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Two Approaches to Estimate the Shapley Value for Convex Partially Defined Games

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Abstract: In the classical approach of von Neumann and Morgenstern to cooperative games, it was assumed that the worth of all coalitions must be given. However, in real-world problems, the worth of some coalitions may be unknown. Therefore, in this study, we consider the Shapley value for convex partially defined games using two approaches. Firstly, we introduce a polytope that includes the set of Shapley values that can be obtained from a given convex partially defined game and select one rational value in some sense from the set. The elements of this polytope are said to be the Shapley payoff vectors. Secondly, we obtain the set of Shapley values that can be obtained from a given convex partially defined game and select one rational value in some sense from the set. Moreover, we axiomatize the proposed two values.

Keywords: cooperative game; partially defined game; Shapley value

MSC: 91A12

1. Introduction

Cooperative game theory provides useful a tool to analyze various cost and/or surplus allocation problems, the distribution of voting power in a parliament, and so on. This theory is employed to analyze problems that involve n entities called players which are usually expressed by characteristic functions that map each subset of players to a real number. The solutions are given by a set of n-dimensional real numbers or value functions that assign a real number to each player. Such a real number can represent the cost borne by the player, the power of influence, allocation of shared profits, and so on. Several solution concepts for cooperative games have been proposed. Representative examples of solution concepts are the core, the Shapley value [1], and the nucleolus [2]. The core can be represented by a set of solutions, while the Shapley value and the nucleolus are one-point solutions. A partially defined game (a PDG, in short) is a cooperative game in which the worth of some coalitions is unknown.

Willson [3] first considered partially defined cooperative games. He proposed a generalized Shapley value obtained by using only the known coalitional worth of a game and then axiomatized the proposed Shapley value. Continuing the study of Willson [3], Housman [4] characterized the generalized Shapley value. However, the generalized Shapley value coincides with the ordinary Shapley value of a game whose coalitional worth is zero if it is unknown and given values otherwise, which seems to be not well justified. Usually, such a game dissatisfies natural properties such as superadditivity.

Masuya and Inuiguchi [5] defined full games as the lower and the upper games for given PDGs. Assuming that the PDG is superadditive, they showed that the lower game is superadditive, but the upper game need not be superadditive.

Moreover, Yu [6] studied a cooperative game with a coalition structure under limited feasible coalitions. Here, PDGs with a coalition structure were considered to develop and axiomatize the Owen value [7] for such games. The Owen value is an extension of the

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Shapley value for cooperative games with a coalition structure. Černý [8] investigated various solutions for PDGs, such as the Shapley value, the nucleolus, and the core. On the other hand, Albizuri et al. [9] considered an extended Shapley value for PDGs whose definition is based on the Harsanyi's dividends approach [10].

The volume of research papers addressing the axiomatization of the Shapley value for PDGs is relatively small. In particular, there are no papers that deal with axiomatizations of a value for convex PDGs. Furthermore, to the best of our knowledge, there are no studies that explore the computational aspects such as running time analyses or upper and lower bound analyses of solutions pertaining to convex PDGs.

Recently, XAI under deep neural network architectures has been studied by many researchers as an application of the Shapley value. As representative examples, we refer to Ancona et al. [11] and Chen et al. [12]. Since explaining the architecture is complex from a computational view point, we believe that the theory of partially defined games is useful in such problems.

In this study, we consider the Shapley value for convex PDGs using two approaches. First, a polytope that includes the set of Shapley values obtained from a given convex PDG is introduced. The elements of this polytope are said to be Shapley payoff vectors. Second, the property of the Shapley value, which is the gravity center of the core if the game is convex, is used.

It is crucial to propose a selection method of the one-point solution from the set of Shapley values that can be obtained from a PDG. Therefore, we propose selection methods of one value from the set of Shapley values and the set of Shapley payoff vectors, and axiomatize the proposed values.

This paper is organized as follows. In Section 2, we introduce a PDG and well-known solution concepts. In Section 3, we introduce the set of Shapley payoff vectors that includes the set of Shapley values obtained from a convex PDG. In Section 4, we investigate the set of marginal vectors for each permutation of players obtained from a given convex PDG. In Section 5, we axiomatize the proposed value. The axiom system consists of six axioms. In Section 6, concluding remarks are given.

2. Partially Defined Cooperative Games and the Shapley Value

Let $N = \{1, 2, \ldots, n\}$ be the set of players and $v\colon 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. A classical cooperative game is a coalitional game with transferable utility (a TU game) characterized by a pair (N, v). A set $S \subseteq N$ is regarded as a coalition of players, and the number $v(S)$ represents a collective payoff that players in S can gain by forming a coalition S. For an arbitrary coalition S, the number $v(S)$ is called the worth of coalition S.

A game (N, v) is superadditive if and only if

$$v(S \cup T) \geq v(S) + v(T),\ \forall S,\ T \subseteq N \text{ such that } S \cap T = \emptyset. \tag{1}$$

Superadditivity is a natural property that gives each player an incentive to form a larger coalition.

A game (N, v) is convex if and only if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T),\ \forall S,\ T \subseteq N. \tag{2}$$

The convexity can also be characterized by

$$v(T \cup i) - v(T) \geq v(S \cup i) - v(S),\ \forall S \subseteq T \subseteq N \setminus i,\ \forall i \in N, \tag{3}$$

where $S \cup i$ denotes $S \cup \{i\}$ for the sake of simplicity. This implies that the marginal contribution of a player to a coalition is nondecreasing as the coalition enlarges in the sense of set-theoretic inclusion if and only if the cooperative game is convex.

In cooperative games, the grand coalition N is assumed to be formed. The problem is how to allocate the collective payoff $v(N)$ to all players. A solution is a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where each component $x_i \in \mathbb{R}$ represents the payoff to player i.

A solution x is efficient in a game (N, v) if $\sum_{i \in N} x_i = v(N)$. A set of requirements $x_i \geq v(\{i\}), \ \forall i \in N$, is called individual rationality. Let $I(N, v)$ denote the set of payoff vectors that satisfy efficiency and individual rationality in (N, v), or "imputations".

Many solution concepts have been proposed. We introduce the Shapley value.

The Shapley value is a one-point solution concept. Let $G(N)$ be the set of all cooperative games with the player set N. For convenience, a cooperative game (N, v) is denoted simply by v because the set of players is fixed as N.

The Shapley value ϕ is defined by

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} (v(S_i^\pi) - v(S_i^\pi \setminus i)), \ \forall i \in N, \tag{4}$$

where $\Pi(N)$ is the set of permutations of N and S_i^π is the set of players that are the predecessors of i, including i under permutation π.

In the Shapley value, the term $v(S_i^\pi) - v(S_i^\pi \setminus i)$ is the marginal contribution of player i to coalition S_i^π. Let $\pi = (\pi(1), \ldots, \pi(n))$ be a permutation of players. Then, the n-dimensional real number $m^\pi(v)$ is defined as follows:

$$m^\pi(v) = (v(S_1^\pi) - v(S_1^\pi \setminus 1), v(S_2^\pi) - v(S_2^\pi \setminus 2), \ldots, v(S_n^\pi) - v(S_n^\pi \setminus n)). \tag{5}$$

That is, $m_i^\pi(v)$ is the marginal contribution of player i in permutation π on game v. Vectors $m^\pi(v)$ for all $\pi \in \Pi(N)$ are called marginal vectors. The convex hull of all the marginal vectors is called the Weber set [13]. The Weber set coincides with the core of v if it is convex, and then the Shapley value $\phi(v)$ is its gravity center.

Now, we present the definition of PDGs by [5]. In classical cooperative games, we assume that the worth of all coalitions is known. However, in real-world problems, the worth of some coalitions may be unknown. For the sake of simplicity, we call such a game a PDG, and a conventional game a full game. A PDG can be characterized by a set of players $N = \{1, 2, \ldots, n\}$, a set of coalitions whose worth is known, say $\mathcal{K} \subseteq 2^N$, and a function $v : \mathcal{K} \to \mathbb{R}$, with $\emptyset \in \mathcal{K}$ and $v(\emptyset) = 0$. We assume that the worth of singleton coalitions and the grand coalition is known and the worth of singleton coalitions is nonnegative, i.e., $\{i\} \in \mathcal{K}$ and $v(\{i\}) \geq 0$ for all $i \in N$ and $N \in \mathcal{K}$.

Moreover, we assume that v is superadditive in the following sense:

$$\begin{aligned} v(S) \quad &\geq \quad \sum_{i=1}^{s} v(T_i), \ \forall S, T_i \in \mathcal{K}, \ i = 1, 2, \ldots, s \text{ such that } \bigcup_{i=1,2,\ldots,s} T_i = S \\ &\qquad \text{and } T_i, \, i = 1, 2, \ldots, s \text{ are disjoint.} \end{aligned} \tag{6}$$

A triple $(N, \mathcal{K}, v)$ identifies a PDG. When we consider only games under a fixed N and $\mathcal{K}$, a PDG $(N, \mathcal{K}, v)$ is simply written as v. Given a PDG $(N, \mathcal{K}, v)$, we define two associated full games $(N, \underline{v})$ and $(N, \overline{v})$:

$$\underline{v}(S) = \max_{\substack{T_i \in \mathcal{K}, \ i=1,2,\ldots,s \\ \cup_i T_i = S, \ T_i \text{ are disjoint}}} \sum_{i=1}^{s} v(T_i), \tag{7}$$

$$\overline{v}(S) = \min_{\hat{S} \in \mathcal{K}, \ \hat{S} \supseteq S} \left(v(\hat{S}) - \underline{v}(\hat{S} \setminus S) \right) \tag{8}$$

As shown in the following theorem, $\underline{v}(S)$ is the minimal payoff of coalition S among superadditive full games that can be obtained from $(N, \mathcal{K}, v)$. On the other hand, $\overline{v}(S)$ is the maximal payoff of coalition S among superadditive full games that can be obtained from $(N, \mathcal{K}, v)$.

Theorem 1 ([5]). *Let $(N, \mathcal{K}, v)$ be a PDG, and $(N, \underline{v})$ and $(N, \overline{v})$ be the full games in (7) and (8). For an arbitrary superadditive full game (N, w) of $(N, \mathcal{K}, v)$, we obtain*

$$\underline{v}(S) \leq w(S), \ \forall S \subseteq N, \tag{9}$$
$$\overline{v}(S) \geq w(S), \ \forall S \subseteq N. \tag{10}$$

Therefore, full games $(N, \underline{v})$ and $(N, \overline{v})$ are called a "lower game" and an "upper game" associated with $(N, \mathcal{K}, v)$, respectively. When there is no confusion about the underlying PDG, these games are simply called the lower game and the upper game.

In this study, we investigate the set of payoff vectors of convex PDGs that can be obtained from a given convex PDG $(N, \mathcal{K}, v)$. As it is difficult to investigate the set of payoff vectors for general $(N, \mathcal{K}, v)$, we assume a property of $\mathcal{K}$ that is defined as follows:

$$\mathcal{K} = \{S \subseteq N \mid |S| \leq k\} \cup \{N\}, \text{ for some } 1 \leq k \leq n-1. \tag{11}$$

We can establish $k = \max\{|S| \mid S \in \mathcal{K}, S \neq N\}$. Therefore, a PDG $(N, \mathcal{K}, v)$ satisfying Equation (11) is called an (N, k)-PDG. Thus, $1 \leq k \leq n-1$ holds. When $k = n-1$, $(N, \mathcal{K}, v)$ is a full game and when $k = 1$, the game is a PDG, where only the worth of the grand coalition and singleton coalitions is known. An (N, k)-PDG $(N, \mathcal{K}, v)$ could be written as (N, k, v). We provide the relation between $(N, k+l, v)$ $(l \geq 1)$ and (N, k, v) as follows:

$$(N, k+l, v)(S) = (N, k, v)(S) \text{ for all } S \subseteq N \text{ such that } |S| \leq k \text{ or } |S| = n. \tag{12}$$

An (N, k)-PDG v is convex in the following sense:

$$\begin{aligned} v(T \cup i) - v(T) \geq v(S \cup i) - v(S), \forall S \subseteq T \subseteq N \setminus i, \ \forall i \in N \\ \text{such that } |T| \leq k-1, \\ v(N) - v(R^* \cup i) \geq (n-k)\Big(v(R^* \cup i) - v(R^*)\Big) \text{ where } R^* \subseteq N \setminus i \text{ and } i \in N \\ \text{such that } v(R^* \cup i) - v(R^*) = \max_{i \in N, |R| = k-1} \Big(v(R \cup i) - v(R)\Big) \end{aligned}$$

An (N, k)-convex PDG satisfies the monotonicity of the marginal contribution of each player. Generally, the set of convex full games obtained from a PDG $(N, \mathcal{K}, v)$ is denoted as follows:

$$V(N, \mathcal{K}, v) = \{w : 2^N \to \mathbb{R} \mid w \text{ is convex}, w(S) = v(S), \ \forall S \in \mathcal{K}\}. \tag{13}$$

Let Γ^k be the set of (N, k)-convex PDGs and Γ be the set of (N, k)-convex PDGs, where k is not fixed. If there is no confusion, a PDG can be written as v or (N, k, v) instead of $(N, \mathcal{K}, v)$.

In this study, since we consider (N, k)-convex PDGs instead of general superadditive PDGs, the formulas of $\underline{v}$ of $v \in \Gamma^k$ are given as follows:

$$\underline{v}(S) = \begin{cases} \max_{R \subset S, |R| = |S|-1} \Big(\underline{v}(R) + \max_{T \subset R, |T| = k-1} (v(T \cup (S \setminus R)) - v(T))\Big), \\ \text{if } \Big(|S| \geq k+1 \text{ and } |S| \neq n\Big), \\ v(S), \text{ if } \Big(1 \leq |S| \leq k \text{ or } |S| = n\Big), \end{cases} \tag{14}$$

3. The Set of Shapley Payoff Vectors: A Relaxation of the Set of Shapley Values Obtained from a PDG

In this section, we define the set of Shapley payoff vectors as a relaxation of the set of Shapley values. Let $\Phi(v)$ be the set of Shapley values that can be obtained from $v \in \Gamma^k$.

Given an arbitrary player $i \in N$ and $v \in \Gamma^k$, let $\underline{\phi}_i(v) = \phi_i(\underline{v}_i)$ such that $\phi_i(\underline{v}_i) = \min_{w \in V(v)} \phi_i(w)$. Moreover, let $\phi^l(v) = (\phi^l_1(v), \ldots, \phi^l_n(v))$, where

$$\phi^l_i(v) = \begin{cases} \underline{\phi}_i(v), & \text{if } i \neq l, \\ v(N) - \sum_{k \in N \setminus i} \underline{\phi}_k(v), & \text{if } i = l, \end{cases} \tag{15}$$

where $l \in \{1, \ldots, n\}$. Then, we define the set of n-dimensional real numbers $\Psi(v) = ch(\phi^1(v), \ldots, \phi^n(v))$, where $ch(x_1, \ldots, x_n)$ is the convex hull of $\{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n \in \mathbb{R}^n$. $\Psi(v)$ includes $\Phi(v)$. Each element of $\Psi(v)$ is said to be a Shapley payoff vector of PDG $v \in \Gamma^k$. The closer a payoff vector becomes to the center of $\Psi(v)$, the more likely it is to be a Shapley value that can be obtained from v.

When a PDG is applied to real-world problems, a one-point solution needs to be selected from the set of payoff vectors $\Psi(v)$. Before proposing a solution, we define a null player in PDG v.

Definition 1 (Null Player)**.** *Let $v \in \Gamma^k$ and $i \in N$. Player i is said to be a null player in v if $v(S) - v(S \setminus i) = 0$ holds for all $S \subseteq N$ such that $|S| \leq k$.*

A null player is a player whose marginal contributions to any known coalitions are zero. A null player in a PDG is not necessarily a null player in a full game that can be obtained from the PDG. Even if the full game is the lower game or the upper game, they are not necessarily a null player. However, if $i \in N$ is a null player in a PDG v, then i is a null player in the full game $\underline{v}_i$.

Let $NP \subseteq N$ be the set of null players in PDG v. Now, we define a one-point solution that is an element of $\Psi(v)$. We define $\hat{\phi} : \Gamma \to \mathbb{R}^n$ as follows:

$$\hat{\phi}_i(v) = \begin{cases} \dfrac{1}{n - |NP|} \displaystyle\sum_{l \in N \setminus NP} \phi^l_i(v), & \text{if } i \in N \setminus NP, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

$\hat{\phi}$ is the average of all extreme points of $\Psi(v)$. Optimistically, as both $\Phi(v)$ and $\Psi(v)$ are polytopes and $\Psi(v)$ includes $\Phi(v)$, $\hat{\phi}(v)$ can be an approximation of the average of all extreme points of $\Phi(v)$.

Next, we axiomatize the proposed value $\hat{\phi}$ using six axioms. First, we define the concept of the symmetry of players.

Definition 2. *Let $v \in \Gamma^k$ and $i, j \in N$. If $v(S \cup i) = v(S \cup j)$ holds for all $S \subseteq N \setminus \{i, j\}$ such that $|S| \leq k - 1$, then i and j are said to be symmetric in v.*

Let $\sigma : \Gamma \to \mathbb{R}^n$. Notice that k is not fixed on the domain of σ. Let $v, w \in \Gamma^k$. We define $v + w$ as $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$ such that $|S| \leq k$ or $|S| = n$. The axioms are given as follows.

Axiom 1 (Efficiency)**.**

$$\sum_{i \in N} \sigma_i(v) = v(N). \tag{17}$$

Axiom 2 (Symmetry)**.** *Let $v \in \Gamma^{n-1}$ and $i, j \in N$ such that i and j are symmetric in v. Then, the following holds:*

$$\sigma_i(v) = \sigma_j(v). \tag{18}$$

Axiom 2 states that if two players are symmetric in a full game, then they are allocated the same amount of payoffs.

Axiom 3 (Additivity)**.** *Let $v_1, v_2 \in \Gamma^k$. If $|V(v_1)| = |V(v_2)| = 1$, then the following holds:*

$$\sigma(v_1 + v_2) = \sigma(v_1) + \sigma(v_2). \tag{19}$$

Axiom 3 is an adaptation of the well-known axiom of additivity to a solution for PDGs.

Definition 3 (Null Coalition)**.** *Let $v \in \Gamma^k$ and $S \subset N$ such that $1 \leq |N \setminus S| \leq k$. If $v(N) - v(N \setminus S) = 0$ holds, then S is said to be a null coalition in v.*

A null coalition is a coalition whose marginal contribution to N is zero if it can be obtained. The worth of all coalitions that form a subset of the null coalition S is zero.

Axiom 4 (Null Coalition)**.** *Let $v \in \Gamma^k$ and $S \subset N$ be a null coalition in v. Then, the following holds:*

$$\sigma_i(v) = 0, \text{ for all } i \in S. \tag{20}$$

Axiom 4 states that the players who belong to a null coalition are not allocated payoffs at all.

Axiom 5 (Aggregate Monotonicity)**.** *Let $v \in \Gamma^k$ and $\alpha \in \mathbb{R}$. Let $v^\alpha \in \Gamma^k$ such that:*

$$v^\alpha(S) = \begin{cases} v(N) + \alpha, & \text{if } S = N, \\ v(S), & \text{if } S \subset N \text{ and } S \leq k. \end{cases} \tag{21}$$

Then, $\sigma_i(v^\alpha) = \sigma_i(v) + \frac{\alpha}{n}$ for all $i \in N$.

Axiom 5 states that the same payoff is allocated to all players additionally to original allocated payoffs if the worth of the grand coalition increases.

Axiom 6 (Symmetric Difference)**.** *Let $(N, k, v) \in \Gamma^k$ and $i, j \in N$. If $v(S) - v(S \setminus i) = v(S) - v(S \setminus j)$ holds for all $S \subseteq N$ such that $S \supseteq \{i, j\}$ and $|S| \leq k$, the following holds:*

$$\sigma_i(N, k, v) - \sigma_i(N, k+1, v) = \sigma_j(N, k, v) - \sigma_j(N, k+1, v). \tag{22}$$

Axiom 6 states that if two players are equivalent with respect to their marginal contributions to $S \subseteq N$ such that $|S| \leq k$, their differences of values are the same if the number of known coalitions changes.

Theorem 2. *$\hat{\phi}$ is the unique function on Γ that satisfies Axioms 1– 6.*

Proof. First, we show that $\hat{\phi}$ satisfies Axioms 1–6. Clearly, $\hat{\phi}$ satisfies Axiom 2. Furthermore, we can easily verify that $\hat{\phi}$ satisfies Axiom 5.

We show that $\hat{\phi}$ satisfies Axiom 3. Let $v_1, v_2 \in \Gamma^k$, $|V(v_1)| = |V(v_2)| = 1$, and $i \in N$. Since v_1 is a convex game, $v_1(S) - v_1(S \setminus i) = v_1(T) - v_1(T \setminus i)$ holds for all $S \supseteq T$ such that $|S| \geq k + 1$, where $T \subset N$ such that $v_1(T) - v_1(T \setminus i)$ satisfying $T \ni i$ is the maximal marginal contribution of player i within known coalitions. Thus, the assumption $|V(v_1)| = 1$ ensures that

$$v_1(N) = v_1(T) + (n-k)(v_1(T) - v_1(T \setminus i)) \text{ for all } i \in T \text{ such that } |T| = k. \tag{23}$$

With respect to v_2, the same result holds. Let $v_2(T') - v_2(T' \setminus i)$ such that $T' \subseteq N \setminus i$ be the maximal marginal contribution of player i within known coalitions.

Then, the following holds:

$$(v_1 + v_2)(N) = v_1(T) + v_2(T') + (n-k)(v_1(T) + v_2(T') - v_1(T \setminus i) + v_2(T' \setminus i)). \tag{24}$$

Decreasing k one by one, all coalitional worth of $v_1 + v_2$ can be obtained uniquely, and we can see that $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for all $S \subseteq N$.

Therefore, we have:

$$\begin{aligned}\hat{\phi}_i(v_1+v_2) &= \frac{1}{n}((n-1)\phi_i(\underline{(v_1+v_2)}_i) + (v_1+v_2)(N) - \sum_{p\in N\setminus i}\phi_p(\underline{(v_1+v_2)}_p)) \\ &= \frac{1}{n}((n-1)\phi_i(v_1+v_2) + v_1(N) + v_2(N) - \sum_{p\in N\setminus i}\phi_p(v_1+v_2)) \\ &= \hat{\phi}_i(v_1) + \hat{\phi}_i(v_2).\end{aligned}$$

The second equality follows since the worth of all coalitions of $v_1 + v_2$ is obtained. The third equality follows from the additivity of the Shapley value.

That is, $\hat{\phi}$ satisfies Axiom 3.

We show that $\hat{\phi}$ satisfies Axiom 4. Let $v \in \Gamma^k$ and $S \subset N$ such that $|N \setminus S| \leq k$. Let $i \in S$. If $v(N) - v(N \setminus S) = 0$ holds, then $v(T) - v(T \setminus i) = 0$ holds for all $i \in S$ and for all $T \subseteq N$ such that $T \ni i$ since v is a convex PDG. Thus, i is a null player of v. Therefore, $\hat{\phi}(v)_i = 0$ holds for all $i \in S$ from the definition of $\hat{\phi}$.

That is, $\hat{\phi}$ satisfies Axiom 4.

Next, we show that $\hat{\phi}$ satisfies Axiom 6. For $i \in N$ and $v \in \Gamma$, we have:

$$\begin{aligned}&\hat{\phi}_i(N,k+1,v) - \hat{\phi}_i(N,k,v) \\ &= \frac{1}{n}\Big\{\Big((n-1)\phi_i(N,k+1,\underline{v}_i) + v(N) - \sum_{p\in N\setminus i}\phi_p(N,k+1,\underline{v}_p)\Big)\Big\} \\ &- \frac{1}{n}\Big\{\Big((n-1)\phi_i(N,k,\underline{v}_i) + v(N) - \sum_{p\in N\setminus i}\phi_p(N,k,\underline{v}_p)\Big)\Big\}\end{aligned}$$

$$= \frac{1}{n}\Big((n-1)(\phi_i(N,k+1,\underline{v}_i) - \phi_i(N,k,\underline{v}_i)) - \sum_{p\in N\setminus i}(\phi_p(N,k+1,\underline{v}_p) - \phi_p(N,k,\underline{v}_p))\Big). \tag{25}$$

Since $(N, k+1, \underline{v}_i)$ and $(N, k, \underline{v}_i)$ are convex PDGs, the marginal contributions of player i to unknown coalitions are $\max_{|S|=k+1, S\ni i}\Big(v(S) - v(S \setminus i)\Big)$ and $\max_{|S|=k, S\ni i}\Big(v(S) - v(S \setminus i)\Big)$, respectively. We define $S^*_{k+1}, S^*_k \subseteq N$ as follows:

$$v(S^*_{k+1}) - v(S^*_{k+1} \setminus i) = \max_{|S|=k+1, S\ni i}\Big(v(S) - v(S\setminus i)\Big), \tag{26}$$

$$v(S^*_k) - v(S^*_k \setminus i) = \max_{|S|=k, S\ni i}\Big(v(S) - v(S\setminus i)\Big), \tag{27}$$

Then, we have:

$$\begin{aligned}&\phi_i(N,k+1,\underline{v}_i) - \phi_j(N,k+1,\underline{v}_j) \\ &= \sum_{S\subset N, S\ni i,j, |S|\geq k+2}\frac{(|S|-1)!(n-|S|)!}{n!}\Big(\underline{v}_i(S^*_{k+1}) - \underline{v}_i(S^*_{k+1}\setminus i)\Big) \\ &- \Big(\underline{v}_j(S^*_{k+1}) - \underline{v}_j(S^*_{k+1}\setminus j)\Big) \\ &+ \frac{1}{n}\{v(N) - \underline{v}_i(N\setminus i) - v(N) + \underline{v}_j(N\setminus j)\}\end{aligned}$$

From the definition of Axiom 6, the following holds:

$$v(S^*_{k+1}) - v(S^*_{k+1} \setminus i) = v(S^*_{k+1}) - v(S^*_{k+1} \setminus j). \tag{28}$$

Moreover, from the definition of Axiom 6 and the definitions of $\underline{v}_i$ and $\underline{v}_j$, the marginal contributions of i and j are same within known coalitions.

Thus, we obtain:

$$v(N) - \underline{v}_i(N \setminus i) = v(N) - \underline{v}_j(N \setminus j). \tag{29}$$

Therefore, from Equations (28) and (29), the following holds:

$$\phi_i(N, k+1, \underline{v}_i) - \phi_j(N, k+1, \underline{v}_j) = 0. \tag{30}$$

Similarly, $\phi_i(N, k, \underline{v}_i) - \phi_j(N, k, \underline{v}_j) = 0$ holds.

Therefore, the following holds:

$$\phi_i(N, k+1, \underline{v}_i) - \phi_j(N, k+1, \underline{v}_j) = \phi_i(N, k, \underline{v}_i) - \phi_j(N, k, \underline{v}_j). \tag{31}$$

Finally, substituting Equation (31) into Equation (25), the following holds:

$$\hat{\phi}_i(N, k+1, v) - \hat{\phi}_j(N, k+1, v) = \hat{\phi}_i(N, k, v) - \hat{\phi}_j(N, k, v). \tag{32}$$

That is, $\hat{\phi}$ satisfies Axiom 6.

Next, to prove that $\hat{\phi}$ satisfies Axiom 1, we assume that N does not include null players without the loss of generality.

We show that $\hat{\phi}$ satisfies Axiom 1.

$$\begin{aligned}
\sum_{i \in N} \hat{\phi}_i(v) &= \frac{1}{n} \sum_{i \in N} \Big((n-1)\phi_i(\underline{v}_i) + v(N) - \sum_{p \in N \setminus i} \phi_p(\underline{v}_p) \Big) \\
&= \frac{1}{n} \sum_{i \in N} \Big(n\phi_i(\underline{v}_i) + v(N) - \sum_{p \in N} \phi_p(\underline{v}_p) \Big) \\
&= \sum_{i \in N} \phi_i(\underline{v}_i) + \frac{1}{n} \sum_{i \in N} \Big(v(N) - \sum_{p \in N} \phi_p(\underline{v}_p) \Big) \\
&= v(N).
\end{aligned}$$

That is, $\hat{\phi}$ satisfies Axiom 1.

Next, we show the uniqueness. Let $\sigma : \Gamma \to \mathbb{R}^n$. For an arbitrary $c_T \in \mathbb{R}$, let $c_T u_T$ be an (N, k)-PDG which is defined as follows:

$$c_T u_T(S) = \begin{cases} c_T, & \text{if } S \supseteq T \text{ and } \Big(|S| \le k \text{ or } |S| = n\Big), \\ 0, & \text{if } S \not\supseteq T \text{ and } |S| \le k, \end{cases} \tag{33}$$

where $T \subseteq N$ such that $|T| \le k$ or $|T| = n$.

For an arbitrary $c_T \in \mathbb{R}(\ \forall 1 \le |T| \le k$ or $|T| = n)$, we consider a value $\sigma(N, k, c_T u_T)$.

First, consider the case of $|T| = n$. When $k = n-1$, all players are symmetric in $(N, k, c_T u_T)$. Hence, from Axiom 1 and 2, $\sigma_i(N, k, c_T u_T) = \frac{c_T}{n} \ \forall i \in N$ holds. Therefore, $\sigma(N, k, c_T u_T)$ is uniquely obtained when $|T| = n$. Assume that $\sigma(N, k, c_N u_N)$ is obtained uniquely when $k = n - l (l \ge 1)$. We show that $\sigma(N, k, c_N u_N)$ is obtained uniquely when $k = n - l - 1$. Then, from Axiom 6, we obtain $(n-1)$ equations that are linearly independent. Moreover, using Axioms 1 and 4, we obtain (n) equations that are linearly independent. Thus, $\sigma(N, k, c_N u_N)$ is obtained uniquely when $k = n - l - 1$.

Next, we consider the case of $1 \le |T| \le n-1$. If $k = n-1$, then $\sigma(N, k, c_T u_T)$ is a full game. All players who belong to T are symmetric in v, and for an arbitrary player $i \in N \setminus T$, $\{i\}$ is a null coalition in $c_T u_T$. Therefore, from Axiom 4, $\sigma_i(N, k, c_T u_T) = 0 \ \forall N \setminus T$. Furthermore, from Axioms 1 and 2, $\sigma_i(N, k, c_T u_T) = \frac{c_T}{|T|} \ \forall i \in T$ holds. Hence, $\sigma(N, k, c_T u_T)$ is uniquely obtained when $k = n-1$.

Assume that $\sigma(N, k, c_T u_T)$ is uniquely obtained when $k = n - l (l \ge 1)$. We show that $\sigma(N, k, c_T u_T)$ is uniquely obtained when $k = n - l - 1$. In this case, for two arbitrary players $i, j \in T$, as $u_T(S) - u_T(S \setminus i) = u_T(S) - u_T(S \setminus j)$ holds for all $S \subseteq N$ such that $S \supseteq \{i, j\}$ and $|S| \leq n - l - 1$, the following holds from Axiom 6:

$$\sigma_i(N, k, c_T u_T) - \sigma_j(N, k, c_T u_T) = \sigma_i(N, k+1, c_T u_T) - \sigma_j(N, k+1, c_T u_T). \quad (34)$$

Therefore, we obtain $(|T| - 1)$ equations that are linearly independent. Moreover, since $N \setminus T$ is a null coalition, $\sigma_i(N, k, c_T u_T) = 0$ for all $i \in N \setminus T$ from Axiom 4. Further, Axiom 1 ensures that we obtain $|T|$ equations that are linearly independent. Therefore, $\sigma_i(N, k, c_T u_T) = \frac{c_T}{|T|}$ holds for all $i \in T$.

That is, $\sigma(N, k, c_T u_T)$ is uniquely obtained when $k = n - l - 1$ and $|T| \neq n$.

Finally, from Proposition 1 of Albizuri et al. [9], the collection of u_T ($T \subseteq N$ such that $T \leq k$ or $T = n$) is a basis of the linear space of (N, k)-PDGs.

However, notice that among all unanimity PDGs u_T, where $T \subseteq N$ such that $T \leq k$ or $T = N$, only $|V(u_N)| \neq 1$ holds.

Therefore, for an arbitrary (N, k)-convex PDG v, Axiom 3 ensures that

$$\begin{aligned} \sigma((N, k, v) - (c_N u_N)) &= \sigma(N, k, \sum_{\substack{T \subset N \\ |T| \leq k}} c_T u_T) \\ &= \sum_{\substack{T \subset N \\ |T| \leq k}} \sigma(N, k, c_T u_T). \end{aligned}$$

From Axiom 5, since $v = (v - c_N u_N)^\alpha$ holds for some $\alpha \in \mathbb{R}$, we have $\sigma_i(v) = \sigma_i(v - c_N u_N) + \frac{\alpha}{n}$ for all $i \in N$.

This completes the proof. □

The proposed value $\hat{\phi}$ is an imputation as we prove below.

Proposition 1. *Given $v \in \Gamma^k$, $\hat{\phi}(v) \in I(v)$.*

Proof. Without the loss of generality, we can assume that N does not include null players. From Theorem 2, $\hat{\phi}(v)$ satisfies efficiency for an arbitrary $v \in \Gamma^k$. Therefore, we show that $\hat{\phi}(v)$ satisfies individual rationality. Here, we have:

$$\begin{aligned} v(N) - \sum_{p \in N} \phi_p(\underline{v}_p) &\geq v(N) - \sum_{p \in N} \phi_p(\underline{v}) \\ &= v(N) - \underline{v}(N) = 0. \end{aligned}$$

The first inequality holds, as $\phi_p(\underline{v}_p) \leq \phi_p(\underline{v})$ holds from the definition of $\underline{v}_p$ and $\underline{v}$. Therefore, $v(N) - \sum_{p \in N \setminus i} \phi_p(\underline{v}_p) \geq \phi_i(\underline{v}_i)$ holds. Therefore,

$$\begin{aligned} \hat{\phi}_i(v) &= \frac{1}{n}\Big((n-1)\phi_i(\underline{v}_i) + v(N) - \sum_{p \in N \setminus i} \phi_p(\underline{v}_p)\Big) \\ &\geq \frac{1}{n}(n\phi_i(\underline{v}_i)) = \phi_i(\underline{v}_i) \geq \underline{v}_i(i) = v(i). \end{aligned}$$

□

We show the independence of all axioms of Theorem 2. Let $\delta^{-l} : \Gamma \to \mathbb{R}^n$ such that $l \in \{1, 2, 3, 4, 5, 6\}$ be a value that satisfies Axioms 1–6 other than Axiom l. The null coalition of $v \in \Gamma^k$ is denoted NC_v. Let $\mathbb{R}_+ = \{r \in \mathbb{R} | r \geq 0\}$.

In the following, we enumerate $\delta^{-1}, \delta^{-2}, \ldots, \delta^{-6}$.

$$\delta_i^{-1}(v) = \begin{cases} \sum\limits_{\substack{S \subseteq N \\ S \ni i, |S| \leq k}} \dfrac{v(S) - v(S \setminus i)}{|S|} + \dfrac{v(N)}{n}, & \text{if } i \in N \setminus NC_v, \\ \dfrac{v(N) - v(N \setminus NC_v)}{n}, & \text{if } i \in NC_v. \end{cases} \tag{35}$$

$$\delta_i^{-2}(v) = \begin{cases} \dfrac{v(N) + \alpha_i}{n}, & \text{if } i \in N \setminus NC_v, \\ \dfrac{v(N) - v(N \setminus NC_v)}{n}, & \text{if } i \in NC_v. \end{cases} \tag{36}$$

where $\alpha_i \in \mathbb{R}_+$ for all $i \notin NC_v$ such that $\sum_{i \in N \setminus NC_v} \alpha_i = |NC_v| v(N)$.

$$\delta_i^{-3}(v) = \begin{cases} \dfrac{1}{\sum\limits_{S \subseteq N \setminus NC_v, |S| \leq k} v(S)} \sum\limits_{\substack{S \subseteq N \setminus NC_v \\ S \ni i, |S| \leq k}} \dfrac{v(S)}{|S|} + \dfrac{v(N)}{n - |NC_v|} + \dfrac{1}{n - |NC_v|}, & \text{if } i \in N \setminus NC_v, \\ \dfrac{v(N) - v(N \setminus NC_v)}{n}, & \text{if } i \in NC_v. \end{cases} \tag{37}$$

$$\delta^{-4}(v) = \Big(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n}\Big). \tag{38}$$

$$\delta_i^{-5}(v) = \begin{cases} \dfrac{v(N)}{n - |NC_v|}, & \text{if } i \in N \setminus NC_v, \\ 0, & \text{if } i \in NC_v. \end{cases} \tag{39}$$

$$\delta^{-6}(v) = \bar{\phi}(v), \tag{40}$$

where $\bar{\phi}(v)$ is defined in the next section.

The proof is straightforward and is omitted.

Example 1. *Let $N = \{1, 2, 3\}$ be the set of players and $v \in \Gamma^k$ such that $k = 1$. We define v as follows:*

$$v(1) = 5, v(2) = 3, v(3) = 0, v(1, 2, 3) = 15.$$

Then, $\underline{v}_i (i = 1, 2, 3)$ are obtained as follows:

$\underline{v}_1(1) = 5,\ \underline{v}_1(2) = 3,\ \underline{v}_1(3) = 0,\ \underline{v}_1(1,2) = 8,\ \underline{v}_1(1,3) = 5,\ \underline{v}_1(2,3) = 10,$
$\underline{v}_1(1,2,3) = 15,$
$\underline{v}_2(1) = 5,\ \underline{v}_2(2) = 3,\ \underline{v}_2(3) = 0,\ \underline{v}_2(1,2) = 8,\ \underline{v}_2(1,3) = 12,\ \underline{v}_2(2,3) = 3,$
$\underline{v}_2(1,2,3) = 15,$
$\underline{v}_3(1) = 5,\ \underline{v}_3(2) = 3,\ \underline{v}_3(3) = 0,\ \underline{v}_3(1,2) = 15,\ \underline{v}_3(1,3) = 5,\ \underline{v}_3(2,3) = 3,$
$\underline{v}_3(1,2,3) = 15,$

Moreover, we have:

$$\underline{\phi}_1(v) = \phi_1(\underline{v}_1) = \frac{37}{6}, \underline{\phi}_2(v) = \phi_2(\underline{v}_2) = \frac{18}{6}, \underline{\phi}_3(v) = \phi_3(\underline{v}_3) = 0.$$

Finally, from the definition of $\hat{\phi}$ in Equation (16), the proposed value is obtained as follows:

$$\hat{\phi}_1(v) = \frac{109}{12}, \hat{\phi}_2(v) = \frac{71}{12}, \hat{\phi}_3(v) = 0. \tag{41}$$

4. The Set of Marginal Vectors and the Shapley Value Obtained by Convex PDGs

In this section, we investigate the marginal vectors and the Shapley values obtained from a convex PDG. We use a property of the Shapley value that is the gravity center of the core whose extreme points are marginal vectors if the game is convex.

We propose the second value selected from $\Psi(v)$. To this end, we show that the set of marginal vectors for each permutation $\pi \in \Pi(N)$. First, we show the lower value of the marginal contribution of each player. Let $v \in \Gamma^k$, $\pi \in \Pi(N)$, and $i \in N$. The sequence of the lower value of a marginal vector $\underline{m}^{\pi}(v)$ is obtained from the convexity of the game as follows:

$$\underline{m}_i^{\pi}(v) = \begin{cases} \max\limits_{\pi' \in \Pi(N), \pi'(k)=i} m_i^{\pi'}(v), & \text{if } \pi^{-1}(i) \geq k+1, \\ m_i^{\pi}(v), & \text{otherwise.} \end{cases} \tag{42}$$

Furthermore, let $T^k = \{\pi(k+1), \pi(k+2), \dots, \pi(n)\}$ and $l \in T^k$. Then, the marginal vector $m^{\pi,l}$ is defined as follows:

$$m_i^{\pi,l}(v) = \begin{cases} m_i^{\pi}(v), & \text{if } i \notin T^k, \\ \underline{m}_i^{\pi}(v), & \text{if } i \in T^k \text{ and } i \neq l, \\ v(N) - \sum\limits_{p \in N \setminus T^k} m_p^{\pi}(v) - \sum\limits_{p \in T^k, p \neq l} \underline{m}_p^{\pi}(v), & \text{if } i \in T^k \text{ and } i = l. \end{cases} \tag{43}$$

Let $\pi \in \Pi(N)$ be an arbitrary permutation of N and $v \in \Gamma^k$. Then, the set of marginal vectors that can be obtained from v is denoted as $M^{\pi}(v)$. We can verify the following theorem below.

Theorem 3. *Let $\pi \in \Pi(N)$ and $v \in \Gamma^k$. Then, $M^{\pi}(v)$ is a polytope whose extreme points are $m^{\pi,l}(v)$ for all $l \in T^k$. That is, the next expression holds:*

$$M^{\pi}(v) = \{m^{\pi}(v) \in \mathbb{R}^n | m^{\pi}(v) = \sum_{t=k+1}^{n} c^t m^{\pi,t}(v), \sum_{t=k+1}^{n} c^t = 1, c^t \geq 0 \\ \textit{for all } t = k+1, \dots n\}.$$

Theorem 3 states that the set of marginal vectors $M^{\pi}(v)$ that can be obtained from game v and a permutation of players $\pi \in \Pi(N)$ is a polytope whose extreme points are $m^{\pi,l}(v)$ for all $l \in T^k$. The number of extreme points of $M^{\pi}(v)$ is $n-k$. Notice that there are $n!$ polytopes that can be obtained from v.

By selecting each point from $M^{\pi}(v)$ for all $\pi \in \Pi(N)$, the convex hull of them is a Weber set that can be obtained from v and π. As the gravity center of the Weber set for a convex game is the Shapley value, the gravity center of the convex hull of the selected points is the Shapley value that can be obtained from v.

However, there are an infinite number of points we can select from $M^{\pi}(v)$ for each $\pi \in \Pi(N)$. Considering the applications of the theory of PDGs, a rational one-point solution should be selected from the set of Shapley values. Although several methods may exist to select a one-point solution from the set of Shapley values, we herein propose a simple method.

We formulate the Shapley value. Let $\hat{m}^{\pi}(v)$ be the gravity center of $M^{\pi}(v)$. Then, $\hat{m}^{\pi}(v)$ can be defined as follows:

$$\hat{m}_i^{\pi}(v) = \begin{cases} m_i^{\pi}(v), & \text{if } i \notin T^k, \\ \sum_{t=k+1}^{n} \frac{m_i^{\pi,t}(v)}{n-k}, & \text{otherwise.} \end{cases} \tag{44}$$

Then, the proposed Shapley value $\bar{\phi} : \Gamma \to \mathbb{R}^n$ is defined as follows:

$$\bar{\phi}_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \hat{m}_i^{\pi}(v), \text{ for all } i \in N. \tag{45}$$

Example 2. *We use the same example that was used in Example 1.*

Let $N = \{1,2,3\}$ be the set of players and $v \in \Gamma^k$ such that $k = 1$. We define v as follows:

$$v(1) = 5, v(2) = 3, v(3) = 0, v(1,2,3) = 15.$$

Moreover, we define permutations $\pi^i \in \Pi(N)$ $(i = 1,2,\ldots,6)$ as follows:

$$\pi^1 = (123), \pi^2 = (132), \pi^3 = (213), \pi^4 = (231), \pi^5 = (312), \pi^6 = (321).$$

Then, we obtain $m^{\pi,l}$ from Equations (43) and (42) as follows:

$$m^{\pi^1,2}(v) = (5,10,0), m^{\pi^1,3}(v) = (5,3,7), m^{\pi^2,3}(v) = (5,7,3), m^{\pi^2,2}(v) = (5,0,10),$$
$$m^{\pi^3,1}(v) = (3,12,0), m^{\pi^3,3}(v) = (3,5,7), m^{\pi^4,3}(v) = (3,7,5), m^{\pi^4,1}(v) = (3,0,12),$$
$$m^{\pi^5,1}(v) = (0,12,3), m^{\pi^5,2}(v) = (0,5,10), m^{\pi^6,2}(v) = (0,10,5),$$
$$m^{\pi^6,1}(v) = (0,3,12).$$

Then, we obtain $\hat{m}^{\pi}$ from Equation (44) as follows:

$$\hat{m}^{\pi^1}(v) = (5, \frac{13}{2}, \frac{7}{2}), \hat{m}^{\pi^2}(v) = (5, \frac{7}{2}, \frac{13}{2}), \hat{m}^{\pi^3}(v) = (3, \frac{17}{2}, \frac{7}{2}),$$
$$\hat{m}^{\pi^4}(v) = (3, \frac{7}{2}, \frac{17}{2}), \hat{m}^{\pi^5}(v) = (0, \frac{17}{2}, \frac{13}{2}), \hat{m}^{\pi^6}(v) = (0, \frac{13}{2}, \frac{17}{2}).$$

Then, from the definition of $\bar{\phi}(v)$ in Equation (45), the proposed value is obtained as follows:

$$\bar{\phi}_1(v) = \frac{22}{3}, \bar{\phi}_2(v) = \frac{16}{3}, \bar{\phi}_3(v) = \frac{7}{3}, \tag{46}$$

5. Axiomatic Approach

In this section, we axiomatize the proposed value $\bar{\phi}(v)$ using six axioms: Efficiency, Symmetry, Additivity, Null Coalition, Aggregate Monotonicity, and Dual Symmetric Difference. To axiomatize the value, the integer k is not fixed so that the set of (N,k)-convex PDGs is denoted by Γ. Let $\sigma : \Gamma \to \mathbb{R}^n$. Let π^{ij} be a permutation exchanging two players $i, j \in N$ in π.

Axiom 7 (Dual Symmetric Difference)**.** *Let $(N,k,v) \in \Gamma^k$ and $i, j \in N$. If $v(N) - v((N \setminus S) \cup i) = v(N) - v((N \setminus S) \cup j)$ holds for all $S \subseteq N$ such that $S \supseteq \{i,j\}$ and $|N \setminus S| = k - 1$, then the following holds:*

$$\sigma_i(N,k,v) - \sigma_i(N,k+1,v) = \sigma_j(N,k,v) - \sigma_j(N,k+1,v). \tag{47}$$

Axiom 7 states that if two players are equivalent with respect to their contributions to N, then their differences of values are same if the number of known coalitions is changed.

We have obtained the following theorem.

Theorem 4. *$\bar{\phi}$ is the unique function on Γ that satisfies Axiom 1, 2, 3, 4, and 7.*

Proof. First, we show that $\bar{\phi}$ satisfies Axioms 1, 2, 3, 4, and 7. Clearly, $\bar{\phi}$ satisfies Axioms 2 and 5, and it is straightforward to show that $\bar{\phi}$ satisfies Axiom 1.

We show that $\bar{\phi}$ satisfies Axiom 3. Since $\bar{\phi}$ is the Shapley value for some convex full game, $\bar{\phi}$ satisfies Axiom 3 from the additivity of the Shapley value.

We show that $\bar{\phi}$ satisfies Axiom 4. Let $v \in \Gamma^k$ and $S \subset N$, such that $1 \le |N \setminus S| \le k$. If $v(N) - v(N \setminus S) = 0$ holds, then for some $\pi \in \Pi(N)$, $\sum_{i \in S} m_i^{\pi}(v) = v(N) - v(N \setminus S) = 0$ holds. Furthermore, since v is convex, $m_i^{\pi}(v)$ is monotonic with respect to set inclusion to which player i belongs. Therefore, the maximal marginal contributions of players who belongs to S are zero; that is, they are null players in all fully defined convex games that can be obtained from v. That is, $\bar{\phi}_i(v) = 0$ for all $i \in S$. Namely, $\bar{\phi}$ satisfies Axiom 4.

Next, we show that $\bar{\phi}$ satisfies Axiom 7. Let $v \in \Gamma^k$ and two players be $i, j \in N$. If $v(N) - v((N \setminus S) \cup i) = v(N) - v((N \setminus S) \cup j)$ holds for all $S \subseteq N$ such that $S \supseteq \{i, j\}$ and $|N \setminus S| = k - 1$, then $\sum_{t=k+1}^{n-1} m_i^{\pi,t}(v) = \sum_{t=k+1}^{n-1} m_j^{\pi^{ij},t}(v)$ such that $\pi(i) \in \{k+1, k+2, \dots, n-1\}$ holds. That is, $\sum_{t=k+1}^{n-1} \frac{m_i^{\pi,t}(v)}{n-k-1} = \sum_{t=k+1}^{n-1} \frac{m_j^{\pi^{ij},t}(v)}{n-k-1}$ holds. Namely, when $\pi(i) \in \{k+1, k+2, \dots, n-1\}$, $\hat{m}_i^{\pi}(v) = \hat{m}_j^{\pi^{ij}}(v)$ holds.

Then, we have:

$$\begin{aligned} \bar{\phi}_i(N,k,v) - \bar{\phi}_i(N,k+1,v) &= \sum_{\pi \in \Pi(N)} \frac{1}{n!} (\hat{m}_i^{\pi}(N,k,v) - \hat{m}_i^{\pi}(N,k+1,v)) \\ &= \sum_{\pi \in \Pi(N)} \frac{1}{n!} \Big(\sum_{t=k+1}^{n-1} \frac{m_i^{\pi,t}(N,k,v)}{n-k-1} - \\ &\quad \sum_{t=k+2}^{n-1} \frac{m_i^{\pi,t}(N,k+1,v)}{n-k-2} - m_i^{\pi}(N,k+1,v) \Big). \end{aligned}$$

From the discussion above, the following holds:

$$\sum_{t=k+1}^{n-1} \frac{m_i^{\pi,t}(N,k,v)}{n-k-1} = \sum_{t=k+1}^{n-1} \frac{m_j^{\pi^{ij},t}(N,k,v)}{n-k-1}, \tag{48}$$

$$\sum_{t=k+2}^{n-1} \frac{m_i^{\pi,t}(N,k+1,v)}{n-k-2} = \sum_{t=k+2}^{n-1} \frac{m_j^{\pi^{ij},t}(N,k+1,v)}{n-k-2}. \tag{49}$$

Moreover, from equation $v(N) - v((N \setminus S) \cup i) = v(N) - v((N \setminus S) \cup j)$, $v((N \setminus S) \cup i) = v((N \setminus S) \cup j)$ holds, so that $v((N \setminus S) \cup i) - v(N \setminus S) = v((N \setminus S) \cup j) - v(N \setminus S)$ holds. That is, $m_i^{\pi}(N,k+1,v) = m_j^{\pi^{ij}}(N,k+1,v)$ holds.

From the discussion above, the following holds:

$$\hat{m}_i^{\pi}(N,k,v) - \hat{m}_i^{\pi}(N,k+1,v) = \hat{m}_j^{\pi^{ij}}(N,k,v) - \hat{m}_j^{\pi^{ij}}(N,k+1,v). \tag{50}$$

Therefore, the following holds:

$$\begin{aligned} &\sum_{\pi \in \Pi(N)} \frac{1}{n!} \Big(\hat{m}_i^{\pi}(N,k,v) - \hat{m}_i^{\pi}(N,k+1,v) \Big) \\ &= \sum_{\pi \in \Pi(N)} \frac{1}{n!} \Big(\hat{m}_j^{\pi^{ij}}(N,k,v) - \hat{m}_j^{\pi^{ij}}(N,k+1,v) \Big). \end{aligned}$$

That is, $\bar{\phi}$ satisfies Axiom 7.

Next, we show the uniqueness. Although the proof of the uniqueness is almost the same as that of $\hat{\phi}$, we show it for confirmation since Axiom 6 has changed to Axiom 7.

Let $\sigma : \Gamma \to \mathbb{R}^n$. For an arbitrary $c_T \in \mathbb{R}$, let $c_T u_T$ be an (N,k)-PDG which is defined as follows:

$$c_T u_T(S) = \begin{cases} c_T, & \text{if } S \supseteq T \text{ and } \Big(|S| \le k \text{ or } |S| = n\Big), \\ 0, & \text{if } S \not\supseteq T \text{ and } |S| \le k, \end{cases} \tag{51}$$

where $T \subseteq N$ such that $|T| \leq k$ or $|T| = n$.

For an arbitrary $c_T \in \mathbb{R}(\ \forall 1 \leq |T| \leq k$ or $|T| = n)$, we consider a value $\sigma(N,k,c_T u_T)$.

First, consider the case of $|T| = n$. When $k = n-1$, all players are symmetric in $(N,k,c_T u_T)$. Hence, from Axioms 1 and 2, $\sigma_i(N,k,c_T u_T) = \frac{c_T}{n} \ \ \forall i \in N$ holds. Therefore, $\sigma(N,k,c_T u_T)$ is uniquely obtained when $|T| = n$. Assume that $\sigma(N,k,c_N u_N)$ is uniquely obtained when $k = n - l (l \geq 1)$. We show that $\sigma(N,k,c_N u_N)$ is uniquely obtained when $k = n-l-1$. Then, from Axiom 7 we obtain $(n-1)$ equations that are linearly independent. Moreover, using Axioms 1 and 4, we obtain (n) equations that are linearly independent. Thus, $\sigma(N,k,c_N u_N)$ is uniquely obtained when $k = n-l-1$.

Next, we consider the case of $1 \leq |T| \leq n-1$. If $k = n-1$, then $\sigma(N,k,c_T u_T)$ is a full game. All players who belong to T are symmetric in v, and for an arbitrary player $i \in N \setminus T$, $\{i\}$ is a null coalition in $c_T u_T$. Therefore, from Axiom 4, $\sigma_i(N,k,c_T u_T) = 0 \ \ \forall N \setminus T$. Furthermore, from Axioms 1 and 2, $\sigma_i(N,k,c_T u_T) = \frac{c_T}{|T|} \ \ \forall i \in T$ holds. Hence, $\sigma(N,k,c_T u_T)$ is uniquely obtained when $k = n-1$.

Assume that $\sigma(N,k,c_T u_T)$ is uniquely obtained when $k = n - l(l \geq 1)$. We show that $\sigma(N,k,c_T u_T)$ is uniquely obtained when $k = n-l-1$. In this case, for two arbitrary players $i,j \in T$, as $u_T(N) - u_T((N \setminus S) \cup i) = u_T(N) - u_T((N \setminus S) \cup j)$ holds for all $S \subseteq N$ such that $S \supseteq \{i,j\}$ and $|N \setminus S| = k-1$, the following holds from Axiom 7:

$$\sigma_i(N,k,c_T u_T) - \sigma_j(N,k,c_T u_T) = \sigma_i(N,k+1,c_T u_T) - \sigma_j(N,k+1,c_T u_T). \tag{52}$$

Therefore, we obtain $(|T|-1)$ equations that are linearly independent. Moreover, since $N \setminus T$ is a null coalition, $\sigma_i(N,k,c_T u_T) = 0$ for all $i \in N \setminus T$ from Axiom 4. Further, Axiom 1 ensures that we obtain $|T|$ equations that are linearly independent. Therefore, $\sigma_i(N,k,c_T u_T) = \frac{c_T}{|T|}$ holds for all $i \in T$.

That is, $\sigma(N,k,c_T u_T)$ is uniquely obtained when $k = n-l-1$ and when $|T| \neq n$.

Finally, from Proposition 1 of Albizuri et al. [9], the collection of u_T ($T \subseteq N$ such that $T \leq k$ or $T = n$) is a basis of the linear space of (N,k)-PDGs.

However, notice that among all unanimity PDGs u_T, where $T \subseteq N$ such that $T \leq k$ or $T = N$, only $|V(u_N)| \neq 1$ holds.

Therefore, for an arbitrary (N,k)-convex PDG v, Axiom 3 ensures that

$$\begin{aligned} \sigma((N,k,v) - (c_N u_N)) &= \sigma(N,k, \sum_{\substack{T \subset N \\ |T| \leq k}} c_T u_T) \\ &= \sum_{\substack{T \subset N \\ |T| \leq k}} \sigma(N,k,c_T u_T). \end{aligned}$$

From Axiom 5, since $v = (v - c_N u_N)^{\alpha}$ holds for some $\alpha \in \mathbb{R}$, we have $\sigma_i(v) = \sigma_i(v - c_N u_N) + \frac{\alpha}{n}$ for all $i \in N$.

This completes the proof. $\square$

Comparing axiom systems between $\hat{\phi}$ and $\bar{\phi}$, only Axioms 6 and 7 are different. Additionally, the relationship between these axioms is analogous to the concept of duality. Furthermore, axioms, excluding the aforementioned, can be viewed as natural extensions of the classic axioms of the Shapley value for TU games.

6. Concluding Remarks

In this study, we proposed two values that are estimations of the Shapley value. First, we introduced a polytope that includes the set of Shapley values that can be obtained from a given convex PDG and selected one value. Additionally, we axiomatized the proposed value that consists of axioms of Efficiency, Symmetry, Additivity, Null Coalition, Aggregate Monotonicity, and Symmetric Difference.

Secondly, we obtained the set of Shapley values that can be obtained from a given convex PDG and selected a specific value as well as the first approach. The axiom system consists of axioms of Efficiency, Symmetry, Additivity, Null Coalition, Aggregate Monotonicity, and Dual Symmetric Difference. The sole distinction between axiom systems of the two proposed values is the axiom of symmetric difference.

Future research should focus on exploring the Shapley value for PDGs where the set of known coalitions is more general. While we have formulated axiom systems for (N,k)-PDGs, the task becomes increasingly challenging. For instance, we used the properties of a convex game, such as the monotonicity of marginal contributions of each player in the sense of set-theoretic inclusion. This enabled us to use axioms such as a null coalition when a PDG is an (N,k)-PDG.

Other topics include investigations of other solutions for PDGs such as the core, the nucleolus [2], and the Banzhaf value [14,15].

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