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Frank Prioritized Aggregation Operators and WASPAS Method Based on Complex Intuitionistic Fuzzy Sets and Their Application in Multi-Attribute Decision-Making

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Abstract: Complex intuitionistic fuzzy (CIF) information covers the degree of membership and the degree of non-membership in the form of polar coordinates with a valuable and dominant characteristic where the sum of the real parts (the same rule for the imaginary parts) of the pair must be contained in the unit interval. In this paper, we first derive the Frank operational laws for CIF information and then examine the prioritized aggregation operators based on Frank operational laws for managing the theory of CIF information. These are the CIF Frank prioritized averaging (CIFFPA) operator, the CIF Frank prioritized ordered averaging (CIFFPOA) operator, the CIF Frank prioritized geometric (CIFFPG) operator, and the CIF Frank prioritized ordered geometric (CIFFPOG) operator with properties of idempotency, monotonicity, and boundedness. Furthermore, we derive the WASPAS (weighted aggregates sum product assessment) under the consideration or presence of the CIF information and try to justify it with the help of a suitable example. Additionally, we illustrate some numerical examples in the presence of the MADM (multi-attribute decision-making) procedures for evaluating the comparison between the proposed operators with some well-known existing operators to show the validity and worth of the proposed approaches.

Keywords: fuzzy sets; intuitionistic fuzzy sets; complex intuitionistic fuzzy sets; frank prioritized aggregation operators; WASPAS techniques; multi-attribute decision-making

MSC: 03B52; 03E72; 47S40; 90C70

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1. Introduction

Multi-attribute decision-making (MADM) procedures are some of the finest or best techniques for evaluating the valuable and dominant preference from the set of feasible ones under the consideration of the available data. Traditionally, the MADM problem is a part of the decision-making procedure which often needs experts to provide evaluation data about the attributes and the alternatives with fuzzy sets (FSs) [1] in which FSs have been applied in different fields [2–4]. Various attempts have been derived by the distinct individuals in proceeding the data values using different extensions such as hesitant soft fuzzy rough sets [5], and fuzzy Mandelbrot sets [6]. Furthermore, intuitionistic FSs (IFSs) are also one of the most valuable and dominant extensions of FSs which was performed by Atanassov [7]. IFSs cover the degree of membership and the degree of non-membership of a given element to the set of discourse with the characteristic in which the sum of the pair must be contained in the unit interval. FSs are the particular cases of IFSs if we remove the degree of non-membership with its applications [8–10]. Furthermore, the utilization of the second term in the grade of truth is very awkward, and so, in many situations, we may face a problem with two-dimensional information, where FSs and IFSs deal only with one-dimensional information. Therefore, Ramot, et al. [11] successfully utilized the second term in the grade of truth and gave their name in the form of complex FS (CFS), where the truth grade in CFS is computed in the form of complex numbers whose real and unreal (imaginary) parts are covered in the unit interval. Various attempts have been performed by various individuals using systems such as the Mamdani complex fuzzy inference system [12]. Additionally, Alkouri and Salleh [13] exposed the new theory of complex IFS (CIFS) with its applications [14], which is the modified version of the three different types of ideas such as FSs, IFSs, and CFSs.

Frank t-norm and t-conorm are used for computing any type of aggregation operators (AOs) which was derived by Frank [15] in 1979. Frank norms have a lot of benefits because the simple algebraic and Lukasiewicz's t-norm and t-conorm [16] are the special cases of Frank t-norm and t-conorm. Furthermore, the idea of prioritizing AOs for the first time was given by Yager [17], and then Yu and Xu [18] who considered prioritized intuitionistic fuzzy AOs. These AOs were computed based on algebraic operational laws. Moreover, the main idea of the weighted aggregated sum product assessment (WASPAS) technique was given by Zavadskas, et al. [19,20] with its applications [21,22], which is the generalization of two different techniques such as weighted sum assessment (WSA) and weighted product assessment (WPS). The WASPAS technique is very strong and valuable because this is the modified version of many techniques and many individuals have utilized it in numerous fields such as the computer sciences, pure mathematics, engineering sciences, artificial intelligence, and decision-making.

The theory of FSs, IFSs, CFSs, and CIFSs has gained a lot of attention from different fuzzy researchers because these structures are very beneficial and valuable for depicting awkward and unreliable information very easily. Various attempts have been derived by distinct individuals in proceeding with the data values using different extensions such as AOs for IFSs [23] and geometric AOs for IFSs [24]. Furthermore, the Frank power AOs based on IFSs [25] are also a combination of the Frank and power AOs which is a very awkward and complicated task. The complex fuzzy credibility of Frank AOs was derived by Yahya, et al. [26]. Under the consideration of hesitant fuzzy information, the theory of Frank AOs was invented by Qin, et al. [27]. In the presence of the dual hesitant set theory, the major theory of Frank AOs was evaluated by Tang, et al. [28]. The prioritized AOs for trapezoidal IFS were derived by Ye [29], and the simple prioritized AOs for IFS were evaluated by Yu and Xu [18]. Ali, et al. [30] derived the idea of prioritized AOs for CIF soft information with their application in decision-making procedures. Yu [31] examined the theory of generalized prioritized AOs for intuitionistic fuzzy environments, and Lin, et al. [32] derived the fuzzy number intuitionistic fuzzy prioritized AOs and their application in decision-making procedures. Furthermore, Garg and Rani [33] exposed the averaging operators for CIFSs. Garg and Rani [34] evaluated the geometric operators for CIFSs, and Mahmood, et al. [35] examined the Aczel–Alsina AOs for CIFSs. Sarfraz, et al. [36] examined the prioritized AOs for IFSs with IF-prioritized Aczel–Alsina averaging. Poryazov, et al. [37] applied AOs for IFSs to the estimation of service compositions in telecommunication systems. Dai [38] derived linguistic complex fuzzy sets with their properties.

Frank and prioritized AOs based on IFSs were derived by different researchers, however, the theory of Frank and prioritized AOs based on CIFSs has not yet been evaluated by researchers in the literature. The investigation of Frank and prioritized AOs based on CIFSs is a very challenging task. In this analysis, we have accepted this task and not only derive the theory of Frank and prioritized AOs based on CIFSs but also derive the combination of Frank and prioritized AOs based on CIFSs, where the simple Frank and prioritized AOs are the special case of the derived theory. Furthermore, we also invent the theory of WASPAS for CIFSs. Inspired by the above discussion, the major investigations of this analysis are listed below:

1. To discover Frank operational laws for managing the theory of CIF information;
2. To derive the CIF Frank prioritized averaging (CIFFPA) operator, the CIF Frank prioritized ordered averaging (CIFFPOA) operator, the CIF Frank prioritized geometric (CIFFPG) operator, and the CIF Frank prioritized ordered geometric (CIFFPOG) operator with their properties;
3. To expose the idea of the weighted aggregates sum product assessment (WASPAS) procedure under the consideration or presence of the CIF information and try to simplify it with the help of a suitable example;
4. To demonstrate an example in the presence of the MADM procedures for evaluating the comparison between the proposed operators with some well-known existing operators to show the validity and worth of the discovered approaches.

This article is arranged in the form as follows: in Section 2, we review the different types of norms, CIFS, and the WASPAS technique; in Section 3, we examine Frank operational laws, CIFFPA operator, CIFFPOA operator, CIFFPG operator, and CIFFPOG operator, and their properties of idempotency, monotonicity, and boundedness; in Section 4, we derive the WASPAS for CIF information and try to justify it with the help of a suitable example; and in Section 5, we illustrate some examples in the presence of the MADM procedures for evaluating CIF information. Furthermore, the comparisons between the proposed operators and some well-known existing operators, such as Xu [23], Xu and Yager [24], Yahya, et al. [26], Yu [31], Lin, et al. [32], Garg and Rani [33], Garg and Rani [34], and Mahmood, et al. [35], are used to show the validity and worth of the discovered approaches and are discussed in Section 6. The final concluding information is shown in Section 7.

2. Preliminaries

In this section, we describe the prevailing theory of Frank norms, algebraic norms, and Lukasiewicz's norms for positive numbers. Furthermore, we also explain the idea of the WASPAS method [19,20] for classical set theory. Moreover, the idea of CIFSs and their related work are also a part of this study. For a clear presentation, the meaning of the symbols used in this paper is shown in Table 1.

Table 1. Meanings of different symbols used in the paper.

Symbols	Meanings	Symbols	Meanings	Symbols	Meanings
$u^{rp}_{⅃}(x)$	Real part of membership grade	$v^{rp}_{⅃}(x)$	Real part of the non-membership grade	X	Universal set
$u^{ip}_{⅃}(x)$	Imaginary part of membership grade	$v^{ip}_{⅃}(x)$	Imaginary part of the non-membership grade	x	Element of the universal set
$r^{rp}(x)$	Real part of the refusal grade	$r^{ip}(x)$	Imaginary part of refusal grade	$r(x)$	Refusal grade
$\overline{\overline{\mathbf{I}\mathfrak{T}}}$	Complex intuitionistic fuzzy set	$\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}$	Complex intuitionistic fuzzy value	$V_s\left(\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}\right)$	Score value
$V_a(\overline{\overline{\mathbf{I}\mathfrak{T}}}_{\mathfrak{H}})$	Accuracy value	$W_{\mathcal{B}}$	Weighted vector	$°\mathrm{F} \geq 0$	Scaler
$\mathfrak{Y}(\mho, \mho^*)$	t-norm	$\mathfrak{Y}^*(\mho, \mho^*)$	t-conorm	$\daleth\daleth \in (1, +\infty)$	Scaler

2.1. WASPAS Method for Classical Set Theory

The major influence of this section is to recall the theory of the WASPAS procedure for classical information. The main procedure of the WASPAS method contains various valuable and dominant steps. Before evaluating the normalization, we arrange a collection of classical data which may be of a benefit type or cost type. If the data are of a benefit type, then good, otherwise, using the below theory, we normalize the information, such as:

$$\mathtt{C}'_{\mathfrak{H}\mathcal{B}} = \begin{cases} \dfrac{\tilde{\mathtt{C}}_{\mathfrak{H}\mathcal{B}}}{\max\limits_{\mathfrak{H}} \tilde{\mathtt{C}}_{\mathfrak{H}\mathcal{B}}} & for\ benefit \\ \dfrac{\min\limits_{\mathfrak{H}} \tilde{\mathtt{C}}_{\mathfrak{H}\mathcal{B}}}{\tilde{\mathtt{C}}_{\mathfrak{H}\mathcal{B}}} & for\ cost. \end{cases} \tag{1}$$

After performing the above evaluation, we calculate the WSA and WPA, such as:

$$T_{\mathfrak{H}}^{WSA} = \sum_{\mathcal{B}=1}^{d} W_{\mathcal{B}} \tilde{\mathtt{C}}'_{\mathfrak{H}\mathcal{B}}; \tag{2}$$

$$T_{\mathfrak{H}}^{WPA} = \sum_{\mathcal{B}=1}^{d} (\tilde{\mathtt{C}}'_{\mathfrak{H}\mathcal{B}})^{W_{\mathcal{B}}}. \tag{3}$$

Using the data in Equations (2) and (3), we calculate the aggregated measure based on convex theory, such as:

$$T_{\mathfrak{H}} = {}^{\circ}\mathrm{F} T_{\mathfrak{H}}^{WSA} + (1 - {}^{\circ}\mathrm{F}) T_{\mathfrak{H}}^{WPA}, {}^{\circ}\mathrm{F} \in [0,1]. \tag{4}$$

Before ranking the alternatives, we discuss the special cases of the WASPAS technique such as: When $^{\circ}\mathrm{F} = 1$, we obtain the data in Equation (2):

1. When $^{\circ}\mathrm{F} = 1$, we obtain the data in Equation (2);
2. When $^{\circ}\mathrm{F} = 0$, we obtain the data in Equation (3).

Finally, we derive the ranking result for examining the best one from the family of finite preferences.

2.2. Existing Ideas

Definition 1 ([15])**.** *For any two positive numbers $\mho$ and $\mho^*$, we have the theory of Frank t-norm and t-conorm, such that:*

$$\mathfrak{Y}(\mho, \mho^*) = \log_{\daleth\daleth}\left(1 + \frac{\left(\daleth\daleth^{\mho} - 1\right)\left(\daleth\daleth^{\mho^*} - 1\right)}{\daleth\daleth - 1}\right), \daleth\daleth \in (1, +\infty). \tag{5}$$

$$\mathfrak{Y}^*(\mho, \mho^*) = 1 - \log_{\daleth\daleth}\left(1 + \frac{\left(\daleth\daleth^{1-\mho} - 1\right)\left(\daleth\daleth^{1-\mho^*} - 1\right)}{\daleth\daleth - 1}\right), \daleth\daleth \in (1, +\infty) \tag{6}$$

Definition 2 ([1])**.** *For any two positive numbers $\mho$ and $\mho^*$, we have the theory of algebraic t-norm and t-conorm if we put the value of $\daleth\daleth \to 1$ in Equations (5) and (6), such that:*

$$\mathfrak{Y}(\mho, \mho^*) = \mho * \mho^*. \tag{7}$$

$$\mathfrak{Y}^*(\mho, \mho^*) = \mho + \mho^* - \mho * \mho^*. \tag{8}$$

Definition 3 ([16]). *For any two positive numbers* $\mho$ *and* $\mho^*$ *, we have the theory of Lukasiewicz t-norm and t-conorm if we put the value of* $\daleth\daleth \to +\infty$ *in Equations (5) and (6), such that:*

$$\mathfrak{Y}(\mho, \mho^*) = \max\{0, \mho + \mho^* - 1\}. \tag{9}$$

$$\mathfrak{Y}^*(\mho, \mho^*) = \min\{\mho + \mho^*, 1\} \tag{10}$$

Definition 4 ([13]). *A numerical or mathematical equation*:

$$\overline{\overline{I\mathfrak{T}}} = \left\{ \left(\left(u_{⅃}^{rp}(x), u_{⅃}^{ip}(x) \right), \left(v_{⅃}^{rp}(x), v_{⅃}^{ip}(x) \right) \right) : x \in X \right\} \tag{11}$$

Stated the CIFS with a truth grade $\left(u_{⅃}^{rp}(x), u_{⅃}^{ip}(x) \right)$ and falsity grade $\left(v_{⅃}^{rp}(x), v_{⅃}^{ip}(x) \right)$ must be implementing the following rules, such that $0 \leq u_{⅃}^{rp}(x) + v_{⅃}^{rp}(x) \leq 1$ and $0 \leq u_{⅃}^{ip}(x) + v_{⅃}^{ip}(x) \leq 1$. The notion of neutral grade is stated by: $r(x) = \left(r^{rp}(x), r^{ip}(x) \right)$ $= \left(1 - \left(u_{⅃}^{rp}(x) + v_{⅃}^{rp}(x) \right), 1 - \left(u_{⅃}^{ip}(x) + v_{⅃}^{ip}(x) \right) \right)$ and the representation of the CIF values (CIFVs) is the following: $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \left(\left(u_{⅃_{\mathfrak{H}}}^{rp}, u_{⅃_{\mathfrak{H}}}^{ip} \right), \left(v_{⅃_{\mathfrak{H}}}^{rp}, v_{⅃_{\mathfrak{H}}}^{ip} \right) \right), \mathfrak{H} = 1, 2, \ldots, ⅃$. As noted in the presence of the above information, we recall the idea of score and accuracy function, such as:

$$V_s\left(\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} \right) = \frac{1}{2} \left(u_{⅃_{\mathfrak{H}}}^{rp} - v_{⅃_{\mathfrak{H}}}^{rp} + u_{⅃_{\mathfrak{H}}}^{ip} - v_{⅃_{\mathfrak{H}}}^{ip} \right) \in [-1, 1]. \tag{12}$$

$$V_a\left(\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} \right) = \frac{1}{2} \left(u_{⅃_{\mathfrak{H}}}^{rp} + v_{⅃_{\mathfrak{H}}}^{rp} + u_{⅃_{\mathfrak{H}}}^{ip} + v_{⅃_{\mathfrak{H}}}^{ip} \right) \in [0, 1] \tag{13}$$

To differentiate the above information, we recall some valuable characteristics: if $V_s\left(\overline{\overline{I\mathfrak{T}}}_1 \right) > V_s\left(\overline{\overline{I\mathfrak{T}}}_2 \right) \Rightarrow \overline{\overline{I\mathfrak{T}}}_1 > \overline{\overline{I\mathfrak{T}}}_2$; If $V_s\left(\overline{\overline{I\mathfrak{T}}}_1 \right) < V_s\left(\overline{\overline{I\mathfrak{T}}}_2 \right) \Rightarrow \overline{\overline{I\mathfrak{T}}}_1 < \overline{\overline{I\mathfrak{T}}}_2$; If $V_s(\overline{\overline{I\mathfrak{T}}}_1) = V_s(\overline{\overline{I\mathfrak{T}}}_2) \Rightarrow$ If $V_a\left(\overline{\overline{I\mathfrak{T}}}_1 \right) > V_a\left(\overline{\overline{I\mathfrak{T}}}_2 \right) \Rightarrow \overline{\overline{I\mathfrak{T}}}_1 > \overline{\overline{I\mathfrak{T}}}_2$; If $V_a\left(\overline{\overline{I\mathfrak{T}}}_1 \right) < V_a\left(\overline{\overline{I\mathfrak{T}}}_2 \right) \Rightarrow \overline{\overline{I\mathfrak{T}}}_1 < \overline{\overline{I\mathfrak{T}}}_2$.

3. CIF Frank Prioritized Aggregation Operators

In this section, we propose the idea of Frank operational laws for CIF information. Furthermore, we examine the theory of the CIFFPA operator, the CIFFPOA operator, the CIFFPG operator, and the CIFFPOG operator, and their properties (idempotency, monotonicity, and boundedness). From now on, we will be using the CIFVs $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \left(\left(u_{⅃_{\mathfrak{H}}}^{rp}, u_{⅃_{\mathfrak{H}}}^{ip} \right), \left(v_{⅃_{\mathfrak{H}}}^{rp}, v_{⅃_{\mathfrak{H}}}^{ip} \right) \right), \mathfrak{H} = 1, 2, \ldots, ⅃$ for constructing any ideas.

Definition 5. *The mathematical form of Frank operational laws is stated as follows: for* $\daleth\daleth \in (1, +\infty)$,

$$\overline{\overline{I\mathfrak{T}}}_1 \oplus \overline{\overline{I\mathfrak{T}}}_2 = \left(\begin{array}{c} \left(1 - \log_{\daleth\daleth}\left(1 + \frac{\left(\daleth\daleth^{1-u_{⅃_1}^{rp}} - 1 \right)\left(\daleth\daleth^{1-u_{⅃_2}^{rp}} - 1 \right)}{\daleth\daleth - 1} \right), 1 - \log_{\daleth\daleth}\left(1 + \frac{\left(\daleth\daleth^{1-u_{⅃_1}^{ip}} - 1 \right)\left(\daleth\daleth^{1-u_{⅃_2}^{ip}} - 1 \right)}{\daleth\daleth - 1} \right) \right), \\ \left(\log_{\daleth\daleth}\left(1 + \frac{\left(\daleth\daleth^{v_{⅃_1}^{rp}} - 1 \right)\left(\daleth\daleth^{v_{⅃_2}^{rp}} - 1 \right)}{\daleth\daleth - 1} \right), \log_{\daleth\daleth}\left(1 + \frac{\left(\daleth\daleth^{v_{⅃_1}^{ip}} - 1 \right)\left(\daleth\daleth^{v_{⅃_2}^{ip}} - 1 \right)}{\daleth\daleth - 1} \right) \right) \end{array} \right). \tag{14}$$

$$\overline{\overline{I\mathfrak{T}}}_1 \otimes \overline{\overline{I\mathfrak{T}}}_2 = \left(\begin{array}{c} \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{u^{rp}_{\lrcorner_1}}-1\right)\left(\daleth\daleth^{u^{rp}_{\lrcorner_2}}-1\right)}{\daleth\daleth-1}\right), \log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{u^{ip}_{\lrcorner_1}}-1\right)\left(\daleth\daleth^{u^{ip}_{\lrcorner_2}}-1\right)}{\daleth\daleth-1}\right)\right), \\ \left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-v^{rp}_{\lrcorner_1}}-1\right)\left(\daleth\daleth^{1-v^{rp}_{\lrcorner_2}}-1\right)}{\daleth\daleth-1}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-v^{ip}_{\lrcorner_1}}-1\right)\left(\daleth\daleth^{1-v^{ip}_{\lrcorner_2}}-1\right)}{\daleth\daleth-1}\right)\right) \end{array} \right) \quad (15)$$

$$\dddot{\mathcal{P}}\overline{\overline{I\mathfrak{T}}}_1 = \left(\begin{array}{c} \left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{rp}_{\lrcorner_1}}-1\right)^{\dddot{\mathcal{P}}}}{(\daleth\daleth-1)^{\dddot{\mathcal{P}}-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{ip}_{\lrcorner_1}}-1\right)^{\dddot{\mathcal{P}}}}{(\daleth\daleth-1)^{\dddot{\mathcal{P}}-1}}\right)\right), \\ \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{rp}_{\lrcorner_1}}-1\right)^{\dddot{\mathcal{P}}}}{(\daleth\daleth-1)^{\dddot{\mathcal{P}}-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{ip}_{\lrcorner_1}}-1\right)^{\dddot{\mathcal{P}}}}{(\daleth\daleth-1)^{\dddot{\mathcal{P}}-1}}\right)\right) \end{array} \right). \quad (16)$$

$$\overline{\overline{I\mathfrak{T}}}_1^{\dddot{\mathcal{P}}} = \left(\begin{array}{c} \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{u^{rp}_{\lrcorner_1}}-1\right)^{\dddot{\mathcal{P}}}}{(\daleth\daleth-1)^{\dddot{\mathcal{P}}-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{u^{ip}_{\lrcorner_1}}-1\right)^{\dddot{\mathcal{P}}}}{(\daleth\daleth-1)^{\dddot{\mathcal{P}}-1}}\right)\right), \\ \left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-v^{rp}_{\lrcorner_1}}-1\right)^{\dddot{\mathcal{P}}}}{(\daleth\daleth-1)^{\dddot{\mathcal{P}}-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-v^{ip}_{\lrcorner_1}}-1\right)^{\dddot{\mathcal{P}}}}{(\daleth\daleth-1)^{\dddot{\mathcal{P}}-1}}\right)\right) \end{array} \right). \quad (17)$$

Definition 6. *The mathematical form of the CIFFPA operator is shown below*:

$$\begin{array}{c} CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_\lrcorner\right) = \\ \left(\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_1 \oplus \left(\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_2 \oplus \ldots \oplus \left(\frac{\overline{\overline{\mathring{A}}}_\lrcorner}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_\lrcorner \\ = \oplus_{\mathfrak{H}=1}^{\lrcorner}\left(\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}. \end{array} \quad (18)$$

With the values of $\overline{\overline{\mathring{A}}}_1 = 1$ and $\overline{\overline{\mathring{A}}}_{\mathfrak{H}} = \prod_{\mathcal{B}=1}^{\mathfrak{H}-1} V_s\left(\overline{\overline{I\mathfrak{T}}}_{\mathcal{B}}\right)$.

Theorem 1. *With the help of the data in Equation (18), we show that the aggregated value of Equation (18) will again be in the form of CIFV, such as:*

$$CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_\lrcorner\right) = \left(\begin{array}{c} \left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\daleth\daleth^{1-u^{rp}_{\lrcorner_{\mathfrak{H}}}}-1\right)^{\frac{\dddot{\overline{\overline{\mathring{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\daleth\daleth^{1-u^{ip}_{\lrcorner_{\mathfrak{H}}}}-1\right)^{\frac{\dddot{\overline{\overline{\mathring{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right), \\ \left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\daleth\daleth^{v^{rp}_{\lrcorner_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\daleth\daleth^{v^{ip}_{\lrcorner_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right) \end{array} \right). \quad (19)$$

Proof. The procedure of Mathematical Induction is used in this proof as follows: if $\Finv = 2$, then we obtain

$$\left(\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_1=\left(\begin{array}{c}\left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{rp}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{ip}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right),\\ \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{rp}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{ip}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

$$\left(\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_2=\left(\begin{array}{c}\left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{rp}_{\Finv 2}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{ip}_{\Finv 2}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right),\\ \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{rp}_{\Finv 2}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{ip}_{\Finv 2}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

Thus,

$$\left(\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_1\oplus\left(\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_2=\left(\begin{array}{c}\left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{rp}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{ip}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right),\\ \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{rp}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{ip}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)\oplus$$

$$\left(\begin{array}{c}\left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{rp}_{\Finv 2}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u^{ip}_{\Finv 2}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right),\\ \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{rp}_{\Finv 2}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v^{ip}_{\Finv 2}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

$$= \left(\begin{array}{c}\left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{2}\left(\daleth\daleth^{1-u^{rp}_{\Finv\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{2}\left(\daleth\daleth^{1-u^{ip}_{\Finv\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right), \\ \left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{2}\left(\daleth\daleth^{v^{rp}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{2}\left(\daleth\daleth^{v^{ip}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

We obtain the correct theory. Furthermore, we assume that we also obtain the correct theory for $\Finv = \mathcal{B}$, such that:

$$CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\mathcal{B}}\right) = \left(\begin{array}{c}\left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}}\left(\daleth\daleth^{1-u^{rp}_{\Finv\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}}\left(\daleth\daleth^{1-u^{ip}_{\Finv\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right), \\ \left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}}\left(\daleth\daleth^{v^{rp}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}}\left(\daleth\daleth^{v^{ip}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

Furthermore, we prove it for $\Finv = \mathcal{B}+1$, such as:

$$\begin{aligned} CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\mathcal{B}+1}\right) &= \left(\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_1 \oplus \left(\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_2 \oplus \ldots \oplus \left(\frac{\overline{\overline{\mathring{A}}}_{\mathcal{B}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{\mathcal{B}} \oplus \left(\frac{\overline{\overline{\mathring{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{\mathcal{B}+1} \\ &= \oplus_{\mathfrak{H}=1}^{\mathcal{B}}\left(\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} \oplus \left(\frac{\overline{\overline{\mathring{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{\mathcal{B}+1} \end{aligned}$$

$$= \left(\begin{array}{c}\left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}}\left(\daleth\daleth^{1-u^{rp}_{\Finv\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}}\left(\daleth\daleth^{1-u^{ip}_{\Finv\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right), \\ \left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}}\left(\daleth\daleth^{v^{rp}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}}\left(\daleth\daleth^{v^{ip}_{\Finv 1}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\left(1-\log_{\urcorner\urcorner}\left(1+\frac{\left(\urcorner\urcorner^{1-u^{rp}_{\daleth\mathcal{B}+1}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\frac{\overline{\overline{\bar{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\urcorner\urcorner}\left(1+\frac{\left(\urcorner\urcorner^{1-u^{ip}_{\daleth\mathcal{B}+1}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\frac{\overline{\overline{\bar{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right)\right), \\ \left(\log_{\urcorner\urcorner}\left(1+\frac{\left(\urcorner\urcorner^{v^{rp}_{\daleth\mathcal{B}+1}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\frac{\overline{\overline{\bar{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\urcorner\urcorner}\left(1+\frac{\left(\urcorner\urcorner^{v^{ip}_{\daleth\mathcal{B}+1}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\frac{\overline{\overline{\bar{A}}}_{\mathcal{B}+1}}{\sum_{\mathfrak{H}=1}^{\mathcal{B}+1}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(1-\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}+1}\left(\urcorner\urcorner^{1-u^{rp}_{\daleth\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\daleth}\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}+1}\left(\urcorner\urcorner^{1-u^{ip}_{\daleth\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\daleth}\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right)\right), \\ \left(\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}+1}\left(\urcorner\urcorner^{v^{rp}_{\daleth 1}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\daleth}\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\mathcal{B}+1}\left(\urcorner\urcorner^{v^{ip}_{\daleth 1}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\daleth}\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

This proves the theorem. □

Proposition 1 (Idempotency). *If we use* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}=\overline{\overline{I\mathfrak{T}}}=\left(\left(u^{rp}_{\daleth},u^{ip}_{\daleth}\right),\left(v^{rp}_{\daleth},v^{ip}_{\daleth}\right)\right)$, *then*

$$CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\daleth}\right)=\overline{\overline{I\mathfrak{T}}}. \tag{20}$$

Proof. Let

$$CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\daleth}\right)=\left(\begin{array}{c}\left(1-\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\daleth}\left(\urcorner\urcorner^{1-u^{rp}_{\daleth\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\daleth}\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\daleth}\left(\urcorner\urcorner^{1-u^{ip}_{\daleth\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\daleth}\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right)\right), \\ \left(\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\daleth}\left(\urcorner\urcorner^{v^{rp}_{\daleth\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\daleth}\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\daleth}\left(\urcorner\urcorner^{v^{ip}_{\daleth\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\daleth}\frac{\overline{\overline{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\daleth}\overline{\overline{\bar{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right)$$

$$
\begin{aligned}
&= \left(\begin{array}{c} \left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\daleth\daleth^{1-u_{\lrcorner}^{rp}}-1\right)^{\frac{\ddot{\bar{\bar{\bar{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\daleth\daleth^{1-u_{\lrcorner}^{ip}}-1\right)^{\frac{\ddot{\bar{\bar{\bar{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}-1}}\right) \right), \\ \left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\daleth\daleth^{v_{\lrcorner}^{rp}}-1\right)^{\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\daleth\daleth^{v_{\lrcorner}^{ip}}-1\right)^{\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}-1}}\right) \right) \end{array} \right) \\
&= \left(\begin{array}{c} \left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u_{\lrcorner}^{rp}}-1\right)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\ddot{\bar{\bar{\bar{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u_{\lrcorner}^{ip}}-1\right)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\ddot{\bar{\bar{\bar{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}-1}}\right) \right), \\ \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v_{\lrcorner}^{rp}}-1\right)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v_{\lrcorner}^{ip}}-1\right)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}}-1}}\right) \right) \end{array} \right) \\
&= \left(\begin{array}{c} \left(1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u_{\lrcorner}^{rp}}-1\right)}{(\daleth\daleth-1)^{1-1}}\right), 1-\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{1-u_{\lrcorner}^{ip}}-1\right)}{(\daleth\daleth-1)^{1-1}}\right) \right), \\ \left(\log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v_{\lrcorner}^{rp}}-1\right)}{(\daleth\daleth-1)^{1-1}}\right), \log_{\daleth\daleth}\left(1+\frac{\left(\daleth\daleth^{v_{\lrcorner}^{ip}}-1\right)}{(\daleth\daleth-1)^{1-1}}\right) \right) \end{array} \right), \sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\bar{\bar{\bar{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\bar{\bar{\bar{A}}}_{\mathfrak{H}}} = 1 \\
&= \left(\begin{array}{c} \left(1-\log_{\daleth\daleth}\left(1+\left(\daleth\daleth^{1-u_{\lrcorner}^{rp}}-1\right)\right), 1-\log_{\daleth\daleth}\left(1+\left(\daleth\daleth^{1-u_{\lrcorner}^{ip}}-1\right)\right) \right), \\ \left(\log_{\daleth\daleth}\left(1+\left(\daleth\daleth^{v_{\lrcorner}^{rp}}-1\right)\right), \log_{\daleth\daleth}\left(1+\left(\daleth\daleth^{v_{\lrcorner}^{ip}}-1\right)\right) \right) \end{array} \right) \\
&= \left(\left(1-\log_{\daleth\daleth}\left(\daleth\daleth^{1-u_{\lrcorner}^{rp}}\right), 1-\log_{\daleth\daleth}\left(\daleth\daleth^{1-u_{\lrcorner}^{ip}}\right) \right), \left(\log_{\daleth\daleth}\left(\daleth\daleth^{v_{\lrcorner}^{rp}}\right), \log_{\daleth\daleth}\left(\daleth\daleth^{v_{\lrcorner}^{ip}}\right) \right) \right) \\
&= \left(\left(1-\left(1-u_{\lrcorner}^{rp}\right), 1-\left(1-u_{\lrcorner}^{ip}\right) \right), \left(v_{\lrcorner}^{rp}, v_{\lrcorner}^{ip} \right) \right) \\
&= \left(\left(u_{\lrcorner}^{rp}, u_{\lrcorner}^{ip} \right), \left(v_{\lrcorner}^{rp}, v_{\lrcorner}^{ip} \right) \right) = \overline{\overline{I\mathfrak{T}}}.
\end{aligned}
$$

This proves the proposition. □

Proposition 2 (Monotonicity). *If* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \left(\left(u_{\lrcorner_{\mathfrak{H}}}^{rp}, u_{\lrcorner_{\mathfrak{H}}}^{ip}\right), \left(v_{\lrcorner_{\mathfrak{H}}}^{rp}, v_{\lrcorner_{\mathfrak{H}}}^{ip}\right)\right) \leq \overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{*} = \left(\left(u_{\lrcorner_{\mathfrak{H}}}^{rp\,*}, u_{\lrcorner_{\mathfrak{H}}}^{ip\,*}\right), \left(v_{\lrcorner_{\mathfrak{H}}}^{rp\,*}, v_{\lrcorner_{\mathfrak{H}}}^{ip\,*}\right)\right)$*, then*

$$
CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}\right) \leq CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1^{*}, \overline{\overline{I\mathfrak{T}}}_2^{*}, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}^{*}\right) \tag{21}
$$

Proof. Consider $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = ((u_{\lrcorner_{\mathfrak{H}}}^{rp}, u_{\lrcorner_{\mathfrak{H}}}^{ip}), (v_{\lrcorner_{\mathfrak{H}}}^{rp}, v_{\lrcorner_{\mathfrak{H}}}^{ip})) \leq \overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{*} = ((u_{\lrcorner_{\mathfrak{H}}}^{rp\,*}, u_{\lrcorner_{\mathfrak{H}}}^{ip\,*}), (v_{\lrcorner_{\mathfrak{H}}}^{rp\,*}, v_{\lrcorner_{\mathfrak{H}}}^{ip\,*}))$. Notice that $u_{\lrcorner_{\mathfrak{H}}}^{rp} \leq u_{\lrcorner_{\mathfrak{H}}}^{rp\,*}, u_{\lrcorner_{\mathfrak{H}}}^{ip} \leq u_{\lrcorner_{\mathfrak{H}}}^{ip\,*}$ and $v_{\lrcorner_{\mathfrak{H}}}^{rp} \geq v_{\lrcorner_{\mathfrak{H}}}^{rp\,*}, v_{\lrcorner_{\mathfrak{H}}}^{ip} \geq v_{\lrcorner_{\mathfrak{H}}}^{ip\,*}$, then we have $u_{\lrcorner_{\mathfrak{H}}}^{rp} \leq u_{\lrcorner_{\mathfrak{H}}}^{rp\,*}$

$$
\Rightarrow 1-u_{\lrcorner_{\mathfrak{H}}}^{rp} \geq 1-u_{\lrcorner_{\mathfrak{H}}}^{rp\,*} \Rightarrow \daleth\daleth^{1-u_{\lrcorner_{\mathfrak{H}}}^{rp}} \geq \daleth\daleth^{1-u_{\lrcorner_{\mathfrak{H}}}^{rp\,*}} \Rightarrow \daleth\daleth^{1-u_{\lrcorner_{\mathfrak{H}}}^{rp}}-1 \geq \daleth\daleth^{1-u_{\lrcorner_{\mathfrak{H}}}^{rp\,*}}-1
$$

$$\Rightarrow \left(\daleth\daleth^{1-u^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}} \geq \left(\daleth\daleth^{1-u^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}} \Rightarrow \prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}$$

$$\geq \prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}} \Rightarrow \frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}} \geq \frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}$$

$$\Rightarrow \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}\right) \geq \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}\right) \Rightarrow 1-$$

$$\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}\right) \leq 1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}\right).$$

In the same way, we find the unreal part, such as:

$$1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{ip}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}\right) \leq 1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{1-u^{ip\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overset{\cdots}{\overline{\overline{\text{Å}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}\right)$$

Furthermore, we have $v^{rp}_{\text{⅃}_{\mathfrak{H}}} \geq v^{rp\,*}_{\text{⅃}_{\mathfrak{H}}} \Rightarrow \daleth\daleth^{v^{rp}_{\text{⅃}_{\mathfrak{H}}}} \geq \daleth\daleth^{v^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}} \Rightarrow \daleth\daleth^{v^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1 \geq \daleth\daleth^{v^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1$

$$\Rightarrow \left(\daleth\daleth^{v^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}} \geq \left(\daleth\daleth^{v^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}} \Rightarrow \prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{v^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}$$

$$\geq \prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{v^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}} \Rightarrow \frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{v^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}} \geq \frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{v^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}$$

$$\Rightarrow 1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{v^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}} \geq 1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{v^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}$$

$$\Rightarrow \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{v^{rp}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}\right) \geq \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\text{⅃}}\left(\daleth\daleth^{v^{rp\,*}_{\text{⅃}_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\text{⅃}}\frac{\overline{\overline{\text{Å}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\text{⅃}}\overline{\overline{\text{Å}}}_{\mathfrak{H}}}-1}}\right).$$

In the same way, we find the unreal part, such as: $\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{\upsilon_{\gimel_{\mathfrak{H}}}^{ip}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)$ $\geq \log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{\upsilon_{\gimel_{\mathfrak{H}}}^{ip\,*}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)$. Then, with the presence of the score function and accuracy function, we can easily obtain our required result with $CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)$ $\leq CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1^{*},\overline{\overline{I\mathfrak{T}}}_2^{*},\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}^{*}\right)$. This proves the proposition. □

Proposition 3 (Boundedness). *If* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{-}=\left(\left(\underset{\mathfrak{H}}{min}\,u_{\gimel_{\mathfrak{H}}}^{rp},\underset{\mathfrak{H}}{min}\,u_{\gimel_{\mathfrak{H}}}^{ip}\right),\left(\underset{\mathfrak{H}}{max}\,\upsilon_{\gimel_{\mathfrak{H}}}^{rp},\underset{\mathfrak{H}}{max}\,\upsilon_{\gimel_{\mathfrak{H}}}^{ip}\right)\right)$ *and* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{+}=\left(\left(\underset{\mathfrak{H}}{max}\,u_{\gimel_{\mathfrak{H}}}^{rp},\underset{\mathfrak{H}}{max}\,u_{\gimel_{\mathfrak{H}}}^{ip}\right),\left(\underset{\mathfrak{H}}{min}\,\upsilon_{\gimel_{\mathfrak{H}}}^{rp},\underset{\mathfrak{H}}{min}\,\upsilon_{\gimel_{\mathfrak{H}}}^{ip}\right)\right)$*, then we have*

$$\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{-}\leq CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)\leq\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{+}. \tag{22}$$

Proof. Using Propositions 1 and 2, we have $CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)\leq CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1^{+},\overline{\overline{I\mathfrak{T}}}_2^{+},\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}^{+}\right)=\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{+}$ and $CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)\geq CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1^{-},\overline{\overline{I\mathfrak{T}}}_2^{-},\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}^{-}\right)=\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{-}$. Then, $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{-}\leq CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)\leq\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{+}$. □

Definition 7. *The mathematical form of the CIFFPOA operator is shown below:*

$$\begin{aligned}CIFFPOA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)&=\left(\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{o(1)}\oplus\left(\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{o(2)}\oplus\ldots\oplus\left(\frac{\overline{\overline{\mathring{A}}}_{\gimel}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{o(\gimel)}\\&=\oplus_{\mathfrak{H}=1}^{\gimel}\left(\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)\overline{\overline{I\mathfrak{T}}}_{o(\mathfrak{H})}.\end{aligned} \tag{23}$$

With the values of $\overline{\overline{\mathring{A}}}_1=1$ and $\overline{\overline{\mathring{A}}}_{\mathfrak{H}}=\prod_{\mathcal{B}=1}^{\mathfrak{H}-1}V_s\left(\overline{\overline{I\mathfrak{T}}}_{\mathcal{B}}\right)$ and $o(\mathfrak{H})\leq o(\mathfrak{H}-1)$.

Theorem 2. *With the help of the data in Equation (23), we expose that the aggregated value of Equation (23) will again be in the form of CIFV, such as:*

$$CIFFPOA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)=\left(\begin{array}{c}\left(\left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{1-u_{\gimel_{o(\mathfrak{H})}}^{rp}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{1-u_{\gimel_{o(\mathfrak{H})}}^{ip}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\right),\\\left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{\upsilon_{\gimel_{o(\mathfrak{H})}}^{rp}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{\upsilon_{\gimel_{o(\mathfrak{H})}}^{ip}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right). \tag{24}$$

Proposition 4 (Idempotency) *If we use* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \overline{\overline{I\mathfrak{T}}} = \left(\left(u^{rp}_{\lrcorner}, u^{ip}_{\lrcorner}\right), \left(v^{rp}_{\lrcorner}, v^{ip}_{\lrcorner}\right)\right)$*, then*

$$CIFFPOA\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}\right) = \overline{\overline{I\mathfrak{T}}}. \tag{25}$$

Proposition 5 (Monotonicity). *If* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \left(\left(u^{rp}_{\lrcorner_{\mathfrak{H}}}, u^{ip}_{\lrcorner_{\mathfrak{H}}}\right), \left(v^{rp}_{\lrcorner_{\mathfrak{H}}}, v^{ip}_{\lrcorner_{\mathfrak{H}}}\right)\right) \leq \overline{\overline{I\mathfrak{T}}}^{*}_{\mathfrak{H}} = \left(\left(u^{rp\,*}_{\lrcorner_{\mathfrak{H}}}, u^{ip\,*}_{\lrcorner_{\mathfrak{H}}}\right), \left(v^{rp\,*}_{\lrcorner_{\mathfrak{H}}}, v^{ip\,*}_{\lrcorner_{\mathfrak{H}}}\right)\right)$*, then*

$$CIFFPOA\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}\right) \leq CIFFPOA\left(\overline{\overline{I\mathfrak{T}}}^{*}_1, \overline{\overline{I\mathfrak{T}}}^{*}_2, \ldots, \overline{\overline{I\mathfrak{T}}}^{*}_{\lrcorner}\right). \tag{26}$$

Proposition 6 (Boundedness). *If* $\overline{\overline{I\mathfrak{T}}}^{-}_{\mathfrak{H}} = \left(\left(\min_{\mathfrak{H}} u^{rp}_{\lrcorner_{\mathfrak{H}}}, \min_{\mathfrak{H}} u^{ip}_{\lrcorner_{\mathfrak{H}}}\right), \left(\max_{\mathfrak{H}} v^{rp}_{\lrcorner_{\mathfrak{H}}}, \max_{\mathfrak{H}} v^{ip}_{\lrcorner_{\mathfrak{H}}}\right)\right)$*and* $\overline{\overline{I\mathfrak{T}}}^{+}_{\mathfrak{H}} = \left(\left(\max_{\mathfrak{H}} u^{rp}_{\lrcorner_{\mathfrak{H}}}, \max_{\mathfrak{H}} u^{ip}_{\lrcorner_{\mathfrak{H}}}\right), \left(\min_{\mathfrak{H}} v^{rp}_{\lrcorner_{\mathfrak{H}}}, \min_{\mathfrak{H}} v^{ip}_{\lrcorner_{\mathfrak{H}}}\right)\right)$*, then we have*

$$C\overline{\overline{I\mathfrak{T}}}^{-}_{\mathfrak{H}} \leq CIFFPOA\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}\right) \leq \overline{\overline{I\mathfrak{T}}}^{+}_{\mathfrak{H}} \tag{27}$$

Definition 8. *The mathematical form of the CIFFPG operator is shown below:*

$$\begin{aligned} CIFFPG\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}\right) &= \overline{\overline{I\mathfrak{T}}}_1^{\left(\frac{\overline{\overline{\mathring{A}}}_1}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)} \otimes \overline{\overline{I\mathfrak{T}}}_2^{\left(\frac{\overline{\overline{\mathring{A}}}_2}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)} \otimes \ldots \otimes \overline{\overline{I\mathfrak{T}}}_{\lrcorner}^{\left(\frac{\overline{\overline{\mathring{A}}}_{\lrcorner}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)} \\ &= \otimes_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{\left(\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)} \end{aligned} \tag{28}$$

With the values of $\overline{\overline{\mathring{A}}}_1 = 1$ and $\overline{\overline{\mathring{A}}}_{\mathfrak{H}} = \prod_{\mathcal{B}=1}^{\mathfrak{H}-1} V_s\left(\overline{\overline{I\mathfrak{T}}}_{\mathcal{B}}\right)$.

Theorem 3. *With the help of the data in Equation (28), we expose that the aggregated value of Equation (28) will again be in the form of CIFV, such as:*

$$CIFFPG\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}\right) = \left(\begin{array}{c} \left(\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\urcorner\urcorner^{u^{rp}_{\lrcorner_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), \log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\urcorner\urcorner^{u^{ip}_{\lrcorner_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right), \\ \left(1-\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\urcorner\urcorner^{1-v^{rp}_{\lrcorner_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right), 1-\log_{\urcorner\urcorner}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\lrcorner}\left(\urcorner\urcorner^{1-v^{ip}_{\lrcorner_{\mathfrak{H}}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\urcorner\urcorner-1)^{\sum_{\mathfrak{H}=1}^{\lrcorner}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\lrcorner}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right) \end{array}\right) \tag{29}$$

Proposition 7 (Idempotency). *If we use* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \overline{\overline{I\mathfrak{T}}} = \left(\left(u^{rp}_{\lrcorner}, u^{ip}_{\lrcorner}\right), \left(v^{rp}_{\lrcorner}, v^{ip}_{\lrcorner}\right)\right)$*, then*

$$CIFFPG\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}\right) = \overline{\overline{I\mathfrak{T}}}. \tag{30}$$

Proposition 8 (Monotonicity). *If* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \left(\left(u^{rp}_{\lrcorner_{\mathfrak{H}}}, u^{ip}_{\lrcorner_{\mathfrak{H}}}\right), \left(v^{rp}_{\lrcorner_{\mathfrak{H}}}, v^{ip}_{\lrcorner_{\mathfrak{H}}}\right)\right) \leq \overline{\overline{I\mathfrak{T}}}^{*}_{\mathfrak{H}} = \left(\left(u^{rp\,*}_{\lrcorner_{\mathfrak{H}}}, u^{ip\,*}_{\lrcorner_{\mathfrak{H}}}\right), \left(v^{rp\,*}_{\lrcorner_{\mathfrak{H}}}, v^{ip\,*}_{\lrcorner_{\mathfrak{H}}}\right)\right)$*, then*

$$CIFFPG\left(\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \ldots, \overline{\overline{I\mathfrak{T}}}_{\lrcorner}\right) \leq CIFFPG\left(\overline{\overline{I\mathfrak{T}}}^{*}_1, \overline{\overline{I\mathfrak{T}}}^{*}_2, \ldots, \overline{\overline{I\mathfrak{T}}}^{*}_{\lrcorner}\right). \tag{31}$$

Proposition 9 (Boundedness). *If* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{-}=\left(\left(\min_{\mathfrak{H}} u_{\Finv_{\mathfrak{H}}}^{rp},\min_{\mathfrak{H}} u_{\Finv_{\mathfrak{H}}}^{ip}\right),\left(\max_{\mathfrak{H}} v_{\Finv_{\mathfrak{H}}}^{rp},\max_{\mathfrak{H}} v_{\Finv_{\mathfrak{H}}}^{ip}\right)\right)$ *and* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{+}=\left(\left(\max_{\mathfrak{H}} u_{\Finv_{\mathfrak{H}}}^{rp},\max_{\mathfrak{H}} u_{\Finv_{\mathfrak{H}}}^{ip}\right),\left(\min_{\mathfrak{H}} v_{\Finv_{\mathfrak{H}}}^{rp},\min_{\mathfrak{H}} v_{\Finv_{\mathfrak{H}}}^{ip}\right)\right)$, *then we have*

$$\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{-}\leq CIFFPG\left(\overline{\overline{I\mathfrak{T}}}_{1},\overline{\overline{I\mathfrak{T}}}_{2},\ldots,\overline{\overline{I\mathfrak{T}}}_{\Finv}\right)\leq\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{+}. \tag{32}$$

Definition 9. *The mathematical form of the CIFFPOG operator is shown below*:

$$\begin{aligned}CIFFPOG\left(\overline{\overline{I\mathfrak{T}}}_{1},\overline{\overline{I\mathfrak{T}}}_{2},\ldots,\overline{\overline{I\mathfrak{T}}}_{\Finv}\right)&=\overline{\overline{I\mathfrak{T}}}_{o(1)}^{\left(\frac{\overline{\overline{\mathring{A}}}_{1}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)}\otimes\overline{\overline{I\mathfrak{T}}}_{o(2)}^{\left(\frac{\overline{\overline{\mathring{A}}}_{2}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)}\otimes\ldots\otimes\overline{\overline{I\mathfrak{T}}}_{o(\Finv)}^{\left(\frac{\overline{\overline{\mathring{A}}}_{\Finv}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)}\\&=\otimes_{\mathfrak{H}=1}^{\Finv}\overline{\overline{I\mathfrak{T}}}_{o(\mathfrak{H})}^{\left(\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}\right)}\end{aligned} \tag{33}$$

With the values of $\overline{\overline{\mathring{A}}}_{1}=1$ and $\overline{\overline{\mathring{A}}}_{\mathfrak{H}}=\prod_{\mathcal{B}=1}^{\mathfrak{H}-1}V_{s}\left(\overline{\overline{I\mathfrak{T}}}_{\mathcal{B}}\right)$ and $o(\mathfrak{H})\leq o(\mathfrak{H}-1)$.

Theorem 4. *With the help of the data in Equation (33), we expose that the aggregated value of Equation (33) will again be in the form of CIFV, such as*:

$$CIFFPOG\left(\overline{\overline{I\mathfrak{T}}}_{1},\overline{\overline{I\mathfrak{T}}}_{2},\ldots,\overline{\overline{I\mathfrak{T}}}_{\Finv}\right)=\left(\begin{array}{c}\left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\Finv}\left(\daleth\daleth^{u_{\Finv_{o(\mathfrak{H})}}^{rp}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\Finv}\left(\daleth\daleth^{u_{\Finv_{o(\mathfrak{H})}}^{ip}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right),\\\left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\Finv}\left(\daleth\daleth^{1-v_{\Finv_{o(\mathfrak{H})}}^{rp}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\Finv}\left(\daleth\daleth^{1-v_{\Finv_{o(\mathfrak{H})}}^{ip}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\Finv}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\Finv}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right) \tag{34}$$

Proposition 10 (Idempotency). *If we use* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}=\overline{\overline{I\mathfrak{T}}}=\left(\left(u_{\Finv}^{rp},u_{\Finv}^{ip}\right),\left(v_{\Finv}^{rp},v_{\Finv}^{ip}\right)\right)$, *then*

$$CIFFPOG\left(\overline{\overline{I\mathfrak{T}}}_{1},\overline{\overline{I\mathfrak{T}}}_{2},\ldots,\overline{\overline{I\mathfrak{T}}}_{\Finv}\right)=\overline{\overline{I\mathfrak{T}}}. \tag{35}$$

Proposition 11 (Monotonicity). *If* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}=\left(\left(u_{\Finv_{\mathfrak{H}}}^{rp},u_{\Finv_{\mathfrak{H}}}^{ip}\right),\left(v_{\Finv_{\mathfrak{H}}}^{rp},v_{\Finv_{\mathfrak{H}}}^{ip}\right)\right)\leq\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{*}=\left(\left(u_{\Finv_{\mathfrak{H}}}^{rp\,*},u_{\Finv_{\mathfrak{H}}}^{ip\,*}\right),\left(v_{\Finv_{\mathfrak{H}}}^{rp\,*},v_{\Finv_{\mathfrak{H}}}^{ip\,*}\right)\right)$, *then*

$$CIFFPOG\left(\overline{\overline{I\mathfrak{T}}}_{1},\overline{\overline{I\mathfrak{T}}}_{2},\ldots,\overline{\overline{I\mathfrak{T}}}_{\Finv}\right)\leq CIFFPOG\left(\overline{\overline{I\mathfrak{T}}}_{1}^{*},\overline{\overline{I\mathfrak{T}}}_{2}^{*},\ldots,\overline{\overline{I\mathfrak{T}}}_{\Finv}^{*}\right). \tag{36}$$

Proposition 12 (Boundedness). *If* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{-}=\left(\left(\min_{\mathfrak{H}} u_{\Finv_{\mathfrak{H}}}^{rp},\min_{\mathfrak{H}} u_{\Finv_{\mathfrak{H}}}^{ip}\right),\left(\max_{\mathfrak{H}} v_{\Finv_{\mathfrak{H}}}^{rp},\max_{\mathfrak{H}} v_{\Finv_{\mathfrak{H}}}^{ip}\right)\right)$ *and* $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{+}=\left(\left(\max_{\mathfrak{H}} u_{\Finv_{\mathfrak{H}}}^{rp},\max_{\mathfrak{H}} u_{\Finv_{\mathfrak{H}}}^{ip}\right),\left(\min_{\mathfrak{H}} v_{\Finv_{\mathfrak{H}}}^{rp},\min_{\mathfrak{H}} v_{\Finv_{\mathfrak{H}}}^{ip}\right)\right)$, *then we have*

$$\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{-}\leq CIFFPOG\left(\overline{\overline{I\mathfrak{T}}}_{1},\overline{\overline{I\mathfrak{T}}}_{2},\ldots,\overline{\overline{I\mathfrak{T}}}_{\Finv}\right)\leq\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}^{+}. \tag{37}$$

4. CIF WASPAS Procedures

The main point of this section is to extend the theory of the WASPAS procedure to CIF information. The procedures of the WASPAS method contain various valuable and dominant steps. Before evaluating the normalization, we arrange a collection of CIF data which may be of a benefit type or cost type. If the data are of a benefit type, then good, otherwise, using the below theory, we normalize the information, such as:

$$\overline{\overline{I\mathfrak{T}}}_{0,\mathfrak{H}}=\left(\left(\max_{\mathfrak{H}}u^{rp}_{\gimel i\mathfrak{H}},\max_{\mathfrak{H}}u^{ip}_{\gimel i\mathfrak{H}}\right),\left(\min_{\mathfrak{H}}v^{rp}_{\gimel i\mathfrak{H}},\min_{\mathfrak{H}}v^{ip}_{\gimel i\mathfrak{H}}\right)\right),i,\mathfrak{H}=1,2,\ldots,\gimel,\gimel \tag{38}$$

$$\overline{\overline{I\mathfrak{T}}}'_{\mathfrak{H}}=\left(\left(u^{rp}_{\gimel i\mathfrak{H}},u^{ip}_{\gimel i\mathfrak{H}}\right),\left(v^{rp}_{\gimel i\mathfrak{H}},v^{ip}_{\gimel i\mathfrak{H}}\right)\right)=\begin{cases}0 & otherwise\\ \frac{u^{rp}_{\gimel i\mathfrak{H}}}{1+u^{rp}_{\gimel 0,\mathfrak{H}}},\frac{u^{ip}_{\gimel i\mathfrak{H}}}{1+u^{ip}_{\gimel 0,\mathfrak{H}}} & if\ u^{rp}_{\gimel i\mathfrak{H}}\le u^{rp}_{\gimel 0,\mathfrak{H}},u^{ip}_{\gimel i\mathfrak{H}}\le u^{ip}_{\gimel 0,\mathfrak{H}}\\ \frac{v^{rp}_{\gimel i\mathfrak{H}}}{1+v^{rp}_{\gimel 0,\mathfrak{H}}},\frac{v^{ip}_{\gimel i\mathfrak{H}}}{1+v^{ip}_{\gimel 0,\mathfrak{H}}} & for\ falsity\ grade\ (real\ a\gimel d\ u\gimel real\ parts)\end{cases} \tag{39}$$

After performing the above evaluation, we calculate the WSA and WPA with the help of derived theory, such as:

$$T^{WSA}_{\mathfrak{H}}=CIFFPA\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)=\left(\begin{array}{c}\left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{1-u^{rp}_{\gimel\mathfrak{H}}}-1\right)^{\frac{\dddot{\overline{\overline{\mathring{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{1-u^{ip}_{\gimel\mathfrak{H}}}-1\right)^{\frac{\dddot{\overline{\overline{\mathring{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right),\\ \left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{v^{rp}_{\gimel\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{v^{ip}_{\gimel\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right) \tag{40}$$

$$T^{WPA}_{\mathfrak{H}}=CIFFPG\left(\overline{\overline{I\mathfrak{T}}}_1,\overline{\overline{I\mathfrak{T}}}_2,\ldots,\overline{\overline{I\mathfrak{T}}}_{\gimel}\right)=\left(\begin{array}{c}\left(\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{u^{rp}_{\gimel\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{u^{ip}_{\gimel\mathfrak{H}}}-1\right)^{\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right),\\ \left(1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{1-v^{rp}_{\gimel\mathfrak{H}}}-1\right)^{\frac{\dddot{\overline{\overline{\mathring{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right),1-\log_{\daleth\daleth}\left(1+\frac{\prod_{\mathfrak{H}=1}^{\gimel}\left(\daleth\daleth^{1-v^{ip}_{\gimel\mathfrak{H}}}-1\right)^{\frac{\dddot{\overline{\overline{\mathring{A}}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}}}{(\daleth\daleth-1)^{\sum_{\mathfrak{H}=1}^{\gimel}\frac{\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}{\sum_{\mathfrak{H}=1}^{\gimel}\overline{\overline{\mathring{A}}}_{\mathfrak{H}}}-1}}\right)\right)\end{array}\right) \tag{41}$$

where, $\overline{\overline{\mathring{A}}}_1=1$ and $\overline{\overline{\mathring{A}}}_{\mathfrak{H}}=\prod_{\mathcal{B}=1}^{\mathfrak{H}-1}V_s\left(\overline{\overline{I\mathfrak{T}}}_{\mathcal{B}}\right)$.

Using the data in Equations (40) and (41), we calculate the aggregated measure based on convex theory, such as:

$$T_{\mathfrak{H}}={}^{\circ}\mathrm{F}V_sT^{WSA}_{\mathfrak{H}}+(1-{}^{\circ}\mathrm{F})V_sT^{WPA}_{\mathfrak{H}},{}^{\circ}\mathrm{F}\in[0,1] \tag{42}$$

Before ranking the alternatives, we discuss the special cases of the WASPAS technique based on CIF information such as:

1. When ${}^{\circ}\mathrm{F}=1$, we obtain the data in Equation (40);
2. When ${}^{\circ}\mathrm{F}=0$, we obtain the data in Equation (41).

In last, we derive the ranking result for examining the best one from the family of finite preferences. Furthermore, we justify the supremacy and worth of the derived theory with the help of some suitable examples, such as:

$$\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \begin{bmatrix} ((0.4,0.2),(0.1,0.2)) & ((0.4,0.2),(0.1,0.2)) & ((0.4,0.2),(0.1,0.2)) & ((0.1,0.2),(0.1,0.2)) \\ ((0.5,0.4),(0.2,0.3)) & ((0.1,0.4),(0.2,0.3)) & ((0.5,0.4),(0.2,0.3)) & ((0.5,0.4),(0.2,0.3)) \\ ((0.6,0.5),(0.2,0.3)) & ((0.6,0.5),(0.2,0.3)) & ((0.2,0.5),(0.2,0.3)) & ((0.6,0.5),(0.2,0.3)) \\ ((0.7,0.8),(0.1,0.1)) & ((0.7,0.8),(0.1,0.1)) & ((0.7,0.8),(0.1,0.1)) & ((0.7,0.8),(0.1,0.1)) \end{bmatrix}$$

Then, we find the positive ideal, such as:

$$\overline{\overline{I\mathfrak{T}}}_{0,\mathfrak{H}} = \{((0.7,0.8),(0.1,0.1)),((0.7,0.8),(0.1,0.1)),((0.7,0.8),(0.1,0.1)),((0.7,0.8),(0.1,0.1))\}$$

With the help of the $\overline{\overline{I\mathfrak{T}}}_{0,\mathfrak{H}}$ and the information in $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}}$, we obtain the below theory, such as:

$$\overline{\overline{I\mathfrak{T}}}'_{\mathfrak{H}} = \begin{bmatrix} \begin{pmatrix} (0.2352,0.1111), \\ (0.0909,0.1818) \end{pmatrix} & \begin{pmatrix} (0.2353,0.1111), \\ (0.0909,0.1818) \end{pmatrix} & \begin{pmatrix} (0.2353,0.1111), \\ (0.0909,0.1818) \end{pmatrix} & \begin{pmatrix} (0.0588,0.1111), \\ (0.0909,0.2727) \end{pmatrix} \\ \begin{pmatrix} (0.2941,0.2222), \\ (0.1818,0.2727) \end{pmatrix} & \begin{pmatrix} (0.0588,0.2222), \\ (0.1818,0.2727) \end{pmatrix} & \begin{pmatrix} (0.2941,0.2222), \\ (0.1818,0.2727) \end{pmatrix} & \begin{pmatrix} (0.2941,0.2222), \\ (0.1818,0.2727) \end{pmatrix} \\ \begin{pmatrix} (0.3529,0.2777), \\ (0.1818,0.2727) \end{pmatrix} & \begin{pmatrix} (0.3529,0.2778), \\ (0.1818,0.2727) \end{pmatrix} & \begin{pmatrix} (0.1176,0.2778), \\ (0.1818,0.2727) \end{pmatrix} & \begin{pmatrix} (0.3529,0.2778), \\ (0.1818,0.2727) \end{pmatrix} \\ \begin{pmatrix} (0.4117,0.4444), \\ (0.0909,0.0909) \end{pmatrix} & \begin{pmatrix} (0.4118,0.4444), \\ (0.0909,0.0909) \end{pmatrix} & \begin{pmatrix} (0.4118,0.4444), \\ (0.0909,0.0909) \end{pmatrix} & \begin{pmatrix} (0.4118,0.4444), \\ (0.0909,0.0909) \end{pmatrix} \end{bmatrix}$$

After performing the above evaluation, we calculate the WSA and WPA with the help of derived theory, such as: $T_1^{WSA} = ((0.2375,0.1154),(0.0933,0.1846))$, $T_2^{WSA} = ((0.229,0.1146),(0.0933,0.1846))$, $T_3^{WSA} = ((0.2373,0.1154),(0.0933,0.1846))$, $T_4^{WSA} = ((0.0456,0.1055),(0.0866,0.1764))$, and $T_1^{WPA} = ((0.2373,0.1140),(0.0944,0.1853))$, $T_2^{WPA} = ((0.2241,0.1136),(0.0944,0.1849))$, $T_3^{WPA} = ((0.2373,0.1141),(0.0944,0.1853))$, $T_4^{WPA} = ((0.0532,0.1058),(0.0865,0.1778))$. Using the data in Equations (40) and (41) with $^{\circ}\mathrm{F} = 0.2$, we calculate the aggregated measure based on convex theory, such as: $T_1 = 0.03621, T_2 = 0.0302, T_3 = 0.036, T_4 = -0.053$. According to the score values of the four alternatives, the ranking results is with $T_1 > T_3 > T_2 > T_4$. Thus, the best optimal is T_1 according to the score values of alternatives.

5. Application in MADM Method

The MADM technique is the valuable and dominant part of the decision-making procedure. The main theme of this section is to utilize the theory of the MADM technique based on the presented information for CIF set theory. To examine the above problem, we collect the finite family of alternatives $\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \overline{\overline{I\mathfrak{T}}}_3, \overline{\overline{I\mathfrak{T}}}_4, \ldots, \overline{\overline{I\mathfrak{T}}}_{\daleth}$ and their attributes $\overline{\overline{I\mathfrak{T}}}_{a-1}, \overline{\overline{I\mathfrak{T}}}_{a-2}, \overline{\overline{I\mathfrak{T}}}_{a-3}, \overline{\overline{I\mathfrak{T}}}_{a-4}, \overline{\overline{I\mathfrak{T}}}_{a-\daleth}$. Based on the above alternatives and their attributes, we compute the matrix of information whose term is computed in the form of CIF values such that the CIFS with a truth grade $\left(u_{\daleth}^{rp}(x), u_{\daleth}^{ip}(x)\right)$ and falsity grade $\left(v_{\daleth}^{rp}(x), v_{\daleth}^{ip}(x)\right)$ must be implementing the following rule: $0 \leq u_{\daleth}^{rp}(x) + v_{\daleth}^{rp}(x) \leq 1$ and $0 \leq u_{\daleth}^{ip}(x) + v_{\daleth}^{ip}(x) \leq 1$. The notion of neutral grade is stated by $r(x) = \left(r^{rp}(x), r^{ip}(x)\right) = \left(1 - \left(u_{\daleth}^{rp}(x) + v_{\daleth}^{rp}(x)\right), 1 - \left(u_{\daleth}^{ip}(x) + v_{\daleth}^{ip}(x)\right)\right)$ and the representation of the CIFVs is with $\overline{\overline{I\mathfrak{T}}}_{\mathfrak{H}} = \left(\left(u_{\daleth_{\mathfrak{H}}}^{rp}, u_{\daleth_{\mathfrak{H}}}^{ip}\right), \left(v_{\daleth_{\mathfrak{H}}}^{rp}, v_{\daleth_{\mathfrak{H}}}^{ip}\right)\right), \mathfrak{H} = 1, 2, \ldots, \daleth$. Furthermore, to proceed with the above information, we compute a technique of decision-making, whose major steps are shown below:

Step 1: Before evaluating the normalization, we arrange a collection of CIF data which may be of a benefit type or cost type. If the data are of a benefit type, then good, otherwise, using the below theory, we normalize the information, such as:

$$C = \begin{cases} \left(\left(u^{rp}_{\daleth_{\mathfrak{H}}}, u^{ip}_{\daleth_{\mathfrak{H}}}\right), \left(v^{rp}_{\daleth_{\mathfrak{H}}}, v^{ip}_{\daleth_{\mathfrak{H}}}\right)\right) & for\ benefit \\ \left(\left(v^{rp}_{\daleth_{\mathfrak{H}}}, v^{ip}_{\daleth_{\mathfrak{H}}}\right), \left(u^{rp}_{\daleth_{\mathfrak{H}}}, u^{ip}_{\daleth_{\mathfrak{H}}}\right)\right) & for\ cost. \end{cases}$$

Step 2: After performing the above evaluation, we calculate the CIFFPA operator and CIFFPG operator with the help of the derived theory.

Step 3: Evaluate the score or accuracy values of the aggregated information.

Step 4: Examine the ranking values in the presence of the score information.

In the last, we aim to show the supremacy and worth of the above procedure with the help of illustrating some numerical examples.

Illustrative Example: An investment enterprise wants to invest in an enterprise to increase or grow its income. There are five potential enterprises as alternatives, which are $\overline{\overline{I\mathfrak{T}}}_1, \overline{\overline{I\mathfrak{T}}}_2, \overline{\overline{I\mathfrak{T}}}_3, \overline{\overline{I\mathfrak{T}}}_4$ and $\overline{\overline{I\mathfrak{T}}}_5$. Four attributes are employed to resolve the problem in order to find the best preference from our five alternatives, including $\overline{\overline{I\mathfrak{T}}}_{a-1}$: growth analysis, $\overline{\overline{I\mathfrak{T}}}_{a-2}$: social-political impact, $\overline{\overline{I\mathfrak{T}}}_{a-3}$: environmental impact, and $\overline{\overline{I\mathfrak{T}}}_{a-4}$: development of society. Furthermore, to proceed with the above information, we compute a technique of decision-making, whose major steps are shown below:

Step 1: Before evaluating the normalization, we arrange a collection of CIF data in the form of Table 1, which may be of a benefit type or cost type. If the data are of a benefit type, then good, otherwise, using the below theory, we normalize the information, such as:

$$C = \begin{cases} \left(\left(u^{rp}_{\daleth_{\mathfrak{H}}}, u^{ip}_{\daleth_{\mathfrak{H}}}\right), \left(v^{rp}_{\daleth_{\mathfrak{H}}}, v^{ip}_{\daleth_{\mathfrak{H}}}\right)\right) & for\ benefit \\ \left(\left(v^{rp}_{\daleth_{\mathfrak{H}}}, v^{ip}_{\daleth_{\mathfrak{H}}}\right), \left(u^{rp}_{\daleth_{\mathfrak{H}}}, u^{ip}_{\daleth_{\mathfrak{H}}}\right)\right) & for\ cost. \end{cases}$$

However, the data in Table 2 is not required to be normalized.

Table 2. Original CIF information matrix.

	$\overline{\overline{\mathbf{I\mathfrak{T}}}}_{a-1}$	$\overline{\overline{\mathbf{I\mathfrak{T}}}}_{a-2}$	$\overline{\overline{\mathbf{I\mathfrak{T}}}}_{a-3}$	$\overline{\overline{\mathbf{I\mathfrak{T}}}}_{a-4}$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_1$	$((0.4, 0.3), (0.1, 0.3))$	$((0.41, 0.31), (0.11, 0.31))$	$((0.42, 0.32), (0.12, 0.32))$	$((0.43, 0.33), (0.13, 0.33))$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_2$	$((0.6, 0.7), (0.2, 0.1))$	$((0.61, 0.71), (0.21, 0.11))$	$((0.62, 0.72), (0.22, 0.12))$	$((0.63, 0.73), (0.23, 0.13))$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_3$	$((0.3, 0.2), (0.3, 0.4))$	$((0.31, 0.21), (0.31, 0.41))$	$((0.32, 0.22), (0.32, 0.42))$	$((0.33, 0.23), (0.33, 0.43))$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_4$	$((0.7, 0.4), (0.2, 0.3))$	$((0.71, 0.41), (0.21, 0.31))$	$((0.72, 0.42), (0.22, 0.32))$	$((0.73, 0.43), (0.23, 0.33))$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_5$	$((0.7, 0.7), (0.1, 0.1))$	$((0.71, 0.71), (0.11, 0.11))$	$((0.72, 0.72), (0.12, 0.12))$	$((0.73, 0.73), (0.13, 0.13))$

Step 2: After performing the above evaluation, we calculate the CIFFPA operator and CIFFPG operator with the help of the derived theory, and see Table 3.

Table 3. Aggregated values.

	CIFFPA	***CIFFPG***
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_1$	$((0.4084, 0.3071), (0.0969, 0.2941))$	$((0.3941, 0.2941), (0.1039, 0.3071))$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_2$	$((0.7369, 0.8279), (0.108, 0.043))$	$((0.4986, 0.6188), (0.2794, 0.1486))$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_3$	$((0.3007, 0.2007), (0.3002, 0.1003))$	$((0.3002, 0.2003), (0.3007, 0.1006))$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_4$	$((0.7448, 0.4393), (0.1694, 0.2647))$	$((0.6776, 0.364), (0.2249, 0.3329))$
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_5$	$((0.8753, 0.8753), (0.0258, 0.0258))$	$((0.5741, 0.5741), (0.1738, 0.1738))$

Step 3: Evaluate the score or accuracy values of aggregated information, and see Table 4.

Table 4. Score values.

	CIFFPA	*CIFFPG*
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_1$	0.1623	0.1386
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_2$	0.7069	0.3447
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_3$	0.0504	0.0496
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_4$	0.375	0.2419
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_5$	0.8495	0.4003

Step 4: Examine the ranking values of the score information, and see Table 5.

Table 5. Ranking information.

Methods	*Ranking Results*
CIFFPA	$\overline{\overline{\mathbf{I\mathfrak{T}}}}_5 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_2 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_4 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_1 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_3$
CIFFPG	$\overline{\overline{\mathbf{I\mathfrak{T}}}}_5 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_2 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_4 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_1 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_3$

The valuable and best preference is $\overline{\overline{\mathbf{I\mathfrak{T}}}}_5$**,** according to the theory of CIFFPA and CIFFPG operators. Furthermore, by excluding the phase term, we have checked the stability and supremacy of the derived information. Thus, we remove the phase information from the data in Table 2 in which their score values are shown in Table 6.

Table 6. Score values (without phase term).

	CIFFPA	*CIFFPG*
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_1$	0.1558	0.1451
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_2$	0.3144	0.1096
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_3$	0.0002	0.00002
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_4$	0.2877	0.2263
$\overline{\overline{\mathbf{I\mathfrak{T}}}}_5$	0.4247	0.2001

Furthermore, we examine the ranking values of the score information, and see Table 7.

Table 7. Ranking information.

Methods	*Ranking Results*
CIFFPA	$\overline{\overline{\mathbf{I\mathfrak{T}}}}_5 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_2 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_4 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_1 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_3$
CIFFPG	$\overline{\overline{\mathbf{I\mathfrak{T}}}}_4 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_5 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_1 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_2 > \overline{\overline{\mathbf{I\mathfrak{T}}}}_3$

The valuable and best preference is $\overline{\overline{\mathbf{I\mathfrak{T}}}}_5$ according to the theory of the CIFFPA operator. Furthermore, the best preference is $\overline{\overline{\mathbf{I\mathfrak{T}}}}_4$ according to the theory of the CIFFPG operator. Additionally, we find the comparisons between the proposed and existing data with the help of data in Table 2.

6. Comparative Analysis

In this section, we select some existing operators based on various prevailing ideas. We then try to compare their obtained results with the obtained results of our proposed works. The comparative analysis is one of the most effective and dominant techniques because without comparison we fail to show the supremacy and validity of the derived theory. For this, we consider different types of information, such as aggregation operators (AOs) for IFSs [23], geometric AOs for IFSs [24], the complex fuzzy credibility Frank AOs [26]. Additionally, Yu [31] examined the theory of generalized prioritized AOs for intuitionistic fuzzy environments, and Lin, et al. [32] derived the fuzzy number intuitionistic fuzzy prioritized AOs and their application in decision-making procedures. Furthermore, Garg and Rani [33] exposed the averaging operators for CIFSs. Garg and Rani [34] evaluated the geometric operators for CIFSs, and Mahmood, et al. [35] examined the Aczel–Alsina aggregation operators for CIFSs. Using data in Table 2, the comparison information is listed in Table 8.

Table 8. Comparative analysis.

Methods	Score Information	Ranking Information
Xu [23]	***************************	***************************
Xu and Yager [24]	***************************	***************************
Yahya, et al. [26]	***************************	***************************
Yu [31]	***************************	***************************
Lin, et al. [32]	***************************	***************************
Garg and Rani [33]	0.1506, 0.5008, 0.0506, 0.3005, 0.6010	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
Garg and Rani [34]	0.1497, 0.4998, 0.0496, 0.2997, 0.5998	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
Mahmood, et al. [35]	0.1506, 0.5007, 0.0505, 0.3005, 0.6009	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
CIFFPA	0.1623, 0.7069, 0.0504, 0.375, 0.8495	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
CIFFPG	0.1386, 0.3447, 0.0496, 0.2419, 0.4003	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$

The valuable and best preference is $\overline{\overline{\mathrm{I}\mathfrak{T}}}_5$, according to the theory of CIFFPA, CIFFPG operators, Garg and Rani [33,34], and Mahmood, et al. [35]. However, the theory of AOs for IFSs [23], geometric AOs for IFSs [24], the complex fuzzy credibility of Frank AOs [26], and Yu [31] examined the theory of generalized prioritized AOs for intuitionistic fuzzy environments with the limitation that fails to evaluate it. Similarly, Lin, et al. [35] derive the fuzzy number intuitionistic fuzzy prioritized AOs and their application in decision-making procedures also with the limitation and restriction, because they fail to evaluate it. It is possible if we use the data in Table 2, however, without phase information, then the comparison information is listed in Table 9.

Table 9. Comparative analysis (without phase terms).

Methods	Score Information	Ranking Information
Xu [23]	0.1504, 0.5005, 0.0504, 0.3003, 0.6006	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
Xu and Yager [24]	0.1498, 0.4999, 0.0497, 0.2998, 0.5999	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
Yahya, et al. [26]	***************************	***************************
Yu [31]	0.1614, 0.6901, 0.0504, 0.3687, 0.8319	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
Lin, et al. [32]	0.1394, 0.3526, 0.0496, 0.2452, 0.4097	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
Garg and Rani [33]	0.1502, 0.2002, 0.0001, 0.2502, 0.3003	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
Garg and Rani [34]	0.1499, 0.1999, 0.0001, 0.2499, 0.2999	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
Mahmood, et al. [35]	0.1502, 0.2001, 0.00009, 0.2502, 0.3003	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
CIFFPA	0.1558, 0.3144, 0.0002, 0.2877, 0.4247	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$
CIFFPG	0.1451, 0.1096, 0.00002, 0.2263, 0.2001	$\overline{\overline{\mathrm{I}\mathfrak{T}}}_4 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_5 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_1 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_2 > \overline{\overline{\mathrm{I}\mathfrak{T}}}_3$

The valuable and best preference is $\overline{\overline{\mathrm{I}\mathfrak{T}}}_5$ according to the theory of the CIFFPA operator, Xu [23], Xu and Yager [24], Yu [31], Lin, et al. [32], Garg and Rani [33,34], and Mahmood, et al. [35]. However, the most valuable and best preference is $\overline{\overline{\mathrm{I}\mathfrak{T}}}_4$ according to the theory of the CIFFPG operator. However, the complex fuzzy credibility Frank AOs [26] have limitations and restrictions, and because it failed to evaluate it. Therefore, the proposed work is effective and valid for evaluating most of the CIFS information.

7. Conclusions

The idea of CIFS is the modified version of the complex fuzzy set theory, which covered the grade of truth and falsity in the form of polar coordinates. Furthermore, the theory of Frank and prioritized aggregation operators is also very famous and valuable because they are the modified version of the simple averaging and geometric aggregation operators. Motivated by the above information, in this manuscript, we examined the following ideas:

1. We evaluated the Frank operational laws for the theory of CIF information;
2. We examined the theory of the CIFFPA, CIFFPOA, CIFFPG, and CIFFPOG operators, and their properties of idempotency, monotonicity, and boundedness;
3. We derived the WASPAS under the presence of the CIFFPA and CIFFPG operators;
4. We demonstrated the MADM procedures based on the invented theory for CIF information;
5. We compared the derived theory with various existing information to show the validity and worth of the discovered approaches.

In the future, we aim to develop new aggregation operators based on Frank operational laws and then we aim to employ them in the field of game theory, neural networks, clustering, pattern recognition, and decision-making to enhance the worth of the derived information.

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