}$-concave, then (42) is inverted.

Proof. Consider $\mathfrak{L} = \omega_{1}, \omega_{i} = \omega_{2}$, $(i = 1, 2, 3, \ldots, k)$, $\mathcal{U} = \omega_{3}$ in (42). Then, for each $\mathrm{i} \in (0,1]$, we have

$$\mathfrak{S}_{*}(\omega_{i}, i) \leq [\mathfrak{S}_{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)} [\mathfrak{S}_{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)},$$
$$\mathfrak{S}^{*}(\omega_{i}, i) \geq [\mathfrak{S}^{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)} [\mathfrak{S}^{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)}.$$

The above inequality can be written as,

$$\begin{aligned} \mathfrak{S}_{*}(\omega_{i}, i)^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} &\leq [\mathfrak{S}_{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} [\mathfrak{S}_{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)}, \\ \mathfrak{S}^{*}(\omega_{i}, i)^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} &\geq [\mathfrak{S}^{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} [\mathfrak{S}^{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)}. \end{aligned} \tag{43}$$

Taking multiplication of all inequalities (43) for $i = 1, 2, 3, \ldots, k$, we have

$$\prod_{i=1}^{k} \mathfrak{S}_{*}(\omega_{i}, i)^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \leq \prod_{i=1}^{k} \left([\mathfrak{S}_{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} [\mathfrak{S}_{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right),$$
$$\prod_{i=1}^{k} \mathfrak{S}^{*}(\omega_{i}, i)^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \geq \prod_{i=1}^{k} \left([\mathfrak{S}^{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} [\mathfrak{S}^{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right),$$

that is

$$\begin{aligned} \prod_{i=1}^{k} \mathfrak{S}_{i}(\omega_{i})^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} &= \left[\prod_{i=1}^{k} \mathfrak{S}_{*}(\omega_{i}, i)^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)},\ \prod_{i=1}^{k} \mathfrak{S}^{*}(\omega_{i}, i)^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right] \\ &\supseteq_{I} \left[\prod_{i=1}^{k} \left([\mathfrak{S}_{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} [\mathfrak{S}_{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right), \right. \\ &\left. \prod_{i=1}^{k} \left([\mathfrak{S}^{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} [\mathfrak{S}^{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right) \right], \\ &\supseteq_{I} \prod_{i=1}^{k} \left(\left[[\mathfrak{S}_{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)}, \quad [\mathfrak{S}^{*}(\mathfrak{L}, i)]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right] \right) \\ &\cdot \prod_{i=1}^{k} \left(\left[[\mathfrak{S}_{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)},\ [\mathfrak{S}^{*}(\mathcal{U}, i)]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right] \right), \\ &= \prod_{i=1}^{k} [\mathfrak{S}_{i}(\mathfrak{L})]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} . \prod_{i=1}^{k} [\mathfrak{S}_{i}(\mathcal{U})]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)}. \end{aligned}$$

Thus,

$$\prod_{i=1}^{k} [\widetilde{\mathfrak{S}}(\omega_{i})]^{\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \supseteq_{\mathbb{F}} \prod_{i=1}^{k} \left([\widetilde{\mathfrak{S}}(\mathfrak{L})]^{\left(\frac{\mathcal{U}-\omega_{i}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \otimes [\widetilde{\mathfrak{S}}(\mathcal{U})]^{\left(\frac{\omega_{i}-\mathfrak{L}}{\mathcal{U}-\mathfrak{L}}\right)\left(\frac{\mathrm{i}_{i}}{W_{k}}\right)} \right),$$

this completes the proof. □

Remark 6. If $\mathfrak{S}_{*}(\omega, \mathrm{i}) = \mathfrak{S}^{*}(\omega, \mathrm{i})$ with $\mathrm{i} = 1$, then from (37), (38), and (39), we achieved the outcomes reduced for convex mapping, see [51].

4. Conclusions

Using fuzzy Aumman integrals, we showed some new Hermite–Hadamard type inequalities for newly defined class up and down log–convex functions in the fuzzy environment. Furthermore, using the up-and-down log–convex fuzzy-number valued mappings, we established Jensen's and Schur's type inequalities. We used a mathematical example to demonstrate the correctness of the newly discovered results. We also demonstrated that the newly obtained results are an extension of previously established results in the literature. It is a novel problem in which future scholars can obtain equivalent inequalities for fractal sets and coordinated convex functions.

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