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Generalized Equilibrium Problems

Mircea Balaj * and Dan Florin Serac

Department of Mathematics, University of Oradea, 410087 Oradea, Romania; dany16121986@yahoo.com
* Correspondence: mbalaj@uoradea.ro

Abstract: If X is a convex subset of a topological vector space and f is a real bifunction defined on $X \times X$, the problem of finding a point $x_0 \in X$ such that $f(x_0, y) \geq 0$ for all $y \in X$, is called an equilibrium problem. When the bifunction f is defined on the cartesian product of two distinct sets X and Y we will call it a generalized equilibrium problem. In this paper, we study the existence of the solutions, first for generalized equilibrium problems and then for equilibrium problems. In the obtained results, apart from the bifunction f, another bifunction is introduced, the two being linked by a certain compatibility condition. The particularity of the equilibrium theorems established in the last section consists of the fact that the classical equilibrium condition ($f(x,x) = 0$, for all $x \in X$) is missing. The given applications refer to the Minty variational inequality problem and quasi-equilibrium problems.

Keywords: generalized equilibrium problem; equilibrium problem; fixed point; variational inequality; quasi-equilibrium problem

MSC: 49J53; 49J40

1. Introduction

In this paper, we consider the following problem

(GEP) $$\text{find } x_0 \in X \text{ such that } f(x_0, y) \geq 0,\ \forall y \in Y,$$

where X is a nonempty convex subset of a Hausdorff topological vector space, Y is a nonempty set, and f is a real bifunction defined on $X \times Y$.

When $X = Y$, the problem above has been termed by Blum and Oetlli [1] *equilibrium problem* (EP), and this terminology is used now by all the researchers working in this field. It should be mentioned that Kassay [2] kept the same name also in the case when the sets X and Y are distinct, but we consider that in this case it would be more suitable to call it *a generalized equilibrium problem*, and this fact motivates the above abbreviation (GEP).

The first result in the existence of solutions for problem (EP) is due to Ky Fan [3] and it refers to the case when X is a compact convex set and the bifunction f is upper semicontinuous in the first variable, quasiconvex in the second variable, and its values on the diagonal of $X \times X$ are non-negative. Problem (EP) has been extensively studied in recent years in both finite and infinite dimensional settings, under various assumptions on the set X, and on the bifunction f. Since in the classical examples of equilibrium problems (minimization problems, fixed point problems, saddle point problems, Nash equilibrium problems, and variational inequality problems) the bifunction f is identically null on the diagonal of $X \times X$, often the existence of solutions for problem (EP) is studied under the assumption that f fulfills this condition (see, for instance [4–7]). Nevertheless, in the past, papers in which no assumption is made about the values of f on the diagonal of $X \times X$ [8–11] have appeared. Theorem 4 of the present paper will also be a result of this type.

Besides the existence of solutions, some algorithms for determining the solution set for so-called general equilibrium problems were given in [12,13]. Recently, the theory of

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equilibrium problems found unexpected applications in the study of a wide class of linear and nonlinear problems arising in traffic networks, image reconstructions, medical imaging, and physical or mechanical structures.

Unlike the equilibrium problems which have extensive literature on existing results, stability of the solutions, and solution methods (see, for instance, the surveys [14,15]) the literature devoted to generalized equilibrium problems is poorer and quite old. We can, however, mention the works [16,17], in which Y is assumed to be a subset of X and [2,18,19], in which Y is an arbitrary set without any topological or algebraic structure.

At the end of this section, we briefly describe the content of the paper. In Section 2, we recall some notions regarding set-valued mappings and bifunctions. Section 3 is devoted to generalized equilibrium problems. We establish here an interesting intersection theorem from which two theorems regarding the existence of solutions for generalized equilibrium problems are derived. In the last section, we obtain two existence theorems for equilibrium problems, first when the convex set X is paracompact and then when the paracompactness condition is missing. In general, the proofs of the existence theorems for equilibrium or generalized equilibrium problems rely on the Knaster–Kuratowski–Mazurkiewicz (KKM, for short) principle or on other intersection results. In our paper, the proofs of the main results use two important tools. The first is the Berge–Klee intersection theorem. The second is the KKM property of continuous functions. In order to highlight the applicability of our results, at the end of this paper we study the existence of solutions of the Minty variational inequality and quasi-equilibrium problems.

2. Preliminaries

In this section, we fix some notations and recall some concepts used in the paper. For subset A of a topological vector space, the standard notations $\operatorname{conv} A$ and $\operatorname{cl} A$ designate the convex hull and the closure of A, respectively.

Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values. The set-valued mapping $F^- : Y \rightrightarrows X$, defined by $F^-(y) = \{x \in X : y \in F(x)\}$, is called the inverse of F and its values are called the fibers of F. If X and Y are topological spaces, F is said to be: (i) upper semicontinuous, if for every open subset G of Y, the set $\{x \in X : F(x) \subseteq G\}$ is open; (ii) lower semicontinuous, if for every open subset G of Y, the set $\{x \in X : F(x) \cap G \neq \emptyset\}$ is open; (iii) continuous, if it is both upper semicontinuous and lower semicontinuous; (iii) closed, if its graph (that is, the set $\mathrm{Gr}F = \{(x, y) \in X \times Y : y \in F(x)\}$) is a closed subset of $X \times Y$; (iv) compact, if its range $F(X)$ is contained in a compact subset of Y.

The following statements about semicontinuous mappings are well known:

Lemma 1. *Let X and Y be topological spaces and let $F : X \rightrightarrows Y$ be set-valued mappings with nonempty values.*

(i) Assume that F is compact. Then, F is closed if and only if it is upper semicontinuous and closed-valued.

(ii) F is lower semicontinuous if and only if for any net $\{x_t\}$ in X converging to $x \in X$ and each $y \in F(x)$, there exists a subnet $\{x_{t_i}\}$ of the net $\{x_t\}$ and a net $\{y_i\}$ in Y converging to y, with $y_i \in F(x_{t_i})$, for each index i.

If Y is a subset of a topological vector space and $X \subseteq Y$, the set-valued mapping F is said to be a KKM mapping if for each nonempty finite subset A of X, $\operatorname{conv} A \subseteq F(A)$. The concept of KKM mapping has been generalized by Chang and Yen [20] as follows:

Definition 1. *Let X be a convex set in a vector space and Y be a topological space. If $F, G : X \rightrightarrows Y$ are two set-valued mappings such that $G(conv\ A) \subseteq F(A)$ for each nonempty finite subset A of X, then F is called a generalized KKM mapping w.r.t. G. The set-valued mapping G is said to have the KKM property if for any set-valued mapping $F : X \rightrightarrows Y$ which is generalized KKM w.r.t. G, the family $\{cl\ F(x) : x \in X\}$ has the finite intersection property.*

If Y is a topological space, the set-valued mapping $F : X \rightrightarrows Y$ is said to be intersectionally closed if

$$\bigcap_{x \in X} \operatorname{cl} F(x) = \operatorname{cl} \left(\bigcap_{x \in X} F(x) \right).$$

Clearly, any set-valued mapping with closed values is intersectionally closed, but the converse is not true. Sufficient conditions for a mapping F to be intersectionally closed can be found in [21] (Proposition 2.3).

Recall that a real function g defined on a convex set X is quasiconvex if for any nonempty finite subset A of X and every $x \in \operatorname{conv} A$, $g(x) \leq \max_{z \in A} g(z)$. We will need two close notions related to bifunctions.

Definition 2 ([22])**.** *Let X be a convex subset of a vector space and $\gamma \in \mathbb{R}$. A bifunction $g : X \times X \to \mathbb{R}$ is said to be γ- diagonally quasiconvex in the first variable, if for any nonempty finite subset A of X and all $x \in \operatorname{conv} A$, $\gamma \leq \max_{z \in A} g(x, z)$.*

For the sake of simplicity, a bifunction 0-diagonally quasiconvex will be simply termed a diagonally quasiconvex bifunction.

Remark 1. *Consider the set-valued mapping $G : X \rightrightarrows X$ defined by*

$$G(z) = \{x \in X : g(x, z) \geq 0\}.$$

It can be seen easily that bifunction g is diagonally quasiconvex in the first variable if and only if G is a KKM set-valued mapping.

Definition 3 ([23])**.** *Let X be a nonempty set, Y be a nonempty convex subset of a vector space, and f, g be two real bifunctions defined on $X \times Y$. We say that g is f-quasiconvex in the second variable if for any $x \in X$ and every nonempty finite subset A of Y,*

$$g(x, y) \leq \max_{z \in A} f(x, z), \quad \text{for all } y \in \operatorname{conv} A.$$

3. Generalized Equilibrium Problems

The proof of Theorem 1 relies on an important intersection property of convex sets established, first in Euclidean spaces by Klee [24], and then in topological vector spaces by Berge [25] and Ghouila-Houri [26]. For this reason, the lemma below is known in the literature as the Berge–Klee intersection theorem.

Lemma 2. *Let $\{X_1, \ldots, X_n\}$ be a family of closed convex subsets of a Hausdorff topological vector space whose union is convex. If the intersection of every $n - 1$ of these sets is nonempty, then their intersection is nonempty.*

Theorem 1 is an intersection result of self-interest. From it, we will derive the main result of this section.

Theorem 1. *Let X be a compact convex subset of a Hausdorff topological vector space, Y be a nonempty set, and $F : Y \rightrightarrows X$ be a set-valued mapping with nonempty closed values. Assume that there exists a KKM set-valued mapping $G : X \rightrightarrows X$ such that*

$$G(F(y)) \subseteq F(y), \text{ for all } y \in Y. \tag{1}$$

Then, $\bigcap_{y \in Y} F(y) \neq \emptyset$.

Proof. We show first that the values of F are convex sets. Indeed, for each $y \in Y$ and any two distinct points $x_1, x_2 \in F(y)$, since G is a KKM set-valued mapping, we have

$$[x_1, x_2] \subseteq G(x_1) \cup G(x_2) \subseteq G(F(y)) \cup G(F(y)) \subseteq F(y).$$

Assume, by way of contradiction, that $\bigcap_{y \in Y} F(y) = \emptyset$. Then, because X is compact, there exists a finite subset A of Y such that $\bigcap_{y \in A} F(y) = \emptyset$. Since F has nonempty values, the cardinality of A is grater than 1. Let $A = \{y_1, y_2, \ldots, y_n\}$, a minimal set with this property, that is,

$$\bigcap_{i=1}^{n} F(y_i) = \emptyset, \text{ and } \bigcap_{i \neq j} F(y_i) \neq \emptyset, \text{ for all } j \in \{1, \ldots, n\}.$$

For each $j \in \{1, \ldots, n\}$, we choose a point $x_j \in \bigcap_{i \neq j} F(y_i)$ and denote by

$$C = \text{ conv } \{x_1, \ldots, x_n\}.$$

Consider the set-valued mapping $\widetilde{F} : Y \rightrightarrows C$ defined by

$$\widetilde{F}(y) = F(y) \cap C.$$

It is clear that the values of $\widetilde{F}$ are closed and convex. Moreover,

$$\bigcap_{i=1}^{n} \widetilde{F}(y_i) = \emptyset, \text{ and } \bigcap_{i \neq j} \widetilde{F}(y_i) \neq \emptyset, \text{ for all } j \in \{1, \ldots, n\}.$$

By the Berge–Klee theorem, there exists a point $\overline{x} \in C \setminus \left(\bigcup_{i=1}^{n} \widetilde{F}(y_i) \right)$. Since $\overline{x} \in C$ and G is a KKM mapping, for some index $j \in \{1, \ldots, n\}$,

$$\overline{x} \in G(x_j).$$

Let us fix an index $k \neq j$. From $x_j \in F(y_k)$ and $\overline{x} \in G(x_j)$, we obtain $\overline{x} \in G\big(F(y_k)\big)$, and then from (1), we obtain $\overline{x} \in F(y_k)$. Since $\overline{x} \in C$, we are led to the following contradiction

$$\overline{x} \in \widetilde{F}(y_k) \subseteq \bigcup_{i=1}^{n} \widetilde{F}(y_i).$$

□

Remark 2. *Let us note that the inclusion in (1) is actually an equality. Indeed, since G is a KKM mapping, any point $x \in X$ is a fixed point for G, and hence for every $x \in F(y)$, we have*

$$x \in G(x) \subseteq G(F(y)).$$

Therefore, $F(y) \subseteq G(F(y))$.

Using a standard topological argument, we can establish the following result, apparently more general than Theorem 1.

Theorem 2. *Let X be a compact convex subset of a Hausdorff topological vector space, Y be a nonempty set, and $F : Y \rightrightarrows X$ be a set-valued mapping with nonempty closed values. Assume that for each nonempty finite subset A of Y there exists a KKM set-valued mapping $G_A : X \rightrightarrows X$ such that*

$$G_A(F(y)) \subseteq F(y), \text{ for all } y \in A.$$

Then, $\bigcap_{y \in Y} F(y) \neq \emptyset$.

Proof. By Theorem 1, for each nonempty finite subset A of Y, $\bigcap_{y \in A} F(y) \neq \emptyset$, hence the family of closed sets $\{F(y) : y \in Y\}$ has the finite intersection property. Since X is compact, $\bigcap_{y \in Y} F(y) \neq \emptyset$. $\square$

We give below the main result of this section.

Theorem 3. *Let X be a compact convex subset of a Hausdorff topological vector space, Y be a nonempty set, and $f : X \times Y \to \mathbb{R}$, $g : X \times X \to \mathbb{R}$ be two bifunctions satisfying the following assumptions:*

(i) for each $y \in Y$, the set $\{x \in X : f(x,y) \geq 0\}$ is nonempty and closed;
(ii) g is diagonally quasiconvex in the first variable;
(iii) for every $x, z \in X$ and $y \in Y$ the following implication holds

$$g(x,z) \geq 0, \ f(z,y) \geq 0 \implies f(x,y) \geq 0.$$

Then, there exists $x_0 \in X$ such that $f(x_0, y) \geq 0$, for all $y \in Y$.

Proof. Consider the set-valued mappings $F : Y \rightrightarrows X$, $G : X \rightrightarrows X$ defined by

$$F(y) = \{x \in X : f(x,y) \geq 0\},$$

$$G(z) = \{x \in X : g(x,z) \geq 0\}.$$

By (i), for each $y \in Y$, $F(y)$ is a nonempty and closed set. In view of (ii) and Remark 1, G is a KKM mapping. If $y \in Y$ and $x \in G\big(F(y)\big)$, then there exists $z \in X$ such that $f(z,y) \geq 0$ and $g(x,z) \geq 0$. By (iii), $f(x,y) \geq 0$, hence $x \in F(y)$. Therefore, $G(F(y)) \subseteq F(y)$. By Theorem 1, there exists $x_0 \in \bigcap_{y \in X} F(y)$, that is, $f(x_0, y) \geq 0$ for all $y \in X$. $\square$

A simple example is outlined below for showing the applicability of the previous result.

Example 1. *Let $X = Y = [0,2]$. Take the bifunctions $f, g : [0,2] \times [0,2] \to \mathbb{R}$ defined as follows:*

$$f(x,y) = \begin{cases} 2x - y^2 - 1 & \text{if } y \in [0,1[\\ x - 2y + 2 & \text{if } y \in [1,2], \end{cases}$$

$$g(x,z) = x - z.$$

It can be easily checked that the first two assumptions of the previous theorem are fulfilled. For all $y \in [0,2]$, $f(\cdot, y)$ is a strictly increasing function. Then, if for some $x, y, z \in [0,2]$, $g(x,z) \geq 0$ and $f(z,y) \geq 0$, then $x \geq z$ and $f(z,y) \geq 0$, whence $f(x,y) \geq 0$.

By Theorem 3, the associated equilibrium problem has at least one solution. Actually, it can be seen immediately that it has a unique solution, $x_0 = 2$. Let us note that on the diagonal of the product set $[0,2] \times [0,2]$ the bifunction f takes both positive and negative values; hence, in this case, the classical results of the existence of solutions cannot be used.

From Theorem 3, we immediately obtain the next result that can be regarded as a version of [9] (Theorem 2.4). Recall that a bifunction $f : X \times X \to \mathbb{R}$ is said to be properly quasimonotone [27] if for any nonempty finite subset A of X, and all $y \in \text{conv } A$, $\min_{x \in A} f(x,y) \leq 0$.

Corollary 1. *Let X be a compact convex subset of a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$ be a bifunction that satisfies the following assumptions:*

(i) for every $y \in Y$, the set $\{x \in X : f(x,y) \geq 0\}$ is nonempty and closed;
(ii) f is properly quasimonotone;

(iii) *for every $x, y, z \in X$ the following implication holds*

$$f(z,x) \le 0,\ f(z,y) \ge 0 \implies f(x,y) \ge 0.$$

Then, there exists $x_0 \in X$ such that $f(x_0, y) \ge 0$, for all $y \in X$.

Proof. Apply Theorem 3 when $X = Y$ and $g(x,z) = -f(z,x)$. □

Remark 3.

(a) *The unique difference between Corollary 1 and Theorem 2.4 [9] consists of the fact that in the mentioned theorem, assumption (iii) is replaced with another implication, namely,*

$(iii)' x, y, z \in X,\ f(x,z) \le 0, f(z,y) < 0 \Longrightarrow f(x,y) < 0.$

However, this difference is a major one because conditions (iii) and $(iii)'$ are not comparable. For instance, let us consider the bifunctions $f_1, f_2 : [0,2] \times [0,2] \to \mathbb{R}$ defined by

$$f_1(x,y) = y - x - 1,\ f_2(x,y) = x(x-y).$$

One can easily see that f_1 satisfies condition (iii) and for f_2, $(iii)'$ holds. On the other side, f_1 does not fulfill condition $(iii)'$ (take, for instance, $x = 0, y = 1, z = 1$), whereas the bifunction f_2 does not verify (iii) (take $x = 1, y = 2, z = 0$).

(b) *In [5,28–30], the existence of solutions of equilibrium problems when the bifunction f has the following triangle inequality property is studied:*

$$f(x,y) \le f(x,z) + f(z,y),\ \text{for all } x, y, z \in X.$$

We show that condition (iii) of Corollary 1 holds whenever f has the triangle inequality property. Let us assume that f has the triangle inequality property and x, y, z are three points from X for which $f(z,x) \le 0$ and $f(z,y) \ge 0$. By way of contradiction, assume that $f(x,y) < 0$. Then,

$$f(z,y) \le f(z,x) + f(x,y) < 0;\ \ \text{a contradiction.}$$

4. Equilibrium Problems

The proof of Theorem 4 uses the following selection result:

Lemma 3 ([31])**.** *Let X be a paracompact Hausdorff space and Y be a topological vector space. If $P : X \rightrightarrows Y$ is a set-valued mapping with nonempty and convex values and open fibers, then P has a continuous selection; that is, there exists a continuous function $p : X \to Y$ such that $p(x) \in P(x)$ for all $x \in X$.*

Theorem 4. *Let X be a paracompact and convex subset of a Hausdorff topological vector space and f, g two real bifunctions defined on $X \times X$ that satisfy the following conditions:*

(i) *for each $y \in X$ the set $\{x \in X : g(x,y) > 0\}$ is nonempty and convex;*

(ii) *for each $x \in X$ the set $\{y \in X : g(x,y) > 0\}$ is open in X;*

(iii) *g is f-quasiconvex in y;*

(iv) *the set-valued mapping $F : X \rightrightarrows X$ defined by*

$$F(y) = \{x \in X : f(x,y) \ge 0\}$$

is intersectionally closed;

(v) *for at least one $y \in X$, $F(y)$ is relatively compact.*

Then, there exists $x_0 \in X$ such that $f(x_0, y) \ge 0$, for all $y \in X$.

Proof. Apart from the set-valued mapping F defined in assumption (iv), we consider the set-valued mapping $G : X \rightrightarrows X$, defined by

$$G(y) = \{x \in X : g(x,y) > 0\}.$$

We claim that F is a generalized KKM map w.r.t. G. Assume, by way of contradiction, that there exists a nonempty finite set $A \subseteq X$ such that $G(\text{conv } A) \nsubseteq F(A)$. If $x \in G(\text{conv } A) \setminus F(A)$, then $f(x,z) < 0$ for all $z \in A$ and there exists $y \in \text{conv } A$ such that $g(x,y) > 0$. Then,

$$\max_{z \in A} f(x,z) < 0 < g(x,y),$$

which contradicts (iii).

The first two assumptions prove that G has nonempty and convex values and open fibers. In view of Lemma 3, G has a continuous selection $p : X \to X$. Since F is a generalized KKM map w.r.t. G, it will be also a generalized KKM map w.r.t. p. It is well known that any continuous function has the KKM property (see for instance [32] (Theorem 1)); hence, the family $\{\text{cl}F(y) : y \in X\}$ has the finite intersection property. Since at least one $y \in X$, $\text{cl}F(y)$ is a compact set, $\bigcap_{y \in X} \text{cl}F(y) \neq \emptyset$. Because $\bigcap_{y \in X} \text{cl}F(y) = \text{cl}\big(\bigcap_{y \in X} F(y)\big)$, there exists a point $x_0 \in \bigcap_{y \in X} F(y)$. $\square$

The condition that the set X be paracompact can be removed, adding instead new conditions.

Theorem 5. *Let X be a convex subset of a Hausdorff topological vector space and $f, g : X \times X \to \mathbb{R}$. Assume that:*

(i) for each $y \in X$, the set $\{x \in X : g(x,y) > 0\}$ is nonempty and convex;
(ii) for each $x \in X$, the set $\{y \in X : g(x,y) > 0\}$ is open in X;
(iii) g is f-quasiconvex in y;
(iv) for every $y \in X$ the set $\{x \in X : f(x,y) \geq 0\}$ is closed in X;
(v) there exists a nonempty compact subset K_0 of X such that:
(v_1) for at least one $y \in K_0$, the set $\{x \in X : f(x,y) \geq 0\}$ is compact;
(v_2) for every $x \in X \setminus K_0$, there exists $y \in K_0$ such that $f(x,y) < 0$.
Then, there exists $x_0 \in K_0$ such that $f(x_0, y) \geq 0$, for all $y \in X$.

Proof. Consider the family of sets

$$\mathcal{K} = \{K : K_0 \subseteq K \subseteq X,\ K \text{ is compact}\}.$$

For $K \in \mathcal{K}$, we denote by $\widehat{K} = \text{conv}K$. By Lemma 1 [33], $\widehat{K}$ is paracompact. Define the set-valued mappings $F, G : \widehat{K} \rightrightarrows X$ as follows:

$$F(y) = \{x \in X : f(x,y) \geq 0\},\ G(y) = \{x \in X : g(x,y) > 0\}.$$

With arguments similar to those used in the previous proof, it is shown that G has a continuous selection $p : \widehat{K} \to X$ and F is a generalized KKM-map w.r.t. p. From (iv) and (v_1), the values of F are closed sets relative to X and for at least one $y \in K_0$, $F(y)$ is compact. Consequently, $\bigcap_{y \in \widehat{K}} F(y) \neq \emptyset$. Hence, there exists an $x_K \in X$ such that, $f(x_K, y) \geq 0$ for all $y \in \widetilde{K}$. From (v_2), we deduce that $x_K \in K_0$.

Since the set K_0 is compact, we may assume that the net $\{x_K\}_{K \in \mathcal{K}}$ converges to some $x_0 \in K_0$. We now show that $f(x_0, y) \geq 0$, for all $y \in X$.

If y is an arbitrary point in X, then clearly the set $K_y = K_0 \cup \{y\}$ belongs to $\mathcal{K}$. For each $K \in \mathcal{K}$ satisfying $K_y \subseteq K$, we have $f(x_K, y) \geq 0$ and, since the set $\{x \in X : f(x,y) \geq 0\}$ is closed, it follows that $f(x_0, y) \geq 0$. $\square$

It is reasonable now to obtain an existing result for a classical optimization problem that can be transformed into an equilibrium problem. Let E be a locally convex Hausdorff topological vector space and E^* be its topological dual. Given a nonempty convex subset

X of E and a set-valued mapping $P : X \rightrightarrows E^*$, the Minty variational inequality problem ((MVIP), for short) associated with X and P consists of finding a point $x_0 \in X$ such that

$$\langle y^*, y - x_0 \rangle \geq 0, \ \forall (y, y^*) \in \text{ Gr } P.$$

Recall that the set-valued mapping P is called monotone if for every $(x, x^*), (y, y^*) \in \text{Gr } P$,

$$\langle y^* - x^*, y - x \rangle \geq 0.$$

Theorem 6. *Assume that P is monotone and has nonempty weak* compact values. Problem (MVIP) has at least a solution whenever the following conditions are satisfied:*

(i) for each $y \in X$, the set $\{x \in X : \exists x^* \in P(x)$ such that $\langle x^*, y - x \rangle > 0\}$ is nonempty and convex;

(ii) there exists a nonempty compact subset K_0 of X such that:

(ii_1) there are $y_0 \in K_0$ and $y_0^* \in P(y_0)$, such that the set $\{x \in X : \langle y_0^*, y_0 - x \rangle \geq 0\}$ is compact;

(ii_2) for every $x \in X \setminus K_0$, there exist $y \in K_0$ and $y^* \in P(y)$ such that $\langle y^*, y - x \rangle < 0$.

Proof. The conclusion follows from Theorem 5 as soon as we prove that the assumptions of this theorem are satisfied, when the bifunctions f and g are defined as follows

$$f(x, y) = \min_{y^* \in P(y)} \langle y^*, y - x \rangle, \ \ g(x, y) = \max_{x^* \in P(x)} \langle x^*, y - x \rangle.$$

Observe that condition (i) is nothing else but a condition similarly noted in Theorem 5. For each $x \in X$, the set $\{y \in X : g(x, y) > 0\}$ is open in X since it can be written as a union of open sets,

$$\{y \in X : g(x, y) > 0\} = \bigcup_{x^* \in P(x)} \{y \in X : \langle x^*, y - x \rangle > 0\}.$$

We show that g is f-quasiconvex in the second variable. Let $x \in X$, A be a nonempty finite subset of X, and $y \in$ conv A. The bifunction g is convex in the second variable because it is the upper envelope of a family of affine functions. Consequently,

$$g(x, y) \leq \max_{z \in A} g(x, z) = \max_{z \in A} \max_{x^* \in P(x)} \langle x^*, z - x \rangle. \tag{2}$$

The pseudomonotonicity of P implies

$$\max_{x^* \in P(x)} \langle x^*, z - x \rangle \leq \min_{z^* \in P(z)} \langle z^*, z - x \rangle = f(x, z). \tag{3}$$

From (2) and (3), we obtain the inequality $g(x, y) \leq \max_{z \in A} f(x, z)$.

Since

$$\{x \in X : f(x, y) \geq 0\} = \bigcap_{y^* \in P(y)} \{x \in X : \langle y^*, y - x \rangle \geq 0\},$$

the sets $\{x \in X : f(x, y) \geq 0\}$ are closed for all $y \in X$. Moreover, in view of (ii_1), the set $\{x \in X : f(x, y_0) \geq 0\}$ is compact. The last assumption of Theorem 5 follows immediately from (ii_2) and thus the proof is complete. □

Remark 4. *A problem closely related to the Minty variational inequality problem is the so-called Stampacchia variational inequality problem. It consists of finding a pair $(x_0, x_0^*) \in$ Gr T such that*

$$\langle x_0^*, y - x_0 \rangle \geq 0, \ \ \forall y \in X.$$

It is well known (see, for instance [34] (Proposition 1)) that if the set-valued mapping P is upper semicontinuous from the line segments in X to the weak topology of E^*, then any solution of the*

Minty variational inequality problem is also solution for the Stampacchia variational inequality problem. Consequently, from Theorem 6, a criterion for the existence of solutions for the Stampacchia variational inequality problem can be easily obtained.

The following proposition provides sufficient conditions for condition (i) of Theorem 6 to be satisfied.

Proposition 1. *Condition (i) of Theorem 6 holds that whenever, besides the monotonicity, P satisfies one of the following assumptions:*

(a) *-P is pseudomonotone (this means that for every $(x, x^*), (y, y^*) \in Gr\,P$, the following implication holds: $\langle x^*, y - x\rangle \leq 0 \Longrightarrow \langle y^*, y - x\rangle \leq 0$).*

(b) *P is convex (that is, for all $x, y \in X$ and $\lambda \in [0,1]$, $\lambda P(x) + (1-\lambda)P(y) \subseteq P(\lambda x + (1-\lambda)y)$).*

Proof. See the proofs of [11] (Theorem 10), for case (a), and [35] (Theorem 4.2), for case (b). □

Considering a bifunction $f : X \times X \to \mathbb{R}$ and a set-valued mapping $T : X \rightrightarrows X$, we can formulate a more general problem than the equilibrium problem. This is called a quasi-equilibrium problem and is formulated as follows:

$$\text{find } x_0 \in X \text{ such that } x_0 \in T(x_0) \text{ and } f(x_0, y) \geq 0 \text{ for all } y \in T(x_0).$$

The literature dedicated to quasi-equilibrium problems has been continuously enriched in recent years. To the best of our knowledge, the theorem below is the first existing result in which besides bifunction f a second bifunction is involved.

Theorem 7. *Let X be a convex subset of a Hausdorff topological vector space, $T : X \rightrightarrows X$ be a compact continuous set-valued mapping with nonempty closed and convex values and $f, g : X \times X \to \mathbb{R}$ be two bifunctions that satisfy the following conditions:*

(i) *for every $x \in X$ and any $y \in T(x)$, the set $\{u \in T(x) : g(u, y) > 0\}$ is nonempty and convex;*

(ii) *for each $x \in X$ the set $\{y \in X : g(x, y) > 0\}$ is open in X;*

(iii) *g is f-quasiconvex in y;*

(iv) *the set $\{(x, y) \in X \times X : f(x, y) \geq 0\}$ is closed in $X \times X$;*

(v) *for every $y \in X$, the set $\{x \in X : f(x, y) \geq 0\}$ is convex.*

Then, there exists $x_0 \in X$ such that $x_0 \in T(x_0)$ and $f(x_0, y) \geq 0$, for all $y \in T(x_0)$.

Proof. We define the set-valued mapping $S : X \rightrightarrows X$ by

$$S(x) = \{u \in T(x) : f(u, y) \geq 0,\ \forall y \in T(x)\}.$$

For an arbitrary $x \in X$, it can be easily checked whether the restrictions of f and g to $T(x) \times T(x)$ satisfy all the assumptions of Theorem 4. From the mentioned theorem, there exists $u \in T(x)$ such that $f(u, y) \geq 0$, for all $y \in T(x)$. Therefore, $S(x) \neq \emptyset$. If $u_1, u_2 \in S(x)$ and $\lambda \in [0,1]$, since $T(x)$ is a convex set, $\lambda u_1 + (1-\lambda)u_2 \in T(x)$. In view of (v), $f(\lambda u_1 + (1-\lambda)u_2, y) \geq 0$ for all $y \in T(x)$; hence, $S(x)$ is a convex set.

We prove now that S is a closed mapping. Let $\{(x_t, u_t)\}$ be a net in the graph of S converging to (x, u). Then, for each index t, $u_t \in T(x_t)$ and, since T is a closed mapping, $u \in T(x)$. For any $y \in T(x)$, since T is lower semicontinuous, there exists a subnet $\{x_{t_i}\}$ of the net $\{x_t\}$ and a net $\{y_i\}$ converging to y, with $y_i \in T(x_{t_i})$, for each index i. As $u_{t_i} \in S(x_{t_i})$, $f(u_{t_i}, y_i) \geq 0$. Assumption (iv) implies, $f(u, y) \geq 0$, hence $u \in S(x)$.

Summing up, S is a closed mapping with nonempty convex values. Moreover, since T is compact, so will be S. Thus, by the Himmelberg fixed point theorem [36], S has a fixed point, and clearly, this satisfies the desired conclusion. □

In the particular case $f = g$, it could be of some interest to compare the assumptions of the previous theorem, with those from [37] (Theorem 2), [11] (Theorem 7), or [38] (Theorem 2.3). We point out only a few differences. Thus, in [37] Theorem 2, the convexity of the sets $\{y \in X : f(x,y) < 0\}$ is required. Then, in [11] (Theorem 7), the sets $\{y \in X : f(x,y) \leq 0\}$ must be closed and convex. Finally, in [38] (Theorem 2.3), the compactness of the convex set X is needed and bifunction f is assumed diagonally quasiconvex in the first variable.

5. Conclusions

In this paper, we obtain first a criterion for the existence of solutions to a generalized equilibrium problem in which two real bifunctions are involved. Then, sufficient conditions for the existence of solutions to an equilibrium problem are given. The obtained theorems are then applied to the Minty variational inequality problem and to quasi-equilibrium problems. The particularity of our equilibrium results consists of the fact that the classical equilibrium condition ($f(x,x) = 0$ for all $x \in X$) is missing. For this reason, we believe that in a future paper, we will be able to apply our results to the study of so-called variational-like inequality problems in which the involved bifunction is not subject to the equilibrium condition.

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