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Fixed Point Results for Hybrid Rational Contractions Under a New Compatible Condition with an Application

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Abstract: In this scholarly discourse, we present proof of the existence of unique fixed points in b-metric spaces for hybrid rational contractions. Moreover, we establish a common fixed point theorem for four self-mappings, assuming S-compatibility for two pairs of self-mappings within the framework of b-metric spaces. As a practical demonstration of the aforementioned results, we apply them to a type of integral equation and derive a theorem that guarantees the existence of solutions.

Keywords: fixed/common fixed point; rational contraction; b-metric space; C-class function; S-compatibility

MSC: 47H10; 54H25

1. Introduction

Fixed point theory is an interdisciplinary field that brings together concepts from topology, geometry, pure and applied analysis. It has proven to be an invaluable tool in the study of nonlinear analysis, economics, engineering, medicine, biology, optimal control, game theory, and other theoretical sciences. One of the key contributions of fixed point theory is its ability to solve all kinds of mathematical problems, such as variational inequalities, differential equations and integral equations, and mainly establish the existence and uniqueness of the solutions to these problems. In this regard, the selection of a generalized and extended metric space plays a crucial role in providing non-trivial conditions that guarantee the existence of solutions for a given equation. As early as 1989, Bakhtin [1] initially introduced a extension version of metric, called b-metric space, and later formally defined by Czerwik [2] in 1993. Czerwik also generalized the well-known Banach Contraction Principle within this generalized metric space. The topological properties of such metric spaces and the fixed point theorems of KKM mappings in metric type spaces were first discussed by Khamsi and Hussain [3]. Van An et al. [4] established the stone type theorem under b-metric space, and provided conditions such that b-metric space can be metrizable. Additionally, Czerwik et al. [5,6] introduced set-valued mappings in b-metric spaces and generalized Nadler's fixed point theorem. In 2012, Aydi and co-authors [7,8] demonstrated fixed point and common fixed point theorems for set-valued quasi-contraction mappings and set-valued weak φ-contraction mappings within the framework of b-metric spaces. Various papers have explored fixed point theory for both single-valued and set-valued operators in b-metric spaces, as documented in references [9–19].

In 1976, the concept of commutative maps was introduced by Jungck [20], which sparked the study of the existence of a common fixed point of such maps in metric spaces. Following this, Sessa [21] introduced the weak version of commuting mappings, known as weak commuting mappings. In 1986, Jungck [22] further generalized weak commutativity by introducing compatible mappings. This research opened new directions in fixed point theory for many researchers. Jungck [23] subsequently extended his own concept by introducing the notion of weak compatibility. Over the last few decades, various generalizations of compatible mappings have been developed, including compatible mapping of type (A) [24], compatible mapping of type (B) [25], compatible mapping of type (C) [26], compatible mapping of type (P) [27], semi-compatible mappings [28], weak semi-compatible mappings [29], conditional semi-compatible mappings [30], faintly compatible mappings [31], occasionally weakly compatible mappings [32–34], and other types of mappings [35,36]. Recently, Zhou et al. [37] introduced a new compatible condition called S_τ-compatibility, which is weaker than the $(E.A.)$ property, and also presented a common fixed point theorem in metric spaces.

This paper aims to demonstrate the presence of a distinct fixed point for a novel hybrid rational contraction within the framework of b-metric spaces. Additionally, we introduce a novel form of compatibility condition known as S-compatible for two pairs of self-mappings, in order to investigate the common fixed point theorem for hybrid rational contractions under this specific compatibility condition. Furthermore, we will showcase the existence of a solution to a particular integral equation as an application of our primary findings.

2. Preliminaries

In 1993, Czerwik [2] introduced the notion of b-metric spaces in the following way.

Definition 1. *Suppose that G is a nonempty set, $s \geq 1$ be a given real number. A function $\rho : G \times G \to \mathbb{R}^+$ is a b-metric if, for all $u, v, w \in G$, the following conditions are satisfied:*

(i) $\rho(u,v) = 0$ *if and only if* $u = v$;

(ii) $\rho(u,v) = \rho(v,u)$;

(iii) $\rho(u,w) \leq s[\rho(u,v) + \rho(v,w)]$.

Then the pair (G,ρ) is called a b-metric space.

A b-metric is a metric if (and only if) $s = 1$, at this point, b-metric is a generalization of the normal metric. In other words, a metric is necessarily a b-metric, but a b-metric is not necessarily a metric. Some examples can be used to illustrate the above conclusions (see Example 1.2, [38]).

Definition 2 ([39])**.** *Let (G,ρ) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*

(1) *b-convergent if and only if there exists $x \in X$ such that $\rho(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim\limits_{n\to+\infty} x_n = x$.*

(2) *b-Cauchy if and only if $\rho(x_n, x_m) \to 0$ as $n, m \to +\infty$.*

A b-metric space (G,ρ) is b-complete if every b-Cauchy sequence in X is b-convergent.

Lemma 1 ([38]). *Let (G, ρ) be a b-metric space with $s \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent sequences and converges to x, y, respectively. Then we have*

$$\frac{1}{s^2}\rho(x, y) \leq \liminf_{n \to +\infty} \rho(x_n, y_n) \leq \limsup_{n \to +\infty} \rho(x_n, y_n) \leq s^2 \rho(x, y).$$

When $x = y$, we can obtain $\lim_{n \to +\infty} \rho(x_n, y_n) = 0$. In addition, for all $z \in X$, we obtain,

$$\frac{1}{s}\rho(x, z) \leq \liminf_{n \to +\infty} \rho(x_n, z) \leq \limsup_{n \to +\infty} \rho(x_n, z) \leq s\rho(x, z).$$

Lemma 2 ([40]). *Let $\{x_n\}$ be a sequence in a b-metric space (G, ρ) such that*

$$\rho(x_n, x_{n+1}) \leq \lambda \rho(x_n, x_{n-1})$$

for some λ with $0 < \lambda < \frac{1}{s}$ and $s > 1$ for $n \in \mathbb{N}$. Then $\{x_n\}$ is a b-Cauchy sequence in (G, ρ).

In 2014, Ansari [41] proposed a type of function called C-class functions which covers a number of contractive conditions.

Definition 3 ([41]). *A continuous function $F : [0, +\infty)^2 \to \mathbb{R}$ is called a C-class function if for any $s, t \in [0, +\infty)$, the following conditions hold:*

1. *$F(s, t) \leq s$;*
2. *$F(s, t) = s$ implies that either $s = 0$ or $t = 0$.*

An extra condition on F can be imposed such that $f(0, 0) = 0$ in some cases if required. The letter C denotes the class of all C-class functions. Some classical examples of C-class functions can be found in [41].

Definition 4 ([42]). *A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:*

(i) *ψ is non-decreasing and continuous;*
(ii) *$\psi(t) = 0$ if and only if $t = 0$.*

We denote the class of the altering distance functions by Ψ.

A minor modification of the altering distance function is stated as follows.

Definition 5. *A function $\phi : \mathbb{R} \to \mathbb{R}$ is called an infinite altering distance function if the following properties are satisfied:*

(i) *ϕ is non-decreasing and continuous;*
(ii) *$\phi(u) = 0$ if and only if $u = 0$.*

We denote the class of the infinite altering distance functions by $\Psi_{\inf}$.

Let Φ be the class of the functions $\varphi : [0, +\infty) \to [0, +\infty)$, then the following conditions are true:

(Φ_1) φ is continuous;
(Φ_2) $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$.

Definition 6 ([37]). *A pair (g, f) of self-mappings defined on G is called to be compatible w.r.t. S_τ (S_τ-compatible, for short) if there exists a sequence $\{u_n\} \in G$ such that*

$$\lim_{n \to +\infty} \tau u_n = t \quad \textit{and} \quad \lim_{n \to +\infty} g\tau u_n = \lim_{n \to +\infty} f\tau u_n = \tau t.$$

Example 1. *Suppose that $G = \mathbb{R}$, $gu = 2u$, $fu = 4 - 2u$ and $\tau u = 1 + u$. Take $\{u_n\} = \frac{1}{n}$. Since $\lim_{n\to+\infty} \tau u_n = 1$ with $\lim_{n\to+\infty} g\tau u_n = 2 = \tau(1)$ and $\lim_{n\to+\infty} f\tau u_n = 2 = \tau(1)$. Then pair (g, f) is S_τ-compatible. However, $\lim_{n\to+\infty} \tau u_n \neq \lim_{n\to+\infty} gu_n$ and $\lim_{n\to+\infty} \tau u_n \neq \lim_{n\to+\infty} fu_n$.*

Apparently, S_τ-compatibility of a pair (g, f) self-maps implies $E.A.$ property of a pair (g, f) of self-maps by taking self-map τ as identity map.

Let $G = \mathbb{R}$, $gu = u$, $fu = u^2$ and $\tau = I_u$(identity function on G). Take $\{u_n\} = \frac{1}{n}$. Here $\lim_{n\to+\infty} \tau u_n = \lim_{n\to+\infty} f\tau u_n = \lim_{n\to+\infty} g\tau u_n = \lim_{n\to+\infty} u_n = 0$. Hence, pair self-maps (g, f) satisfies $(E.A.)$ property.

Based on the definition mentioned above, we introduce a new compatible condition for two pairs $(A, B), (J, T)$ of self-mappings called S-compatiblity as follows.

Definition 7. *Suppose that $(A, B), (J, T)$ are two pairs of self-mappings defined on G. Then (A, B) is said to be compatible w.r.t. (J, T) ($S_{(J,T)}$-compatible, for short), if there exists a sequence $\{u_n\} \in G$ such that*

$$\begin{aligned}
&\lim_{n\to+\infty} Ju_n = \lim_{n\to+\infty} Tu_n = t, \\
&\lim_{n\to+\infty} AJu_n = \lim_{n\to+\infty} BJu_n = Jt, \\
&\lim_{n\to+\infty} ATu_n = \lim_{n\to+\infty} BTu_n = Tt.
\end{aligned}$$

Definition 8. *Let $(A, B), (J, T)$ be two pairs of self-mappings defined on X. Then $(A, B), (J, T)$ are said to be S-compatible, if (A, B) is compatible w.r.t. (J, T) and (J, T) is compatible w.r.t. (A, B).*

Example 2. *Suppose that $G = \mathbb{R}^+$ and define $A, B, J, T : G \to G$ by $Au = 2u$, $Bu = u^2$, $Ju = \frac{u}{1+u}$, and $Tu = \sqrt{u}$ for all $u \in G$. For a sequence $\{u_n\} \in G$, where $u_n = \frac{1}{n}, n \in \mathbb{N}$. Then $\lim_{n\to+\infty} Ju_n = \lim_{n\to+\infty} Tu_n = 0$ and also $\lim_{n\to+\infty} AJu_n = \lim_{n\to+\infty} BJu_n = J(0) = 0$ and $\lim_{n\to+\infty} ATu_n = \lim_{n\to+\infty} BTu_n = T(0) = 0$. Therefore, (A, B) is compatible w.r.t. (J, T) for sequence $\{u_n\}$. Similarly, after simple calculation, we also obtain that (J, T) is compatible w.r.t. (A, B) for sequence $\{u_n\}$. Hence, $(A, B), (J, T)$ are S-compatible.*

3. Fixed/Common Fixed Point Theorems for Hybrid Rational Contractions

In this section, in the framework of b-metric space, some results of the fixed points and common fixed points for rational contractive mappings are given.

Theorem 1. *Suppose that (G, ρ, s) is a complete b-metric space, $g : G \to G$ is a mapping satisfying*

$$\psi(s^{1+\varepsilon}\rho(gu, gv)) \leq F(\psi(X(u, v)), \varphi(X(u, v))) + LY(u, v), \tag{1}$$

where X, Y: $G \times G \to \mathbb{R}^+$ are two mappings satisfying

$$\begin{aligned}
X(u, v) = \quad & \frac{1}{m+n+q+p+2sw}\bigg\{m\rho(u, v) + n\frac{\rho(u,gu)\rho(u,gv)+\rho(v,gv)\rho(v,gu)}{\rho(u,gv)+\rho(v,gu)} \\
& +q\rho(u, gu) + p\rho(u, gv) + w[\rho(u, gv) + \rho(v, gu)]\bigg\}
\end{aligned}$$

and

$$Y(u, v) = \min\{\rho(u, gu), \rho(v, gu), \rho(g^2u, g^2v)\},$$

for all elements $u, v \in G$, $L \in \mathbb{R}$, $\varepsilon, m, n, q, p, w \geq 0$ with $m + n + q + p + 2sw > 0$, and $F \in \mathcal{C}$, $\psi \in \Psi_{\inf}$, $\varphi \in \Phi_u$. Then g has a unique fixed point.

Proof. Step I. We prove that the sequence $\{u_n\}$ is a convergent in G.

Let $u_0 \in G$, we can construct a Picard sequence $\{u_n\}$ in G such that $u_{n+1} = gu_n = g^{n+1}u_0$ for all $n \in \mathbb{N} \cup \{0\}$.

If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then $g(u_{n_0}) = u_{n_0}$. This implies u_{n_0} is a fixed point of g. Hence, assume that for all $n \in \mathbb{N} \cup \{0\}$, $u_n \neq u_{n+1}$, i.e., $\rho(u_n, u_{n+1}) > 0$. Applying $u = u_{n-1}$ and $v = u_n$ in (1), we obtain

$$\begin{aligned} \psi(s^{1+\varepsilon}(\rho(u_n, u_{n+1}))) &= \psi(s^{1+\varepsilon}\rho(gu_{n-1}, gu_n)) \\ &\leq F(\psi(X(u_{n-1}, u_n)), \varphi(X(u_{n-1}, u_n))) + LY(u_{n-1}, u_n) \\ &\leq \psi(X(u_{n-1}, u_n)) + LY(u_{n-1}, u_n), \end{aligned} \tag{2}$$

where

$$\begin{aligned} X(u_{n-1}, u_n) &= \frac{1}{m+n+q+p+2sw}\Bigg\{m\rho(u_{n-1}, u_n) + n\frac{\rho(u_{n-1}, gu_{n-1})\rho(u_{n-1}, gu_n) + \rho(u_n, gu_n)\rho(u_n, gu_{n-1})}{\rho(u_{n-1}, gu_n) + \rho(u_n, gu_{n-1})} \\ &\quad + q\rho(u_{n-1}, gu_{n-1}) + p\rho(u_n, gu_n) + w[\rho(u_{n-1}, gu_n) + \rho(u_n, gu_{n-1})]\Bigg\} \\ &= \frac{1}{m+n+q+p+2sw}\Bigg\{m\rho(u_{n-1}, u_n) + n\frac{\rho(u_{n-1}, u_n)\rho(u_{n-1}, u_{n+1}) + \rho(u_n, u_{n+1})\rho(u_n, u_n)}{\rho(u_{n-1}, u_{n+1}) + \rho(u_n, u_n)} \\ &\quad + q\rho(u_{n-1}, u_n) + p\rho(u_n, u_{n+1}) + w[\rho(u_{n-1}, u_{n+1}) + \rho(u_n, u_n)]\Bigg\} \\ &\leq \frac{1}{m+n+q+p+2sw}\Bigg\{m\rho(u_{n-1}, u_n) + n\rho(u_{n-1}, u_n) + q\rho(u_{n-1}, u_n) \\ &\quad + p\rho(u_n, u_{n+1}) + sw\rho(u_{n-1}, u_n) + sw\rho(u_n, u_{n+1})\Bigg\} \\ &= \frac{m+n+q+sw}{m+n+q+p+2sw}\rho(u_{n-1}, u_n) + \frac{p+sw}{m+n+q+p+2sw}\rho(u_n, u_{n+1}) \end{aligned}$$

and

$$\begin{aligned} Y(u_{n-1}, u_n) &= \min\{\rho(u_{n-1}, gu_{n-1}), \rho(u_n, gu_{n-1}), \rho(g^2u_{n-1}, g^2u_n)\} \\ &= \min\{\rho(u_{n-1}, u_n), \rho(u_n, u_n), \rho(u_{n+1}, u_{n+2})\} = 0. \end{aligned}$$

According to (2), we can obtain

$$\begin{aligned} s^{1+\varepsilon}(\rho(u_n, u_{n+1})) &\leq \frac{m+n+q+sw}{m+n+q+p+2sw}\rho(u_{n-1}, u_n) + \frac{p+sw}{m+n+q+p+2sw}\rho(u_n, u_{n+1}) \\ &\leq \frac{m+n+q+sw}{m+n+q+p+2sw}\rho(u_{n-1}, u_n) + s^{1+\varepsilon}(\frac{p+sw}{m+n+q+p+2sw})\rho(u_n, u_{n+1}). \end{aligned}$$

So,

$$\rho(u_n, u_{n+1}) \leq \frac{1}{s^{1+\varepsilon}}\rho(u_{n-1}, u_n). \tag{3}$$

By induction, we can infer that

$$\rho(u_{n+1}, u_n) \leq (\frac{1}{s^{1+\varepsilon}})\rho(u_n, u_{n-1}) \leq (\frac{1}{s^{1+\varepsilon}})^2\rho(u_{n-1}, u_{n-2}) \leq \cdots \leq (\frac{1}{s^{1+\varepsilon}})^n\rho(u_1, u_0).$$

Hence, we have

$$\lim_{n\to+\infty} \rho(u_n, u_{n+1}) = 0. \tag{4}$$

By Lemma 2, $\{u_n\}$ is a b-Cauchy sequence. Further, according to the completeness of G, we have $\{u_n\}$ converges to a point $u^* \in G$.

Step II. We prove that u^* is a fixed point of g.

Again, applying $u = u_n, v = u^*$ in (1), we have

$$\psi(s^{1+\varepsilon}\rho(u_{n+1}, gu^*)) = \psi(s^{1+\varepsilon}\rho(gu_n, gu^*)) \leq F(\psi(X(u_n, u^*)), \varphi(X(u_n, u^*))) + LY(u_n, u^*), \tag{5}$$

where

$$\begin{aligned} X(u_n, u^*) = & \frac{1}{m+n+q+p+2sw}\bigg\{m\rho(u_n, u^*) + n\frac{\rho(u_n, gu_n)\rho(u_n, gu^*) + \rho(u^*, gu^*)\rho(u^*, gu_n)}{\rho(u_n, gu^*) + \rho(u^*, gu_n)} \\ & + q\rho(u_n, gu_n) + p\rho(u^*, gu^*) + w[\rho(u_n, gu^*) + \rho(u^*, gu_n)]\bigg\} \\ = & \frac{1}{m+n+q+p+2sw}\bigg\{m\rho(u_n, u^*) + n\frac{\rho(u_n, u_{n+1})\rho(u_n, gu^*) + \rho(u^*, gu^*)\rho(u^*, u_{n+1})}{\rho(u_n, gu^*) + \rho(u^*, u_{n+1})} \\ & + q\rho(u_n, u_{n+1}) + p\rho(u^*, gu^*) + w[\rho(u_n, gu^*) + \rho(u^*, u_{n+1})]\bigg\} \end{aligned} \tag{6}$$

and

$$\begin{aligned} Y(u_n, u^*) &= \min\{\rho(u_n, gu_n), \rho(u^*, gu_n), \rho(g^2u_n, g^2u^*)\} \\ &= \min\{\rho(u_n, u_{n+1}), \rho(u^*, u_{n+1}), \rho(u_{n+2}, g^2u^*)\}. \end{aligned} \tag{7}$$

Taking the upper limit as $n \to +\infty$ in (6) and (7), then

$$\lim_{n\to+\infty} \sup X(u_n, u^*) \leq \frac{p+sw}{m+n+q+p+2sw}\rho(u^*, gu^*), \lim_{n\to+\infty} \sup Y(u_n, u^*) = 0. \tag{8}$$

From the properties of ψ and F, we have

$$\begin{aligned} &\psi(s\rho(u_{n+1}, gu^*)) \\ &\leq \psi(s^{1+\varepsilon}\rho(u_{n+1}, gu^*)) \\ &\leq F(\psi(X(u_n, u^*)), \varphi(X(u_n, u^*))) + LY(u_n, u^*) \\ &\leq \psi(X(u_n, u^*)) + LY(u_n, u^*). \end{aligned}$$

Taking the upper limit as $n \to +\infty$ in the above inequalities and using (8), we obtain

$$\begin{aligned} &\psi(\frac{p+sw}{m+n+q+p+2sw}\rho(u^*, gu^*)) \\ &\leq \psi(s^{1+\varepsilon}\rho(u^*, gu^*)) \\ &\leq F(\psi(\frac{p+sw}{m+n+q+p+2sw}\rho(u^*, gu^*)), \varphi(\frac{p+sw}{m+n+q+p+2sw}\rho(u^*, gu^*))) \\ &\leq \psi(\frac{p+sw}{m+n+q+p+2sw}\rho(u^*, gu^*)), \end{aligned}$$

which yields that $\psi(\frac{p+sw}{m+n+q+p+2sw}\rho(u^*, gu^*)) = 0$ or $\varphi(\frac{p+sw}{m+n+q+p+2sw}\rho(u^*, gu^*)) = 0$. We derive $\rho(u^*, gu^*) = 0$ implies $u^* = gu^*$.

Step III. we will prove that u^* is a unique fixed point of g.

Assume that i is also a fixed point of g, that is $gi = i$. Then, we have

$$\psi(s^{1+\varepsilon}\rho(u^*, i)) = \psi(r^{1+\varepsilon}\rho(gu^*, gi)) \leq F(\psi(X(u^*, i)), \varphi(X(u^*, i))) + LY(u^*, i), \tag{9}$$

where X, Y: $G \times G \to \mathbb{R}^+$ are two mappings satisfying

$$\begin{aligned} X(i,u^*) &= \frac{1}{m+n+q+p+2sw}\Big\{m\rho(i,u^*) + n\frac{\rho(i,gi)\rho(i,gu^*)+\rho(u^*,gu^*)\rho(u^*,gi)}{\rho(i,gu^*)+\rho(u^*,gi)} \\ &\quad + q\rho(i,gi) + p\rho(u^*,gu^*) + w[\rho(i,gu^*)+\rho(u^*,gi)]\Big\} \\ &= \frac{m+2w}{m+n+q+p+2sw}\rho(i,u^*) \end{aligned}$$

and

$$Y(i,u^*) = \min\{\rho(i,gi), \rho(u^*,gi), \rho(g^2i, g^2u^*)\} = 0.$$

Then (9) becomes

$$\begin{aligned} \psi(\frac{m+2w}{m+n+q+p+2sw}\rho(u^*,i)) &\le \psi(\rho(u^*,i)) \\ &\le \psi(s^{1+\varepsilon}\rho(u^*,i)) \\ &\le F(\psi(\frac{m+2w}{m+n+q+p+2sw}\rho(u^*,i)), \varphi(\frac{m+2w}{m+n+q+p+2sw}\rho(u^*,i))) \\ &\le \psi(\frac{m+2w}{m+n+q+p+2sw}\rho(u^*,i)), \end{aligned}$$

which yields that $\psi(\frac{m+2w}{m+n+q+p+2sw}\rho(u^*,i)) = 0$ or $\varphi(\frac{m+2w}{m+n+q+p+2sw}\rho(u^*,i)) = 0$. We derive $\rho(u^*,i) = 0$, it implies $u^* = i$. So, u^* is a unique fixed point of g. □

Example 3. *Let $G = [0,1]$ equipped with $\rho(u,v) = (u-v)^2$, it is obvious that (G,ρ) is a complete b-metric space respected to $s = 2$. Suppose that $\psi(t) = \frac{1}{2}t$, $F(r,t) = r$ and $g(u) = \frac{1}{2}u$, we have*

$$\begin{aligned} \psi(2^{1+1}\rho(gu,gv)) &= \frac{1}{2}(u-v)^2 \\ &\le (\frac{1}{2})^2 2\rho(u,v). \end{aligned}$$

So g satisfies (1) respected to ρ, $L \in \mathbb{R}$, $F \in \mathcal{C}$, $\psi \in \Psi_{\inf}$, $\varphi \in \Phi_u$ and $m = 2, \varepsilon = 1, n = q = p = w = 0$. By Theorem 1, we have g has a unique fixed point.

Corollary 1. *Suppose that (G,ρ,s) is a complete b-metric space, and g is a self-mapping defined on G satisfying for all elements $u,v \in G$,*

$$\psi(s^{1+\varepsilon}\rho(gu,gv)) \le F(\psi(X(u,v)), \varphi(X(u,v))),$$

where X: $G \times G \to \mathbb{R}^+$ is a mapping satisfying

$$\begin{aligned} X(u,v) = &\frac{1}{m+n+q+p+2sw}\Big\{m\rho(u,v) + n\frac{\rho(gu,u)\rho(u,gv)+\rho(gu,v)\rho(gv,v)}{\rho(u,gv)+\rho(gu,v)} \\ &+ q\rho(gu,u) + p\rho(gv,v) + w(\rho(u,gv)+\rho(gu,v))\Big\}, \end{aligned}$$

and $F \in C$, $\psi \in \Psi_{\inf}$, $\varphi \in \Phi_u$, $\varepsilon, m, n, p, q, w \ge 0$ with $m+n+q+p+2sw > 0$. If g is continuous, then g has a unique fixed point.

With choice $F(r,t) = (m+n+q+p+2sw)r$, for some $m+n+q+p+2sw \in (0,1)$, we have the following corollary.

Corollary 2. *Suppose that (G, ρ, s) is a complete b-metric space, $g : G \to G$ is a mapping satisfying*

$$\psi(s^{1+\varepsilon}\rho(gu, gv)) \leq \psi(X(u,v)) + LY(u,v),$$

where X, Y: $G \times G \to \mathbb{R}^+$ are two mappings satisfying

$$X(u,v) = \frac{1}{m+n+q+p+2sw}\left\{m\rho(u,v) + n\frac{\rho(u,gu)\rho(u,gv) + \rho(v,gv)\rho(v,gu)}{\rho(u,gv) + \rho(v,gu)} + q\rho(u,gu) + p\rho(v,gv) + w[\rho(u,gv) + \rho(v,gu)]\right\}$$

and

$$Y(u,v) = \min\{\rho(u,gu), \rho(v,gu), \rho(g^2u, g^2v)\},$$

for all $u, v \in G$, $L \in \mathbb{R}$, $\varepsilon \geq 0$, and $\psi \in \Psi_{\inf}$. Then g has a unique fixed point.

With choice $F(r,t) = (m+n+q+p+2sw)r$, for some $m+n+q+p+2sw \in (0,1)$, $\psi(t) = t$ and $L = 0$, we have the following corollary.

Corollary 3. *Suppose that (G, ρ, s) is a complete b-metric space, and $g : G \to G$ is a mapping satisfying*

$$s^{1+\varepsilon}\rho(gu, gv)) \leq \left\{m\rho(u,v) + n\frac{\rho(u,gu)\rho(u,gv) + \rho(v,gv)\rho(v,gu)}{\rho(u,gv) + \rho(v,gu)} + q\rho(u,gu) + p\rho(v,gv) + w[\rho(u,gv) + \rho(v,gu)]\right\}$$

for all $u, v \in G$, $\varepsilon, m, n, q, p, w \geq 0$ such that $m+n+q+p+2sw \in (0, \frac{1}{s^{1+\varepsilon}})$. Then g has a unique fixed point.

In the following, we will illustrate some common fixed point theorems for two pairs of self-mappings with S-compatible condition.

Theorem 2. *Suppose that (G, ρ, s) is a complete b-metric space, and (A, B), (J, T) are two pairs of self-mappings defined on G satisfying*

(1) *$A(G) \subseteq J(G)$ and $T(G) \subseteq B(G)$;*
(2) *for all elements $u, v \in G$, $\varepsilon, m, n, q, p, w \geq 0$ such that $m+n+q+p+2sw > 0$,*

$$\psi(s^{1+\varepsilon}\rho(Au, Tv)) \leq F(\psi(X(u,v)), \varphi(Y(u,v))), \tag{10}$$

where $X : G \times G \to \mathbb{R}^+$ is a mapping satisfying

$$X(u,v) = \frac{1}{m+n+q+p+2sw}\left\{m\rho(Bu, Jv) + n\frac{\rho(Au,Bu)\rho(Bu,Tv) + \rho(Au,Jv)\rho(Jv,Tv)}{\rho(Bu,Tv) + \rho(Au,Jv)} + q\rho(Au,Bu) + p\rho(Jv,Tv) + w(\rho(Bu,Tv) + \rho(Au,Jv))\right\}, \tag{11}$$

and $F \in \mathcal{C}$, $\psi \in \Psi_{\inf}$, $\varphi \in \Phi_u$. Suppose that ρ is continuous. If (J, T), (A, B) are S-compatible and J is continuous, then A, B, J and T have a unique common fixed point.

Proof. Let $u_0 \in G$. Since $A(G) \subseteq J(G)$ and $T(G) \subseteq B(G)$, then there exist $u_1, u_2 \in G$ such that $Au_0 = Ju_1 = v_0, Tu_1 = Bu_2 = v_1$. Repeating this process, we can obtain a sequence

$\{u_n\}$ in G such that $Au_{n-1} = Ju_n = v_{n-1}$ and $Tu_n = Bu_{n+1} = v_n$ for $n \in \mathbb{N}$. From (10) and (11), we have

$$\begin{aligned}
&\psi(s^{1+\varepsilon}\rho(Au_n, Tu_{n+1}))\\
&\leq F(\psi(X(u_n, u_{n+1})), \varphi(X(u_n, u_{n+1})))\\
&\leq \psi(X(u_n, u_{n+1}))\\
&= \psi(\frac{1}{m+n+q+p+2sw}\Big\{m\rho(Bu_n, Ju_{n+1})\\
&\quad + n\frac{\rho(Au_n, Bu_n)\rho(Bu_n, Tu_{n+1}) + \rho(Au_n, Ju_{n+1})\rho(Ju_{n+1}, Tu_{n+1})}{\rho(Bu_n, Tu_{n+1}) + \rho(Au_n, Ju_{n+1})}\\
&\quad + q\rho(Au_n, Bu_n) + p\rho(Ju_{n+1}, Tu_{n+1}) + w(\rho(Bu_n, Tu_{n+1}) + \rho(Au_n, Ju_{n+1}))\Big\})
\end{aligned}$$

From the construction of sequence $\{v_n\}$, it yields that

$$\begin{aligned}
&\psi(s^{1+\varepsilon}\rho(v_n, v_{n+1}))\\
&\leq \psi(\frac{1}{m+n+q+p+2sw}[m\rho(v_{n-1}, v_n) + n\frac{\rho(v_n, v_{n-1})\rho(v_{n-1}, v_{n+1}) + \rho(v_n, v_n)\rho(v_{n+1}, \rho_n)}{\rho(v_{n-1}, \rho_{n+1}) + \rho(v_n, v_n)}\\
&\quad + q\rho(v_n, v_{n-1}) + p\rho(v_{n+1}, v_n) + w(\rho(v_{n-1}, v_{n+1}) + \rho(v_n, v_n))])\\
&\leq \psi(\frac{1}{m+n+q+p+2sw}[m\rho(v_{n-1}, v_n) + n\rho(v_n, v_{n-1}) +\\
&\quad + q\rho(v_n, v_{n-1}) + p\rho(v_{n+1}, v_n) + sw(\rho(v_{n-1}, v_n) + \rho(v_n, v_{n+1}))])
\end{aligned}$$

By the definition of ψ, we have

$$\begin{aligned}
&s^{1+\varepsilon}\rho(v_n, v_{n+1})\\
&\leq \frac{m+n+q+sw}{m+n+q+p+2sw}\rho(v_{n-1}, v_n) + \frac{p+sw}{m+n+q+p+2sw}\rho(v_n, v_{n+1})\\
&\leq \frac{m+n+q+sw}{m+n+q+p+2sw}\rho(v_{n-1}, v_n) + \frac{(p+sw)s^{1+\varepsilon}}{m+n+q+p+2sw}\rho(v_n, v_{n+1}).
\end{aligned}$$

Then, it implies that

$$s^{1+\varepsilon}(1 - \frac{p+sw}{m+n+q+p+2sw})\rho(v_n, v_{n+1}) \leq \frac{m+n+q+sw}{m+n+q+p+2sw}\rho(v_{n-1}, v_n).$$

Sequentially, we obtain

$$\rho(v_n, v_{n+1}) \leq \frac{1}{s^{1+\varepsilon}}\rho(v_{n-1}, v_n).$$

Therefore, $\{v_n\}$ is a Cauchy sequence. Completeness of (G, ρ, s) implies $\{v_n\}$ converges to some point $t \in G$ or $\lim_{n\to+\infty} Au_n = \lim_{n\to+\infty} Ju_n = \lim_{n\to+\infty} Tu_n = \lim_{n\to+\infty} Bu_n = t$. The S-compatibility of (J, T) and (A, B) yields the following results:

$$\lim_{n\to+\infty} TAu_n = \lim_{n\to+\infty} JAu_n = At. \tag{12}$$

$$\lim_{n\to+\infty} TBu_n = \lim_{n\to+\infty} JBu_n = Bt. \tag{13}$$

$$\lim_{n\to+\infty} ATu_n = \lim_{n\to+\infty} BTu_n = Tt. \tag{14}$$

$$\lim_{n\to+\infty} AJu_n = \lim_{n\to+\infty} BJu_n = Jt. \tag{15}$$

Now by the definition of ψ and s,

$$\begin{aligned}
&\psi(\rho(ATu_n, Tu_n)) \\
&\leq \psi(s^{1+\varepsilon}\rho(ATu_n, Tu_n)) \\
&\leq F(\psi(X(Tu_n, u_n)), \varphi(X(Tu_n, u_n))) \\
&\leq \psi(X(Tu_n, u_n)) \\
&= \psi(\frac{1}{m+n+q+p+2sw}[m\rho(BTu_n, Ju_n) \\
&\quad + n\frac{\rho(ATu_n, BTu_n)\rho(BTu_n, Tu_n) + \rho(ATu_n, Ju_n)\rho(Ju_n, Tu_n)}{\rho(BTu_n, Tu_n) + \rho(ATu_n, Ju_n)} \\
&\quad + q\rho(ATu_n, BTu_n) + p\rho(Ju_n, Tu_n) + w(\rho(BTu_n, Tu_n) + \rho(ATu_n, Ju_n))]).
\end{aligned}$$

Taking limit as $n \to +\infty$ in the above inequalities, together with (14), we have

$$\psi(\rho(Tt, t)) \leq \psi[\frac{m+2w}{m+n+q+p+2sw}\rho(Tt, t)].$$

By the definition of ψ, we obtain

$$\rho(Tt, t) \leq \frac{m+2w}{m+n+q+p+2sw}\rho(Tt, t).$$

Then,

$$\rho(Tt, t)\frac{n+q+p+2w(s-1)}{m+n+q+p+2sw} \leq 0.$$

Since $m, n, q, p, w \geq 0$ and $s \geq 1$, this yields that $Tt = t$. Again by the definition of ψ and s, we have

$$\begin{aligned}
&\psi(\rho(Au_n, TBu_n)) \\
&\leq \psi(s^{1+\varepsilon}\rho(Au_n, TBu_n)) \\
&\leq F(\psi(X(u_n, Bu_n)), \varphi(X(u_n, Bu_n))) \\
&\leq \psi(M(u_n, Bu_n)) \\
&= \psi(\frac{1}{m+n+q+p+2sw}[m\rho(Bu_n, JBu_n) \\
&\quad + n\frac{\rho(Au_n, Bu_n)\rho(Bu_n, TBu_n) + \rho(Au_n, JBu_n)\rho(JBu_n, TBu_n)}{\rho(Bu_n, TBu_n) + \rho(Au_n, JBu_n)} \\
&\quad + q\rho(Au_n, Bu_n) + p\rho(JBu_n, TBu_n) + w(\rho(Bu_n, TBu_n) + \rho(Au_n, JBu_n))]).
\end{aligned}$$

Taking limit as $n \to +\infty$ in the above inequalities, together with (13), we have

$$\psi(\rho(t, Bt)) \leq \psi[\frac{m+2w}{m+n+q+p+2sw}\rho(t, Bt)].$$

Hence $Bt = t$.

If we put $u = u_n$ and $v = Au_n$ in (10), then with the help of (12), we have $At = t$. Since J is continuous, this yields $\lim_{n\to\infty} JBu_n = Jt$. With (13), it is easy to see that $Bt = Jt$. Hence $Tt = Jt = At = Bt = t$ and then t is common fixed point of A, B, J and T.

For the uniqueness, suppose that i is another common fixed point of A, B, J and T, that is $Ai = Bi = Ji = Ti = i$. From (10), we have

$$\begin{aligned}
&\psi(\rho(t,i)) \\
&= \psi(\rho(At,Ti)) \\
&\leq \psi(s^{1+\varepsilon}\rho(At,Ti)) \\
&\leq F(\psi(X(t,i)),\varphi(X(t,i))) \\
&\leq \psi(X(t,i)) \\
&= \psi(\frac{1}{m+n+q+p+2sw}[m\rho(Bt,Jw) + n\frac{\rho(At,Bt)\rho(Bt,Ti)+\rho(At,Ji)\rho(Ji,Ti)}{\rho(Bt,Ti)+\rho(At,Ji)} \\
&\quad + q\rho(At,Bt) + p\rho(Ji,Ti) + w(\rho(Bt,Ti)+\rho(At,Ji))]) \\
&= \psi(\frac{m+2w}{m+n+q+p+2sw}\rho(t,i)),
\end{aligned}$$

which implies that $\rho(t,i) \leq \frac{m+2w}{m+n+q+p+2sw}\rho(t,i)$. Hence, $\rho(i,t) = 0$, that is $i = t$. So t is the unique common fixed point of A, B, J and T. $\square$

Example 4. *Let $G = [0,2]$ equipped with $\rho(u,v) = (u-v)^2$, it is obvious that (G,ρ) is a complete b-metric space respected to $s = 2$. If $Au = \frac{1}{16}u$, $Bu = \frac{1}{4}u$, $Tu = \frac{1}{8}u$ and $Ju = \frac{1}{2}u$, obviously, $A(G) \subseteq J(G)$, $T(G) \subseteq B(G)$ and J is continuous. Consider a sequence $\{u_n\} \in X$ where $u_n = \frac{1}{n}, n \in \mathbb{N}$, it is obvious that*

$$\begin{aligned}
\lim_{n\to+\infty} Ju_n &= \lim_{n\to+\infty} Tu_n = 0, \\
\lim_{n\to+\infty} AJu_n &= \lim_{n\to+\infty} BJu_n = J0, \\
\lim_{n\to+\infty} ATu_n &= \lim_{n\to+\infty} BTu_n = T0, \\
\lim_{n\to+\infty} Au_n &= \lim_{n\to+\infty} Bu_n = 0, \\
\lim_{n\to+\infty} JAu_n &= \lim_{n\to+\infty} TAu_n = A0, \\
\lim_{n\to+\infty} JBu_n &= \lim_{n\to+\infty} TBu_n = B0.
\end{aligned}$$

So (T,J) and (A,B) are S-compatible. Suppose that $\psi(t) = 2t$ and $F(r,t) = r$, we have

$$\begin{aligned}
\psi(2^{1+1}\rho(Au,Tv)) &= \frac{1}{32}(u-2v)^2 \\
&\leq \frac{1}{2}(u-2v)^2 \\
&= \psi(\rho(u,v)).
\end{aligned}$$

So g satisfies (10) respected to $\varphi \in \Phi_u$ and $m = 2, \varepsilon = 1, n = q = p = w = 0$. By Theorem 2, then A, B, J and T have a unique common fixed point.

Theorem 3. *Let (G,ρ,s) be a complete b-metric space, and $(A,B),(J,T)$ be two pairs of self-mappings defined on G satisfying*

(1) *$A(G) \subseteq J(G)$ and $T(G) \subseteq B(G)$;*
(2) *for all elements $u,v \in G$, $L \in \mathbb{R}$, $\varepsilon, m, n, q, p, w \geq 0$ with $m+n+q+p+2sw > 0$,*

$$\psi(s^{1+\varepsilon}\rho(Au,Tv)) \leq F(\psi(X(u,v)),\varphi(X(u,v))) + LY'(u,v), \tag{16}$$

where $X, Y' \colon G \times G \to \mathbb{R}^+$ *are two mappings satisfying*

$$X(u,v) = \frac{1}{m+n+q+p+2sw}\bigg\{ m\rho(Bu,Jv) + n\frac{\rho(Au,Bu)\rho(Bu,Tv) + \rho(Au,Jv)\rho(Jv,Tv)}{\rho(Bu,Tv)+\rho(Au,Jv)} + q\rho(Au,Bu) + p\rho(Jv,Tv) + w(\rho(Bu,Tv)+\rho(Au,Jv))\bigg\}$$

and

$$Y'(u,v) = \min\{\rho(Au,Bu), \rho(Jv,Tv), \rho(Au,Jv), \rho(Bu,Tv)\},$$

where $F \in \mathcal{C}$, $L \in \mathbb{R}$, $\psi \in \Psi_{\inf}$, $\varphi \in \Phi_u$. *Suppose that* ρ *is continuous. If* $(J,T), (A,B)$ *are S-compatible and* J *is continuous, then* A, B, J *and* T *have a unique common fixed point.*

Proof. The conclusion follows by the analysis similar to that in the proof of Theorem 2. □

Corollary 4. *Let* (G,ρ,s) *be a complete b-metric space, and* A, B, T *and* J *be four self-mappings of defined on* G *satisfying* $A(G) \subseteq J(G)$, $T(G) \subseteq B(G)$ *and for all elements* $u, v \in G$, $\varepsilon, m, n, q, p, w \geq 0$ *with* $m+n+q+p+2sw > 0$,

$$\psi(s^{1+\varepsilon}\rho(Au,Tv)) \leq \psi(X(u,v))$$

where

$$X(u,v) = \frac{1}{m+n+q+p+2sw}\bigg\{ m\rho(Bu,Jv) + n\frac{\rho(Au,Bu)\rho(Bu,Tv) + \rho(Au,Jv)\rho(Tv,Jv)}{\rho(Bu,Tv)+\rho(Au,Jv)} + q\rho(Au,Bu) + p\rho(Tv,Jv) + w(\rho(Bu,Tv)+\rho(Au,Jv))\bigg\},$$

and $\psi \in \Psi_{\inf}$. *Suppose that* ρ *is continuous, If* $(J,T), (A,B)$ *are S-compatible and* J *is continuous, then* A, B, T *and* J *have a unique common fixed point.*

Proof. Common fixed point of mappings A, B, T and J can be obtained just by choosing $F(r,t) = r$ in Theorem 3. □

Corollary 5. *Let* (G,ρ,s) *be a complete b-metric space, and* A, B, T *and* J *be four self-mappings of* G *satisfying the following*

(1) $A(G) \subseteq J(G)$ *and* $T(G) \subseteq B(G)$;
(2) *for all elements* $u \in G$, $\varepsilon, m, n, q, p, w \geq 0$ *with* $m+n+q+p+2sw > 0$,

$$s^{1+\varepsilon}\rho(Au,Tv)) \leq k\bigg\{ m\rho(Bu,Jv) + n\frac{\rho(Au,Bu)\rho(Bu,Tv) + \rho(Au,Jv)\rho(Tv,Jv)}{\rho(Bu,Tv)+\rho(Au,Jv)} + q\rho(Au,Bu) + p\rho(Tv,Jv) + w(\rho(Bu,Tv)+\rho(Au,Jv))\bigg\},$$

where $k = \frac{1}{m+n+q+p+2sw}$. *Suppose that* ρ *is continuous. If* $(J,T), (A,B)$ *are S-compatible and* J *is continuous, then* A, B, T *and* J *have a unique common fixed point.*

Proof. On applying the definition of $\psi(t) = t$ in Corollary 4 and common fixed point of mappings A, B, T and J can be easily obtained. □

Corollary 6. *Let (G, ρ, s) be a complete b-metric space, and let A and B be two self-mappings defined on G satisfying*

(1) $A(G) \subseteq B(G)$;
(2) *for all elements $u, v \in G$, $\varepsilon, m, n, q, p, w \geq 0$ with $m + n + q + p + 2sw > 0$,*

$$\psi(s^{1+\varepsilon}\rho(Au, Av)) \leq F(\psi(X(u, v)), \varphi(X(u, v))),$$

where X: $G \times G \to \mathbb{R}^+$ is a mapping satisfying

$$X(u,v) = \frac{1}{m+n+q+p+2sw}\bigg\{m\rho(Bu,Bv) + n\frac{\rho(Au,Bu)\rho(Bu,Av) + \rho(Au,Bv)\rho(Av,Bv)}{\rho(Bu,Av) + \rho(Au,Bv)}$$
$$+ q\rho(Au,Bu) + p\rho(Av,Bv) + w(\rho(Bv,Av) + \rho(Au,Bv))\bigg\},$$

and $F \in \mathcal{C}$, $\psi \in \Psi_{\inf}$, $\varphi \in \Phi_u$. Suppose that ρ is continuous. If A and B are continuous, then A and B have a unique common fixed point.

Proof. The conclusion can be easily deduced from the Corollary 4, instead of considering the case involving one pair self-mappings A and B defined on G. $\square$

4. Existence for a Solution to an Integral Equation

Consider the integral equation

$$u(t) = p(t) + \int_0^T \lambda(t,r)g(r,u(r))dr, \quad t \in [0,T] \tag{17}$$

where $T > 0$. The purpose of this section is to give an existence theorem for a solution of (17) that belongs to $G = C(I, R)$ (the set of continuous real functions defined on $I = [0, T]$, by using the obtained result in Corollary 3. Obviously, this space with the b-metric given by

$$\rho(u, v) = \max_{t \in I} \mid u(t) - v(t) \mid^p$$

for all $u, v \in G$ is a complete b-metric space with $s = 2^{p-1}$ and $p \geq 1$.

We will consider (17) under the following assumptions:

(i) $g, p : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous.
(ii) $\lambda : [0, T] \times \mathbb{R} \to [0, \infty)$ is continuous.
(iii) There exist $q > 0, 0 < a < \frac{1}{2^{2p-2}}$ such that for all $u, v \in G$,

$$\mid g(r,v) - g(r,u) \mid \leq a^{\frac{1}{p}} q \max_{t \in I}\{\mid v - u \mid\}.$$

(iv) $\max_{t \in I}(\int_0^T \mid \lambda(t,r) \mid dr)^p \leq \frac{1}{2^{2p-2}q^p}$.

Theorem 4. *Under assumptions (i)–(iv), (17) has a solution in G, where $G = C([0, T], \mathbb{R})$.*

Proof. We define $H : G \to G$ by

$$H(u(t)) = p(t) + \int_0^T \lambda(t,r)g(r,u(r))dr.$$

We have

$$
\begin{aligned}
2^{p-1} \mid Hu(t) - Hv(t) \mid^p &= 2^{p-1} \mid \int_0^T \lambda(t,r)[g(r,u(r)) - g(r,v(r))]dr \mid^p \\
&\leq 2^{p-1}(\int_0^T \mid \lambda(t,r)[g(r,u(r)) - g(r,v(r))] \mid dr)^p \\
&\leq 2^{p-1}aq^p(\max_{r\in I} \mid u(r) - v(r) \mid)^p(\int_0^T \mid \lambda(t,r) \mid dr)^p \\
&\leq 2^{p-1}q^p a\rho(u,v)\frac{1}{2^{2p-2}q^p} \\
&= \frac{1}{2^{p-1}}a\rho(u,v)
\end{aligned}
$$

Thus, from Corollary 3, by taking $\varepsilon = 1$, we deduce the existence of $u \in G$ such that $u = H(u)$. $\square$

5. Conclusions

This paper proposes a novel form of hybrid rational contraction in b-metric spaces and establishes the corresponding fixed point results. Additionally, the concept of S-compatibility is introduced to establish a common fixed point theorem for two pairs of self-mappings in b-metric spaces. Furthermore, the application of these main results to integral equations is explored to demonstrate the existence of solutions. Additionally, potential future research directions are suggested, including: (i) modifying or altering certain conditions in the main theorems, (ii) extending the results to other metric spaces such as fuzzy metric space [43], (iii) considering the uniqueness of solutions to integral equations, (iv) utilizing the main results and techniques to solve fractional differential equations [44,45], and (v) investigating common fixed points of more than four self-mappings, such as six or eight self-mappings. Especially, the possibility of obtaining five or seven self-mappings is also raised.

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