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# Optimal Control of SLBRS with Recovery Rates

**Xiangqing Zhao [1,*] and Wanmei Hou [2]**

[1] Department of Mathematics, Suqian University, Suqian 223800, China
[2] School of Marxism, Suqian University, Suqian 223800, China; 29008@squ.edu.cn
* Correspondence: 23139@squ.edu.cn

**Abstract:** In the information age, frequent information exchange has provided a breeding ground for the spread of computer viruses. The significant losses caused by computer virus attacks have long rung the alarm for information security. From academia to businesses, and even to government, everyone remains highly vigilant about information security. Researchers have put forward various approaches to combat computer viruses, involving innovative efforts in both the hardware and software aspects, as well as theoretical innovation and practical exploration. This article is dedicated to theoretical exploration, specifically investigating the stability of a computer virus model, known as SLBRS, from the perspective of optimal control. Firstly, a control system is introduced with the aim of minimizing the costs related to network detoxification and diminishing the percentage of computers impacted by the virus. Secondly, we employ the Pontryagin maximum principle to analyze the optimality of a control strategy for the proposed system. Thirdly, we validate the effectiveness of our theoretical analysis through numerical simulation. In conclusion, both theoretical analysis and numerical simulation reveal that the utilization of optimal control analysis to stabilize the SLBRS has been demonstrated to be advantageous in restoring contaminated computer network environments.

**Keywords:** computer virus; SLBRS; optimal control; Pontryagin principle; simulation

**MSC:** 49J15; 93-10; 93C15



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## 1. Introduction

In the information age, frequent information exchange has become an integral part of our daily lives, greatly facilitating the transmission of computer viruses in the cyber environment. The proliferation of network computer viruses has posed significant global information security threats, leading to substantial losses in various sectors, including finance, education, and energy.

From 1987 to 1988, F. Cohen and W. Murray discovered certain similarities between computer viruses and biological infectious diseases [1,2]. Consequently, they suggested applying the principles of infectious disease dynamics and qualitative and quantitative analysis methods to study the patterns of computer virus transmission. Unfortunately, they did not propose specific models for the spread of computer viruses at that time. It was not until 1991 that J. O. Kephart and S. R. White adopted the recommendations of F. Cohen and W. Murray [3]. Based on the similarity between computer viruses and biological viruses, they introduced the SIS (susceptible–infected–susceptible) computer virus propagation model for the first time, pioneering the application of biological virus propagation models to the field of computer viruses. Since then, extensive research has been conducted on computer viruses; a rough summary is provided in the Table 1 below.

**Table 1.** Early computer virus transmission models.

| Model | Year | Authors | Character | Reference |
|---|---|---|---|---|
| SIS | 1991 | Kephart, White | Susceptible, infected computers involved | [3] |
| SIR | 2001 | Tian, Zheng | Computers with permanent immunity | [4] |
| SIRS | 2004 | Chen, Carley | Computers with temporary immunity | [5] |
| SEIR | 2006 | Yuan, Chen | Computers in a dormant state | [6] |
| SEIRS | 2007 | Mishra, Saini | Computers with temporary immunity or in dormant state | [7] |
| SAIC | 2008 | Piqueira, Vasconcelos | The infected computers exhibit logarithmic growth | [8] |
| SAIR | 2009 | Piqueira, Araujo | Coexistence of multiple viruses | [9] |
| SEIQRS | 2010 | Mishra, Jha | Infected computers are isolated | [10] |

The computer virus models described above are based on a common assumption borrowed from epidemiological virus modeling, where an infected computer that remains in latency will not infect other computers. However, in the context of computer viruses, it is a whole different story:

**Difference 1**: Upon infection, a computer typically gains the immediate capability to propagate the infection.

**Difference 2**: Computers that have recovered may develop temporary immunity.

Considering the differences between computer viruses and biological viruses (**Differences 1–2**), Yang and Wen [11] and Yang, Zhang, and Li [12] proposed a mathematical model with characteristics of computer viruses known as SLBRS. They categorize the computers in the system into four groups, namely, S(t), L(t), B(t), and R(t):

**S(t)—susceptible computers:** computers not yet infected by the virus but susceptible to being infected by latent or outbreak computers, subsequently transitioning into latent computers.

**L(t)—latent computers**: computers that have been infected by the virus but do not exhibit apparent destructive behavior yet retain the potential to spread the infection.

**B(t)—outbreak computers:** computers that are infected by the virus, exhibiting apparent destructive effects and having the potential to spread the infection.

**R(t)—recovery computers**: computers that have been cleared of the virus by third-party security software or firewall products, possessing temporary immunity.

For the convenience of mathematical calculations, we assume that S(t), L(t), B(t), and R(t) represent the proportions of susceptible, latent, outbreak, and recovered computers in the network, respectively. Therefore, S(t), L(t), B(t), and R(t) satisfy the following normalization condition:

$$S(t) + L(t) + B(t) + R(t) = 1.$$

The transition relationships among the different states S(t), L(t), B(t), and R(t) are determined by the following assumptions:

**Assumption 1**: susceptible computers S(t) are infected with a virus at a certain rate $\beta S(L + B)$ and transform into latent computers L(t).

**Assumption 2**: latent computers L(t) transition into outbreak computers B(t) at an outbreak rate $\alpha$.

**Assumption 3**: outbreak computers B(t) are cured at an antivirus software recovery rate $\gamma_1$ and transform into recovery computers R(t).

**Assumption 4**: recovery computers R(t) lose immunity at a certain rate $\sigma$ and transform back into susceptible computers S(t).

**Assumption 5**: newly added computers are all susceptible computers S(t) with an access rate p, and each state of computer has an exit rate $\mu$.

Based on the above assumptions (**Assumptions 1–5**), the state transition relationships among computers in different states of SLBRS are illustrated in Figure 1:

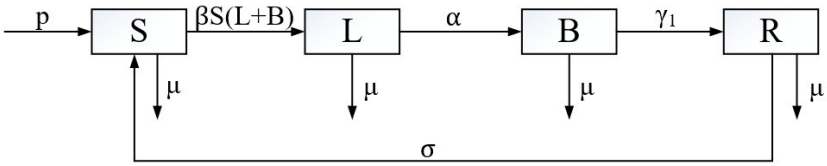


**Figure 1.** State transition diagram of SLBRS.

In reality, latent computers L(t) may recover due to users' virus prevention and control habits. Yang considered this factor in [13] and made the following additional assumption:

**Assumption 6:** latent computers L(t) are cured and transformed into recovery computers R(t) at a certain rate $\gamma_2$ due to third-party protective software or firewalls. Generally speaking, the possibility of clearing the virus by reinstalling the system is lower than using antivirus software, thus

$$\gamma_2 < \gamma_1.$$

Based on these assumptions (**Assumptions 1–6**), the state transition relationships among computers in different states of SLBRS can be illustrated in Figure 2:

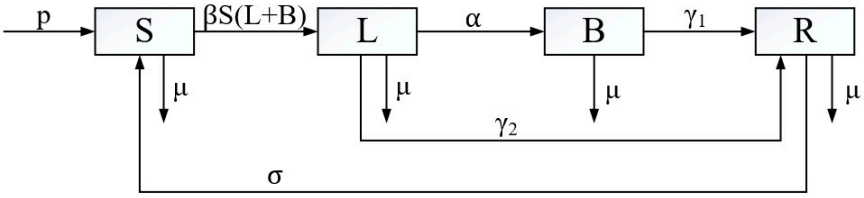


**Figure 2.** State transition diagram of SLBRS with graded recovery rates.

In consideration of various virus prevention and control measures that can be taken for outbreak computers B(t) in reality, the following assumption is further proposed:

**Assumption 7:** apart from using antivirus software, the virus can also be cleared by reinstalling the system, thereby transitioning the infected computer B(t) into a susceptible computer S(t) at a recovery rate of $\gamma_3$. Obviously, due to the numerical advantage of outbreak computers over latent computers, the probability of latent computers transitioning to the recovered state must be lower than the probability of outbreak computers transitioning to the recovered state. Therefore, the following assumption is also reasonable:

$$\gamma_3 < \gamma_2 < \gamma_1.$$

Under these assumptions (**Assumptions 1–7**), the state transition relationships of SLBRS are illustrated as follows:

Figure 3 can be formulated as:

$$\begin{cases} \frac{dS(t)}{dt} = p - \beta S(B+L) + \gamma_3 B + \sigma R - \mu S, \\ \frac{dL(t)}{dt} = \beta S(B+L) - \alpha L - \gamma_2 L - \mu L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_3 B - \mu B - \gamma_1 B, \\ \frac{dR(t)}{dt} = \gamma_2 L - \sigma R - \mu R + \gamma_1 B, \end{cases} \tag{1}$$

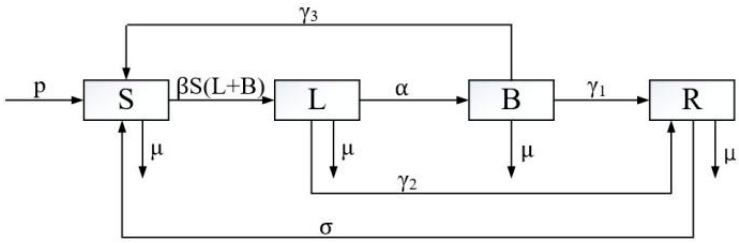


**Figure 3.** State transition diagram of SLBRS with constant recovery rates.

The fact is that the severity of computer virus attacks varies over time, and protective measures should be flexible and adaptable in response. Therefore, it is more reasonable to consider that recovery rates vary with time [14]. In this paper, as an example, the authors take the recovery rate $\gamma_1$ as a continuous function of time $\mu_1(t)$, representing the frequency of running antivirus software on outbreak computers at time t. Consequently, we propose SLBRS with variable recovery rates as follows:

Figure 4 can be reformulated by:

$$\begin{cases} \frac{dS(t)}{dt} = p - \beta S(B+L) + \gamma_3 B + \sigma R - \mu S, \\ \frac{dL(t)}{dt} = \beta S(B+L) - \alpha L - \gamma_2 L - \mu L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_3 B - \mu B - u_1(t)B, \\ \frac{dR(t)}{dt} = \gamma_2 L - \sigma R - \mu R + u_1(t)B. \end{cases} \tag{2}$$

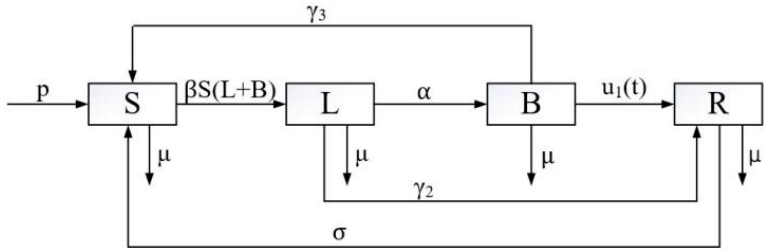


**Figure 4.** State transition diagram of SLBRS with time-variable recovery rate.

Following the introduction of the SLBRS model and its variations (1) and (2) in [4–13], the focus of analysis concerning them has primarily revolved around stability. This paper will delve further into studying the system's behavior from an optimal control perspective, aiming to explore the mechanisms of computer virus transmission and prevention. The organization of the remaining sections is as follows:

In Section 2, we will utilize the Hurwitz criterion to explore the stability conditions of both non-toxic and toxic equilibria of the SLBRS (1) and prove the stability of these two types of equilibrium.

In Section 3, we will focus on establishing the fundamental theoretical results concerning the optimal control of SLBRS. This will encompass the following key aspects: the existence of an optimal control strategy, the necessary conditions for optimal control, and the uniqueness of the optimal control system.

In Section 4, we will perform simulations to offer a numerical illustration of the practical implications of the theoretical discussions.

In Section 5, we will discuss the advantages of optimal control analysis in this paper and the necessity of multi-control input analysis.

In Section 6, we will conclude the paper by summarizing the progress made in the study of SLBRS.

## 2. The Stability of SLBRS

Before investigating the optimal control of SLBRS, let us first conduct a brief analysis of its stability. For the sake of convenience, under the normalization condition

$$S(t) + L(t) + B(t) + R(t) = 1,$$

the system of Equation (1) is transformed into:

$$\begin{cases} \frac{dL(t)}{dt} = \beta(1 - L - B - R)(L + B) - \alpha L - \gamma_2 L - \mu L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_1 B - \gamma_3 B - \mu B, \\ \frac{dR(t)}{dt} = \gamma_1 B + \gamma_2 L - \sigma R - \mu R, \end{cases} \tag{3}$$

### *2.1. Stability of the Non-Toxic Equilibrium*

According to the definition of a non-toxic equilibrium, there are no infected computers in the system. By (3), we deduced that L = 0, B = 0, and R = 0. Furthermore, utilizing the normalization condition, we obtain the non-toxic equilibrium $E_0 = (1,\ 0,\ 0,\ 0)$. We define the basic reproduction number as follows:

$$R_0 = \frac{[(\alpha + \gamma_2 + \mu - \beta)(\gamma_1 + \gamma_3 + \mu) - \beta\alpha](\sigma + \mu)}{\{(\alpha + \gamma_1 + \gamma_2 + \gamma_3 + \sigma + 3\mu - \beta) * [(\alpha + \gamma_1 + \gamma_2 + \gamma_3 + 2\mu - \beta)(\sigma + \mu) + (\alpha + \gamma_2 + \mu - \beta)(\gamma_1 + \gamma_3 + \mu) - \beta\alpha]\}}$$

**Theorem 1.** *For system (1), when $R_0 < 1$, the non-toxic equilibrium $E_0$ is locally asymptotically stable; when $R_0 > 1$, the non-toxic equilibrium $E_0$ is unstable.*

**Proof of Theorem 1.** The Hurwitz criterion is as follows: If the coefficients of characteristic equation

$$D(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_1 = 0,\ a_0 > 0.$$

of the Jacobian matrix of the linearized system are positive, and $a_1 a_2 - a_0 a_3 > 0$ then the linearized system is asymptotic stable. First, we linearize (3) at $E_0$. The Jacobian matrix of functions of the right-hand side of (3) is:

$$\begin{aligned} J_{(E_0)} &= \begin{bmatrix} \beta - 2\beta L - 2\beta B - \beta R - \alpha - \gamma_2 - \mu & \beta - 2\beta L - 2\beta B - \beta R & -\beta L - \beta B \\ \alpha & -\gamma_1 - \gamma_3 - \mu & 0 \\ \gamma_2 & \gamma_1 & -\sigma - \mu \end{bmatrix}_{(E_0)} \\ &= \begin{bmatrix} \beta - \alpha - \gamma_2 - \mu & \beta & 0 \\ \alpha & -\gamma_1 - \gamma_3 - \mu & 0 \\ \gamma_2 & \gamma_1 & -\sigma - \mu \end{bmatrix}, \end{aligned}$$

Then, the linearized equations of (3) at E0 are as follows:

$$\begin{cases} \frac{dL(t)}{dt} = (\beta - \alpha - \gamma_2 - \mu)L + \beta B, \\ \frac{dB(t)}{dt} = \alpha L - (\gamma_1 + \gamma_3 + \mu)B, \\ \frac{dR(t)}{dt} = \gamma_2 L + \gamma_1 B - (\sigma + \mu)R, \end{cases}$$

The characteristic equation of the Jacobian matrix $J_{(E_0)}$ is:

$$\begin{vmatrix} \lambda - \beta + \alpha + \gamma_2 + \mu & -\beta & 0 \\ -\alpha & \lambda + \gamma_1 + \gamma_3 + \mu & 0 \\ -\gamma_2 & -\gamma_1 & \lambda + \sigma + \mu \end{vmatrix} = 0,$$

Expressed as a polynomial, it is:

$$p_1(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_0 = 1,$$
$$a_1 = \alpha + \gamma_1 + \gamma_2 + \gamma_3 + \sigma + 3\mu - \beta,$$
$$a_2 = (\alpha + \gamma_1 + \gamma_2 + \gamma_3 + 2\mu - \beta)(\sigma + \mu) + (\alpha + \gamma_2 + \mu - \beta)(\gamma_1 + \gamma_3 + \mu) - \beta\alpha,$$
$$a_3 = [(\alpha + \gamma_2 + \mu - \beta)(\gamma_1 + \gamma_3 + \mu) - \beta\alpha](\sigma + \mu).$$

when $R_0 < 1$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_0a_3 > 0$. By the Hurwitz criterion, $E_0$ is locally asymptotically stable.

The proof is finished. □

*2.2. Stability of the Toxic Equilibrium*

According to the definition of a toxic equilibrium, there exist infected computers in the system. By (3), we deduced the toxic equilibrium

$$E_1 = (L_*, B_*, R_*),$$

where

$$L_* = \frac{(\gamma_1 + \gamma_3 + \mu)(\sigma + \mu)(\mu S_* - p)}{\sigma\gamma_2(\gamma_1 + \gamma_3 + \mu) + \alpha[(\sigma + \mu)\gamma_3 + \sigma\gamma_1] - \beta S_*(\gamma_1 + \gamma_3 + \alpha + \mu)(\sigma + \mu)},$$
$$B_* = \frac{\alpha L_*}{(\gamma_1 + \gamma_3 + \mu)},$$
$$R_* = \frac{[\alpha\gamma_1 + (\gamma_1 + \gamma_3 + \mu)\gamma_2]L_*}{(\gamma_1 + \gamma_3 + \mu)(\sigma + \mu)}.$$

Furthermore, by (1), we obtain

$$S_* = \frac{(\gamma_1 + \gamma_3 + \mu)(\alpha + \gamma_2 + \mu)}{\beta(\gamma_1 + \gamma_3 + \mu + \alpha)}.$$

We define the basic reproduction number as follows:

$$R_1 = \frac{\beta\alpha(L_* + B_*)\left[\frac{L_*}{B_*}(\sigma+\mu) + \gamma_1 + \frac{\gamma_2 L_*}{B_*} + (\sigma+\mu)\right]}{\left\{\left[\beta(L_* + B_* + \frac{S_* B_*}{L_*}) + (\gamma_1+\gamma_3+\sigma+2\mu)\right] * \left[(\beta(L_* + B_* + \frac{S_* B_*}{L_*}) + \frac{\alpha L_*}{B_*})(\sigma+\mu) + \beta(L_* + B_*)(\frac{\alpha L_*}{B_*} + \gamma_2 + \alpha)\right]\right\}}$$

**Theorem 2.** *For system (1), when $R_1 < 1$, the toxic equilibrium $E_1$ is locally asymptotically stable; when $R_1 > 1$, the toxic equilibrium is unstable.*

**Proof of Theorem 2.** First, we linearize (3) at $E_1$. Then, the Jacobian matrix linearized system is

$$J_{(E_0)} = \begin{bmatrix} \beta - 2\beta L - 2\beta B - \beta R - \alpha - \gamma_2 - \mu & \beta - 2\beta L - 2\beta B - \beta R & -\beta L - \beta B \\ \alpha & -\gamma_1 - \gamma_3 - \mu & 0 \\ \gamma_2 & \gamma_1 & -\sigma - \mu \end{bmatrix}_{(E_1)}$$
$$= \begin{bmatrix} -\beta(L_* + B_* - S_*) - \alpha - \gamma_2 - \mu & -\beta(L_* + B_* - S_*) & -\beta(L_* + B_*) \\ \alpha & -\gamma_1 - \gamma_3 - \mu & 0 \\ \gamma_2 & \gamma_1 & -\sigma - \mu \end{bmatrix},$$

We simplify the characteristic equation $|\lambda E - J| = 0$ to obtain

$$p_2(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_0 = 1,$$

$$a_1 = \beta(L_* + B_* + \frac{S_*B_*}{L_*}) + (\gamma_1 + \gamma_3 + \sigma + 2\mu),$$

$$a_2 = \left[\beta\left(L_* + B_* + \frac{S_*B_*}{L_*}\right) + \frac{\alpha L_*}{B_*}\right](\sigma + \mu) + \beta(L_* + B_*)\left(\frac{\alpha L_*}{B_*} + \gamma_2 + \alpha\right),$$

$$a_3 = \beta\alpha(L_* + B_*)\left[\frac{L_*}{B_*}(\sigma + \mu) + \gamma_1 + \frac{\gamma_2 L_*}{B_*} + (\sigma + \mu)\right].$$

When $R_1 < 1$,

$$\begin{aligned} R_1 &= \frac{\beta\alpha(L_* + B_*)\left[\frac{L_*}{B_*}(\sigma+\mu) + \gamma_1 + \frac{\gamma_2 L_*}{B_*} + (\sigma+\mu)\right]}{\left\{\left[\beta(L_* + B_* + \frac{S_*B_*}{L_*}) + (\gamma_1+\gamma_3+\sigma+2\mu)\right] * \left[(\beta(L_* + B_* + \frac{S_*B_*}{L_*}) + \frac{\alpha L_*}{B_*})(\sigma+\mu) + \beta(L_* + B_*)(\frac{\alpha L_*}{B_*} + \gamma_2 + \alpha)\right]\right\}} \\ &= \frac{a_3}{a_1 \cdot a_2} < 1, \end{aligned}$$

We have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_0a_3 > 0$. By the Hurwitz criterion, $E_1$ is locally asymptotically stable.

The proof is finished. □

## 3. Optimal Control of SLBRS

In the past, significant emphasis has been placed on mathematically modeling computer viruses and analyzing the stability of these models [3–13]. Comparatively, fewer studies have delved into the control strategies of computer virus models, as is evident in [14–17]. Nevertheless, these limited investigations have offered valuable insights to our understanding of restoration strategies of virus containment networks.

The focal point of this paper is to investigate the optimal control problem of SLBRS, aiming at diminishing the percentage of infected computers within a network and reducing network maintenance expenses.

### *3.1. The Formulation of the Optimal Control Problem*

Now, let us establish an optimal control model based on SLBRS. Firstly, in Section 1, we transformed the constant recovery rate $\gamma_1$ into a time-varying function $u_1(t)$. The state transition relationships of S(t), L(t), B(t), and R(t) are illustrated in Figure 4, and the relevant mathematical model can be found in (2). Secondly, we will select an appropriate control set. Thirdly, we will choose a suitable objective function based on the intention of control.

Let us assume T is a predefined time. We stipulate the allowable control set as follows:

$$U = \left\{u_1(t) \in L^2(0,\ T) :\ 0 \le u_1(t) \le 1,\ 0 \le t \le T\right\}. \tag{4}$$

Our goal is to reduce the prevalence of virus-infected computers within the network while minimizing the system's maintenance expenses. Therefore, the objective function could be formulated as follows:

$$J(u_1) = \int_0^T \left[B(t) + \frac{\varepsilon u_1^2(t)}{2}\right] dt, \tag{5}$$

where $\varepsilon$ is a positive constant.

Thus, the optimal control problem can be formulated as:

$$\min_{u_1 \in U} J(u_1)$$

Subject to the differential equations in (2).

In order to facilitate subsequent analysis of the problem and corresponding to the objective function, we introduce the following Lagrangian function:

$$L(B,\ u_1) = B(t) + \frac{\varepsilon u_1^2(t)}{2}$$

and the following Hamiltonian function:

$$\begin{aligned} H(t) = \quad & L(B,\ u_1) + \lambda_1[p - \beta S(L+B) + \gamma_3 B + \sigma R - \mu S] \\ & + \lambda_2[\beta S(L+B) - \alpha L - \gamma_2 L - \mu L] + \lambda_3[\alpha L - \gamma_3 B \\ & - \mu B - u_1(t)B] + \lambda_4[\gamma_2 L - \sigma R - \mu R + u_1(t)B]. \end{aligned} \tag{6}$$

*3.2. Optimal Control Results and Their Proofs*

In this section, we will establish the existence of an optimal control strategy, the necessary conditions for the existence of optimal control, and the uniqueness of the optimal control system. This will serve as the foundation for numerical simulations.

**Theorem 3.** *An optimal control input, denoted as* $u_1^*(t)$*, exists for the control system (5) with the given initial data:*

$$S(0) = S^0 \geq 0,\ L(0) = L^0 \geq 0,\ B(0) = B^0 \geq 0,\ R(0) = R^0 \geq 0,$$

such that

$$\min_{u_1 \in U} J(u_1) = J(u_1^*)$$

**Proof of Theorem 3.** According to [18], it is sufficient to verify the following four conditions:

**Condition 1.** $U = \left\{ u_1(t) \in L^2(0,\ T) :\ 0 \leq u_1(t) \leq 1,\ 0 \leq t \leq T \right\} \neq \Phi$;

**Condition 2.** $U = \left\{ u_1(t) \in L^2(0,\ T) :\ 0 \leq u_1(t) \leq 1,\ 0 \leq t \leq T \right\}$ is closed and convex set;

**Condition 3.** The right-hand side of state equations

$$\begin{cases} \frac{dS(t)}{dt} = p - \beta S(B+L) + \gamma_3 B + \sigma R - \mu S, \\ \frac{dL(t)}{dt} = \beta S(B+L) - \alpha L - \gamma_2 L - \mu L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_3 B - \mu B - u_1(t)B, \\ \frac{dR(t)}{dt} = \gamma_2 L - \sigma R - \mu R + u_1(t)B. \end{cases}$$

are restricted by linear functions in terms of $S,\ L,\ B,\ R$. We need only to show the condition for the first equation (the second is similar). Indeed, by the normalization condition

$$S(t) + L(t) + B(t) + R(t) = 1$$

we have $0 \leq B \leq 1$. Thus,

$$\begin{aligned} & p - \beta S(B+L) + \gamma_3 B + \sigma R - \mu S \\ & \leq p - \beta(L+B) + \gamma_3 B + \sigma R - \mu S \\ & = p - \beta L + (\gamma_3 - \beta)B + \sigma R - \mu S. \end{aligned}$$

**Condition 4.** The Lagrangian function $L(B,\ u_1)$ exhibits concavity over the set U, and there exists $\rho > 1$, $\eta_1 > 0$, and $\eta_2$ such that

$$L(B,\ u_1) \geq \eta_1(|u|)\rho + \eta_2.$$

The proof is completed. □

Next, through the utilization of the Pontryagin maximum principle, we will derive the essential condition for the optimal control input:

**Theorem 4.** *For the given optimal control input* $u_1^*(t)$ *and the related states* $S^*$, $L^*$, $B^*$, $R^*$ *of state Equation (2), there exist co-states* $\lambda_1$, $\lambda_2$, $\lambda_3$, *and* $\lambda_4$ *such that*

$$\frac{d\lambda_1}{dt} = \lambda_1[\beta(L+B)+\mu] - \lambda_2\beta(L+B), \tag{7}$$

$$\frac{d\lambda_2}{dt} = \lambda_1\beta S - \lambda_2(\beta S - \alpha - \gamma_2 - \mu) - \lambda_3\alpha - \lambda_4\gamma_2, \tag{8}$$

$$\frac{d\lambda_3}{dt} = -1 + \lambda_1(\beta S - \gamma_3) - \lambda_2\beta S + \lambda_3[\gamma_3 + \mu + u_1(t)] - \lambda_4 u_1(t), \tag{9}$$

$$\frac{d\lambda_4}{dt} = -\lambda_1\sigma + \lambda_4(\sigma+\mu), \tag{10}$$

with the transversal conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0. \tag{11}$$

The optimal control is as follows:

$$u_1^*(t) = \max\left\{\min\left\{\frac{\lambda_3-\lambda_4}{\varepsilon}B^*,\ 1\right\},\ 0\right\}.$$

**Proof of Theorem 4.** We differentiate the Hamiltonian function (6), and we obtain the following co-state system:

$$\frac{d\lambda_1}{dt} = -H_{S^*}(t), \frac{d\lambda_2}{dt} = -H_{L^*}(t), \frac{d\lambda_3}{dt} = -H_{B^*}(t), \frac{d\lambda_4}{dt} = -H_{R^*}(t),$$

which implies (7)–(10).

We deduce from the optimal conditions

$$\left.\frac{\partial H}{\partial u_1}\right|_{u_1(t)=u_1^*(t)} = \varepsilon u_1(t) - \lambda_3 B + \lambda_4 B.$$

and the admissible condition

$$U = \left\{u_1(t) \in L^2(0,\ T) :\ 0 \le u_1(t) \le 1,\ 0 \le t \le T\right\}$$

that

$$u_1^*(t) = \max\left\{\min\left\{\frac{\lambda_3-\lambda_4}{\varepsilon}B^*,\ 1\right\},\ 0\right\}.$$

By assembling the state equations of (2), co-state Equations (7)–(10), and the transversal conditions of (11), we derive the optimal system as follows:

$$\begin{cases} \frac{dS(t)}{dt} = p - \beta S(B+L) + \gamma_3 B + \sigma R - \mu S, \\ \frac{dL(t)}{dt} = \beta S(B+L) - \alpha L - \gamma_2 L - \mu L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_3 B - \mu B - u_1(t)B, \\ \frac{dR(t)}{dt} = \gamma_2 L - \sigma R - \mu R + u_1(t)B, \end{cases} \tag{12}$$

and

$$\begin{cases} \frac{d\lambda_1}{dt} = \lambda_1[\beta(L+B)+\mu] - \lambda_2\beta(L+B), \\ \frac{d\lambda_2}{dt} = \lambda_1\beta S - \lambda_2(\beta S - \alpha - \gamma_2 - \mu) - \lambda_3\alpha - \lambda_4\gamma_2, \\ \frac{d\lambda_3}{dt} = -1 + \lambda_1(\beta S - \gamma_3) - \lambda_2\beta S + \lambda_3[\gamma_3 + \mu + u_1(t)] - \lambda_4 u_1(t), \\ \frac{d\lambda_4}{dt} = -\lambda_1\sigma + \lambda_4(\sigma+\mu), \end{cases} \tag{13}$$

with initial values

$$S^0 \geq 0,\ L^0 \geq 0,\ B^0 \geq 0,\ R^0 \geq 0 \tag{14}$$

and the transversal conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0. \tag{15}$$

The proof is completed. □

Finally, we show the uniqueness of the optimal system (12)–(15):

**Theorem 5.** *Given control time T, the solution of the optimal system (12)–(15) is unique.*

**Proof of Theorem 5.** Assume that both (S, L, B, R; $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$) and ($\overline{S}$, $\overline{L}$, $\overline{B}$, $\overline{R}$; $\overline{\lambda}_1$, $\overline{\lambda}_2$, $\overline{\lambda}_3$, $\overline{\lambda}_4$) are solutions of (12)–(15). Let

$$S = e^{\lambda t}a,\ L = e^{\lambda t}b,\ B = e^{\lambda t}c,\ R = e^{\lambda t}d;$$

$$\lambda_1 = e^{-\lambda t}w,\ \lambda_2 = e^{-\lambda t}x,\ \lambda_3 = e^{-\lambda t}y,\ \lambda_4 = e^{-\lambda t}z$$

and

$$\overline{S} = e^{\lambda t}\overline{a},\ \overline{L} = e^{\lambda t}\overline{b},\ \overline{B} = e^{\lambda t}\overline{c},\ \overline{R} = e^{\lambda t}\overline{d};$$

$$\overline{\lambda}_1 = e^{-\lambda t}\overline{w},\ \overline{\lambda}_2 = e^{-\lambda t}\overline{x},\ \overline{\lambda}_3 = e^{-\lambda t}\overline{y},\ \overline{\lambda}_4 = e^{-\lambda t}\overline{z},$$

where $\lambda$ is a constant that will be determined later.

From (11), we obtain

$$u_1(t) = \max\left\{\min\left\{\frac{(y-z)c}{\varepsilon},\ 1\right\},\ 0\right\},$$

$$\overline{u}_1(t) = \max\left\{\min\left\{\frac{(\overline{y}-\overline{z})\overline{d}}{\varepsilon},\ 1\right\},\ 0\right\}.$$

From (12), we obtain

$$\lambda e^{\lambda t}a + e^{\lambda t}a' = p - \beta e^{2\lambda t}a(b+c) + \gamma_3 e^{\lambda t}c + \sigma e^{\lambda t}d - \mu e^{\lambda t}a \tag{16}$$

and

$$\lambda e^{\lambda t}\overline{a} + e^{\lambda t}\overline{a'} = p - \beta e^{2\lambda t}\overline{a}(\overline{b}+\overline{c}) + \gamma_3 e^{\lambda t}\overline{c} + \sigma e^{\lambda t}\overline{d} - \mu e^{\lambda t}\overline{a}. \tag{17}$$

From (13), we obtain

$$w' - \lambda w = (w+x)\beta e^{\lambda t}(b+c) + w\mu,$$

and

$$\overline{w'} - \lambda\overline{w} = (\overline{w}+\overline{x})\beta e^{\lambda t}(\overline{b}+c) + \overline{w}\mu,$$

From (16) and (17), we obtain

$$\begin{aligned} \lambda(a-\overline{a}) + (a'-\overline{a'}) = \quad & -\beta e^{\lambda t}[a(b+c) - \overline{a}(\overline{b}+\overline{c})] \\ & +\gamma_3(\gamma-\overline{\gamma}) + \sigma(d-\overline{d}) - \mu(a-\overline{a}), \end{aligned}$$

We then integrate from 0 to T, and we obtain

$$\begin{aligned}
&\tfrac{1}{2}(a(T)-\bar{a}(T))2+\lambda\int_0^T(a-\bar{a})^2dt\\
&=-\beta\int_0^T e^{\lambda t}[a(b+c)-\bar{a}(\bar{b}+\bar{c})](a-\bar{a})dt+\gamma_3\int_0^T(\gamma-\bar{\gamma})(a-\bar{a})dt+\sigma\int_0^T(d-\bar{d})(a-\bar{a})dt-\mu\int_0^T(a-\bar{a})^2dt\\
&=-\beta\int_0^T e^{\lambda t}(a\bar{b}-ab+\bar{a}c-ac)(\bar{a}-a)dt+\gamma_3\int_0^T(\bar{\gamma}-\gamma)(\bar{a}-a)dt\\
&\quad+\sigma\int_0^T(\bar{d}-d)(\bar{a}-a)dt-\mu\int_0^T(\bar{a}-a)^2dt\\
&\le C_1e^{\lambda t}\int_0^T\left[(\bar{a}-a)^2+(\bar{b}-b)^2+(d-c)^2\right]dt+C_2\int_0^T(\bar{a}-a)^2+(\bar{b}-b)^2+(\bar{c}-c)^2+(\bar{d}-d)^2\\
&\quad+(w-\bar{w})^2+(x-\bar{x})^2+(y-\bar{y})^2+(z-\bar{z})^2dt
\end{aligned}$$

where $C_1$ and $C_2$ are constants.

Similarly, we estimate the following:

$$\frac{1}{2}(b-\bar{b})^2(T),\ \frac{1}{2}(c-\bar{c})^2(T),\ \frac{1}{2}(d-\bar{d})^2(T).$$

Finally, we obtain

$$\begin{aligned}
&\tfrac{1}{2}(\bar{a}-a)^2(T)+\tfrac{1}{2}(\bar{b}-b)^2(T)+\tfrac{1}{2}(\bar{c}-c)^2(T)+\tfrac{1}{2}(\bar{d}-d)^2(T)+\tfrac{1}{2}(\bar{w}-w)^2(0)+\tfrac{1}{2}(\bar{x}-x)^2(0)+\tfrac{1}{2}(\bar{y}-y)^2(0)\\
&+\tfrac{1}{2}(z-\bar{z})^2(0)+\lambda\int_0^T\left[(\bar{a}-a)^2+(\bar{b}-b)^2+(\bar{c}-c)^2+(\bar{d}-d)^2\right]dt+\lambda\int_0^T\left[(\bar{x}-x)^2+(\bar{y}-y)^2+(\bar{z}-z)^2+(\bar{w}-w)^2\right]dt\\
&\le(C_3+C_4^{3\lambda t})\int_0^T\left[(\bar{a}-a)^2+(\bar{b}-b)^2+(\bar{c}-c)^2+(\bar{d}-d)^2\right]dt+(C_3+C_4^{3\lambda t})\int_0^T\left[(\bar{w}-w)^2+(\bar{x}-x)^2+(\bar{y}-y)^2+(\bar{z}-z)^2\right]dt.
\end{aligned}$$

Taking $\lambda$ such that

$$\lambda>(C_3+C_4e^{3\lambda T})$$

and

$$T<\frac{1}{3\lambda}\ln(\frac{\lambda-C_3}{C_4})$$

then

$$\begin{aligned}
&(\lambda-(C_3+C_4e^{3\lambda T}))\int_0^T\left[(\bar{a}-a)^2+(\bar{b}-b)^2+(c-\bar{d})^2+(\bar{d}-d)^2\right]dt\\
&+(\lambda-(C_3+C_4e^{3\lambda T}))\int_0^T\left[(\bar{x}-x)^2+(\bar{y}-y)^2+(\bar{z}-z)^2+(\bar{w}-w)^2\right]dt\\
&\le 0,
\end{aligned}$$

which implies that

$$a=\bar{a},b=\bar{b},c=\bar{c},d=\bar{d};$$

and

$$x=\bar{x},y=\bar{y},z=\bar{z},w=\bar{w}.$$

Thus,

$$S=\bar{S},L=\bar{L},B=\bar{B},R=\bar{R};$$

$$\lambda_1=\bar{\lambda}_1,\ \lambda_2=\bar{\lambda}_2,\ \lambda_3=\bar{\lambda}_3,\ \lambda_4=\bar{\lambda}_4.$$

The proof is completed. □

**Remark 1.** *The benefit of employing Pontryagin's maximum principle to prove Theorem 2 is that, alongside completing the proof, it sets the stage for our subsequent numerical simulations. In fact, the state equation, co-state equation, transversality condition, and initial values generated during the proof process serve as the very foundation for our upcoming numerical simulations.*

## 4. Numerical Simulation

In this section, we will conduct numerical simulations to examine the stability of (1) and the controllability of (2) separately. To facilitate a comparative analysis, we will use common initial values and parameter values for models (1) and (2) in the following section.

We take the initial value as follows:

$$S^0 = 0.4,\ L^0 = 0.3,\ B^0 = 0.2,\ R^0 = 0.1.$$

The parameter selection principle abides by the conditions specified in Theorems 1 and 2, ensuring that the basic reproduction number is less than 1. For detailed parameter selection, refer to Table 2.

**Table 2.** Values of parameters.

| Parameters | Values | Parameters | Values |
|---|---|---|---|
| p | 0.10 | $\varepsilon$ | 2.00 |
| $\alpha$ | 0.60 | $\beta$ | 0.90 |
| $\gamma_2$ | 0.10 | $\gamma_1$ | 0.15 |
| $\sigma$ | 0.05 | $\gamma_3$ | 0.05 |
| $\mu$ | 0.10 | | |

### *4.1. Stability of SLBRS*

Stability simulation is a relatively straightforward process. By utilizing the provided parameters and initial values, we write a simple code employing ODE45 and execute it. The outcomes of the simulation are illustrated in Figures 5 and 6.

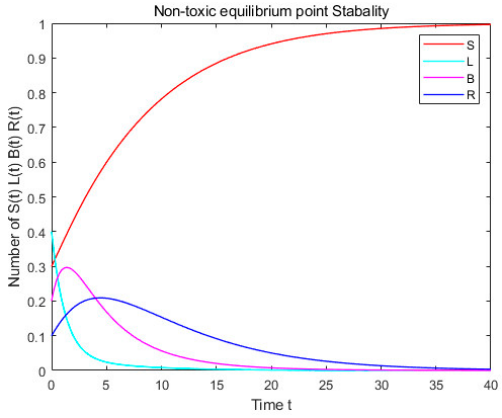


**Figure 5.** Stability of non-toxic equilibrium.

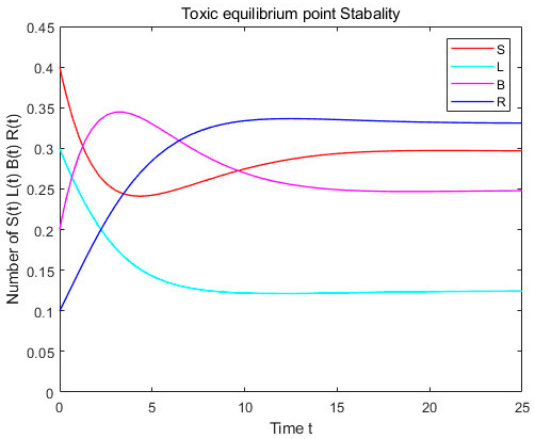


**Figure 6.** Stability of toxic equilibrium.

Figure 5 demonstrates that all the states S(t), L(t), B(t), and R(t) asymptotically stabilize to the non-toxic equilibrium, in line with the conclusion of Theorem 1. Similarly, from Figure 6, it is evident that all the states S(t), L(t), B(t), and R(t) asymptotically stabilize to the toxic equilibrium, consistent with the conclusion of Theorem 2.

### *4.2. Controllability of SLBRS*

The simulation of optimal control is quite complex. On one hand, there are many equations that need to be solved (in fact, the optimal controllability simulation algorithm involves solving the state equations, co-state equations, and control input equation). On the other hand, for ensuring simulation accuracy, it is not only necessary to employ high-order differences but also to utilize a combination of forward and backward differences.

For clarity, let us briefly outline the algorithm:

**Step 1. Initialization**: determine the time step, set iteration termination conditions, and initialize the state, co-state, and control variables.

**Step 2. Iteration Process:** we solve the state equations using forward differences and the co-state equations using backward differences.

**Step 2.1**. **State Equation Solution**: use fourth-order forward Runge-Kutta difference to solve the state equation based on known control and initial value conditions.

**Step 2.2. Co-state Equation Solution:** employ fourth-order backward Runge-Kutta difference, solving the co-state equation backward from the final time.

**Step 2.3. Control Update**: update control variables based on the results of the state and co-state equations.

**Step 2.4. Iterative Update**: based on the updated control, repeat the solution of state and co-state equations until the termination conditions are met.

**Step 3. Simulation Results**: obtain the optimal control strategy, state trajectory, and other necessary numerical results.

The evolution trends of types S(t), L(t), B(t), and R(t) are presented in Figures 7–10. The simulation results align well with the theoretical analysis results (Theorems 3–5).

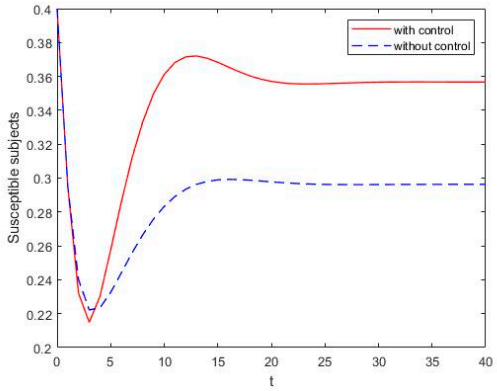


**Figure 7.** Evolution of susceptible computer S(t).

Figures 7 and 9 demonstrate that, under the effect of control, the proportion of uninfected computers (S(t) and R(t)) gradually increases and stabilizes. This indicates the restoration of the virus-contaminated network environment.

Figures 8 and 10 reveal that, under the influence of control, the proportion of infected computers (L(t) and B(t)) gradually decreases and stabilizes. This, from another perspective, reflects the ongoing restoration of the virus-infected network environment.

The evolution of control input is as follows.

The trend of the optimal control curve is consistent with reality (see Figure 11): Initially, the computer virus attack intensity is high, and the level of protection is correspondingly strong. However, as time passes, the intensity of the attack weakens, and the level of protection decreases in parallel.

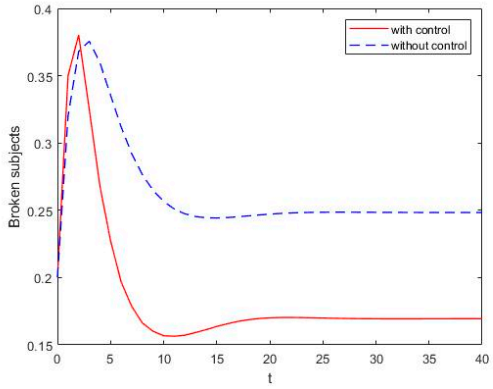


**Figure 8.** Evolution of outbreak computer B(t).

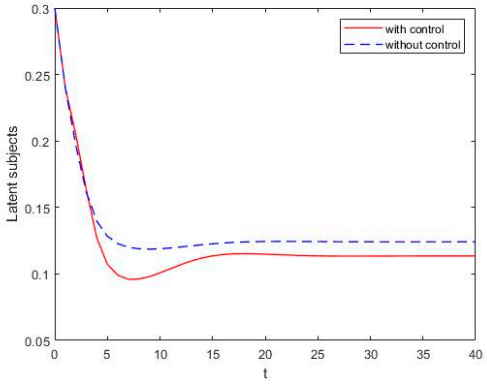


**Figure 9.** Evolution of latent computer L(t).

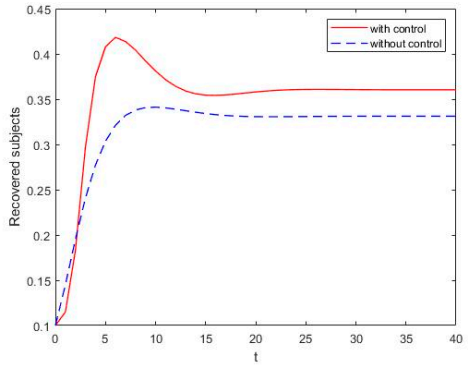


**Figure 10.** Evolution of recovered computer R(t).

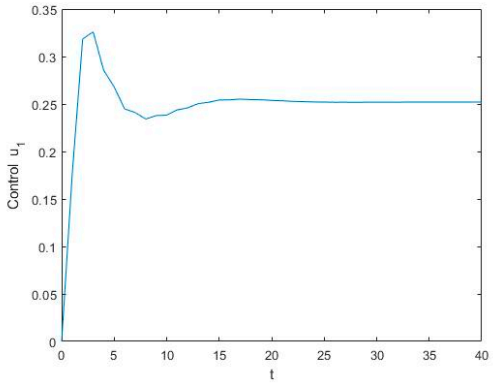


**Figure 11.** Optimal control function $u_1^*(t)$.

**Remark 2.** *We use the forward Runge-Kutta method to solve the state equations and the backward Runge-Kutta method to solve the co-state equations, and the simulation results are superior to simply using either forward or backward Runge-Kutta methods for solving them.*

## 5. Discussion

At this point in the paper, let us address a few issues for discussion.

Firstly, is it necessary to introduce control to SLBRS?

To address this question, we directly compare the stable values of SLBRS with and without control input (Equations (1) and (2)). Utilizing the same parameters and initial states, we calculate the stable values of each state variable in (1) and (2) (see Table 3 for details).

**Table 3.** The comparison of Equations (1) and (2).

| Model | S* | L* | B* |
|---|---|---|---|
| Equation (1) | 0.30 | 0.12 | 0.25 |
| Equation (2) | 0.36 | 0.11 | 0.17 |

Table 2 illustrates that toxicity-free nodes (S*, R*) increase, and toxic nodes (L*, B*) decrease. This observation strongly suggests that optimal control is beneficial for the restoration of a contaminated network.

Secondly, there are several issues regarding the selection of the optimal control input function:

(1) What if the recovery rate ($\gamma_2$ or $\gamma_3$) is used as the control input? It is similar to $\gamma_1$, so we omit the details.

(2) What if more than one recovery rate is utilized in the control inputs? The fact is that the more control inputs are employed, the greater the ability to control the system.

Thirdly, there are questions regarding model improvement.

The selected model in this paper does not account for time-delay factors. However, in reality, the transformation from latent to active computers and the recovery of virus-infected computers both take a certain amount of time. Therefore, in future work, consideration of time-delay factors can be introduced to establish an optimal computer virus control model with time-delay factors. This is expected to yield results that are more practically meaningful. Please refer to [15] for further details.

This paper employs optimal control theory to study the SLBRS model and has obtained research conclusions that align with expectations. Introducing current popular research directions such as stochastic control and adaptive control into the study of SLBRS would undoubtedly open up new possibilities. For more details, please refer to the latest literature [19–24].

## 6. Conclusions

In this paper, we have introduced the SLBRS computer virus model with triple recovery rates. Subsequently, we have investigated its stability through both linearization and optimal control. The primary findings are as follows.

Firstly, using the Hurwitz criterion, we have demonstrated the stability of both the non-toxic equilibrium point and the toxic equilibrium point. Furthermore, we have validated these findings through simulation.

Secondly, the existence and uniqueness of the optimal solution have been rigorously established and confirmed through the application of the Pontryagin maximum principle.

Thirdly, for numerical simulations, we have employed an iterative algorithm. The results of these simulations illustrate that the optimal control strategy can effectively minimize the outbreak of a virus in the network, all the while reducing network maintenance costs.

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