mathematics

MDPI

Article

New Ways to Calculate the Probability in the Bertrand Problem

Javier Rodrigo [1], Mariló López [2,*] and Sagrario Lantarón [2]

1 Departamento de Matemática Aplicada, Universidad Pontificia Comillas, 28015 Madrid, Spain; jrodrigo@comillas.edu
2 Departamento de Matemática e Informática Aplicadas a las Ingenierías Civil y Naval, Universidad Politécnica de Madrid, 28040 Madrid, Spain; sagrario.lantaron@upm.es
* Correspondence: marilo.lopez@upm.es

Abstract: We give two new ways of calculating the probability of a chord of circumference randomly selected being larger than the side of an equilateral triangle inscribed in the circumference (this problem is known as the Bertrand paradox). The first one employs an immersion in R^4, and the second one uses a direct probability measure over the set of chords.

Keywords: probability measures; Bertrand paradox; invariances

MSC: 28D05; 46N30; 49K45

1. Introduction

Probability problems usually carry paradoxes due to the different types of probability measures one can use. Sometimes several probability measures may be considered suitable for a given problem, but they yield different results.

One of the most known paradoxes in probability theory is Bertrand's paradox [1]. The problem is the following: Given a circle with an equilateral triangle inscribed and given a chord of the circle randomly selected, what is the probability of the event that the chord is larger than a side of the triangle?

Since the probability of choosing a diameter of the circle is 0, for practical purposes, the question can be reformulated as: given a circle with an equilateral triangle inscribed and given a chord of the circle not containing the center and randomly selected, what is the probability that the chord is larger than a side of the triangle?

Bertrand gave three different solutions to the problem that carried three different results: $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{4}$. These ways of solving the problem are known in the literature as B1, B2, and B3, respectively (some permutations of these notations are observed in the existing literature). B2 and B3 use the identification of the chords with their midpoint, but they differ in the way the probability is calculated. B3 applies a ratio between areas (the area covered for the midpoints of the chords satisfying the conditions and the area covered for the midpoints of the possible chords), but B2 applies a ratio between lengths after fixing a radius. Beside this, B1 employs a ratio between angles (we see more detailed explanations of these methods in Section *State of Art in Bertrand Paradox*, where a revision of the literature on the Bertrand problem is performed).

The paradox is that there are three fair ways of choosing the chords, where the same probability of being chosen is given to each chord, so respecting the principle of indifference exposed below but they give three different results.

In order to clarify this apparent contradiction, different approaches have been performed in the literature. A first approach is to consider measures of probability that respect the principle of indifference, that is to say, the same probability is assigned to two objects for which there is no reason to believe they have different prevalence [2]. Jaynes [3] proposes to admit only measures of probability that are invariant under translations or dilations of the circle, since the problem seems not to be affected by these transformations.

check for updates

Citation: Rodrigo, J.; López, M.; Lantarón, S. New Ways to Calculate the Probability in the Bertrand Problem. *Mathematics* **2024**, *12*, 3. https://doi.org/10.3390/math12010003

Academic Editor: Velizar Pavlov

Received: 24 October 2023
Revised: 30 November 2023
Accepted: 15 December 2023
Published: 19 December 2023

None of these approaches seems to carry a disambiguation since there are different results for the probability with procedures that respect the two principles of indifference and invariance (for further analysis of Bertrand's paradox, see [4–8] and the more recent paper of Chechile [9]. They give new solutions to the problem, such as 0.609 or 0.75 in [8] and $1 - \frac{\sqrt{3}}{2}$ in [9]).

In this paper, we give two methods for calculating the probability in Bertrand's problem. The first one proposes an immersion of the set of chords in a manifold of R^4, and it gives the same result as B1. The second one applies an intrinsic measure to the set of chords and yields a probability of 0.323545.

In this way, the two methods developed seem to point to B1 as the most reliable method to calculate the probability of the problem within the classical proposals. However, as mentioned at the beginning of the introduction, the resulting probability is subject to the chosen probability measure.

The first method is developed in Section 2. It carries a discussion that is performed in Section *Remarks on the Proof of Proposition 1 and on Bertrand's Proofs*: we note that we have not obtained the same result as B1 in a casual way, since the first method can be obtained from B1 by means of a bijection that preserves the relation between areas and then preserves the probability, given as a quotient of areas.

We also remark that some of the existing methods in the literature are based on identifying the chords with one of their elements, as the midpoint in B3, by means of a bijection. The problem is that the bijections do not always preserve the area, and this fact may change the probability. This leads us to the necessity of establishing a second method that uses an intrinsic measure of the set of the chords, a measure that does not depend on the particular immersion of this set in R^k for some value of k. This second method is developed in Section 3, and it is an approach to Bertrand's problem that represents one of the novelties of the work carried out. A computational study that supports the theoretical results obtained is developed in Section 4, and some concluding remarks are given in Section 5.

State of Art in Bertrand Paradox

The seminal work of Bertrand about his paradox is presented in Bertrand [1]. This work was included in the realm of a set of paradoxes in probability theory based on uncertainty about what must be considered a random choice.

As stated before, Bertrand gave a problem about the probability of randomly selecting a chord of a circle with given conditions, and he found three different solutions, all of them apparently correct.

In the first solution (B1), a chord is defined by its two endpoints, and the origin of the coordinate axes is fixed as one of these endpoints. This way, assuming that the other endpoint is uniformly distributed over the circumference (ensuring a random selection of the chord), the set of chords having a length larger than a side of the equilateral triangle corresponds to an arc of the circumference whose length is $\frac{1}{3}$ of the length of the total circumference, and the probability of $\frac{1}{3}$ is derived.

In the second solution (B2), a chord is defined by its midpoint, so the set of chords is represented by the unit circle (it can be assumed without loss of generality that the circle of the Bertrand problem has radius 1). A radius of the unit circle is fixed, and a uniform distribution of the points of this radius is assumed. With these conditions, the chords whose length is larger than a side of the equilateral triangle are in bijective correspondence with the points of the radius whose distance to the center is less than $\frac{1}{2}$. This gives, as a result, a probability of $\frac{1}{2}$.

The third solution (B3) also represents a chord by its midpoint, so the set of chords is again in bijection with the unit circle. The set of chords satisfying the condition of the Bertrand problem corresponds to a circle with the same center as the unit circle and radius $\frac{1}{2}$. To pick up a chord at random is equivalent to randomly selecting a point of the unit circle, where we assume a uniform distribution to guarantee randomness. To select a chord with the condition of the problem is to select a point of the circle of radius $\frac{1}{2}$. This yields the probability value of $\frac{1}{4}$.

From [1] to nowadays, many attempts to attack the Bertrand problem have been addressed. Some of them adopt a philosophical point of view, with discussions of what must be considered as random or as a random election. This is the approximation of [10], where different theories of probability are treated, or [11], where the aforementioned principle of indifference is philosophically analyzed.

The other approach is the mathematical one, as in [8,9,12]. In [8], Chiu and Larson derive the probability distributions of the chord lengths in the three classical solutions. They also give the probability distributions for five solutions to the problem they provide, with different interpretations of what it means to choose a chord in a random way. Some of these new solutions yield the same probability: 0.609.

In [9], Chechile states a novel method to sample a chord of the circle in a random way. It is based on four steps: first a generation of a random angle, then a random value from a beta distribution, after the construction of an auxiliary circle to define two chords, and finally a random election of one of these chords by flipping a coin. With this sophisticated method, he achieved a probability value of about 0.13, as said before.

Drory ([12]) proposes a mixture of physical and mathematical methods, including physical realizations of the problem of selecting a chord, with 700 simulations that reinforce the first solution of Bertrand. He also gives a valuable review of the literature on the problem.

Within this context, in this paper we adopt the mathematical way, with new representations of the problem that are developed in the next sections.

2. The Set of Chords as a Manifold of $\mathbf{R}^4$

Along this section, we can assume without loss of generality that the circle has a radius of 1 and is centered at the origin.

This way, as a chord is determined by its endpoints, we may model the chords as the image set of the mapping $\overline{f}(\alpha,\ \beta) = (\cos\alpha,\ sen\alpha,\ \cos\beta,\ sen\beta)$, with $0 \le \beta < \alpha \le 2\,\pi$, that is to say, the following two-dimensional manifold in R^4:

$$C = \{(\cos\alpha,\ sen\alpha,\ \cos\beta,\ sen\beta)/0 \le \beta < \alpha \le 2\,\pi\}$$

We can pick a chord at random by giving two values for the angles α, β.

Proposition 1. *For a given uniform measure of probability, the probability of choosing a chord in the circumference whose length is greater than the side of an equilateral triangle inscribed in the circumference is* $\frac{1}{3}$.

Proof of Proposition 1. The chords whose length is greater than a side of the equilateral triangle inscribed in the circumference correspond to the following subset of C:

$$F = \left\{ \begin{array}{l} (\cos\alpha,\ sen\alpha,\ \cos\beta,\ sen\beta) \in C/d((\cos\alpha,\ sen\alpha),\ (\cos\beta,\ sen\beta)) = \\ = \sqrt{(\cos\alpha - \cos\beta)^2 + (sen\alpha - sen\beta)^2} = \\ \sqrt{2 - 2(\cos\alpha\ \cos\beta + sen\alpha\ sen\beta)} = \sqrt{2 - 2\ \cos(\alpha - \beta)} \ge \sqrt{3} \end{array} \right\} =$$
$$= \left\{(\cos\alpha,\ sen\alpha,\ \cos\beta,\ sen\beta) \in C/\cos(\alpha-\beta) \le -\tfrac{1}{2}\right\} =$$
$$= \left\{(\cos\alpha,\ sen\alpha,\ \cos\beta,\ sen\beta) \in C/\tfrac{2\,\pi}{3} + \beta \le \alpha \le \tfrac{4\,\pi}{3} + \beta\right\}$$

Since $\frac{2\,\pi}{3} + \beta \le \alpha \le \min\left\{\frac{4\,\pi}{3} + \beta, 2\,\pi\right\}$ and $0 \le \beta < 2\,\pi$, we have that:

$$F = \left\{ (\cos\alpha,\ sen\alpha,\ \cos\beta,\ sen\beta) \in C / \tfrac{2\,\pi}{3} + \beta \leq \alpha \leq \tfrac{4\,\pi}{3} + \beta \right\} =$$
$$= \left\{ (\cos\alpha,\ sen\alpha,\ \cos\beta,\ sen\beta) / 0 \leq \beta \leq \tfrac{2\,\pi}{3},\ \tfrac{2\,\pi}{3} + \beta \leq \alpha \leq \tfrac{4\,\pi}{3} + \beta \right\} \cup$$
$$\cup \left\{ (\cos\alpha,\ sen\alpha,\ \cos\beta,\ sen\beta) / \tfrac{2\,\pi}{3} \leq \beta \leq \tfrac{4\,\pi}{3},\ \tfrac{2\,\pi}{3} + \beta \leq \alpha \leq 2\,\pi \right\}$$

We consider a uniform probability distribution. This way, we choose as a measure of probability the ratio between the areas of F and C: $\frac{A(F)}{A(C)}$

We have that:

$$\overline{f}_\alpha(\alpha,\ \beta) = (-sen\alpha,\ \cos\alpha,\ 0,\ 0),$$
$$\overline{f}_\beta(\alpha,\ \beta) = (0,\ 0, -sen\beta,\ \cos\beta),$$

so

$$\|\overline{f}_\alpha(\alpha,\ \beta)\| = \sqrt{sen^2\alpha + cos^2\alpha} = 1,$$
$$\|\overline{f}_\beta(\alpha,\ \beta)\| = \sqrt{sen^2\beta + cos^2\beta} = 1$$

$$\overline{f}_\alpha(\alpha,\ \beta) \circ \overline{f}_\beta(\alpha,\ \beta) = (-sen\alpha,\ \cos\alpha,\ 0,\ 0) \circ (0,\ 0, -sen\beta,\ \cos\beta)$$
$$= 0$$

and

$$\sqrt{\|\overline{f}_\alpha(\alpha,\ \beta)\|^2\ \|\overline{f}_\beta(\alpha,\ \beta)\|^2 - \left(\overline{f}_\alpha(\alpha,\ \beta) \circ \overline{f}_\beta(\alpha,\ \beta)\right)^2} = 1.$$

This implies that:

$$A(C) = \iint_D \sqrt{\|\overline{f}_\alpha(\alpha,\ \beta)\|^2\ \|\overline{f}_\beta(\alpha,\ \beta)\|^2 - \left(\overline{f}_\alpha(\alpha,\ \beta) \circ \overline{f}_\beta(\alpha,\ \beta)\right)^2}\ ,$$
$$= \iint_D 1\ d\alpha\ d\beta$$

where

$$D = \{(\alpha,\ \beta) / 0 \leq \beta < \alpha \leq 2\,\pi\}.$$

This way:

$$A(C) = \iint\limits_D 1\ d\alpha\ d\beta = \int_0^{2\,\pi} \left(\int_\beta^{2\,\pi} 1\ d\alpha \right) d\beta = \int_0^{2\,\pi} (2\,\pi - \beta)\ d\beta = 2\,\pi^2$$

On the other hand:

$$A(F) = \iint_{D_1} \sqrt{\|\overline{f}_\alpha(\alpha,\ \beta)\|^2\ \|\overline{f}_\beta(\alpha,\ \beta)\|^2 - \overline{f}_\alpha(\alpha,\ \beta) \circ \overline{f}_\beta(\alpha,\ \beta)}$$
$$= \iint_{D_1} 1\ d\alpha\ d\beta$$

where:

$$D_1 = \left\{ (\alpha,\ \beta) / 0 \leq \beta < 2\,\pi,\ \frac{2\,\pi}{3} + \beta \leq \alpha \leq \min\left\{ \frac{4\,\pi}{3} + \beta,\ 2\,\pi \right\} \right\}$$

This implies that:

$$A(F) = \iint_{D_1} 1\ d\alpha\ d\beta = \int_0^{\frac{2\,\pi}{3}} \left(\int_{\frac{2\,\pi}{3}+\beta}^{\frac{4\,\pi}{3}+\beta} 1\ d\alpha \right) d\beta + \int_{\frac{2\,\pi}{3}}^{\frac{4\,\pi}{3}} \left(\int_{\frac{2\,\pi}{3}+\beta}^{2\,\pi} 1\ d\alpha \right) d\beta$$
$$= \tfrac{4\,\pi^2}{9} + \int_{\frac{2\,\pi}{3}}^{\frac{4\,\pi}{3}} \left(\tfrac{4\,\pi}{3} - \beta \right) d\beta = \tfrac{4\,\pi^2}{9} + \tfrac{8\,\pi^2}{9} - \left[\tfrac{\beta^2}{2} \right]_{\frac{2\,\pi}{3}}^{\frac{4\,\pi}{3}} = \tfrac{2\,\pi^2}{3}$$

Hence:

$$P(\{length(chord) > length(equilateral\ triangle)\}) =$$
$$\frac{A(F)}{A(C)} = \frac{\frac{2\,\pi^2}{3}}{2\,\pi^2} = \frac{1}{3}$$

□

Remarks on the Proof of Proposition 1 and on Bertrand's Proofs

We have assumed at the beginning of the proof of Proposition 1 that the circle has a radius of 1 and is centered at the origin. This can be carried out since, in the general case:

$$\begin{aligned}&\overline{f}(\alpha,\ \beta) = (a + r\ \cos\alpha,\ b + r\ sen\alpha,\ a + r\ \cos\beta,\ b + r\ sen\beta) \Rightarrow\\&\overline{f}_{\alpha}(\alpha,\ \beta) = (-r\ sen\alpha,\ r\ \cos\alpha,\ 0,\ 0)\\&\overline{f}_{\beta}(\alpha,\ \beta) = (0,\ 0, -r\ sen\beta,\ r\ \cos\beta)\end{aligned}$$

Therefore, the areas are multiplied by a factor r that is cancelled in the calculus of the probability. This way, the method is scale invariant and also invariant by translations.

Since $A(C) = A(D)$, $A(F) = A(D_1)$, we have obtained the probability as:

$$P(\{length(\text{chord}) > length(\text{side of the equilateral triangle})\}) = \frac{A(D_1)}{A(D)}$$

This way, the chords with the required length are those whose α angle runs from $\frac{2\,\pi}{3} + \beta$ to $\min\left\{\frac{4\,\pi}{3} + \beta,\ 2\,\pi\right\}$ as the β angle runs from 0 to $\frac{4\,\pi}{3}$. This has the flavor of the B1 argument, in the version of [12].

This way, the probability we have obtained is the one obtained in B1 because the bijection between C and D does not change the areas of the sets.

This is not a general case. We can represent the way Bertrand modeled the chords by their midpoint as a bijection from $D - \{(\alpha,\ \beta)/\alpha = \beta + \pi\}$ onto: $C(0,\ 1) - \{(0,\ 0)\}$ that maps $D_1 - \{(\alpha,\ \beta)/\alpha = \beta + \pi\}$ onto $C\left(0,\ \frac{1}{2}\right) - \{(0,\ 0)\}$ in a bijective way.

The bijection is: $(\alpha,\ \beta) \to \left(\frac{\cos\alpha + \cos\beta}{2},\ \frac{sen\alpha + sen\beta}{2}\right)$.

This bijection does not preserve areas, and then it does not preserve the ratio between the area of the region representing the chords with the required length and the area of the region representing the set of all possible chords. This way, the probability obtained in B1 is different from that obtained in B2.

As in B3, it is only considered the radius of the middle point of the chord for the different angles; B3 can be seen as a transformation to polar coordinates $(r,\ \theta)$.

In this scenario, the chords with the required length correspond to: $F = \{(r,\ \theta)/0 \le r \le \frac{1}{2},\ 0 \le \theta \le 2\,\pi\}$ with $A(F) = \frac{1}{2}\ 2\,\pi = \pi$, and the set of all possible chords corresponds to: $P = \{(r,\ \theta)/0 \le r \le 1,\ 0 \le \theta \le 2\,\pi\}$, with $A(P) = 2\,\pi$ carrying a probability of: $\frac{A(F)}{A(P)} = \frac{\pi}{2\,\pi} = \frac{1}{2}$.

Again, the bijection from $C(0,\ 1) - \{(0,\ 0)\}$ to $P = \{(r,\ \theta)/0 \le r \le 1,\ 0 \le \theta \le 2\,\pi\}$ that maps $C\left(0,\ \frac{1}{2}\right) - \{(0,\ 0)\}$ to F changes the ratio between the areas, and then it changes the probability.

The bijection that gives the chord need not be defined by means of the midpoint: for a given $t \in (0,\ 1)$, we can define the chord by its point: $(1-t)\ (\cos\alpha,\ sen\alpha) + t\ (\cos\beta,\ sen\beta)$. As the point $(\alpha,\ \beta)$ runs in D, the points $(1-t)\ (\cos\alpha,\ sen\alpha) + t\ (\cos\beta,\ sen\beta)$ fill the set $\left\{(x,\ y)/(2\,t-1)^2 \le x^2 + y^2 \le 1\right\}$ (region between the circumferences of radius $|2\,t-1|$, 1) and, as the point $(\alpha,\ \beta)$ runs in F, the points $(1-t)\ (\cos\alpha,\ sen\alpha) + t\ (\cos\beta,\ sen\beta)$ fill the set:

$\left\{(x,\ y)/(2\,t-1)^2 \le x^2 + y^2 \le 3\,t^2 - 3\,t + 1\right\}$ (the region between the circumferences of radius $|2\,t-1|$, $\sqrt{3\,t^2 - 3\,t + 1}$).

So, if we take polar coordinates (B3), the probability of choosing a chord with the required length is $\frac{\left(\sqrt{3\,t^2-3\,t+1} - |2\,t-1|\right)\,2\,\pi}{(1-|2\,t-1|)\,2\,\pi} = \frac{\sqrt{3\,t^2-3\,t+1} - |2\,t-1|}{1-|2\,t-1|}$.

This probability ranges from $\varepsilon > 0$ such that $\varepsilon \to 0$ for $t \to 0$ or $t \to 1$, to $\frac{1}{2}$ for $t = \frac{1}{2}$.

Taking this argument to the extreme, we can find representations of the chords that give a probability of taking a chord with the required length as close to 0 or 1 as we please.

To perform this, we establish a bijection from $D_1 - \{(\alpha,\ \beta)/\alpha = \beta + \pi\}$ to a set:

$C_\varepsilon \subset C(0,\,1) - \{(0,\,0)\}$ such that $A(C_\varepsilon) = \varepsilon$ and we complete it with a bijection from $D - (D_1 \cup \{(\alpha,\,\beta)/\alpha = \beta + \pi\})$ to $C(0,\,1) - (C_\varepsilon \cup \{(0,\,0)\})$. By pasting the two bijections, we obtain a bijection from $D - \{(\alpha,\,\beta)/\alpha = \beta + \pi\}$ to $C(0,\,1) - \{(0,\,0)\}$ that maps $D_1 - \{(\alpha,\,\beta)/\alpha = \beta + \pi\}$ onto C_ε. This way, the problem of choosing a chord with the required properties is equivalent to the problem of choosing a point of C_ε when we randomly choose a point of $C(0,\,1) - \{(0,\,0)\}$ (throwing a dart, for instance).

The probability of this last problem is $P = \frac{A(C_\varepsilon)}{A(C(0,\,1))} = \frac{\varepsilon}{\pi}$, with $P = \frac{\varepsilon}{\pi} \to 0$ when $\varepsilon \to 0$, $P = \frac{\varepsilon}{\pi} \to 1$ when $\varepsilon \to \pi$: in the first case, we have "hidden" the favorable region, and in the second case, we have oversized it.

Again, the bijection has changed the areas (that is why the Jacobian is introduced to calculate an area by means of a change of variable), and then it has changed the probabilities. A way to avoid this problem is to define the probability measure in the set of chords better than in a representation of it. We will treat this approach in the next section.

3. An Inherent Measure of the Probability in the Bertrand Problem

To define a probability measure in the set of chords of the unit circle, we first need an analytical definition of this set. We set:

$$Ch = \{\{(1-t)\,(\cos\alpha,\, sen\alpha) + t\,(\cos\beta,\, sen\beta)/t \in [0,\,1]\}/0 \le \beta < \alpha \le 2\,\pi\},$$

where Ch stands for the set of chords. An element of Ch, $Ch_{\alpha,\,\beta}$ can be seen as the image set of the following mapping from [0, 1] to R^2:

$$g_{\alpha,\,\beta}(t) = (1-t)\,(\cos\alpha,\, sen\alpha) + t\,(\cos\beta,\, sen\beta)$$

We can select a chord at random by choosing a value of the angles α, β.

The distance between two chords

$$Ch_{\alpha_1,\,\beta_1} = (1-t)\,(\cos\alpha_1,\, sen\alpha_1) + t\,(\cos\beta_1,\, sen\beta_1)$$

$$Ch_{\alpha_2,\,\beta_2} = (1-t)\,(\cos\alpha_2,\, sen\alpha_2) + t\,(\cos\beta_2,\, sen\beta_2)$$

will be the distance between the corresponding mappings induced by the norm in R^2:

$$\left(\int_0^1 \|Ch_{\alpha_1,\,\beta_1} - Ch_{\alpha_2,\,\beta_2}\|^2\right)^{\frac{1}{2}}.$$

Taking into account this distance and the determinant of the Gram matrix, if we consider the function $\overline{F}(t,\,\alpha,\,\beta) = (1-t)\,(\cos\alpha,\, sen\alpha) + t\,(\cos\beta,\, sen\beta)$, we can define the measure of a subset of Ch:

$$A = \{\{(1-t)\,(\cos\alpha,\, sen\alpha) + t\,(\cos\beta,\, sen\beta)/t \in [0,\,1]\}/(\alpha,\,\beta) \in D\},\ \text{as}:$$

$$\mu(A) = \iint\limits_D \left(\int_0^1 \|\overline{F}_\alpha\|^2\,dt \int_0^1 \|\overline{F}_\beta\|^2\,dt - \left(\int_0^1 \overline{F}_\alpha \circ \overline{F}_\beta\,dt\right)^2\right)^{\frac{1}{2}} d\alpha\,d\beta \qquad (1)$$

Using this measure to calculate the probability of an event and assuming a uniform distribution of probability, we have the following solution to the Bertrand problem.

Proposition 2. *For the uniform measure of probability based on the measure defined in (1), the probability of choosing a chord in the circumference whose length is greater than the side of an equilateral triangle inscribed in the circumference is approximately 0.323545.*

Proof of Proposition 2. We have that $\overline{F}_\alpha(t,\ \alpha,\ \beta) = (1-t)\ (-sen\alpha,\ \cos\alpha)$, $\overline{F}_\beta(t,\ \alpha,\ \beta) = t\ (-sen\beta,\ \cos\beta)$, so

$$\begin{aligned}&\left\|\overline{F}_\alpha(t,\ \alpha,\ \beta)\right\|^2 = (1-t)^2,\ \left\|\overline{F}_\beta(t,\ \alpha,\ \beta)\right\|^2 = t^2,\\ &\overline{F}_\alpha(t,\ \alpha,\ \beta) \circ \overline{F}_\beta(t,\ \alpha,\ \beta) =\\ &t\ (1-t)\ (sen\alpha\ sen\beta + \cos\alpha\ \cos\beta) = t\ (1-t)\ \cos(\alpha-\beta)\end{aligned}$$

so we have that $\int_0^1 \left\|\overline{F}_\alpha(t,\ \alpha,\ \beta)\right\|^2 dt = \int_0^1 (1-t)^2\, dt = -\left[\frac{(1-t)^3}{3}\right]_0^1 = \frac{1}{3}$,

$$\int_0^1 \left\|\overline{F}_\beta(t,\ \alpha,\ \beta)\right\|^2 dt = \int_0^1 t^2\, dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3},$$

$$\begin{aligned}&\int_0^1 \overline{F}_\alpha(t,\ \alpha,\ \beta) \circ \overline{F}_\beta(t,\ \alpha,\ \beta)\, dt = \int_0^1 t\ (1-t)\ \cos(\alpha-\beta)\ dt\\ &= \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 \cos(\alpha-\beta) = \frac{1}{6}\ \cos(\alpha-\beta)\end{aligned}$$

Then,

$$\left(\int_0^1 \left\|\overline{F}_\alpha\right\|^2 dt \int_0^1 \left\|\overline{F}_\beta\right\|^2 dt - \left(\int_0^1 \overline{F}_\alpha \circ \overline{F}_\beta\, dt\right)^2\right)^{\frac{1}{2}} = \left(\frac{1}{9} - \frac{1}{36}\ \cos^2(\alpha-\beta)\right)^{\frac{1}{2}}.$$

This way, the measure of the set of chords is:

$$\begin{aligned}&\mu(Ch) = \iint_{D'} \left(\int_0^1 \left\|\overline{F}_\alpha\right\|^2 dt \int_0^1 \left\|\overline{F}_\beta\right\|^2 dt - \int_0^1 \overline{F}_\alpha \circ \overline{F}_\beta\, dt\right)^{\frac{1}{2}} d\alpha\, d\beta\\ &= \iint_{D'} \left(\frac{1}{9} - \frac{1}{36}\ \cos^2(\alpha-\beta)\right)^{\frac{1}{2}} d\alpha\, d\beta\end{aligned}$$

where $D' = \{(\alpha,\ \beta)/0 \le \alpha < \beta \le 2\,\pi\}$, so:

$$\begin{aligned}&\mu(Ch) = \iint_{D'} \left(\frac{1}{9} - \frac{1}{36}\ \cos^2(\alpha-\beta)\right)^{\frac{1}{2}} d\alpha\, d\beta\\ &= \int_0^{2\,\pi} \left(\int_\beta^{2\,\pi} \left(\frac{1}{9} - \frac{1}{36}\ \cos^2(\alpha-\beta)\right)^{\frac{1}{2}} d\alpha\right) d\beta\end{aligned}$$

This iterated integral cannot be obtained in an exact way (the inner integral is an elliptic one), but it can be approximated to 6.14699.

The measure of the set of chords R satisfying the condition is:

$$\begin{aligned}&\mu(R) = \iint_{D_1} \left(\int_0^1 \left\|\overline{F}_\alpha\right\|^2 dt \int_0^1 \left\|\overline{F}_\beta\right\|^2 dt - \int_0^1 \overline{F}_\alpha \circ \overline{F}_\beta\, dt\right)^{\frac{1}{2}} d\alpha\, d\beta\\ &= \iint_{D_1} \left(\frac{1}{9} - \frac{1}{36}\ \cos^2(\alpha-\beta)\right)^{\frac{1}{2}} d\alpha\, d\beta\end{aligned}$$

where D_1 is defined as in Proposition 1. This way:

$$\begin{aligned}&\mu(R) = \iint_{D_1} \left(\frac{1}{9} - \frac{1}{36}\ \cos^2(\alpha-\beta)\right)^{\frac{1}{2}} d\alpha\, d\beta =\\ &= \int_0^{\frac{2\,\pi}{3}} \left(\int_{\frac{2\,\pi}{3}+\beta}^{\frac{4\,\pi}{3}+\beta} \left(\frac{1}{9} - \frac{1}{36}\ \cos^2(\alpha-\beta)\right)^{\frac{1}{2}} d\alpha\right) d\beta +\\ &\int_{\frac{2\,\pi}{3}}^{\frac{4\,\pi}{3}} \left(\int_{\frac{2\,\pi}{3}+\beta}^{2\,\pi} \left(\frac{1}{9} - \frac{1}{36}\ \cos^2(\alpha-\beta)\right)^{\frac{1}{2}} d\alpha\right) d\beta \approx 1.9888\end{aligned}$$

This implies that the probability P of the required event is: $P = \frac{\mu(R)}{\mu(A)} \approx \frac{1.9888}{6.14699} = 0.323545$, as desired. □

Remark 1. *The probability obtained in Proposition 2 is very close to the probability of the first solution of Bertrand.*

4. Computational Statistics: Monte Carlo Method

We have used the Monte Carlo method to obtain a simulation of the Bertrand problem through randomized experiments. As a random variable, we take the choice of a chord of the circle of radius equal to one. This variable is uniformly distributed.

Each experiment consists of selecting 10^5 chords, choosing the initial point and the final point for every chord sampled from a uniform distribution on the $[0, 2\pi]$ interval. We count the event 'the length of the chord is greater than the side of the inscribed equilateral triangle'. For each experiment, the probability of the event is obtained. We have carried out 10^5 experiments, with the average probability of the event being 0.333331539330861.

This computational simulation supports the theoretical results obtained.

The simulation has been implemented in MATLAB R2023a. The code used is detailed in Appendix A.

5. Conclusions

We have given two new ways to calculate the probability for the so-called Bertrand problem, a classical problem in probability that carries discussion because of its appearance of paradox: different methods give different values for the probability of selecting adequate chords from a set of chords.

This is because the way to represent the set of possible chords and the set of favorable chords may modify the proportion between the measures of these sets.

For the first method, we have immersed the set of chords in the space R^4 (some existing methods, such as B2, map the set of chords to a subset of R^2). In the second obtained method, we have not represented the objective set by means of a bijection in a subset of R^k for some k. Instead of this, we have applied an intrinsic measure of the set of chords. This represents the main novelty of this paper since it avoids the problem of some of the existing methods about distortion of the probability when a bijection is applied because the bijections do not have to preserve areas. This way, the lack of randomness in some previous studies is sidestepped.

For this reason, we believe that this method is more reliable than the previous one since it directly acts on the set of chords, but we think it is debatable to consider it a definitive solution to Bertrand's problem.

Author Contributions: Conceptualization, J.R., M.L. and S.L.; methodology, J.R. and M.L.; software, S.L.; validation, S.L.; formal analysis, J.R.; investigation, J.R. and M.L.; resources, J.R., M.L. and S.L.; writing—original draft preparation, J.R., M.L. and S.L.; writing—review and editing, J.R., M.L. and S.L.; supervision, J.R., M.L. and S.L. All authors have read and agreed to the published version of the manuscript.

Funding: The first and second authors have been partially supported by the Ministerio de Economía y Competitividad of Spain, under grant PID2019-104735RB-C42.

Data Availability Statement: The data presented in this study are available on request from the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Software for Computational Experiments

We present the technical aspects of the computational experiments performed.

The following code was created with the programming languages MATLAB and Octave (Octave is free software). We simulate n experiments, obtaining m chords randomly in each one. To implement this code, the user has to save it in a plain text file with the m extension and run it in Octave (any version) or MATLAB R2023a software.

```
    radius = input('Circle radius:  ');
    side = sqrt(3)*radius;
    % n:  number of experiments
    N = input('number of experiments:  ');
    % m:  number of chords for each experiment
    m = input('number of chords:  ');
    probmedia = 0;
    for j = 1:n
        nmayor = 0;
        nmenor = 0;
        for i = 1:m
            % polar angle of one end of the chord
        theta1 = 2*pi*rand( );
        P1 = [radius*cos(theta1), radius*sin(theta1)];
        % polar angle of the other end of the chord
        theta2 = 2*pi*rand( );
        P2 = [radius*cos(theta2), radius*sin(theta2)];
        % chord:  chord length
        % function dist:
            % calculates the distance between two points
        chord = dist(P1,P2);
        if abs(chord-side) > 1e-6
            if chord < side
            % count of chords with length less
            % than the side of the triangle
            nmenor = nmenor + 1;
            else
        % count of chords with length greater
        % than the side of the triangle
        nmayor = nmayor + 1;
        end
    end
end
%prob:  probability obtained forthejth experiment
prob=nmayor/(nmenor + nmayor);
probmedia = probmedia + prob;
end
%probmedia:  average probability obtained at the n experiments
Probmedia = probmedia/n
```

References

1. Bertrand, J.L.F. *Calcul des Probabilités*; Gauthier-Villars: Paris, France, 1889; pp. 4–5.
2. Keynes, J.M. *A Treatise on Probability*; Macmillan: London, UK, 1963.
3. Jaynes, E.T. The well-posed problem. *Found. Phys* **1973**, *3*, 477–493. [CrossRef]
4. Northrop, E.P. *Riddles in Mathematics: A Book of Paradoxes*; Van Nostrand: New York, NY, USA, 1944.
5. Garwood, F.; Holroyd, E.M. The distance of a "random chord" of a circle from the centre. *Math. Gaz.* **1966**, *50*, 283–286. [CrossRef]
6. Tissier, P.E. Bertrand's Paradox. *Math. Gaz.* **1984**, *68*, 15–19. [CrossRef]
7. van Fraassen, B. *Laws and Symmetry*; Clarendon Press: Oxford, UK, 1989.
8. Chiu, S.S.; Larson, R.C. Bertrand's Paradox Revisited: More Lessons about that Ambiguous Word Random. *J. Ind. Syst. Eng.* **2009**, *3*, 1–26.
9. Chechile, R.A. Bertrand's Paradox Resolution and Its Implications for the Bing–Fisher Problem. *Mathematics* **2023**, *11*, 3282. [CrossRef]
10. Gillies, D. *Philosophical Theories of Probability*; Routledge: London, UK, 2000.

11. Shackel, N. Bertrand's Paradox and the Principle of Indifference. *Philos. Sci.* **2007**, *74*, 150175. [CrossRef]
12. Drory, A. Failure and Uses of Jaynes' Principle of Transformation Groups. *Found. Phys.* **2015**, *45*, 439–460. [CrossRef]