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On Uniformly S-Multiplication Modules and Rings

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Abstract: In this article, we introduce and study the notions of uniformly S-multiplication modules and rings that are generalizations of multiplication modules and rings. Some examples are given to distinguish the new conceptions with the old classical ones.

Keywords: uniformly S-multiplication module; uniformly S-multiplication ring; idealization

MSC: 13A15

1. Introduction

Throughout this article, R is always a commutative ring with an identity. For a subset U of an R-module M, we denote by $\langle U\rangle$ the submodule of M generated by U. A subset S of R is said to be multiplicative if $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Let N be a submodule of M, and denote by $(N :_R M) = \{r \in R \mid rM \subseteq N\}$.

The notion of multiplication rings was introduced by Krull [1] early in 1925. A ring R is called a multiplication ring if, for every pair of ideals $J \subseteq K$ of R, there exists an ideal I of R such that $J = IK$. Note that an integral domain is a multiplication ring if and only if it is a Dedekind domain (see [2]). Some characterizations of multiplication rings were given by Mott [3]. In 1974, Mehdi [4] first introduced the notion of multiplication modules. An R-module M is said to be a multiplication module if, for every pair of submodules $L \subseteq N$ of M, there exists an ideal I of R such that $L = IN$. Latter in 1988, Barnard [5] alternatively called an R-module M a multiplication if each submodule N of M is of the form $N = IM$ for some ideal I of R, or equivalently, $N = (N :_R M)M$. Some more studies on multiplication modules can be found in [5–7].

At the beginning of this century, Anderson et al. [8] introduced the notion of S-Noetherian rings, which are a generalization of classical Noetherian rings in terms of a multiplicative set S. Since then, some well-known notions of rings and modules have been investigated. In 2020, Anderson, Arabaci, Tekir, and Koç [9] introduced and studied the notion of S-multiplication modules. An R-module M is called an S-multiplication module if, for each submodule N of M, there exist $s \in S$ and an ideal I of R such that $sN \subseteq IM \subseteq N$. They generalized some known results on multiplication modules to S-multiplication modules and studied S-multiplication modules in terms of S-prime submodules. Recently, Chhiti and Moindze [10] studied the notion of S-multiplication rings. A ring R is called an S-multiplication ring if each ideal of R is of the S-multiplication type. They generalized some properties of multiplication rings to S-multiplication rings and then studied the transfer of S-multiplication rings to trivial ring extensions and amalgamated algebras.

In 2021, the second author of this paper first introduced and studied the uniformly S-torsion theory in [11]. Recently, the first author et al. [12] considered the notions of uniformly S-Noetherian rings and modules, which can be seen as "uniform" versions of S-Noetherian rings and modules. The motivation of this article is to introduce and study the notions of uniformly S-multiplication modules and rings, which are "uniform" versions of the S-multiplication modules and rings given in [9,10]. This paper is arranged as follows. In Section 2, we introduce and study the notion of uniformly S-multiplication

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modules. We transfer the uniformly S-multiplication modules to finite direct products, localizations, u-S-isomorphisms, and idealizations. In Section 3, we investigate uniformly S-multiplication rings. We also study uniformly S-multiplication rings under finite direct products, localizations, and idealizations. Furthermore, we connect and distinguish the notions of multiplication modules and rings, uniformly S-multiplication modules and rings, and S-multiplication modules and rings.

2. Uniformly S-Multiplication Modules

Recall from [5] that an R-module M is said to be a multiplication module if each submodule N of M is of the form $N = IM$ for some ideal I of R, or equivalently, $N = (N :_R M)M$. Let S be a multiplicative subset of R. Recently, Anderson et al. [9] introduced the concept of S-multiplication modules; an R-module M is called an S-multiplication module if, for each submodule N of M, there exist $s \in S$ and an ideal I of R such that $sN \subseteq IM \subseteq N$. Note that the "$s$" in this definition is not uniform, i.e., it is decided by the submodule N. To keep it in "uniformity", we introduce the following notion.

Definition 1. *Let M be an R-module and let S be a multiplicative subset of R. Then, M is called a u-S-multiplication (uniformly S-multiplication) module (with respect to s) if there exists an element $s \in S$ such that, for each submodule N of M, there is an ideal I of R satisfying $sN \subseteq IM \subseteq N$.*

From the definition, one can easily verify that an R-module M is a u-S-multiplication if and only if there exists $s \in S$ such that, for each submodule N of M, we have $sN \subseteq (N :_R M)M \subseteq N$.

If S is composed of units, then an R-module is a u-S-multiplication if and only if it is an S-multiplication; if $0 \in S$, then every R-module is a u-S-multiplication. In general, we have the following implications.

multiplication module	$\Longrightarrow$	u-S-multiplication module	$\Longrightarrow$	S-multiplication module

Proposition 1. *Let M_i be an R_i-module and let $S_i \subseteq R_i$ be a multiplicative subset $(i = 1, 2)$. Set $R = R_1 \times R_2, S = S_1 \times S_2$, and $M = M_1 \times M_2$. Then, M is a u-S-multiplication module if and only if M_1 is a u-S_1-multiplication module and M_2 is a u-S_2-multiplication module.*

Proof. For the "only if" part, suppose M is a u-S-multiplication module with respect to some $s = (s_1, s_2) \in S_1 \times S_2$. Then, $(s_1, s_2)(N_1 \times \{0\}) \subseteq [(N_1 \times \{0\}) : M]M$ for any R_1-submodule N_1 of M_1. Therefore, $s_1 N_1 \subseteq (N_1 : M)M$. It follows that M_1 is a u-S-multiplication module with respect to some $s_1 \in S_1$. Similarly, M_2 is a u-S-multiplication module with respect to some $s_2 \in S_2$.

For the "if" part, suppose M_1 is a u-S-multiplication module with respect to some $s_1 \in S_1$ and M_2 is a u-S-multiplication module with respect to some $s_2 \in S_2$. Set $s = (s_1, s_2) \in S$. Let N be an R-module. Then, $N = N(R_1 \times R_2) \cong N_1 \times N_2$, where $N_i = NR_i$ $(i = 1, 2)$. Therefore, $s_i N_i \subseteq (N_i : M_i)M_i$ for each $i = 1, 2$. Consequently, $(s_1, s_2)(N_1 \times N_2) \subseteq [(N_1 \times N_2) : (M_1 \times M_2)](M_1 \times M_2)$. It follows that $M = M_1 \times M_2$ is a u-S-multiplication module with respect to s. □

Note that u-S-multiplication modules need not be a multiplication module. Indeed, let R_1 and R_2 be two commutative rings and let M_1 be a multiplication R_1-module; however, M_2 is not a multiplication R_2-module. Set $R = R_1 \times R_2, S = \{1\} \times \{0\}$ and $M = M_1 \times M_2$. Then. M is not a multiplication R-module, but it is a u-S-multiplication R-module by Proposition 1.

The following example shows that an S-multiplication module need not be a u-S-multiplication module.

Example 1 ([9], Example 3)**.** *Consider the $\mathbb{Z}$-module $E(p) = \{\gamma := \frac{r}{p^m} + \mathbb{Q} \in \mathbb{Q}/\mathbb{Z} \mid r \in \mathbb{Z}, m \geq 0\}$, where p is a prime number. Take the multiplicative closed subset $S = \{p^n : n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then, the $\mathbb{Z}$-module $E(p)$ is an S-multiplication module (see ([9], Example 3)).*

We claim that $E(p)$ is not a u-S-multiplication. Indeed, assume that $E(p)$ is a u-S-multiplication with respect to $p^n \in S$ for some $n \geq 0$. All proper submodules of $E(p)$ are of the form $G_t = \{\gamma := \frac{r}{p^t} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid \gamma \in \mathbb{Z}\}$ for every $t \in \mathbb{N} \cup \{0\}$. Assume that $t \geq n + 1$. Then, $(G_t :_{\mathbb{Z}} E(p)) = 0$. Therefore, $0 \neq p^n G_t \neq (G_t :_{\mathbb{Z}} E(p))E(p) = 0_{E(p)}$. Hence, $E(p)$ is not a u-S-multiplication module.

Let S be a multiplicative subset of R. The saturation S^* of S is defined as $S^* = \{s \in R \mid s_1 = ss_2$ for some $s_1, s_2 \in S\}$. A multiplicative subset S of R is called saturated if $S = S^*$. Note that S^* is always a saturated multiplicative subset containing S.

Proposition 2. *Let M be an R-module. Then, the following statements hold.*

(1) If $S \subseteq T$ are multiplicative subsets of R and M is a u-S-multiplication module, then M is a u-T-multiplication module.
(2) M is a u-S-multiplication module if and only if M is a u-S^*-multiplication module, where S^* is the saturation of S.

Proof. (1): Obvious. (2): Let M be a u-S-multiplication module. Since $S \subseteq S^*$, by (i), M is a u-S^*-multiplication module. For the converse, assume that M is an S^*-multiplication module with some $s \in S^*$. Then, $sN \subseteq (N :_R M)M$ for any submodule N of M. Suppose $s_1 = ss_2$ with some $s_1, s_2 \in S$. Then, $s_1 N = ss_2 N \subseteq s_2(N :_R M)M \subseteq (N :_R M)M$. Therefore, M is a u-S-multiplication module with respect to $s_1 \in S$. □

Let $\mathfrak{p}$ be a prime ideal of R. We say an R-module E is a *u-$\mathfrak{p}$-multiplication* shortly provided that E is a u-$(R \setminus \mathfrak{p})$-multiplication.

Theorem 1. *Let M be an R-module. Then, the following statements are equivalent.*

(1) M is a multiplication module.
(2) M is a u-$\mathfrak{p}$-multiplication module for each $\mathfrak{p} \in \mathrm{Spec}(R)$.
(3) M is a u-$\mathfrak{m}$-multiplication module for each $\mathfrak{m} \in \mathrm{Max}(R)$.
(4) M is a u-$\mathfrak{m}$-multiplication module for each $\mathfrak{m} \in \mathrm{Max}(R)$ with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$.

Proof. $(1) \Rightarrow (2)$: Follows by their definitions.

$(2) \Rightarrow (3)$: This follows the assumption that every maximal ideal is a prime ideal.

$(3) \Rightarrow (4)$: This is trivial.

$(4) \Rightarrow (1)$: Suppose M is a u-$\mathfrak{m}$-multiplication module with respect to some $s_{\mathfrak{m}} \notin \mathfrak{m}$ for each $\mathfrak{m} \in \mathrm{Max}(R)$ with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$. Take a maximal ideal $\mathfrak{m}$ of R with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$. Since M is a u-$\mathfrak{m}$-multiplication module with respect to $s_{\mathfrak{m}}$, we have $s_{\mathfrak{m}} N \subseteq (N :_R M)M$ for every submodule N of M. Then, $N_{\mathfrak{m}} = (s_{\mathfrak{m}} N)_{\mathfrak{m}} \subseteq ((N :_R M)M)_{\mathfrak{m}} \subseteq N_{\mathfrak{m}}$. If $M_{\mathfrak{m}} = 0_{\mathfrak{m}}$, certainly $N_{\mathfrak{m}} = ((N :_R M)M)_{\mathfrak{m}}$. Thus, we conclude that $N_{\mathfrak{m}} = ((N :_R M)M)_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R, and this yields $N = (N :_R M)M$. Therefore, M is a multiplication module. □

Recall from [11] that an R-sequence $M \xrightarrow{f} N \xrightarrow{g} L$ is called *u-S-exact* provided that there is an element $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. An R-homomorphism $f : M \to N$ is a *u-S-monomorphism* (respectively, a *u-S-epimorphism* or an *S-isomorphism*) provided $0 \to M \xrightarrow{f} N$ (respectively, $M \xrightarrow{f} N \to 0$ or $0 \to M \xrightarrow{f} N \to 0$) is u-S-exact. It is easy to verify that an R-homomorphism $f : M \to N$ is a u-S-monomorphism (respectively, u-S-epimorphism) if and only if $\mathrm{Ker}(f)$ (respectively, $\mathrm{Coker}(f)$) is a u-S-torsion module.

Proposition 3. *Let M and M' be R-modules. Suppose M is u-S-isomorphic to M'. Then, M is a u-S-multiplication module if and only if M' is a u-S-multiplication module.*

Proof. Let $f : M \to M'$ be a u-S-isomorphism. Then, there exists $s \in S$ such that $s\mathrm{Ker}(f) = s\mathrm{Coker}(f) = 0$ and M is a u-S-multiplication module with respect to s. Let N be a submodule of M'. Then, there is an ideal I of R such that $sf^{-1}(N) \subseteq IM \subseteq f^{-1}(N)$. Therefore, $f(sf^{-1}(N)) \subseteq f(IM) \subseteq f(f^{-1}(N))$, i.e., $sN \subseteq I\mathrm{Im}(f) \subseteq N$. Since $s\mathrm{Coker}(f) = sM'/\mathrm{Im}(f) = 0$, we have $sM' \subseteq \mathrm{Im}(f)$. Note that $s^2N \subseteq sI\mathrm{Im}(f) \subseteq sIM'$. Consequently, $s^2N \subseteq (sI)M' \subseteq N$. It follows that M' is a u-S-multiplication module with respect to s^2. The converse follows by ([13], Proposition 1.1). □

Proposition 4. *Let M and M' be R-modules. Suppose that S is a multiplicative subset of R and $f : M \twoheadrightarrow M'$ is a u-S-epimorphism. If M is a u-S-multiplication module, then M' is a u-S-multiplication module. Conversely, suppose that M' is an S-multiplication module and $t\mathrm{Ker}(f) = 0$ for some $t \in S$; then, M is a u-S-multiplication module.*

Proof. By Proposition 3, we can assume that f is an epimorphism. Suppose M is a u-S-multiplication module with respect to some $s \in S$. Then, $sN \subseteq (N :_R M)M \subseteq N$ for any submodule N of M. Therefore, $f(sN) \subseteq f((N : M)M) \subseteq f(N)$. Let N' be a submodule of M'. Then, $N := f^{-1}(N')$ is a submodule of M. It follows that $sN' = sf(N) \subseteq (N : M)f(M) = (N : M)M' \subseteq N'$. Thus, $sN' \subseteq (N : M)M' \subseteq N'$ for any submodule N' of M'. Hence, M' is a u-S-multiplication module with respect to s.

On the other hand, suppose that $M' = f(M)$ is a u-S-multiplication module with respect to s. Then, for any submodule N of M, there is an ideal I of R with $sf(N) \subseteq If(M) \subseteq f(N)$. Hence, $sN + \mathrm{Ker}(f) \subseteq N + \mathrm{Ker}(f)$. Since $t\mathrm{Ker}(f) = 0$, we have $(st)N \subseteq (tI)M \subseteq tN \subseteq N$. Consequently, M is a u-S-multiplication module with respect to st. □

Proposition 5. *Let R be a commutative ring and let S and T be multiplicative subsets of R. Set $\widetilde{S} = \{\frac{s}{1} \in T^{-1}R | s \in S\}$, a multiplicative subset of $T^{-1}R$. Suppose M is a u-S-multiplication R-module. Then, $T^{-1}M$ is a u-$\widetilde{S}$-multiplication $T^{-1}R$-module.*

Proof. Suppose M is a u-S-multiplication R-module with respect to some $s \in S$. Then, for any submodule N of M, there is an ideal I of R such that $sN \subseteq IM \subseteq N$. Let L be an submodule of $T^{-1}M$. Then, $L = T^{-1}N'$ for some submodule N' of M. It follows that $\frac{s}{1}L = T^{-1}(sN') \subseteq (T^{-1}I)(T^{-1}M) \subseteq T^{-1}N' = L$. Therefore, $T^{-1}M$ is a u-$\widetilde{S}$-multiplication $T^{-1}R$-module with respect to $\frac{s}{1} \in \widetilde{S}$. □

A multiplicative subset S of R is said to satisfy the maximal multiple condition if there exists an $s \in S$ such that $t|s$ for each $t \in S$. Both finite multiplicative subsets and the multiplicative subsets that consist of units satisfy the maximal multiple condition.

Proposition 6. *Let M be an R-module and let S be a multiplicative subset of R satisfying the maximal multiple condition. Then, the following statements hold:*

(1) M is a u-S-multiplication module.
(2) M is an S-multiplication module.
(3) $S^{-1}M$ is a multiplication $S^{-1}R$-module.

Proof. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): It follows by ([9], Corollary 2).

(3) $\Rightarrow$ (1): Assume that $S^{-1}M$ is a multiplication $S^{-1}R$-module. Take a submodule N of M. We have $S^{-1}N = (S^{-1}I)(S^{-1}M) = S^{-1}(IM)$ for any submodule N of M. Choose $s \in S$ such that $t|s$ for every $t \in S$. Note that for each $n \in N$, we have $\frac{n}{1} \in S^{-1}N = S^{-1}(IM)$, and so there exists $t \in S$ such that $tn \in IM$ and, hence, $sn \in IM$. Thus, $sN \subseteq IM$. Similarly, we have $sIM \subseteq N$. Therefore, we obtain $s^2N \subseteq (sI)M \subseteq N$. Hence, M is a u-S-multiplication module with respect to s^2. □

Recall from [12] the conception of u-S-Noetherian modules. Let $\{M_j\}_{j\in\Gamma}$ be a family of R-modules and let N_j be a submodule of M_j generated by $\{m_{i,j}\}_{i\in\Lambda_j} \subseteq M_j$ for each $j \in \Gamma$.

A family of R-modules $\{M_j\}_{j\in\Gamma}$ is *u-S-generated* (with respective to s) by $\{\{m_{i,j}\}_{i\in\Lambda_j}\}_{j\in\Gamma}$ provided that there exists an element $s \in S$ such that $sM_j \subseteq N_j$ for each $j \in \Gamma$, where $N_j = \langle\{m_{i,j}\}_{i\in\Lambda_j}\rangle$. We say a family of R-modules $\{M_j\}_{j\in\Gamma}$ is *u-S-finite* (with respective to s) if the set $\{m_{i,j}\}_{i\in\Lambda_j}$ can be chosen as a finite set for each $j \in \Gamma$.

Definition 2 ([12])**.** *Let R be a ring and let S be a multiplicative subset of R. An R-module M is called a u-S-Noetherian R-module provided the set of all submodules of M is u-S-finite. A ring R is called a u-S-Noetherian if R itself is a u-S-Noetherian R-module .*

Let R be a ring, let S be a multiplicative subset of R, and let M be an R-module. Denote by $M^\bullet$ an ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M. An ascending chain $M^\bullet$ is called *stationary with respective to* s if there exists $k \geq 1$ such that $sM_n \subseteq M_k$ for any $n \geq k$. Following ([12], Theorem 2.7), M is u-S-Noetherian if and only if there exists an element $s \in S$ such that any ascending chain of submodules of M is stationary with respective to s.

Proposition 7. *Let R be a u-S-Noetherian ring and let M be a u-S-multiplication R-module. Then, M is a u-S-Noetherian R-module.*

Proof. We may assume R is a u-S-Noetherian ring and M is a u-S-multiplication R-module with respect to $s \in S$. Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending chain of submodules of M. Set $A_i = (M_i : M)$. Then, $A_1 \subseteq A_2 \subseteq \cdots$ is an ascending chain of ideals of R. Then there exists n such that $sA_k \subseteq A_n \subseteq A_k$ for any $k \geq n$. Since M is a u-S-multiplication, $sM_i \subseteq (M_i : M)M = A_iM$ for all i. Hence, $s^2M_k \subseteq sA_kM \subseteq A_nM \subseteq M_n$. It follows that M is a u-S-Noetherian R-module with respect to s^2. □

Let M be an R-module. The idealization construction $R(+)M = R \oplus M$ of M is a commutative ring with componentwise additions and multiplications $(a,m)(b,m') = (ab, am' + bm)$ for each $a, b \in R; m, m' \in M$ (see [14]). If S is a multiplicative subset of R and N is a submodule of M, then $S(+)N$ is a multiplicative subset of $R(+)M$. Now, we transfer the uniformly S-multiplication properties to idealization constructions.

Theorem 2. *Let M be an R-module, let N be a submodule of M, and let S be a multiplicative subset of R. Then, the following statements are equivalent.*

(1) *N is a u-S-multiplication R-module.*
(2) *$0(+)N$ is a u-$S(+)0$-multiplication ideal of $R(+)M$.*
(3) *$0(+)N$ is a u-$S(+)M$-multiplication ideal of $R(+)M$.*

Proof. $(1) \Rightarrow (2)$: Suppose N is a u-S-multiplication R-module with respect to some $s \in S$. Let J be an ideal of $R(+)M$ contained in $0(+)N$. Then, $J = 0(+)N'$ for some submodule N' of N. Since N is a u-S-multiplication R-module with respect to s, there exists an ideal I of R such that $sN' \subseteq IN \subseteq N'$. Hence,

$$(s,0)J = (s,0)0(+)N = 0(+)sN' \subseteq 0(+)IN = I(+)M \cdot 0(+)N \subseteq 0(+)N' = J.$$

It follows that $0(+)N$ is a u-$S(+)0$-multiplication ideal of $R(+)M$.

$(2) \Rightarrow (3)$: Since $S(+)0 \subseteq S(+)M$, (3) follows by Proposition 2.

$(3) \Rightarrow (1)$: Suppose that $0(+)N$ is a u-$S(+)M$-multiplication ideal of $R(+)M$ with respective to some $(s,m) \in S(+)M$. Let N' be a submodule of N. Then, $0(+)N'$ is an ideal of $R(+)M$ with $0(+)N' \subseteq 0(+)N$. Since $0(+)N$ is a u-$S(+)M$-multiplication ideal of $R(+)M$ with respect to (s,m), then there exists J' of $R(+)M$ such that $(s,m)0(+)N' \subseteq J' \cdot 0(+)N \subseteq 0(+)N'$. Set $J = J' + 0(+)M$. Then, $J = I(+)M$ for some ideal I of R. Note that

$$J' \cdot 0(+)N = J' \cdot 0(+)N + 0(+)M \cdot 0(+)N = (J' + 0(+)M) \cdot 0(+)N = J \cdot 0(+)N.$$

So $(s,m)0(+)N' \subseteq J \cdot 0(+)N \subseteq 0(+)N'$. This implies that $sN' \subseteq IN \subseteq N'$. So N is a u-S-multiplication R-module with respect to s. □

3. Uniformly *S*-Multiplication Rings

Let R be a ring and let S be a multiplicative subset of R. Recall from [10] that an ideal I of R is an S-multiplication ideal if I is an S-multiplication R-module, and a ring R is an S-multiplication ring if each ideal of R is an S-multiplication. Equivalently, for each pair of ideals $J \subseteq K$ of R, there exist $s \in S$ and an ideal I of R satisfying $sJ \subseteq IK \subseteq J$. Now, we introduce the notion of uniformly S-multiplication rings.

Definition 3. *Let R be a ring and let S be a multiplicative subset of R. Then, R is called a u-S-multiplication (uniformly S-multiplication) ring (with respect to s) if there exists $s \in S$ such that each ideal of R is a u-S-multiplication with respect to s, equivalently, if there exists $s \in S$ such that, for each pair of ideals $J \subseteq K$ of R, there exists an ideal I of R satisfying $sJ \subseteq IK \subseteq J$.*

If S is composed of units, then a ring R is a u-S-multiplication if and only if it is an S-multiplication; if $0 \in S$, then every ring R is a u-S-multiplication. In general, we have the following implications.

multiplication ring $\Longrightarrow$ u-S-multiplication ring $\Longrightarrow$ S-multiplication ring

Proposition 8. *Let $S \subseteq T$ be two multiplicative subsets of R and S^* the saturation of S. Then the following statements hold.*

(1) If R is a u-S-multiplication ring, then R is a u-T-multiplication ring.
(2) R is a u-S-multiplication ring if and only if R is a u-S^*-multiplication ring.

Proof. (1) It immediately follows from the definition of u-S-multiplication rings.

(2) Suppose R is an S^*-multiplication ring with some $s \in S^*$. Then for any pair of ideals $J \subseteq K$, there exists ideal I of R such that $sJ \subseteq IK \subseteq J$. Suppose $s_1 = ss_2$ with some $s_1, s_2 \in S$. Then $s_1 J \subseteq IK \subseteq J$. So R is a u-S-multiplication ring with respect to $s_1 \in S$. □

Corollary 1. *Every multiplication ring is a u-S-multiplication ring.*

Proof. Remark that a multiplication ring is exactly a u-$\{1\}$-multiplication ring. Therefore, the result follows by Proposition 8(1). □

The proof of following result is similar to that of Proposition 1, and so we omit it.

Proposition 9. *Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Then, R is a u-S-multiplication ring if and only if R_1 is a u-S_1-multiplication ring and R_2 is a u-S_2-multiplication ring.*

The following example shows that u-S-multiplication rings are not necessary multiplication rings.

Example 2. *Let R_1 be a multiplication ring and let R_2 be a non-multiplication ring. Set $R = R_1 \times R_2$ and $S = \{1\} \times \{0\}$. Then, R is not a multiplication ring but a u-S-multiplication ring by Proposition 9.*

Trivially, every u-S-multiplication ring is an S-multiplication. Moreover, we have the following result.

Proposition 10. *Let S be a multiplicative subset of R that satisfies the maximal multiple condition. Then, R is a S-multiplication ring if and only if R is a u-S-multiplication ring.*

Proof. If R is a u-S-multiplication ring, R is trivially an S-multiplication. On the other hand, suppose R is an S-multiplication ring. Then, each ideal I of R is an S-multiplication. Therefore, for each pair of ideals $J \subseteq K$ of R, there exist $t \in S$ and an ideal I of R such that $tJ \subseteq IK \subseteq J$. Since S satisfies the maximal multiple condition, there exists $s \in S$ such that $t|s$. Thus, $sJ \subseteq tJ \subseteq IK \subseteq J$. It follows that R is a u-S-multiplication ring with respect to s. □

Let R be a ring and let S be a multiplicative subset of R. For any $s \in S$, there is a multiplicative subset $S_s = \{1, s, s^2, \dots\}$ of S. We denote by M_s the localization of M at S_s for an R-module M.

Proposition 11. *Suppose R is a u-S-multiplication ring. Then, there is an $s \in S$ such that R_s is a multiplication ring.*

Proof. Suppose R is a u-S-multiplication ring with respect to some $s \in S$. Let $J \subseteq K$ be a pair of ideals of R_s. Then, there are two ideals $J' \subseteq K'$ of R such that $J = J'_s$ and $K = K'_s$. There exists an ideal I' of R satisfying $sJ' \subseteq I'K' \subseteq J'$. By localizing at s, we have $J \subseteq IK \subseteq J$, where $I = I'_s$. It follows that R_s is a multiplication ring. □

It follows from Proposition 9.13 in [2] that an integral domain is a multiplication ring if and only if it is a Dedekind domain. The following example shows that rings with each ideal u-S-multiplication are not necessary u-S-multiplication rings, and thus S-multiplication rings are u-S-multiplication rings in general.

Example 3. *Let D be an integral domain such that D_s is not a Dedekind domain for any $0 \neq s \in D$ (e.g., $D = k[x_1, x_2, \dots]$, the polynomial ring with infinite variables over a field k). Set $S = D - \{0\}$. Then D is not a u-S-multiplication ring by Proposition 11. However, every ideal of D is a u-S-multiplication, and thus, D is an S-multiplication ring. Indeed, let K be an ideal of R and let J be a sub-ideal of K. Suppose $K = 0$. Then, $J = 0$, and thus, $sJ \subseteq IK \subseteq J$ always holds. Otherwise, let $0 \neq s \in K$ and $I = J$. Then, we also have $sJ \subseteq IK \subseteq J$. It follows that K is a u-S-multiplication ideal of R.*

Remark 1. *Note that the converse of Proposition 11 is not true in general. Indeed, let D be a valuation domain with valuation group $\mathbb{Z} \times \mathbb{Z}$. It follows by ([15], Chapter II, Exercise 3.4) that the maximal ideal $\mathfrak{m}$ of R is principally generated, say generated as $s \neq 0$. Let $S = D - \{0\}$. Then, D is not a u-S-multiplication ring by Example 3. However D_s is a discrete valuation domain, and hence, it is a multiplication ring.*

Let $\mathfrak{p}$ be a prime ideal of R. We say a ring R is a *u-$\mathfrak{p}$-multiplication* provided that R is a u-$(R \setminus \mathfrak{p})$-multiplication.

Theorem 3. *Let R be a ring. Then, the following statements are equivalent:*

(1) R is a multiplication ring.
(2) R is a u-$\mathfrak{p}$-multiplication ring for each $\mathfrak{p} \in \mathrm{Spec}(R)$.
(3) R is a u-$\mathfrak{m}$-multiplication ring for each $\mathfrak{m} \in \mathrm{Max}(R)$.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$: Trivial.

$(3) \Rightarrow (1)$: Suppose R is a u-$\mathfrak{m}$-multiplication ring with respect to some $s_{\mathfrak{m}} \notin \mathfrak{m}$ for each $\mathfrak{m} \in \mathrm{Max}(R)$. Let $J \subseteq K$ be a pair of ideals of R. Then, there exists an ideal $I^{\mathfrak{m}}$ of R such that $s_{\mathfrak{m}} J \subseteq I^{\mathfrak{m}} K \subseteq J$. Since $\{s^{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{Max}(R)\}$ generates R, there exist finite elements $s^{\mathfrak{m}_1}, \dots, s^{\mathfrak{m}_n}$ such that $J = \langle s^{\mathfrak{m}_1}, \dots, s^{\mathfrak{m}_n} \rangle J \subseteq (\sum_{i=1}^{n} I^{\mathfrak{m}})K \subseteq J$. Setting $I = \sum_{i=1}^{n} I^{\mathfrak{m}}$, we have $IK = J$. Consequently, R is a multiplication ring. □

Proposition 12. *Let R be a ring, let M be an R-module, and let S be a multiplicative subset of R. Suppose $R(+)M$ is a u-$S(+)M$-multiplication ring with respect to some $(s, m) \in S(+)M$. Then,*

R is a u-S-multiplication ring with respect to s, and each submodule of M is a u-S-multiplication R-module with respect to s.

Proof. Let M' be a submodule of M and let N be a submodule of M'. Then, $0(+)N$ is a sub-ideal of $0(+)M'$. Hence, there exists an ideal I' of $R(+)M$ such that $(s,m)0(+)N \subseteq I'0(+)M' \subseteq 0(+)N$. Set $I = \{r \in R \mid \text{there exists } (r,m) \in I'\}$. Then, $sN \subseteq IM' \subseteq N$, and hence, M' is a u-S-multiplication R-module with respect to s.

Let $J \subseteq K$ be a pair of ideals of R. Then, $J(+)M \subseteq K(+)M$ is a pair of ideals of $R(+)M$. Hence, there exists an ideal L' of $R(+)M$ such that $(s,m)J(+)M \subseteq L'K(+)M \subseteq J(+)M$. Set $L = \{r \in R \mid \text{there exists } (r,m) \in L'\}$. Then, $sJ \subseteq LK \subseteq J$. Hence, R is a u-S-multiplication ring with respect to s. □

Remark 2. *We do not know whether the converse of Proposition 12 is true. That is, suppose R is a u-S-multiplication ring with respect to s and each submodule of M is a u-S-multiplication R-module with respect to s. Do we have $R(+)M$ is a u-$S(+)M$-multiplication ring with respect to some $(s,m) \in S(+)M$?*

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References

1. Krull, W. Multiplication rings. *Sitzber Heidelb. Akad. Wiss. Abhand-Lungen* **1925**, *5*, 13–18.
2. Larsen, M.D.; Mccarthy, P.J. *Multiplicative Theory of Ideals*; Academic Press Inc.: New York, NY, USA, 1972.
3. Mott, J.L. Equivalent conditions for a ring to be a multiplication ring. *Can. J. Math.* **1964**, *16*, 429–434. [CrossRef]
4. Mehdi, F. On multiplication modules. *Math. Stud. Vol.* **1974**, *2*, 149–153.
5. Bast, Z.A.; Smith, P.F. Multiplication modules. *Commun. Algebra* **1988**, *16*, 755–779. [CrossRef]
6. Naoum, A.G. Flat modules and multiplication modules. *Period. Math. Hungar.* **1990**, *21*, 309–317. [CrossRef]
7. Smith, P.F. Multiplication modules and projective modules. *Period. Math. Hungar.* **1994**, *29*, 163–168. [CrossRef]
8. Anderson, D.D.; Dumitrescu, T. S-Noetherian rings. *Comm. Algebra* **2002**, *30*, 4407–4416. [CrossRef]
9. Anderson, D.D.; Arabaci, T.; Tekir, U.; Koç, S. On S-multiplication modules. *Comm. Algebra* **2020**, *48*, 3398–3407. [CrossRef]
10. Chhiti, M.; Moindze, S. On S-multiplication rings. *J. Korean Math. Soc.* **2023**, *60*, 327–339.
11. Zhang, X.L. Characterizing S-flat modules and S-von Neumann regular rings by uniformity. *Bull. Korean Math. Soc.* **2022**, *59*, 643–657.
12. Qi, W.; Kim, H.; Wang, F.G.; Chen, M.Z.; Zhao, W. Uniformly S-Noetherian rings. *arXiv* **2022**, arXiv:2201.07913.
13. Zhang, X.L. On uniformly S-absolutely pure modules. *arXiv* **2021**, arXiv:2108.06851.
14. Anderson, D.D.; Winders, M. Idealization of a module. *J. Commut. Algebra* **2009**, *1*, 3–56. [CrossRef]
15. Fuchs, L.; Salce, L. *Modules over Valuation Domains, Lecture Notes in Pure and Applied Mathematics 97*; Marcel Dekker Inc.: New York, NY, USA; Basel, Switzerland, 1985.