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Review

Several Functions Originating from Fisher–Rao Geometry of Dirichlet Distributions and Involving Polygamma Functions

Feng Qi [1,2,3] **and Ravi Prakash Agarwal** [4,*]

[1] School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, China; honest.john.china@gmail.com
[2] School of Mathematics and Physics, Hulunbuir University, Hulunbuir 021008, China
[3] Independent Researcher, Dallas, TX 75252-8024, USA
[4] Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX 78363-8202, USA
* Correspondence: ravi.agarwal@tamuk.edu

Abstract: In this paper, the authors review and survey some results published since 2020 about (complete) monotonicity, inequalities, and their necessary and sufficient conditions for several newly introduced functions involving polygamma functions and originating from the estimation of the sectional curvature of the Fisher–Rao geometry of the Dirichlet distributions in the two-dimensional case.

Keywords: sectional curvature; polygamma function; inequality; complete monotonicity; necessary and sufficient condition; lower bound; majorization

MSC: primary 33B15; secondary 26A48; 26D07; 26D15; 53B12; 58D17

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1. Fisher–Rao Geometry of Dirichlet Distributions

In the electronic arXiv preprints [1,2] and their formally published version [3], the authors investigated the geometry induced by the Fisher–Rao metric on the parameter space of the Dirichlet distributions in statistics theory, showed that the parameter space is a Hadamard manifold (that is, the manifold is geodesically complete and has negative sectional curvature everywhere), and demonstrated that the Fréchet mean of a set of the Dirichlet distributions is uniquely defined in the geometry. The papers [1–3] have been cited in [4–16]. This means that the papers [1–3] have attracted great interest from more and more mathematicians in a short time. The research has been related, or connected, or applied to several areas or subjects in mathematics and applied sciences such as differential geometry, machine learning, mathematical software, methodology, probability, and statistics theory.

In the two-dimensional case of beta distributions, let $M = \{(u, v) : u, v > 0\}$ denote the first quadrant on $\mathbb{R}^2$ and let

$$\mathrm{d}\, s^2 = \psi'(u)\,\mathrm{d}\, u^2 + \psi'(v)\,\mathrm{d}\, v^2 - \psi'(u+v)(\mathrm{d}\, u + \mathrm{d}\, v)^2$$

be the Fisher–Rao metric defined on M. The sectional curvature $K(u, v)$ of the Hadamard manifold M was given in [3] (Proposition 14) by

$$\begin{aligned} K(u,v) &= -\frac{1}{4}\frac{f(u+v)f'(u)f'(v) - f(u)f'(u+v)f'(v) - f(v)f'(u+v)f'(u)}{[f(u+v) - f(u) - f(v)]^2} \\ &= \frac{1}{4}\frac{\psi''(u)\psi''(v)\psi''(u+v)\left[\frac{\psi'(u)}{\psi''(u)} + \frac{\psi'(v)}{\psi''(v)} - \frac{\psi'(u+v)}{\psi''(u+v)}\right]}{[\psi'(u)\psi'(u+v) + \psi'(v)\psi'(u+v) - \psi'(u)\psi'(v)]^2}, \end{aligned}$$

where $f = \frac{1}{\psi'}$ and $\psi = \frac{\Gamma'}{\Gamma}$ is the logarithmic derivative of the classical Euler gamma function $\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \,\mathrm{d}\, t$ for $\Re(x) > 0$ (see [17] (Section 6.4)). The asymptotic behavior of the sectional curvature $K(u,v)$ was given in [3] (Proposition 15) by

$$\lim_{v\to 0^+} K(u,v) = \lim_{v\to 0^+} K(v,u) = \frac{1}{2}\left(\frac{3}{2} - \frac{\psi'(u)\psi'''(u)}{[\psi''(u)]^2}\right), \tag{1}$$

$$\lim_{v\to \infty} K(u,v) = \lim_{v\to \infty} K(v,u) = \frac{\psi'(u) + u\psi''(u)}{4[u\psi'(u) - 1]^2}, \tag{2}$$

$$\lim_{(u,v)\to(0^+,0^+)} K(u,v) = 0, \qquad \lim_{(u,v)\to(\infty,\infty)} K(u,v) = -\frac{1}{2}, \tag{3}$$

$$\lim_{(u,v)\to(0^+,\infty)} K(u,v) = \lim_{(u,v)\to(\infty,0^+)} K(u,v) = -\frac{1}{4}. \tag{4}$$

Based on these limits and strong numerical evidence, the authors conjectured that the negative sectional curvature $K(u,v)$ is bounded from below by $-\frac{1}{2}$.

2. Complete Monotonicity

A real function $f(u)$ defined on an interval $I \subset \mathbb{R}$ is said to be of complete monotonicity if and only if $(-1)^m f^{(m)}(u) \geq 0$ for all $m \geq 0$ and $u \in I$. (See [18] (Chapter XIII), [19] (Chapter 1), and [20] (Chapter IV)).

In the paper [13], the author showed the analyticity of the sectional curvature $K(u,v)$ as a two-variable function on M, alternatively recovered the above limits in (1)–(4), separately considered the functions at the very right ends in (1) and (2), showed that the function

$$\mathcal{H}(u) = \frac{1}{2}\left(\frac{3}{2} - \frac{\psi'(u)\psi'''(u)}{[\psi''(u)]^2}\right) \tag{5}$$

appearing in (1) is decreasing from $(0,\infty)$ onto $\left(-\frac{1}{4}, 0\right)$, found sufficient and necessary conditions on α for the function

$$\mathfrak{H}_\alpha(u) = \psi'(u) + u\psi''(u) + \alpha\left[u\psi'(u) - 1\right]^2 \tag{6}$$

and its additive inverse $-\mathfrak{H}_\alpha(u)$ to be of complete monotonicity on $(0,\infty)$, and derived a sharp two-sided inequality

$$-2 < \frac{\psi'(u) + u\psi''(u)}{[u\psi'(u) - 1]^2} < -1 \tag{7}$$

in the sense that the scalars -2 and -1 cannot be replaced by any larger and smaller ones, respectively.

The two-sided inequality (7) bounds the function at the very right end of Equation (2).

In order to prove the analyticity mentioned above, the author utilized the following known results:

1. For $\Re(u) > 0$ and $n \geq 1$, the polygamma function $\psi^{(n)}(u)$ has the integral representation

$$\psi^{(n)}(u) = (-1)^{n+1} \int_0^\infty \frac{v^n}{1 - \mathrm{e}^{-v}} \mathrm{e}^{-uv} \,\mathrm{d}\, v. \tag{8}$$

   See [17] (p. 260, 6.4.1).

2. For $n \geq 2$ and $u > 0$, the two-sided inequality

$$\frac{n-1}{n} < \frac{\left[\psi^{(n)}(u)\right]^2}{\psi^{(n-1)}(u)\psi^{(n+1)}(u)} < \frac{n}{n+1}$$

   is valid. See [21] (Corollary 2.3), [22] (Section 3.5), or [23] (Equation (1.4)).

3. A real function $\varphi(u)$ is called to be sub-additive on an interval I if $\varphi(u+v) < \varphi(u)+\varphi(v)$ is valid for all $u,v\in I$ with $u+v\in I$. If $\varphi(u+v) > \varphi(u)+\varphi(v)$, then the real function $\varphi(u)$ is said to be super-additive on the interval I. A real-variable function $\varphi:[0,\infty)\to\mathbb{R}$ is called star-shaped if $\varphi(\nu t) < \nu\varphi(t)$ for $\nu\in[0,1]$ and $t\ge 0$. Among star-shaped, convex, and super-additive functions, the following relations hold:
   (a) If the function φ is convex on $[0,\infty)$ with $\varphi(0)\le 0$, then it is star-shaped.
   (b) If the function $\varphi:[0,\infty)\to\mathbb{R}$ is star-shaped, then it is super-additive.

   See [24] (Chapter 16) and [25] (Section 3.4). By these connections, we conclusively obtain that the reciprocal $\frac{1}{\psi'(u)}$ is super-additive.
4. The polygamma function $\psi^{(n)}(u)$ for $n\ge 0$ is a single-valued analytic function over the entire complex plane, except at the points $u=-m$ where it possesses poles of order $n+1$. See [17] (p. 260, 6.4.1).

In order to alternatively recover the limits in (1)–(4), the first author of this article wrote the sectional curvature $K(u,v)$ as

$$K(u,v)=\frac{[u^2\psi''(u)]\psi''(v)[(u+v)^2\psi''(u+v)]\left[\frac{\psi'(u)}{\psi''(u)}-\frac{\psi'(u+v)}{\psi''(u+v)}\right]+[u^2\psi''(u)]\psi'(v)[(u+v)^2\psi''(u+v)]}{4(\psi'(v)u(u+v)[\psi'(u+v)-\psi'(u)]+[u\psi'(u)][(u+v)\psi'(u+v)])^2},$$

made use of the limits

$$\lim_{u\to\infty}\left[u^k\psi^{(k)}(u)\right]=(-1)^{k-1}(k-1)!,\quad k\ge 1 \tag{9}$$

and

$$\lim_{u\to 0^+}\left[u^k\psi^{(k-1)}(u)\right]=(-1)^{k}(k-1)!,\quad k\ge 1 \tag{10}$$

in [23] (p. 769) and several other articles authored by the first author and his coauthors in recent decades, employed the relation

$$\psi^{(k-1)}(u+1)=\psi^{(k-1)}(u)+(-1)^{k-1}\frac{(k-1)!}{u^k},\quad \Re(u)>0,\quad k\ge 1 \tag{11}$$

in [17] (p. 260, 6.4.6), and utilized the asymptotic expansion

$$\psi^{(n)}(u)\sim(-1)^{n-1}\left[\frac{(n-1)!}{u^n}+\frac{n!}{2u^{n+1}}+\sum_{k=1}^{\infty}B_{2k}\frac{(2k+n-1)!}{(2k)!u^{2k+n}}\right],\quad |\arg u|<\pi,\quad u\to\infty \tag{12}$$

in [17] (p. 260, 6.4.11), where B_{2k} is generated by

$$\frac{u}{e^u-1}=1-\frac{u}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{u^{2k}}{(2k)!},\quad |u|<2\pi. \tag{13}$$

The decreasing property of the function $\mathcal{H}(u)$ in (5), the sufficient and necessary conditions for the functions $\pm\mathfrak{H}_\alpha(u)$ in (6) to be of complete monotonicity on $(0,\infty)$, and the two-sided inequality (7) are new, important, and ultimate results of the paper [13].

Theorem 1 ([13] (Theorem 4))**.** *The function $\mathcal{H}(u)$ in (5) is decreasing from $(0,\infty)$ onto $\left(-\frac{1}{4},0\right)$. The function $\mathfrak{H}_\alpha(u)$ in (6) is of complete monotonicity on $(0,\infty)$ if and only if $\alpha\ge 2$, while the function $-\mathfrak{H}_\alpha(u)$ is of complete monotonicity on $(0,\infty)$ if and only if $\alpha\le 1$. The two-sided inequality (7) is valid and sharp in the sense that the numbers -2 and -1 cannot be replaced by any larger and smaller ones, respectively.*

Proof. This is a sketch of the original proof of [13] (Theorem 4). The decreasing property of the function $\mathcal{H}(u)$ in (5) follows from [23] (Theorem 2), which reads that the function $\frac{[\psi^{(n+1)}(u)]^2}{\psi^{(n)}(u)\psi^{(n+2)}(u)}$ for $n\ge 1$ decreases from $(0,\infty)$ onto $\left(\frac{n}{n+1},\frac{n+1}{n+2}\right)$.

Making use of the integral representation (8) and integrating by parts yield

$$u\psi'(u) > 1, \tag{14}$$

$$u[u\psi'(u) - 1] \to \frac{1}{2}, \quad u \to \infty, \tag{15}$$

and

$$u\psi''(u) = -\int_0^\infty \frac{e^v(2e^v - v - 2)v}{(e^v - 1)^2} e^{-uv} \,d v. \tag{16}$$

Adding (8) for $n = 1$ to (16) gives

$$\psi'(u) + u\psi''(u) = -\int_0^\infty \frac{v(e^v - 1 - v)e^v}{(e^v - 1)^2} e^{-uv} \,d v < 0, \quad u \in (0, \infty). \tag{17}$$

If the function $\mathfrak{H}_\alpha(u)$ is of complete monotonicity on $(0, \infty)$, then its first derivative

$$\mathfrak{H}'_\alpha(u) = 2\alpha[u\psi'(u) - 1][\psi'(u) + u\psi''(u)] + 2\psi''(u) + u\psi'''(u) \le 0$$

which is equivalent to $\alpha \ge 2$, where we used (14), (15), (17), and the asymptotic expansion (12). Similarly, if the function $-H_\alpha(u)$ is of complete monotonicity on $(0, \infty)$, then $\alpha \le 1$, where we used (14), (17), and the limit (10). In short, the necessary condition for $H_\alpha(u)$ to be of complete monotonicity on $(0, \infty)$ is $\alpha \ge 2$, while the necessary condition for $-H_\alpha(u)$ to be of complete monotonicity on $(0, \infty)$ is $\alpha \le 1$.

Using Formula (11) and direct computing result in

$$\begin{aligned}
H_1(u) &= \mathfrak{H}_2(u) - \mathfrak{H}_2(u+1) \\
&= 4\left(\frac{1}{u^2} + \frac{2}{u} + 2\right)\psi'(u) - \frac{4u^3 + 7u^2 + 6u + 2}{u^4} - \psi''(u) - 2(2u+1)[\psi'(u)]^2, \\
H_2(u) &= H_1(u) - H_1(u+1) \\
&= 4[\psi'(u)]^2 + \frac{\begin{bmatrix} 4u^6 + 20u^5 + 41u^4 + 48u^3 + 37u^2 + 16u + 4 \\ -4(u+1)^2(2u^3 + 5u^2 + 4u + 2)u^2\psi'(u) \end{bmatrix}}{u^4(u+1)^4},
\end{aligned}$$

and

$$H_3(u) = \frac{(u+1)^2(u+2)^2}{4}[H_2(u) - H_2(u+1)] = \psi'(u) - \frac{2u^4 + 13u^3 + 29u^2 + 27u + 8}{2u(u+1)^2(u+2)^2}.$$

Then, by virtue of the integral representation (8) for $n = 1$ and

$$\frac{1}{u^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-ut} \,d t$$

in [17] (p. 255, 6.1.1), we obtain

$$H_3(u) = \frac{1}{2} \int_0^\infty \left[\sum_{k=3}^\infty \frac{2(3 \times 2^{k-2} - 1) + (k-3)(2 \times 3^{k-1} - 2^{k-1})}{k!} v^k\right] \frac{e^{-(u+2)v}}{e^v - 1} \,d v.$$

Consequently, the function $H_3(u)$, and then the difference $H_2(u) - H_2(u+1)$, is a completely monotonic function on $(0, \infty)$. Hence, we have

$$(-1)^n[H_2(u) - H_2(u+1)]^{(n)} = (-1)^n[H_2(u)]^{(n)} - (-1)^n[H_2(u+1)]^{(n)} \ge 0$$

for $n \geq 0$. By induction, it follows that

$$(-1)^n[H_2(u)]^{(n)} \geq (-1)^n[H_2(u+1)]^{(n)} \geq (-1)^n[H_2(u+2)]^{(n)} \geq \cdots \\ \geq (-1)^n[H_2(u+m)]^{(n)} \geq \cdots \geq \lim_{m\to\infty}(-1)^n[H_2(u+m)]^{(n)}.$$

It is not difficult to see that $[H_2(u)]^{(n)} \to 0$ as $u \to \infty$ for all $n \geq 0$. Accordingly, we obtain $(-1)^n[H_2(u)]^{(n)} \geq 0$ on $(0,\infty)$ for all $n \geq 0$; that is, the function $H_2(u)$ is of complete monotonicity on $(0,\infty)$. Hence, we have

$$(-1)^n[H_1(u) - H_1(u+1)]^{(n)} = (-1)^n[H_1(u)] - (-1)^n[H_1(u+1)]^{(n)} \geq 0$$

for $n \geq 0$. By induction, it follows that

$$(-1)^n[H_1(u)]^{(n)} \geq (-1)^n[H_1(u+1)]^{(n)} \geq (-1)^n[H_1(u+2)]^{(n)} \geq \cdots \\ \geq (-1)^n[H_1(u+m)]^{(n)} \geq \cdots \geq \lim_{m\to\infty}(-1)^n[H_1(u+m)]^{(n)}.$$

It is not difficult to see that $[H_1(u)]^{(n)} \to 0$ as $u \to \infty$. This means that $(-1)^n[H_1(u)]^{(n)} \geq 0$ on $(0,\infty)$ for all $n \geq 0$. In other words,

$$(-1)^n[\mathfrak{H}_2(u) - \mathfrak{H}_2(u+1)]^{(n)} = (-1)^n[\mathfrak{H}_2(u)]^{(n)} - (-1)^n[\mathfrak{H}_2(u+1)]^{(n)} \geq 0$$

which inductively reduces to

$$(-1)^n[\mathfrak{H}_2(u)]^{(n)} \geq (-1)^n[\mathfrak{H}_2(u+1)]^{(n)} \geq (-1)^n[\mathfrak{H}_2(u+2)]^{(n)} \geq \cdots \\ \geq (-1)^n[\mathfrak{H}_2(u+m)]^{(n)} \geq \cdots \geq \lim_{m\to\infty}(-1)^n[\mathfrak{H}_2(u+m)]^{(n)}$$

for all $n \geq 0$. From the integral representation (8) and Formulas (14) and (16), it is clear that $[\mathfrak{H}_2(u)]^{(n)} \to 0$ as $u \to \infty$ for all $n \geq 0$. This means that $(-1)^n[\mathfrak{H}_2(u)]^{(n)} \geq 0$ on $(0,\infty)$ for all $n \geq 0$. In other words, the function $\mathfrak{H}_2(u)$ is of complete monotonicity on $(0,\infty)$.

When $\alpha > 2$, since

$$\mathfrak{H}_\alpha(u) = \mathfrak{H}_2(u) + (\alpha - 2)\left[u\psi'(u) - 1\right]^2$$

and, by virtue of (14),

$$u\psi'(u) - 1 = \int_0^\infty \frac{\mathrm{e}^v(\mathrm{e}^v - 1 - v)}{(\mathrm{e}^v - 1)^2}\,\mathrm{e}^{-uv}\,\mathrm{d}\,v \tag{18}$$

is of complete monotonicity on $(0,\infty)$, from the fact that the product of any finitely many completely monotonic functions is still of complete monotonicity, it follows that, when $\alpha > 2$, the function $\mathfrak{H}_\alpha(u)$ is of complete monotonicity on $(0,\infty)$. As a result, the condition $\alpha \geq 2$ is sufficient for the function $\mathfrak{H}_\alpha(u)$ to be of complete monotonicity on $(0,\infty)$.

The complete monotonicity of $\mathfrak{H}_2(u)$ implies $\mathfrak{H}_2(u) > 0$, which is equivalent to the left-hand side of the two-sided inequality (7), on $(0,\infty)$.

Similarly, we can prove that the function $-\mathfrak{H}_1(u)$ is of complete monotonicity on $(0,\infty)$, and then the function $\mathfrak{H}_\alpha$ is of complete monotonicity on $(0,\infty)$ for all $\alpha < 1$. Consequently, the condition $\alpha \leq 1$ is sufficient for the function $-\mathfrak{H}_\alpha$ to be of complete monotonicity on $(0,\infty)$.

The completely monotonic property of $-\mathfrak{H}_1(u)$ means $\mathfrak{H}_1(u) < 0$, which is equivalent to the right-hand side of the two-sided inequality (7), on $(0,\infty)$.

The sharpness of the two-sided inequality (7) on $(0,\infty)$ is concluded from the second limit in (3) and the limits in (4). The proof of Theorem 1 is complete. □

In [26] (Section 4), alternative proofs of sufficient and necessary conditions in Theorem 1 were given.

3. A Generalization of a Two-Sided Inequality

In [26], the two-sided inequality (7) was generalized as follows.

Theorem 2 ([26] (Theorem 1.1))**.** *For $\beta \in \mathbb{R}$, let*

$$H_\beta(u) = \frac{\psi'(u) + u\psi''(u)}{[u\psi'(u) - 1]^\beta} \tag{19}$$

on $(0, \infty)$. Then the following conclusions are valid:

1. *The function $H_\beta(u)$ is decreasing on $(0, \infty)$ if and only if $\beta \geq 2$, with the limits*

$$\lim_{u \to 0^+} H_\beta(u) = \begin{cases} -1, & \beta = 2; \\ 0, & \beta > 2 \end{cases} \quad \text{and} \quad \lim_{u \to \infty} H_\beta(u) = \begin{cases} -2, & \beta = 2; \\ -\infty, & \beta > 2. \end{cases}$$

2. *If $\beta \leq 1$, the function $H_\beta(u)$ is increasing on $(0, \infty)$, with the limits*

$$H_\beta(u) \to \begin{cases} -\infty, & u \to 0^+; \\ 0, & u \to \infty. \end{cases}$$

3. *The two-sided inequality (7) is true and sharp in the sense that the numbers -2 and -1 cannot be replaced by any larger and smaller ones, respectively.*

The proof of Theorem 2 depends on the convolution theorem for the Laplace transforms in [20] (pp. 91–92), Bernstein's theorem [20] (p. 161, Theorem 12b), [27] (Theorem 6.1), and the following newly-established lemma.

Lemma 1 ([26] (Lemma 2.3))**.** *Let*

$$h(v) = \begin{cases} \dfrac{\mathrm{e}^v(\mathrm{e}^v - 1 - v)}{(\mathrm{e}^v - 1)^2}, & v \neq 0; \\ \dfrac{1}{2}, & v = 0. \end{cases} \tag{20}$$

Then the following conclusions are valid:

1. *The function $h(v)$*
   (a) *satisfies the identity $h(v) + h(-v) = 1$ on $(-\infty, \infty)$.*
   (b) *is infinitely differentiable on $(-\infty, \infty)$, increasing from $(-\infty, \infty)$ onto $(0, 1)$, convex on $(-\infty, 0)$, concave on $(0, \infty)$, and logarithmically concave on $(-\infty, \infty)$.*
2. *The function $\frac{h(2v)}{h^2(v)}$ is increasing from $(-\infty, 0)$ onto $(0, 2)$ and decreasing from $(0, \infty)$ onto $(1, 2)$.*
3. *The two-sided inequality $1 < \frac{h(2v)}{h^2(v)} < 2$ is valid on $(0, \infty)$ and sharp in the sense that the lower bound 1 and the upper bound 2 cannot be replaced by any larger scalar and any smaller scalar, respectively.*
4. *For any fixed $v > 0$, the function $h(uv)h((1-u)v)$ is increasing in $u \in \left(0, \frac{1}{2}\right)$.*

The starting point of the proof of Theorem 2 is the integral representation (18), that is,

$$u\psi'(u) - 1 = \int_0^\infty h(v) e^{-uv} \,\mathrm{d}\, v > 0, \tag{21}$$

whose first derivative is equal to the integral representation (17). Moreover, the limit (10) was also used in the proof of Theorem 2.

4. A Generalization of Two Theorems

In [28], the author introduced a new function:

$$\Phi(u) = u\psi'(u) - 1, \quad u \in (0, \infty). \tag{22}$$

By this notation, the functions $\mathfrak{H}_\alpha(u)$ and $H_\beta(u)$ in (6) and (19) and the two-sided inequality (7) can be reformulated in terms of $\Phi(u)$ and its first derivative as

$$\mathfrak{H}_\alpha(u) = \Phi'(u) + \alpha\Phi^2(u), \quad H_\beta(u) = \frac{\Phi'(u)}{\Phi^\beta(u)}, \quad -2 < \frac{\Phi'(u)}{\Phi^2(u)} < -1.$$

In [28], the author generalized the functions $\mathfrak{H}_\alpha(u)$ and $H_\beta(u)$ as

$$\mathfrak{J}_{k,\lambda_k}(u) = \Phi^{(2k+1)}(u) + \lambda_k\big[\Phi^{(k)}(u)\big]^2 \tag{23}$$

and

$$J_{k,\mu_k}(u) = \frac{\Phi^{(2k+1)}(u)}{\big[(-1)^k\Phi^{(k)}(u)\big]^{\mu_k}} \tag{24}$$

for $k \in \{0\} \cup \mathbb{N}$ and $\lambda_k, \mu_k \in \mathbb{R}$ on $(0, \infty)$. These functions are analogues of some functions surveyed in the expository article [22].

The main results obtained in [28] are the following two theorems.

Theorem 3 ([28] (Theorem 2))**.** *Let $k \in \{0\} \cup \mathbb{N}$ and $\lambda_k \in \mathbb{R}$.*

1. *The function $\mathfrak{J}_{k,\lambda_k}(u)$ is of complete monotonicity on $(0, \infty)$ if and only if $\lambda_k \ge \frac{(2k+2)!}{k!(k+1)!}$.*
2. *The function $-\mathfrak{J}_{k,\lambda_k}(u)$ is of complete monotonicity on $(0, \infty)$ if and only if $\lambda_k \le \frac{1}{2}\frac{(2k+2)!}{k!(k+1)!}$.*

Theorem 4 ([28] (Theorem 3))**.** *Let $k \in \{0\} \cup \mathbb{N}$ and $\mu_k \in \mathbb{R}$.*

1. *The function $J_{k,\mu_k}(u)$ is decreasing on $(0, \infty)$ if and only if $\mu_k \ge 2$, with the limits*

$$\lim_{u\to 0^+} J_{k,\mu_k}(u) = \begin{cases} -\dfrac{1}{2}\dfrac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2; \\ 0, & \mu_k > 2 \end{cases}$$

and

$$\lim_{u\to\infty} J_{k,\mu_k}(u) = \begin{cases} -\dfrac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2; \\ -\infty, & \mu_k > 2. \end{cases}$$

2. *If $\mu_k \le 1$, the function $J_{k,\mu_k}(u)$ is increasing on $(0, \infty)$, with the limits*

$$J_{k,\mu_k}(u) \to \begin{cases} -\infty, & u \to 0^+; \\ 0, & u \to \infty. \end{cases}$$

3. *The two-sided inequality*

$$-\frac{(2k+2)!}{k!(k+1)!} < \frac{\Phi^{(2k+1)}(u)}{\big[\Phi^{(k)}(u)\big]^2} < -\frac{1}{2}\frac{(2k+2)!}{k!(k+1)!}$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.

There were two proofs of Theorem 3. The proofs of these two theorems relied on Lemma 1, the convolution theorem for the Laplace transforms in [20] (pp. 91–92), Bernstein's theorem [20] (p. 161, Theorem 12b), [27] (Theorem 6.1), the limits

$$(-1)^k u^{k+1} \Phi^{(k)}(u) \to \begin{cases} k!, & u \to 0^+ \\ \frac{k!}{2}, & u \to \infty \end{cases} \tag{25}$$

for $k \geq 0$ in [28] (Lemma 2), and the following lemma.

Lemma 2 ([28] (Lemma 6))**.** *For $k, m \in \mathbb{N}$, the function*

$$U_{k,m}(u) = \frac{1}{(u+1)^m} \frac{u^{k+m} + (u+2)^{k+m}}{u^k + (u+2)^k}$$

is decreasing on $[0, \infty)$, with $U_{k,m}(0) = 2^m$ and $\lim_{u\to\infty} U_{k,m}(u) = 1$. Equivalently, the function

$$V_{k,m}(u) = \frac{(1-u)^{k+m} + (1+u)^{k+m}}{(1-u)^k + (1+u)^k}$$

is increasing in $u \in [0, 1]$, with $V_{k,m}(0) = 1$ and $V_{k,m}(1) = 2^m$.

The proof of Lemma 2 is included in the proofs of [29] (Lemma 2.5) and [30] (Lemma 2.6). The proofs of Theorems 3 and 4 and the limits in (25) used the integral representation in (21).

By the way, the preprint [30] has been accepted by Demonstratio Mathematica on 21 December 2023 when the authors are proofreading this article.

5. Further Generalizations

For $m, n \in \{0\} \cup \mathbb{N}$ and $\omega_{m,n} \in \mathbb{R}$, let

$$Y_{m,n}(u) = \frac{\Phi^{(m+n+1)}(u)}{\Phi^{(m)}(u)\Phi^{(n)}(u)} \tag{26}$$

and

$$\mathcal{Y}_{m,n;\omega_{m,n}}(u) = \Phi^{(m+n+1)}(u) + \omega_{m,n}\Phi^{(m)}(u)\Phi^{(n)}(u).$$

It is clear that $\mathcal{Y}_{0,0;\omega_{0,0}}(u) = \mathfrak{H}_{\omega_{0,0}}(u)$, $\mathcal{Y}_{k,k;\omega_{k,k}}(u) = \mathfrak{J}_{k,\omega_{k,k}}(u)$, and $J_{k,2}(u) = Y_{k,k}(u)$, which are defined in (6), (23), and (24), respectively.

In the paper [30], the convolution theorem for the Laplace transforms in [20] (pp. 91–92), Bernstein's theorem in [20] (p. 161, Theorem 12b), the limits in (25), and a monotonicity rule for the ratio of two Laplace transforms in [31] (Lemma 4) were utilized. Moreover, the following two lemmas were newly established.

Lemma 3 ([30] (Lemma 3))**.** *Let $u, v \in \mathbb{R}$ such that $0 < 2u < v$.*

1. *When $v > 2u > 2\left(2 + \frac{1}{\ln 2}\right) = 6.885390\ldots$, the function*

$$F(u,v) = 2\left(\frac{1}{u} - \frac{1}{v-u}\right) + \frac{1}{2}\left(\frac{2^{v-u}}{v-u} - \frac{2^u}{u}\right) - \frac{2^{v-u} - 2^u}{(v-u)u}$$

is positive.

2. *For $k, m \in \mathbb{N}$ such that $6 \leq 2m < k$, the sequence $F(m, k)$ is positive.*

Lemma 4 ([30] (Lemma 4))**.** *For fixed $u \in (0, 1)$, the ratio $\frac{h(uv)}{h^u(v)}$ is increasing in v and maps from $(0, \infty)$ onto $\left(\frac{1}{2^{1-u}}, 1\right)$, where $h(v)$ is defined by (20).*

The main results of the paper [30] are the following decreasing property, complete monotonicity, and sufficient and necessary conditions.

Theorem 5 ([30] (Theorem 1))**.** *For $m, n \in \{0\} \cup \mathbb{N}$, the function $Y_{m,n}(u)$ defined in (26) is decreasing in u from $(0, \infty)$ onto the interval $\left(-\frac{2(m+n+1)!}{m!n!}, -\frac{(m+n+1)!}{m!n!}\right)$. Consequently, for $m, n \in \{0\} \cup \mathbb{N}$, the two-sided inequality*

$$-\frac{2(m+n+1)!}{m!n!} < Y_{m,n}(u) < -\frac{(m+n+1)!}{m!n!} \tag{27}$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.

Theorem 6 ([30] (Theorem 2))**.** *Let $m, n \in \{0\} \cup \mathbb{N}$ and $\omega_{m,n} \in \mathbb{R}$.*

1. *The function $(-1)^{m+n+1}\mathcal{Y}_{m,n;\omega_{m,n}}(u)$ is of complete monotonicity on $(0, \infty)$ if and only if $\omega_{m,n} \le \frac{(m+n+1)!}{m!n!}$.*
2. *The function $(-1)^{m+n}\mathcal{Y}_{m,n;\omega_{m,n}}(u)$ is of complete monotonicity on $(0, \infty)$ if and only if $\omega_{m,n} \ge \frac{2(m+n+1)!}{m!n!}$.*
3. *The two-sided inequality (27) is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.*

The proofs of these two theorems are also based on the proof of [13] (Theorem 4) (that is, Theorem 1 above). In other words, the starting point of the proofs of Theorems 5 and 6 is the integral representation (21) for $\Phi(u)$.

6. Three New Functions Involving Polygamma Functions

In [29], with the help of the function $\Phi(u)$ in (22), the author introduced three new functions

$$G(u) = u\Phi(u) - \frac{1}{2}, \tag{28}$$

$$\mathcal{G}_{k,\theta_k}(u) = G^{(2k+1)}(u) + \theta_k\left[G^{(k)}(u)\right]^2, \tag{29}$$

and

$$\mathfrak{G}_{k,\tau_k}(u) = \frac{G^{(2k+1)}(u)}{\left[(-1)^k G^{(k)}(u)\right]^{\tau_k}} \tag{30}$$

on $(0, \infty)$, where $k \in \{0\} \cup \mathbb{N}$ and $\theta_k, \tau_k \in \mathbb{R}$.

The main results of the paper [29] were stated in the following theorems.

Theorem 7 ([29] (Theorem 3.1))**.** *For $k \in \{0\} \cup \mathbb{N}$ and $\theta_k \in \mathbb{R}$,*

1. *the function $\mathcal{G}_{k,\theta_k}(u)$ is of complete monotonicity on $(0, \infty)$ if and only if $\theta_k \ge \frac{3(2k+2)!}{k!(k+1)!}$;*
2. *the function $-\mathcal{G}_{k,\theta_k}(u)$ is of complete monotonicity on $(0, \infty)$ if and only if $\theta_k \le 0$.*

Theorem 8 ([29] (Theorem 4.1))**.** *Let $k \in \{0\} \cup \mathbb{N}$ and $\tau_k \in \mathbb{R}$.*

1. *The function $\mathfrak{G}_{k,\tau_k}(u)$ is decreasing on $(0, \infty)$ if and only if $\tau_k \ge 2$.*
2. *The function $\mathfrak{G}_{k,\tau_k}(u)$ is increasing on $(0, \infty)$ if $\tau_k \le 1$.*
3. *The function $\mathfrak{G}_{k,\tau_k}(u)$ is increasing on $(0, \infty)$ only if*

$$\tau_k \le \begin{cases} \psi'(1), & k = 0; \\ -\dfrac{\psi'''(1)}{\psi'(1)\psi''(1)}, & k = 1; \\ \dfrac{k-1}{k}\dfrac{\psi^{(k-1)}(1)\psi^{(2k+1)}(1)}{\psi^{(k)}(1)\psi^{(2k)}(1)}, & k \ge 2. \end{cases}$$

4. *The limits*
$$\lim_{u\to0^+}\mathfrak{G}_{k,\tau_k}(u)=\begin{cases}-2^{\tau_0}, & k=0\\ 6\psi''(1), & k=1\\ \dfrac{2(2k+1)}{(k-1)^{\tau_k}k^{\tau_k-1}}\dfrac{\psi^{(2k)}(1)}{\left|\psi^{(k-1)}(1)\right|}, & k\ge2\end{cases}$$
and
$$\lim_{u\to\infty}\mathfrak{G}_{k,\tau_k}(u)=\begin{cases}-\infty, & \tau_k>2\\ -\dfrac{3(2k+2)!}{k!(k+1)!}, & \tau_k=2\\ 0, & \tau_k<2\end{cases}$$
are valid.
5. *The two-sided inequality*
$$-\frac{3(2k+2)!}{k!(k+1)!}<\mathfrak{G}_{k,2}(u)<\begin{cases}-4, & k=0\\ 6\psi''(1), & k=1\\ \dfrac{2(2k+1)}{(k-1)^2k}\dfrac{\psi^{(2k)}(1)}{\left|\psi^{(k-1)}(1)\right|}, & k\ge2\end{cases}$$
is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and smaller numbers, respectively.

The proofs of these two theorems employed the recursive relation (11), the asymptotic expansion (12), the convolution theorem for the Laplace transforms in [20] (pp. 91–92), [27] (Theorem 6.1), Lemma 2 recited above, the limits
$$\lim_{u\to0^+}\left[(-1)^kG^{(k)}(u)\right]=\begin{cases}\dfrac{1}{2}, & k=0\\ 1, & k=1\\ (-1)^kk(k-1)\psi^{(k-1)}(1), & k\ge2\end{cases}\tag{31}$$
and
$$\lim_{u\to\infty}\left[(-1)^ku^{k+1}G^{(k)}(u)\right]=\frac{k!}{6},\quad k\ge0\tag{32}$$
in [29] (Lemma 2.2), and the following lemma.

Lemma 5 ([29] (Lemma 2.1))**.** *Let*
$$w(v)=\begin{cases}\dfrac{\mathrm{e}^v[(v-2)\,\mathrm{e}^v+v+2]}{(\mathrm{e}^v-1)^3}, & v\ne0;\\ \dfrac{1}{6}, & v=0.\end{cases}\tag{33}$$
Then the following conclusions are valid:
1. *The function $w(v)$ is decreasing from $(0,\infty)$ onto $\left(0,\frac{1}{6}\right)$.*
2. *The function $w(v)$ is logarithmically concave on $(-\infty,\infty)$.*
3. *The function $\frac{w(2v)}{w^2(v)}$ is even on $(-\infty,\infty)$, decreasing from $(0,\infty)$ onto $(0,6)$, and increasing from $(-\infty,0)$ onto $(0,6)$.*
4. *For any fixed $v>0$, the function $w(uv)w((1-u)v)$ is increasing in $u\in\left(0,\frac{1}{2}\right)$.*

Lemmas 1, 4, and 5 are connected via the relation $h'(v)=w(v)$.

The proofs of the limits (31) and (32) and Theorems 7 and 8 employed the integral representation

$$G(u)=\int_0^\infty w(v)\,\mathrm{e}^{-uv}\,\mathrm{d}\,v, \tag{34}$$

which can be derived from the proof of [13] (Theorem 4).

7. Further Consideration of a Function

In the paper [32,33], the author noticed that the completely monotonic function $G(t)$ defined in (28) and expressed by (34) had been studied in [34] (Theorem 1), which reads that the function $u^\alpha G(u)$ is of complete monotonicity on $(0,\infty)$ if and only if $\alpha\le 0$. In other words, the completely monotonic degree of the function $\psi'(u)-\frac{1}{u}-\frac{1}{2u^2}$ with respect to u on $(0,\infty)$ is 2. For more information on the notion of completely monotonic degrees, please refer to [27] and the closely related references therein.

In the paper [32], the author introduced two functions

$$\mathscr{G}_{m,n}(u)=\frac{G^{(m+n+1)}(u)}{G^{(m)}(u)G^{(n)}(u)} \tag{35}$$

and

$$\mathscr{G}_{m,n;\lambda_{m,n}}(u)=G^{(m+n+1)}(u)+\lambda_{m,n}G^{(m)}(u)G^{(n)}(u)$$

on $(0,\infty)$, where $m,n\in\{0\}\cup\mathbb{N}$ and $\lambda_{m,n}$ is a scalar dependent of m,n. It is clear that $\mathscr{G}_{k,k}(u)=\mathfrak{G}_{k,\tau_k}(u)$ and $\mathscr{G}_{k,k;\lambda_{k,k}}(u)=\mathcal{G}_{k,\theta_k}(u)$ with $\tau_k=2$ and $\lambda_{k,k}=\theta_k$, where $\mathcal{G}_{k,\theta_k}(u)$ and $\mathfrak{G}_{k,\tau_k}(u)$ are defined in (29) and (30).

The convolution theorem for the Laplace transforms in [20] (pp. 91–92), Bernstein's theorem in [20] (p. 161, Theorem 12b), the limits in (31) and (32), and the monotonicity rule for the ratio of two Laplace transforms in [31] (Lemma 4) were employed once again. Meanwhile, Lemma 5 was generalized as the following theorem.

Lemma 6 ([32] (Lemma 2.3))**.** *The function $w(v)$ defined in (33) has the following properties:*

1. *The function $w(v)$ is infinitely differentiable, positive, and even on $(-\infty,\infty)$, is increasing on $(-\infty,0)$, and is decreasing on $(0,\infty)$.*
2. *For fixed $u\in(0,1)$, the ratio $\frac{w^u(v)}{w(uv)}$ is even in $v\in(-\infty,\infty)$ and decreasing in v from $(0,\infty)$ onto $\left(0,6^{1-u}\right)$.*

Lemmas 1, 4, 5, and 6 are linked to each other via $h'(v)=w(v)$.

The main results of [32] are the two theorems below.

Theorem 9 ([32] (Theorem 3.1))**.** *For $m,n\in\{0\}\cup\mathbb{N}$, the function $\mathscr{G}_{m,n}(u)$ in (35) is decreasing in $u\in(0,\infty)$ and maps from $(0,\infty)$:*

1. *if $(m,n)=(0,0)$, onto the interval $(-6,-4)$;*
2. *if $(m,n)\in\{(1,0),(0,1)\}$, onto the interval $\left(-12,-4\psi'(1)\right)$;*
3. *if $(m,n)\in\{(2,0),(0,2)\}$, onto the interval $\left(-18,\frac{6\psi''(1)}{\psi'(1)}\right)$;*
4. *if $(m,n)=(1,1)$, onto the interval $\left(-36,6\psi''(1)\right)$;*
5. *if $(m,n)\in\{(2,1),(1,2)\}$, onto the interval $\left(-72,-\frac{6\psi'''(1)}{\psi'(1)}\right)$;*
6. *if $m,n\ge 2$, onto the interval*

$$\left(-\frac{6(m+n+1)!}{m!n!},\frac{(m+n+1)(m+n)}{mn(m-1)(n-1)}\frac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)}\right).$$

Consequently, for $m,n\in\{0\}\cup\mathbb{N}$, the two-sided inequality

$$-\frac{6(m+n+1)!}{m!n!} < \mathscr{G}_{m,n}(u) < \begin{cases} -4, & (m,n)=(0,0) \\ -4\psi'(1), & (m,n)\in\{(1,0),(0,1)\} \\ \dfrac{6\psi''(1)}{\psi'(1)}, & (m,n)\in\{(2,0),(0,2)\} \\ 6\psi''(1), & (m,n)=(1,1) \\ -\dfrac{6\psi'''(1)}{\psi'(1)}, & (m,n)\in\{(2,1),(1,2)\} \\ \dfrac{(m+n+1)(m+n)}{mn(m-1)(n-1)}\dfrac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)}, & m,n\ge 2 \end{cases}$$

is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.

Theorem 10 ([32] (Theorem 4.1))**.** *Let $m,n \in \{0\}\cup\mathbb{N}$.*

1. *The function $(-1)^{m+n+1}\mathscr{G}_{m,n;\lambda_{m,n}}(u)$ is of complete monotonicity on $(0,\infty)$ if and only if $\lambda_{m,n} \le 0$;*
2. *The function $(-1)^{m+n}\mathscr{G}_{m,n;\lambda_{m,n}}(u)$ is of complete monotonicity on $(0,\infty)$ if and only if $\lambda_{m,n} \ge \frac{6(m+n+1)!}{m!n!}$.*

The proofs of Theorems 9 and 10 start out from the integral representation (34).

8. Lower Bound of Sectional Curvature

In [1] (Proposition 5) and [3] (Theorem 6), the sectional curvature $K(u,v)$ was proved to be negative and bounded from below.

Conjecture 1 ([2] (pp. 12–13) and [3] (p. 14))**.** *For $u,v>0$, the sectional curvature $K(u,v)$:*

1. *has a lower bound $-\frac{1}{2}$, namely, $K(u,v) > -\frac{1}{2}$;*
2. *is decreasing in both u and v.*

In the papers [12,35], the author considered the function

$$\mathcal{K}(u) = K(u,u) = \frac{1}{4}\frac{\psi''(u)}{[\psi'(u)]^2}\frac{2\psi'(u)\psi''(2u)-\psi'(2u)\psi''(u)}{[\psi'(u)-2\psi'(2u)]^2} \tag{36}$$

on $(0,\infty)$ and proved the sharp two-sided inequality

$$0 > \mathcal{K}(u) > -\frac{1}{2} \tag{37}$$

which confirms the first conjecture in Conjecture 1 along the ray line $u=v>0$ on M.

For proving the two-sided inequality (37), the author used the duplication formula

$$\psi(2u) = \frac{1}{2}\psi(u) + \frac{1}{2}\psi\left(u+\frac{1}{2}\right) + \ln 2$$

in [17] (p. 259, 6.3.8). Moreover, the author used the integral representation (8), the convolution theorem for the Laplace transforms in [20] (pp. 91–92), the limits (9), (10), and (25), [27] (Theorem 6.1), Lemma 2 mentioned above, and the author also established the following three lemmas.

Lemma 7 ([12] (Lemma 2.3))**.** *For $v > u > 0$, the function*

$$W_v(u) = \frac{\left[e^{-(v-u)/2} - e^{-u/2}\right](v-u)}{\left(1 - e^{-u/2}\right)\left[1 - e^{-(v-u)/2}\right]}$$

is increasing in $u \in (0, v)$, with limits

$$\lim_{u \to v^-} W_v(u) = 2 \quad \text{and} \quad \lim_{u \to 0^+} W_v(u) = -\infty.$$

Lemma 8 ([12] (Lemma 2.4))**.** *For $k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $a \geq 0$, we have*

$$\lim_{u \to \infty}\left(u^{k+1}\left[\psi^{(k)}(u+a) - \psi^{(k)}(u)\right]\right) = (-1)^k k! a.$$

For $k, \ell \in \mathbb{N}$ and $a \geq 0$, we have

$$\lim_{u \to \infty}\left(u^{k+\ell+1}\left[\psi^{(k)}(u)\psi^{(\ell+1)}(u) - \psi^{(k+1)}(u)\psi^{(\ell)}(u)\right]\right) = (-1)^{k+\ell}(k-1)!(\ell-1)!(k-\ell)$$

and

$$\lim_{u \to \infty}\left(u^{k+\ell+1}\left[\psi^{(k)}(u)\psi^{(\ell)}(u+a) - \psi^{(\ell)}(u)\psi^{(k)}(u+a)\right]\right) = (-1)^{k+\ell}(k-1)!(\ell-1)!(k-\ell)a.$$

Lemma 9 ([35] (Lemma 2.3))**.** *Let*

$$g(v) = \begin{cases} \dfrac{v}{1 - e^{-v}}, & v \neq 0; \\ 1, & v = 0. \end{cases} \tag{38}$$

Then the following conclusions are valid:

1. *The function $g(v)$:*
   (a) *satisfies the identity $g(v) - g(-v) = v$ on $(-\infty, \infty)$;*
   (b) *is infinitely differentiable on $(-\infty, \infty)$, increasing from $(-\infty, \infty)$ onto $(0, \infty)$, convex on $(-\infty, \infty)$, and logarithmically concave on $(-\infty, \infty)$.*
2. *The function $\frac{g(2v)}{g^2(v)}$ is increasing from $(-\infty, 0)$ onto $(0, 1)$ and decreasing from $(0, \infty)$ onto $(0, 1)$.*
3. *The two-sided inequality $0 < \frac{g(2v)}{g^2(v)} < 1$ is valid on $(0, \infty)$ and sharp in the sense that the lower bound 0 and the upper bound 1 cannot be replaced by any larger scalar and any smaller scalar, respectively.*
4. *For any fixed $v > 0$, the function $g(uv)g((1-u)v)$ is increasing in $u \in \left(0, \frac{1}{2}\right)$.*

Lemmas 1, 4–6, and 9 are connected to each other via the differential relations $g'(v) = h(v)$ and $g''(v) = w(v)$.

It is easy to see that $g(-v) = \frac{v}{e^v - 1}$ is the generating function of the classical Bernoulli numbers B_j for $j \geq 0$; see the series expansion (13). A more general function $\frac{v}{\beta^v - \alpha^v}$ for $\beta > \alpha > 0$ and its reciprocal have been being systematically investigated and extensively applied by the first author and his coauthors from the late 1990s to present. The first two papers about this topic are available at https://doi.org/10.1006/jmaa.1997.5318 (accessed 19 September 2023) and https://doi.org/10.1090/S0002-9939-98-04442-6 (accessed 19 September 2023), published while the first author was a PhD student at the University of Science and Technology of China. The latest two papers are published in *Applied and Computational Mathematics* at https://doi.org/10.30546/1683-6154.22.4.2023.443 (accessed on 19 September 2023) and in the *Electronic Research Archive* with the title "Three identities and a determinantal formula for differences between Bernoulli polynomials and numbers" with the doi code https://doi.org/10.3934/era.2024011 (accessed 19 September 2023) by Cao, López-Bonilla and the first author of this article.

The two-sided inequality (37) is a consequence of the following conclusions in [12,35].

Theorem 11 ([12] (Theorem 3.1)). *Let $p > m \ge n > q \ge 0$ be integers such that $m+n = p+q$ and let*

$$F_{p,m,n,q;c}(u) = \begin{cases} \left|\psi^{(m)}(u)\right|\left|\psi^{(n)}(u)\right| - c\left|\psi^{(p)}(u)\right|, & q = 0 \\ \left|\psi^{(m)}(u)\right|\left|\psi^{(n)}(u)\right| - c\left|\psi^{(p)}(u)\right|\left|\psi^{(q)}(u)\right|, & q \ge 1 \end{cases}$$

for $c \in \mathbb{R}$ and $u \in (0,\infty)$.

1. *For $q \ge 0$, the function $F_{p,m,n,q;c}(u)$ is of complete monotonicity in $u \in (0,\infty)$ if and only if*

$$c \le \begin{cases} \dfrac{(m-1)!(n-1)!}{(p-1)!}, & q = 0; \\ \dfrac{(m-1)!(n-1)!}{(p-1)!(q-1)!}, & q \ge 1. \end{cases}$$

2. *For $q \ge 1$, the function $-F_{p,m,n,q;c}(u)$ is of complete monotonicity in $u \in (0,\infty)$ if and only if $c \ge \frac{m!n!}{p!q!}$.*
3. *The two-sided inequality*

$$-\frac{(m+n-1)!}{(m-1)!(n-1)!} < \frac{\psi^{(m+n)}(u)}{\psi^{(m)}(u)\psi^{(n)}(u)} < 0 \tag{39}$$

for $m,n \in \mathbb{N}$ and the two-sided inequality

$$\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!} < \frac{\psi^{(m)}(u)\psi^{(n)}(u)}{\psi^{(p)}(u)\psi^{(q)}(u)} < \frac{m!n!}{p!q!} \tag{40}$$

for $m,n,p,q \in \mathbb{N}$ with $p > m \ge n > q \ge 1$ and $m+n = p+q$ are valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller scalars, respectively.

Theorem 12 ([12] (Theorem 3.2) and [35] (Theorem 1.1)). *For $k \in \mathbb{N}$ and $u \in (0,\infty)$, let*

$$\mathcal{F}_{k,\eta_k}(u) = \psi^{(2k)}(u) + \eta_k\left[\psi^{(k)}(u)\right]^2 \quad \text{and} \quad \mathfrak{F}_{k,\vartheta_k}(u) = \frac{\psi^{(2k)}(u)}{\left[(-1)^{k+1}\psi^{(k)}(u)\right]^{\vartheta_k}}.$$

Then the following conclusions are true:

1. *The function $\mathcal{F}_{k,\eta_k}(u)$ is of complete monotonicity on $(0,\infty)$ if and only if $\eta_k \ge \frac{1}{2}\frac{(2k)!}{(k-1)!k!}$.*
2. *The function $-\mathcal{F}_{k,\eta_k}(u)$ is of complete monotonicity on $(0,\infty)$ if and only if $\eta_k \le 0$.*
3. *The function $\mathfrak{F}_{k,\vartheta_k}(u)$ is decreasing on $(0,\infty)$ if and only if $\vartheta_k \ge 2$.*
4. *The function $\mathfrak{F}_{k,\vartheta_k}(u)$ is increasing on $(0,\infty)$ if and only if $\vartheta_k \le \frac{2k+1}{k+1}$.*
5. *The following limits are valid:*

$$\lim_{u \to 0^+} \mathfrak{F}_{k,\vartheta_k}(u) = \begin{cases} -\dfrac{(2k)!}{[(k)!]^{\frac{2k+1}{k+1}}}, & \vartheta_k = \dfrac{2k+1}{k+1} \\ 0, & \vartheta_k > \dfrac{2k+1}{k+1} \\ -\infty, & \vartheta_k < \dfrac{2k+1}{k+1} \end{cases}$$

and

$$\lim_{u\to\infty}\mathfrak{F}_{k,\vartheta_k}(u)=\begin{cases}-\dfrac{(2k-1)!}{[(k-1)!]^2}, & \vartheta_k=2\\ -\infty, & \vartheta_k>2\\ 0, & \vartheta_k<2.\end{cases}$$

6. *The two-sided inequality*

$$-\frac{1}{2}\frac{(2k)!}{(k-1)!k!}<\frac{\psi^{(2k)}(u)}{[(-1)^{k+1}\psi^{(k)}(u)]^2}<0$$

is valid on $(0,\infty)$ *and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and smaller numbers, respectively.*

Theorem 13 ([12] (Theorem 4.1) and [35] (Theorem 1.2))**.** *If and only if* $\nu\ge 2$*, the function*

$$I_\nu(u)=\nu\big[\psi'(u)-2\psi'(2u)\big]^2-2\psi'(u)\psi''(2u)+\psi'(2u)\psi''(u)$$

is of complete monotonicity on $(0,\infty)$*. Consequently, the two-sided inequality*

$$0<\frac{2\psi'(u)\psi''(2u)-\psi'(2u)\psi''(u)}{[\psi'(u)-2\psi'(2u)]^2}<2$$

is valid on $(0,\infty)$ *and sharp in the sense that the lower bound* 0 *and the upper bound* 2 *cannot be replaced by any greater number and any smaller number.*

Finally, the author concluded the following theorem.

Theorem 14 ([12] (Theorem 5.1) and [35] (Theorem 1.3))**.** *Let* $\mathcal{K}(u)$ *be defined by* (36)*. For* $u>0$*, the two-sided inequality* (37) *is valid on* $(0,\infty)$ *and sharp in the sense that the lower bound* $-\frac{1}{2}$ *and the upper bound* 0 *cannot be replaced by any larger scalar and any smaller scalar, respectively.*

In [11] (Sections 1 and 5), via the notion of majorization, the author reformulated and alternatively proved Theorem 11 above (that is, [12] (Theorem 3.1)) once again.

9. First Results by Majorization

Let

$$\vartheta=(\vartheta_1,\vartheta_2,\dots,\vartheta_n)\quad\text{and}\quad\theta=(\theta_1,\theta_2,\dots,\theta_n)\in\mathbb{R}^n.$$

An n-tuple ϑ is said to strictly majorize θ (denoted by $\vartheta\succ\theta$) if

$$\big(\vartheta_{[1]},\vartheta_{[2]},\dots,\vartheta_{[n]}\big)\ne\big(\theta_{[1]},\theta_{[2]},\dots,\theta_{[n]}\big),\quad\sum_{i=1}^{k}\vartheta_{[i]}\ge\sum_{i=1}^{k}\theta_{[i]},\quad\sum_{i=1}^{n}\vartheta_i=\sum_{i=1}^{n}\theta_i$$

for $1\le k\le n-1$, where $\vartheta_{[1]}\ge\vartheta_{[2]}\ge\cdots\ge\vartheta_{[n]}$ and $\theta_{[1]}\ge\theta_{[2]}\ge\cdots\ge\theta_{[n]}$ are rearrangements of ϑ and θ in descending order. See [24] (p. 8 and p. 80, Definition A.1) or the papers [36–38].

In [11], the author introduced the following two functions

$$Q_{m,n}(u)=\frac{\psi^{(m+n)}(u)}{\psi^{(m)}(u)\psi^{(n)}(u)}\quad\text{and}\quad\mathcal{Q}_{m,n;p,q}(u)=\frac{\psi^{(m)}(u)\psi^{(n)}(u)}{\psi^{(p)}(u)\psi^{(q)}(u)}\tag{41}$$

for $m,n,p,q\in\mathbb{N}$ such that $(p,q)\succ(m,n)$ on $(0,\infty)$. It is clear that $Q_{k,k}(u)=\mathfrak{F}_{k,2}(u)$ for $k\in\mathbb{N}$.

In order to study properties of the functions $Q_{m,n}(u)$ and $\mathcal{Q}_{m,n;p,q}(u)$, the author employed the integral representation (8), the convolution theorem for the Laplace transforms [20] (pp. 91–92), Bernstein's theorem [20] (p. 161, Theorem 12b), the limits in (9) and (10), the monotonicity rule for the ratio of two functions [39] (pp. 10–11, Theorem 1.25), and the monotonicity rule for the ratio of two Laplace transforms (see [31] (Lemma 4) and [40] (Section 3)). The author generalized Lemma 9 as follows.

Lemma 10 ([11] (Lemma 6))**.** *Let $g(v)$ be defined in (38). Then the following conclusions are valid:*

1. *For fixed $u \in (0,1)$, the ratio $\frac{g^u(v)}{g(uv)}$ is decreasing in v from $(0,\infty)$ onto $(0,1)$.*
2. *For $u \in \left(0,\frac{1}{2}\right)$ and $v \in (0,\infty)$, the mixed second-order partial derivative*

$$\frac{\partial^2 \ln[g(uv)g((1-u)v)]}{\partial u \partial v} > 0.$$

Lemmas 1, 4–6, 9, and 10 are connected to each other via the differential relations $g'(v) = h(v)$ and $g''(v) = w(v)$.

In order to study properties of the functions $Q_{m,n}(u)$ and $\mathcal{Q}_{m,n;p,q}(u)$, the author showed the following lemma.

Lemma 11 ([11] (Lemma 8))**.** *For $m,n,p,q \in \mathbb{N}$ such that $(p,q) \succ (m,n)$, the function*

$$\frac{u^{m-1}(1-u)^{n-1} + (1-u)^{m-1}u^{n-1}}{u^{p-1}(1-u)^{q-1} + (1-u)^{p-1}u^{q-1}}$$

is increasing in $u \in \left(0,\frac{1}{2}\right)$.

More importantly, the author created a new monotonicity rule for the ratio of two parametric integrals.

Lemma 12 ([11] (Lemma 9 and Remark 15) and [41] (Remark 7.2))**.** *Let the functions $U(u)$, $V(u) > 0$, and $W(u,v) > 0$ be integrable in $u \in (a,b)$. If the ratios $\frac{\partial W(u,v)/\partial v}{W(u,v)}$ and $\frac{U(u)}{V(u)}$ are both increasing or both decreasing in $u \in (a,b)$, then the ratio*

$$R(v) = \frac{\int_a^b U(u)W(u,v)\,\mathrm{d}\,u}{\int_a^b V(u)W(u,v)\,\mathrm{d}\,u}$$

is increasing in v; if one of the ratios $\frac{\partial W(u,v)/\partial v}{W(u,v)}$ and $\frac{U(u)}{V(u)}$ is increasing and the other is decreasing in $u \in (a,b)$, then the ratio $R(v)$ is decreasing in v.

Lemma 12 has been applied in [33,42–44] and generalized in [45–47]. There have been a number of papers, plenty of studies in the literature, and many mathematicians contributing to various monotonicity rules, and we just take two examples [48,49] here.

Aside from the alternative proof of Theorem 11 above (that is, [12] (Theorem 3.1)), the remaining main results in [11] are included in the following theorems.

Theorem 15 ([11] (Theorem 11))**.** *For $m,n \in \mathbb{N}$, the function $Q_{m,n}(u)$ defined in (41) is decreasing from $(0,\infty)$ onto $\left(-\frac{(m+n-1)!}{(m-1)!(n-1)!},0\right)$. Consequently, the two-sided inequality (39), that is,*

$$-\frac{(m+n-1)!}{(m-1)!(n-1)!} < Q_{m,n} < 0, \quad m,n \in \mathbb{N},$$

is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.

Theorem 16 ([11] (Theorem 12))**.** *For $m,n,p,q \in \mathbb{N}$ with the majorizing relation $(p,q) \succ (m,n)$, the ratio $\mathcal{Q}_{m,n;p,q}(u)$ defined in (41) is decreasing from $(0,\infty)$ onto the interval*

$\left(\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!}, \frac{m!n!}{p!q!}\right)$. *Consequently, for* $m,n,p,q \in \mathbb{N}$ *with* $(p,q) \succ (m,n)$*, the two-sided inequality* (40)*, that is,*

$$\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!} < \mathcal{Q}_{m,n;p,q}(u) < \frac{m!n!}{p!q!},$$

is valid on $(0,\infty)$ *and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller scalars, respectively.*

10. Second Results by Majorization

In [42], the author considered the function

$$\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(u) = (-1)^{i+j}\Phi^{(i)}(u)\Phi^{(j)}(u) - (-1)^{\ell+m}\Omega_{i,j;\ell,m}\Phi^{(\ell)}(u)\Phi^{(m)}(u) \tag{42}$$

on $(0,\infty)$, where $i,j,\ell,m \geq 0$ are integers such that $(i,j) \succ (\ell,m)$, the scalar $\Omega_{i,j;\ell,m}$ is dependent of $i,j;\ell,m$, and $\Phi(u)$ is defined by (22).

For discovering the sufficient and necessary conditions for the function in (42) to be of complete monotonicity on $(0,\infty)$, the author made use of Lemma 11, the convolution theorem for Laplace's transforms [20] (pp. 91–92), Bernstein's theorem [20] (p. 161, Theorem 12b), and the last property in Lemma 1. Meanwhile, the author proved a new lemma below.

Lemma 13 ([42] (Lemma 4))**.** *For* $i,j,\ell,m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ *with* $(i,j) \succ (\ell,m)$*, the inequality* $i!j! > \ell!m!$ *is valid.*

The main results of the paper [42] were stated in the following theorem.

Theorem 17 ([42] (Theorem 1))**.** *Let* $i,j,\ell,m \geq 0$ *be integers such that* $(i,j) \succ (\ell,m)$*.*

1. *If* $\Omega_{i,j;\ell,m} \leq 1$*, the function* $\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(u)$ *defined in* (42) *is of complete monotonicity on the interval* $(0,\infty)$*;*
2. *The function* $-\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(u)$ *is of complete monotonicity on* $(0,\infty)$ *if and only if* $\Omega_{i,j;\ell,m} \geq \frac{i!j!}{\ell!m!}$*;*
3. *The two-sided inequality*

$$1 < \frac{\Phi^{(i)}(u)\Phi^{(j)}(u)}{\Phi^{(\ell)}(u)\Phi^{(m)}(u)} < \frac{i!j!}{\ell!m!}$$

is valid on $(0,\infty)$ *and the right-hand-side inequality is sharp in the sense that the number* $\frac{i!j!}{\ell!m!}$ *cannot be replaced by any smaller one.*

The proof of Theorem 17 starts off from the integral representation (21) for $\Phi(u)$.

11. Third Results by Majorization

In the paper [33], the author introduced

$$\mathbf{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(u) = (-1)^{i+j}G^{(i)}(u)G^{(j)}(u) - (-1)^{\ell+m}\Lambda_{i,j;p,q}G^{(p)}(u)G^{(q)}(u) \tag{43}$$

on $(0,\infty)$, where $i,j,p,q \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ such that $(i,j) \succ (p,q)$, the quantity $\Lambda_{i,j;p,q}$ is a scalar dependent of $i,j;p,q$, and $G(u)$ is defined in (28).

Making use of the convolution theorem of Laplace transforms in [20] (pp. 91–92), Bernstein's theorem in [20] (p. 161, Theorem 12b), Lemmas 11 and 13, [27] (Theorem 6.1), and the logarithmic concavity of $w(t)$ in Lemma 5, the author discovered the following sufficient and necessary conditions.

Theorem 18 ([33] (Theorem 3.1))**.** *Let* $i,j,p,q \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ *such that* $(i,j) \succ (p,q)$*.*

1. *If* $\Lambda_{i,j;p,q} \le 1$, *the function* $\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(u)$ *defined by* (43) *is of complete monotonicity on* $(0,\infty)$.
2. *The function* $-\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(u)$ *is of complete monotonicity on* $(0,\infty)$ *if and only if* $\Lambda_{i,j;p,q} \ge \frac{i!j!}{p!q!}$.
3. *The two-sided inequality*
$$1 < \frac{G^{(i)}(u)G^{(j)}(u)}{G^{(p)}(u)G^{(q)}(u)} < \frac{i!j!}{p!q!}$$
is valid on $(0,\infty)$ *and the right-hand-side inequality is sharp in the sense that the number* $\frac{i!j!}{p!q!}$ *cannot be replaced by any smaller one.*

The integral representation (34) is the starting point of the proof of Theorem 18 above.

12. Yang–Tian's Investigations on Qi's Guesses and Problems

There are a number of guesses and problems proposed in the eleven papers in [11–13,26,28–30,32,33,35,42].

For $n \ge 2$ and two non-negative integer tuples $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_n) \in \mathbb{N}_0^n$ and $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{N}^n$, let
$$P_{\vartheta,\theta;C_{\vartheta,\theta}}(u) = \prod_{r=1}^{n} \psi^{(\vartheta_r)}(u) - C_{\vartheta,\theta} \prod_{r=1}^{n} \psi^{(\theta_r)}(u) \quad \text{and} \quad Q_{\vartheta,\theta}(u) = \frac{\prod_{r=1}^{n} \psi^{(\vartheta_r)}(u)}{\prod_{r=1}^{n} \psi^{(\theta_r)}(u)}$$
on $(0,\infty)$, where we denote $\psi^{(0)}(u) = -1$ for our own convenience. It is clear that
$$P_{(2k,0),(k,k);C_{(2k,0),(k,k)}}(u) = \mathcal{F}_{k,-C_{(2k,0),(k,k)}}(u), \qquad Q_{(2k,0),(k,k)}(u) = \mathfrak{F}_{k,2}(u),$$
$$Q_{(m+n,0),(m,n)}(u) = Q_{m,n}(u), \qquad Q_{(m,n),(p,q)}(u) = Q_{m,n;p,q}(u).$$

The author proposed in [11] (Remark 19) a problem as follows.

Problem 1 ([11] (Remark 19))**.** *For* $\vartheta \succ \theta$, *discuss sufficient and necessary conditions on* $C_{\vartheta,\theta} \in \mathbb{R}$ *such that the function* $P_{\vartheta,\theta;C_{\vartheta,\theta}}(u)$ *and its additive inverse* $-P_{\vartheta,\theta;C_{\vartheta,\theta}}(u)$ *are completely monotonic on* $(0,\infty)$, *respectively.*

Meanwhile, the author also proposed in [11] (Remark 19) a guess as follows.

Guess 1 ([11] (Remark 19))**.** *If* $\vartheta \succ \theta$, *the function* $Q_{\vartheta,\theta}(u)$ *is increasing from* $(0,\infty)$ *onto the interval*
$$\left(\prod_{r=1}^{n} \frac{\vartheta_r!}{\theta_r!}, \prod_{r=1}^{n} \frac{(\vartheta_r - 1)!}{(\theta_r - 1)!}\right).$$

In [50] (Corollaries 6 and 8), Yang and Tian solved Problem 1. See also [50] (Remark 9).

In [51] (Theorem 3 and Corollary 1), Yang and Tian gave an answer to Guess 1. In [51] (Theorem 1), they generalized the above Theorem 15 (that is, [11] (Theorem 11)). In [51] (Theorem 2), they provided an equivalence of the above Theorem 16 (that is, [11] (Theorem 12)). In [51] (Theorem 4), Yang and Tian generalized a part of [12] (Theorem 3.2).

13. An Open Problem Related to the Lower Bound of Sectional Curvature

We now propose an open problem related to the conjecture that the negative sectional curvature $K(u,v)$ is lower-bounded by $-\frac{1}{2}$.

Problem 2. *For fixed* $\alpha > 0$ *and* $n \in \mathbb{N}$, *the function* $\frac{1}{1-\psi^{(n)}(u+\alpha)/\psi^{(n)}(u)}$ *is convex on* $(0,\infty)$.

This problem looks simple. However, factually, it is not easy or trivial in practice. This problem was even posted at the sites https://mathoverflow.net/q/396837 (accessed on 18 September 2023) and https://www.researchgate.net/post/How_to_prove_convexity_of_a_simple_function_involving_a_ratio_of_two_polygamma_functions (accessed on 18 September 2023).

If Problem 2 were solved for the special case $\alpha = \frac{1}{2}$, then the sectional curvature $\mathcal{K}(u)$ defined in (36) would be decreasing in $u > 0$, and then the second claim in Conjecture 1 would be confirmed along the half-line $u = v > 0$ on M.

14. Conclusions

Several of the articles [11–13,26,28–30,32,33,35,42] have been cited by the papers [4,16,46,50–58], in which the first author is not an author. The articles [4,59] are siblings.

Except for the guess and problem proposed in [11] (Remark 19), which have been answered and solved in [50,51], many other guesses and problems proposed in the articles [11–13,26,28–30,32,33,35,42] still remain open and unsolved.

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