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On a Generalized Gagliardo–Nirenberg Inequality with Radial Symmetry and Decaying Potentials

Mirko Tarulli [1,2,3,†] and George Venkov [4,*,†]

1 Mathematics and Science Department, American University in Bulgaria, 1 Georgi Izmirliev Sq., 2700 Blagoevgrad, Bulgaria; mtarulli@aubg.edu
2 Institute of Mathematics and Informatics, Bulgarian Academy of Science, Acad. Georgi Bonchev Str., Block 8, 1113 Sofia, Bulgaria
3 Department of Mathematics, University of Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy
4 Department of Mathematical Analysis and Differential Equations, Faculty of Applied Mathematics and Informatics, Technical University of Sofia, 1756 Sofia, Bulgaria
* Correspondence: gvenkov@tu-sofia.bg
† These authors contributed equally to this work.

Abstract: We present a generalized version of a Gagliardo–Nirenberg inequality characterized by radial symmetry and involving potentials exhibiting pure power polynomial behavior. As an application of our result, we investigate the existence of extremals for this inequality, which also correspond to stationary solutions for the nonlinear Schrödinger equation with inhomogeneous nonlinearity, competing with H^s-subcritical nonlinearities, either of a local or nonlocal nature.

Keywords: fractional Laplacian; radially symmetric potential; nonhomogeneous potential; Gagliardo–Nirenberg inequality; nonlocal nonlinearity

MSC: 30L15; 35A23; 35R11; 46B50; 46E35

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1. Introduction

Consider the Cauchy problem associated with the fractional NLS, posed on $\mathbb{R}^d$ with $d \geq 2$:

$$\begin{cases} i\partial_t u - (-\Delta)^s u - |x|^{-\gamma}|u|^{\tilde{q}} u = f(x,u), & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $\tilde{q} \geq 0$ and γ are nonlinear parameters, the fractional Laplacian is defined, via Fourier transform, by $\left(\widehat{(-\Delta)^s u}\right)(\xi) = (2\pi\xi)^s \hat{u}(\xi)$, provided $s \leq \frac{d}{2}$, $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $u_0(x)$ is an initial datum assumed to be in some function space, and $f(x,u)$ denotes a general nonlinearity. The stationary points of the above evolution equation satisfy the following nonlinear fractional Laplacian equation:

$$(-\Delta)^s u + |x|^{-\gamma}|u|^{\tilde{q}} u = f(x,u). \tag{2}$$

We consider nonlinearities of type

$$f(x,u) = |u(x)|^{\tilde{p}} u(x), \quad \text{for} \quad 0 < \tilde{p} < \frac{4s}{d-2s} \tag{3}$$

and

$$f(x,u) = \int_{\mathbb{R}^d} \frac{|u(x)|^{p-2} u(x) |u(y)|^p}{|x-y|^{d-\alpha}} dy, \quad \text{for} \quad 0 < p < \frac{d+\alpha}{d-2s}, \quad 0 < \alpha < d. \tag{4}$$

A substantial body of literature exists regarding the radial symmetry of solutions to elliptic equations of type (2) with $s \geq 1$, with the research tradition dating back to the seminal work [1]. As a result, it is not feasible to provide an exhaustive list of works in this context. However, for our aim, we cite some significant compactness and existence results linked to (2) available in key sources like [2–7], among others. Conversely, there appears to be a notable gap in the literature regarding the analysis of similar properties for (2) when $0 < s < 1$. We recall, in this direction [8–13]. We concentrate our attention on [14–16], addressing the references therein for a comprehensive overview of the topics. The phenomenon of symmetry breaking for (2) with nonlinearity of type (3) is investigated in [14], establishing several compact embedding theorems for Sobolev-type spaces involving radial functions with polynomial weight. In [15], the existence of radial ground states of (2) in the case (3) is demonstrated for $q = 2$ and $\gamma = 1$. Finally, in [16], a set of embeddings for the fractional space in the presence of a radial potential is proven by using Lions-type theorems and a refined Sobolev inequality with the Morrey norm. These embeddings are utilized also to inspect the existence of ground state solutions for (2) in the case (3) with $q = 2$ and $\gamma \neq 0$. Motivated by that, we generalize the above outcomes, extending the range of the parameters p, q, γ and s associated with the corresponding embeddings for function spaces. In addition, we improve the Gagliardo–Nirenberg-type inequalities with symmetry related to (2), generalizing them to the nonlocal frame, and, as a direct consequence, we shed light on the extremals of the corresponding minimization problems (see Remarks 1, 3, 4, 5 and 7 for a complete overview of the details). We note also that our work contains a compact embedding result that extends to $s \leq \frac{1}{2}$ the outcomes obtained in [14]. More precisely, one can observe that the inequality due to Strauss (see [17]) suggests the presence of a continuous representative, thereby establishing its validity solely for $s > \frac{1}{2}$. However, this is not a strong restriction. This limitation possesses a structural aspect only, indicating that functions within $H^s(\mathbb{R}^d)$ with s small lack pointwise representations. Moreover, this notion aligns very well with the classical Sobolev embedding theorem, which says that $H^s(\mathbb{R}^d)$ embeds into $C^{0,s-d/2}(\mathbb{R}^d)$ (see for example [18,19]). We bypass this obstacle by using a set of inequalities well suited to handle the case $s \leq \frac{1}{2}$, in combination with a continuity argument (see Remarks 2 and 6).

2. Preliminaries

Before stating our main achievements, we introduce some necessary notations as well as several useful results. We say that a function u is rapidly decreasing, that is, $u \in \mathcal{S}(\mathbb{R}^d)$ with

$$\mathcal{S}(\mathbb{R}^d) = \left\{ u \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \left| x^\alpha D^\beta u(x) \right| < +\infty \right\},$$

for all multi-indices $\alpha, \beta \in \mathbb{N}$. The Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ with $L^1_{loc}(\mathbb{R}^d)$ Fourier transform endowed with the norm

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi.$$

We recall also that the fractional Laplacian, for $0 < s < 1$, can be defined by

$$(-\Delta)^s u(x) = C_{d,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{d+2s}} dy,$$

with $C_{d,s}$, a normalization constant (see [19–21]). Thus, in this regime, we have

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} = \left\| (-\Delta)^{\frac{s}{2}} u \right\|^2_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dx dy. \quad (5)$$

We denote by $L^q_\gamma(\mathbb{R}^d)$ the weighted Lebesgue space with the norm

$$\|u\|^q_{L^q_\gamma(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \frac{|u|^q}{|x|^\gamma} dx.$$

Moreover, we introduce also

$$\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) := \dot{H}^s(\mathbb{R}^d) \cap L^q_\gamma\left(\mathbb{R}^d\right),$$

with the norm

$$\|u\|^2_{\dot{H}^{s,q,\gamma}(\mathbb{R}^d)} := \|u\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|u\|^2_{L^q_\gamma(\mathbb{R}^d)}.$$

In addition, let $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ be the set of radial functions in $\dot{H}^{s,q,\gamma}(\mathbb{R}^d)$. We define $\mathcal{B}_R(0) = \{x \in \mathbb{R}^d \| x| < R\}$. Let be a set $E \subset \Omega \subseteq \mathbb{R}^d$. We denote by $E^c = \Omega \backslash E$ the complement of E in Ω. For any two positive real numbers a, b, we write $a \lesssim b$ (resp. $a \gtrsim b$) to denote $a \leq Cb$ (resp. $Ca \geq b$), with $C > 0$, disclosing the constant only when it is essential. Concerning compactness, we have (see [22]):

Proposition 1 (Riesz–Kolmogorov). *Let Ω be an open subset of $\mathbb{R}^d$, $1 \leq p < \infty$, and let $S \subset L^p(\Omega)$ be such that*

1. $\sup_{u \in S} \|u\|_{L^p(\Omega)} < \infty$;
2. *for every $\varepsilon > 0$, there exists compact $K \subset \Omega$ such that* $\sup_{u \in \mathcal{K}} \int_{K^c} |u|^p dx \leq \varepsilon^p$;
3. *for every compact $K \subset \Omega$,* $\lim_{y \to 0} \sup_{u \in \mathcal{K}} \|u(\cdot + y) - u(\cdot)\|_{L^p(K)} = 0$.

Then $\mathcal{K}$ is precompact in $L^p(\Omega)$.

We need the following generalization of the Strauss lemma (see [14], Theorem 3.1):

Proposition 2. *Let $d \geq 2, s > \frac{1}{2}, q > 1$, and*

$$-d(q-1) \leq \gamma < (d-1).$$

Then

$$|x|^\sigma |u(x)| \leq C\|u\|^\eta_{\dot{H}^s(\mathbb{R}^d)} \|u\|^{1-\eta}_{L^q_\gamma(\mathbb{R}^d)}, \tag{6}$$

for any $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$, where

$$\sigma = \frac{2s(d-1) - (2s-1)\gamma}{(2s-1)q + 2}, \quad \eta = \frac{2}{(2s-1)q + 2}.$$

Notice that a particular case of the previous (6) is the inequality

$$\sup_{|x|>0} |x|^{\frac{d-2s}{2}} |u(x)| \lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}, \tag{7}$$

valid for all $u \in \dot{H}^s_{rad}(\mathbb{R}^d)$. We have also (see [14,20,23])

Proposition 3. *Let $d \geq 2$ and $0 < s < d/2$. Then*

$$\left(\int_{\mathbb{R}^d} |u(x)|^r |x|^{-\beta r} \, dx \right)^{\frac{1}{r}} \leq C\|u\|_{\dot{H}^s(\mathbb{R}^d)}, \tag{8}$$

for any $u \in \dot{H}^s_{rad}(\mathbb{R}^d)$, where $r \geq 2$ and

$$-(d-1)\left(\frac{1}{2} - \frac{1}{r}\right) \leq \beta < \frac{d}{r}, \quad \frac{1}{r} = \frac{1}{2} + \frac{\beta - s}{d}. \tag{9}$$

A particular case of the above inequality (8) is the following estimate contained in [20].

Proposition 4. *Assume $d \geq 2, 0 < s \leq 1/2$ and $\frac{1}{2} - s \leq \frac{1}{p} \leq \frac{1}{2} - \frac{s}{d}$. Then for $R > 0$, the inequality*

$$\int_{B_R^c(0)} |u(x)|^p dx \leq C R^{d-p\left(\frac{d}{2}-s\right)} \|u\|^p_{\dot{H}^s_{\text{rad}}(\mathbb{R}^d)}, \tag{10}$$

with $C = C(d,s,p) > 0$, if fulfilled for any $u \in \dot{H}^s_{rad}(\mathbb{R}^d)$.

The following result deals with the local Hölder continuity property of functions in $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ (see [14]).

Proposition 5. *Let $B_R^c(0)$, with $R > 0$ and $s > \frac{1}{2}$. Then, the continuous representation of $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ is Hölder continuous in $B_R^c(0)$, and moreover, there exists a constant $C > 0$ such that*

$$|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^{\frac{2qs-q}{2qs+2-q}} \|u\|_{\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)}. \tag{11}$$

Moreover, we have the following (see [22]).

Proposition 6. *Let $1 < p < \infty$, and let u_j, $j \in \mathbb{N}$ be a sequence weakly convergent to u in $L^p(\Omega)$, with $\Omega \subseteq \mathbb{R}^d$. Then, $u_j \in L^p(\Omega)$ is bounded and*

$$\|u\|_{L^p(\Omega)} \leq \varliminf_{j\to\infty} \|u_j\|_{L^p(\Omega)}.$$

Let us recall the following generalized Leibnitz fractional rule (see [24]).

Proposition 7. *Suppose $1 < p < \infty, s \geq 0$ and*

$$\frac{1}{\ell} = \frac{1}{\ell_i} + \frac{1}{\tilde{\ell}_i},$$

with $i = 1,2, 1 < \ell_1 \leq \infty, 1 < \tilde{\ell}_2 \leq \infty$. Then

$$\left\|(-\Delta)^{\frac{s}{2}}(fg)\right\|_{L^\ell(\mathbb{R}^d)} \leq C\left(\left\|(-\Delta)^{\frac{s}{2}}(f)\right\|_{\ell_1} \|g\|_{L^{\tilde{\ell}_1}(\mathbb{R}^d)} + \|f\|_{\ell_2}\left\|(-\Delta)^{\frac{s}{2}}g\right\|_{L^{\tilde{\ell}_2}(\mathbb{R}^d)}\right), \tag{12}$$

where the constants $C > 0$ depend on all of the parameters above but not on f and g.

We have the following Hardy–Littlewood–Sobolev inequality (see Lemma 2.4 in [25]):

Proposition 8. *For $0 < \alpha < d$ and $p > 1$, there exists a sharp constant $C = C(d,p,\alpha) > 0$ such that*

$$\left\|\int_{\mathbb{R}^d} |x|^{\alpha-d} * u(x) dy\right\|_{L^q(\mathbb{R}^d)} \leq C\|u\|_{L^p(\mathbb{R}^d)}, \tag{13}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ and $p < \frac{d}{\alpha}$.

We also have the Hausdorff–Young inequality (see for example [26]).

Proposition 9. *Assuming that f in $L^p(\mathbb{R}^d)$, we have then*

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}, \tag{14}$$

with $1 \leq p \leq 2$.

The next tool is a Brezis–Lieb lemma for the nonlocal term (see Theorem in [27]).

Lemma 1. *Let $d > 1, 0 < \alpha < d, 1 \leq p \leq \frac{2d}{d+\alpha}$ and $u_j, j \in \mathbb{N}$ be a bounded sequence in $L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d)$. If $u_j \to u$ almost everywhere on $\mathbb{R}^d$ as $j \to \infty$, then*

$$\lim_{j\to\infty}\left(\int_{\mathbb{R}^d}\left(|x|^{\alpha-d} * |u_j|^p\right)|u_j|^p dx - \int_{\mathbb{R}^d}\left(|x|^{\alpha-d} * |u_j - u|^p\right)|u_j - u|^p dx\right) = \int_{\mathbb{R}^d}\left(|x|^{\alpha-d} * |u|^p\right)|u|^p dx. \quad (15)$$

3. Main Results

We start with the following.

Theorem 1 (Continuous Embedding I)**.** *Let $d \geqslant 2$ and $\frac{1}{2} < s < \frac{d}{2}$ and $q > 1$. Then we have that*

$$\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \quad (16)$$

with

$$p \in [p^*_{s,\gamma}, p^*_s], \quad \frac{1}{q} > \frac{d-2s}{2d-2\gamma} \quad (17)$$

or

$$p \in (p^*_s, p^*_{s,\gamma}], \quad \frac{1}{q} < \frac{d-2s}{2d-2\gamma}, \quad (18)$$

where

$$p^*_{s,\gamma} := q + \frac{((2s-1)q+2)\gamma}{2s(d-1)-(2s-1)\gamma}, \qquad p^*_s := \frac{2d}{d-2s} \quad (19)$$

and

$$\gamma \in \left(0, \frac{2s(d-1)}{2s-1}\right). \quad (20)$$

In addition, we have the following.

Theorem 2 (Compact Embedding I)**.** *Let $d \geqslant 2$, $\frac{1}{2} < s < \frac{d}{2}$ and $q > 1$. Then we have that*

$$\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^d), \quad (21)$$

for $p \neq p^*_{s,\gamma}$ and $p \neq p^*_s$, where $p^*_{s,\gamma}$, p^*_s, q as in (17) *or* (18)*, with $p^*_{s,\gamma}$, p^*_s defined as in* (19) *and $0 < \gamma < d$ as in* (20)*.*

Remark 1. *The embeddings* (16) *and* (21) *in the case* (17) *were available in* [14]*. We improved here the lower bound of the range of admissibility for p. The embeddings in the case* (18) *were given in* [16] *with $q = 2$; we extended them to $q > 1$.*

We prove also the following.

Theorem 3 (Continuous Embedding II)**.** *Let $d \geqslant 2$ and $0 < s \leq \frac{1}{2}$, $q > 1$ and $p^*_{s,\gamma}$, p^*_s defined as in* (19)*. Then we have that*

$$\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \quad (22)$$

$$p \in [p^*_{s,\gamma}, p^*_s], \quad \frac{1}{q} > \frac{d-2s}{2d-2\gamma} \quad (23)$$

or

$$p \in (p^*_s, p^*_{s,\gamma}], \quad \frac{1}{2} - s < \frac{1}{q} < \frac{d-2s}{2d-2\gamma}. \quad (24)$$

The previous result is supported further by

Theorem 4 (Compact Embedding II)**.** *Let $d \geqslant 2$, $0 < s < \frac{d}{2}$, $0 < \gamma < d$ and $q > 1$ such that*

$$q > 2 - \frac{\gamma}{d-1}. \tag{25}$$

Then we have that

$$\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^d), \tag{26}$$

for $p \neq p^*_{s,\gamma}$ and $p \neq p^*_s$, where $p^*_{s,\gamma}$, p^*_s, q as in (17), (18) or (23), (24), with $p^*_{s,\gamma}$, p^*_s defined as in (19).

Remark 2. *The embeddings (22) and (26) in the case (17) were obtained in [16] with $q = 2$; we generalized them to $q > 1$. Let us underline that Theorem 4 is new in the literature and breaks down the dichotomy $s > \frac{1}{2}$ and $s \leq \frac{1}{2}$. In addition, we bypass the application of Proposition 5, which is mandatory to achieve the crucial equicontinuity property in order to apply the method appearing in [14,28]. This property, which is based on the representations of a radial function with Fourier transform in $L^1_{loc}(\mathbb{R}^d)$ by means of the Jost functions (see [29]), relies on the fact that $s > \frac{1}{2}$ (see the proof of Lemma 4.1 in [14]). We pay only the extra restriction (25). However, it perfectly handles the embedding in the case (17) of the work [14] and extends it to the case (18).*

Finally, we have the following.

Theorem 5. *Let $d \geqslant 2$, $\frac{1}{2} < s < \frac{d}{2}$, $1 \leq q, p < \infty$, $-\infty < \gamma < 0$ and $p^*_{s,\gamma}$, p^*_s defined as in (19). Then we have that*

$$\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \tag{27}$$

with

$$p \in [p^*_{s,\gamma}, p^*_s], \quad \frac{1}{q} > \frac{d-2s}{2d+2|\gamma|} \tag{28}$$

or

$$p \in (p^*_s, p^*_{s,\gamma}], \quad \frac{1}{q} < \frac{d-2s}{2d+2|\gamma|}, \quad |\gamma| < d(q-p). \tag{29}$$

Moreover, the embedding is compact for $p \neq p^*_{s,\gamma}$ and $p \neq p^*_s$.

Remark 3. *The compact embedding (27) in the case (28) was available in [14]; we also extended here the lower bound of the range of admissibility for p. The compact embedding in the case (29) was proven in [16] with $q = 2$. We improved it to $q > 1$.*

As a consequence of the above results we obtain the following.

Theorem 6. *Let $d \geq 2$, $0 < s < \frac{d}{2}$, $-d(q-1) < \gamma < d$, $1 \leq q, p < \infty$ and $p^*_{s,\gamma}$, p^*_s defined as in (19). There exists a constant $C = C(d, s, \gamma, q, p) > 0$ such that the scaling-invariant inequality*

$$\int_{\mathbb{R}^d} |u(x)|^p dx \leq C \|u\|^{\frac{2p(d-\gamma)-2dq}{2d-2\gamma-q(d-2s)}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{2d-p(d-2s)}{2d-2\gamma-q(d-2s)}} \tag{30}$$

holds for all functions $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ if

$$p \in [p^*_{s,\gamma}, p^*_s], \quad \frac{1}{q} > \frac{d-2s}{2d-2\gamma},$$
$$p \in (p^*_s, p^*_{s,\gamma}], \quad s > \frac{1}{2}, \quad \frac{1}{q} < \frac{d-2s}{2d-2\gamma},$$

so that (18) *or* (29) *is fulfilled with the extra condition*

$$\gamma \in \left(0, \frac{2s(d-1)}{2s-1}\right).$$

Furthermore, the inequality (30) *remains valid if*

$$p \in [p_s^*, p_{s,\gamma}^*], \quad s \le \frac{1}{2}, \quad \frac{1}{2} - s < \frac{1}{q} < \frac{d-2s}{2d-2\gamma},$$

with $\gamma > 0$.

We also have the following.

Corollary 1. *Let* $d \ge 2, 0 < s < \frac{d}{2}, -d(q-1) < \gamma < d, 1 \le q, p < \infty$ *and* $0 < \alpha < d$. *There exists a constant* $C = C(d, s, \gamma, q, p) > 0$ *such that the scaling-invariant inequality*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{d-\alpha}} dxdy \le C \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{4p(d-\gamma)-2q(d+\alpha)}{2d-2\gamma-q(d-2s)}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx\right)^{\frac{2(d+\alpha)-2p(d-2s)}{q(2d-2\gamma-q(d-2s))}} \tag{31}$$

holds for all functions $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ *if*

$$p \in [p_{s,\alpha,\gamma}^*, p_s^*], \quad \frac{1}{q} > \frac{d-2s}{2d-2\gamma},$$

$$p \in [p_{s,\alpha}^*, p_{s,\alpha,\gamma}^*], \quad s > \frac{1}{2}, \quad \frac{1}{q} < \frac{d-2s}{2d-2\gamma}, \quad (|\gamma| < d(q-p), \gamma < 0),$$

are fulfilled with the extra condition,

$$\gamma \in \left(0, \frac{2s(d-1)}{2s-1}\right),$$

where

$$p_{s,\alpha,\gamma}^* := \frac{(d+\alpha)q}{2d} + \frac{(d+\alpha)((2s-1)q+2)\gamma q}{4ds(d-1)-(2s-1)\gamma}, \quad p_{s,\alpha}^* := \frac{d+\alpha}{d-2s}.$$

Furthermore, the inequality (31) *remains valid if* $\gamma > 0$,

$$p \in [p_{s,\alpha}^*, p_{s,\alpha,\gamma}^*], \quad s \le \frac{1}{2}, \quad \frac{1}{2} - s < \frac{1}{q} < \frac{d-2s}{2d-2\gamma}.$$

Remark 4. *The inequality* (30) *in the cases* (17) *and* (28) *was available in* [14] *(and seminally in* [15]*, for* $q = 2$ *and* $\gamma = 1$*); we improved the lower bound of the domain of admissibility for* p. *Moreover, we extended it in the ranges given in* (18) *and* (29)*, respectively. The inequality* (31) *appears for the first time in the literature.*

Let us introduce now the Weinstein-type functionals

$$W_1^{p,q,s,\gamma}(u) := \frac{\|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p+2)(d-\gamma)-2dq}{2d-2\gamma-q(d-2s)}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^{q+2}}{|x|^\gamma} dx\right)^{\frac{2d-(p+2)(d-2s)}{q(2d-2\gamma-q(d-2s))}}}{\int_{\mathbb{R}^d} |u|^{p+2} dx} \tag{32}$$

and

$$W_2^{p,q,s,\alpha,\gamma}(u) := \frac{\|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{4p(d-\gamma)-2q(d+\alpha)}{2d-2\gamma-q(d-2s)}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^{q+2}}{|x|^\gamma} dx\right)^{\frac{2(d+\alpha)-2p(d-2s)}{q(2d-2\gamma-q(d-2s))}}}{\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{d-\alpha}} dxdy}. \tag{33}$$

Finally, by concentration–compactness arguments, we are in a position to show also the following.

Theorem 7. *Let $\frac{1}{2} < s < 1$, $p^*_{s,\gamma} < p < \frac{2d}{d-2s}$, with $p^*_{s,\gamma}$, q and γ as in Theorem 6. Then, there exists a function $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ such that $W_1^{p,q,s,\gamma}(u) = m$ with $m > 0$ and so that*

$$m = \inf\Big\{ W_1^{p,q,s,\gamma}(u); u \neq 0, u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \Big\}.$$

Analogously, it is possible to prove the following.

Corollary 2. *Let $\frac{1}{2} < s < 1$, $p^*_{s,\alpha,\gamma} < p < \frac{d+\alpha}{d-2s}$, with $p^*_{s,\alpha,\gamma}$, q, α and γ as in Corollary 1. Then, there exists a function $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ such that $W_2^{p,q,s,\alpha,\gamma}(u) = m$ with $m > 0$ and so that*

$$m = \inf\Big\{ W_2^{p,q,s,\alpha,\gamma}(u); u \neq 0, u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \Big\}.$$

Remark 5. *Theorem 7 and Corollary 2 are new in the literature.*

Outline of the paper. The paper is organized as follows. After introducing some preliminaries in Section 2 and presenting the main results in Section 3, through Section 4, we prove, in Theorems 1 and 3, the continuous embedding of the function spaces $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ into the Lebesgue spaces $L^p(\mathbb{R}^d)$. The principal target of Section 5 is to unveil that the previous embeddings are compact. This is achieved with Theorems 2, 4 and 5. We underline that in Theorem 4, we introduce a new method to prove the compactness of the embedding of $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$. This approach allows us to handle both $s > \frac{1}{2}$ and $s \leq \frac{1}{2}$, avoiding the use of Proposition 5. In Section 6, we give the proof of the Gagliardo–Nirenberg inequalities (30) and (31). Finally, in Section 7, we prove Theorem 7 and Corollary 2 and thus the existence of positive radial solutions in $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ for (2).

4. Embedding in Function Spaces: Continuity

We provide the proof of Theorems 1 and 3. We start with the following.

Proof of Theorem 1. Let us choose $R > 0$. We shall estimate the L^p norm of $u \in \dot{H}^{s,q,\gamma}_{rad}$ separately in $\mathcal{B}_R(0)$ and in $\mathcal{B}^c_R(0)$, respectively. Since $p < \frac{2d}{d-2s}$, in $B_R(0)$, we have, by using the Sobolev embedding, the following.

$$\int_{B_R(0)} |u(x)|^p dx \lesssim R^{1-p\left(\frac{1}{2}-\frac{s}{d}\right)} \|u\|^p_{L^{p^*_s}(\mathbb{R}^d)} \lesssim R^{1-p\left(\frac{1}{2}-\frac{s}{d}\right)} \|u\|^p_{\dot{H}^s(\mathbb{R}^d)}. \tag{34}$$

To handle the estimate in $\mathcal{B}^c_R(0)$, we follow the lines of the one given in [14] by using now the inequality (6). More precisely, we have

$$\begin{aligned} \int_{\mathcal{B}^c_R(0)} |u(x)|^p dx &\leq \sup_{|x|>R} \left(|u(x)||x|^{\frac{\gamma}{p-q}} \right)^{p-q} \int_{\mathcal{B}^c_R(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \\ &\lesssim \|u\|^{\frac{2(p-q)}{(2s-1)q+2}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{(1-\eta)(p-q)}{q}} \int_{\mathcal{B}^c_R(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \\ &\lesssim \|u\|^{\frac{2(p-q)}{(2s-1)q+2}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2}+1}. \end{aligned} \tag{35}$$

Note that, in order to apply (6), one needs that

$$\frac{\gamma}{p-q} \leq \sigma = \frac{2s(d-1-\gamma)+\gamma}{(2s-1)q+2}, \tag{36}$$

which is fulfilled since $q < p^*_{s,\gamma} < p$ and $q < \frac{2d-2\gamma}{d-2s}$. We shall look now at the embedding (16) in the case (18). On $B^c_R(0)$, for any $p > \frac{2d}{d-2s}$, we can estimate

$$\int_{B^c_R(0)} |u(x)|^p dx \lesssim \left\| |u(x)||x|^{\frac{(d-2s)}{2}} \right\|^p_{L^\infty(\mathcal{B}^c_R(0))} \int_{B^c_R(0)} |x|^{-\frac{p(d-2s)}{2}} dx \lesssim CR^{d-p\left(\frac{d}{2}-s\right)} \|u\|^p_{\dot{H}^s(\mathbb{R}^d)}, \tag{37}$$

by an application of the Hölder inequality together with (7). To achieve a bound in $\mathcal{B}_R(0)$, we observe that $p^*_s < q < p^*_{s,\gamma}$ due to $q > \frac{2d-2\gamma}{d-2s}$, and hence we can assume that $q < p < p^*_{s,\gamma}$. Then, we obtain

$$\begin{aligned}\int_{\mathcal{B}_R(0)} |u(x)|^p dx &\le \left\| |u(x)||x|^{\frac{\gamma}{p-q}} \right\|^{p-q}_{L^\infty(\mathcal{B}_R(0))} \int_{\mathcal{B}_R(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \\ &\lesssim \|u\|^{\frac{2(p-q)}{(2s-1)q+2}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2}+1}.\end{aligned} \tag{38}$$

Bear in mind that in this framework, to apply the inequality (6), we need the elementary bound $|x|^{\frac{\gamma}{p-q}} \lesssim |x|^\sigma$, for $|x| \lesssim 1$, which is guaranteed if

$$\frac{\gamma}{p-q} \ge \sigma = \frac{2s(d-1-\gamma)+\gamma}{(2s-1)q+2}. \tag{39}$$

This completes the proof. □

Our next target is the following.

Proof of Theorem 3. Letting $R > 0$, we control the L^p norm of $u \in \dot{H}^{s,q,\gamma}_{rad}$ in $\mathcal{B}_R(0)$ in the same way that we did in the proof of Theorem 1 because of $p < \frac{2d}{d-2s}$. The estimate in $B^c_R(0)$ can be handled by using now the inequality (8). In fact, we achieve, by selecting $q < p^*_{s,\gamma} < p < r$ and by a direct application of the Hölder inequality,

$$\begin{aligned}\int_{\mathcal{B}^c_R(0)} |u|^p dx &\le \left(\int_{\mathcal{B}^c_R(0)} |u(x)|^r |x|^{\gamma\frac{r-p}{p-q}} dx \right)^{\frac{p-q}{r-q}} \left(\int_{\mathcal{B}^c_R(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}} \\ &\lesssim \left(\int_{\mathcal{B}^c_R(0)} |u(x)|^r |x|^{-r\beta} dx \right)^{\frac{p-q}{r-q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}} \\ &\lesssim \|u\|^{r\frac{p-q}{r-q}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}},\end{aligned} \tag{40}$$

where in the second line of the above inequality we applied (8), with r and β solution of the system

$$\frac{1}{r} = \frac{1}{2} + \frac{\beta - s}{d}, \quad \frac{r-p}{p-q} = \frac{\beta}{\gamma}.$$

The previous identities read as

$$r = \frac{2(\gamma p - d(p-q))}{2\gamma - (d-2s)(p-q)}, \quad \beta = \frac{1}{2}\frac{\gamma(2d - p(d-2s))}{\gamma p - d(p-q)}, \tag{41}$$

because of the relations (9). It is easy to see that $\beta < 0$ because $q < \frac{2d-2\gamma}{d-2s}$ and $p < \frac{2d}{d-2s}$. In addition, we require also that

$$\frac{1}{2}\frac{\gamma(2d - p(d-2s))}{\gamma p - d(p-q)} \ge \frac{1-d}{2}\frac{\gamma(p-2) - 2s(p-q)}{\gamma p - d(p-q)}, \tag{42}$$

due to the second condition in (41), which is satisfied when $p \geq p^*_{s,\gamma}$. Notice that we can rewrite

$$p^*_{s,\gamma} = \begin{cases} \dfrac{2}{1-2s} + \dfrac{(2s(d-1)((1-2s)q-2)}{(1-2s)(2s(d-1)+\gamma(1-2s))}, & s \neq \dfrac{1}{2} \\[2ex] \dfrac{q(d-1)+2\gamma}{d-1}, & s = \dfrac{1}{2}. \end{cases} \tag{43}$$

Let us examine now the case

$$\frac{(1-2s)(2d-2\gamma)}{d-2s} < (1-2s)q < 2, \quad q < p \leq p^*_{s,\gamma}. \tag{44}$$

In this regime, we bound the L^p norm of $u \in \dot{H}^{s,q,\gamma}_{rad}$ in $\mathcal{B}^c_R(0)$ by using the inequality (10), because of $p > \frac{2d}{d-2s}$. As for the region $\mathcal{B}_R(0)$, we argue exactly as in (40), that is

$$\begin{aligned} \int_{\mathcal{B}_R(0)} |u|^p dx &\leq \left(\int_{\mathcal{B}_R(0)} |u(x)|^r |x|^{\gamma \frac{r-p}{p-q}} dx \right)^{\frac{p-q}{r-q}} \left(\int_{\mathcal{B}_R(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}} \\ &\lesssim \|u\|^{r\frac{p-q}{r-q}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}}, \end{aligned}$$

by taking notice now that $\beta < 0$ since $q > \frac{2d-2\gamma}{d-2s}$ and that the second condition in (41) is fulfilled if one has

$$\frac{1}{2} \frac{\gamma(2d-p(d-2s))}{\gamma p - d(p-q)} \leq \frac{1-d}{2} \frac{\gamma(p-2)-2s(p-q)}{\gamma p - d(p-q)},$$

which means

$$\frac{1}{p} \geq \frac{2s(d-1)+\gamma(1-2s)}{2qs(d-1)+2\gamma} = \frac{1}{p^*_{s,\gamma}}.$$

The proof is then completed. □

5. Embedding in Function Spaces: Compactness

This section is divided into two parts. The first concerns the compactness results for functions in $\dot{H}^s(\mathbb{R}^d)$, with $s > \frac{1}{2}$. The second is devoted to shedding light on the compact embeddings for $s \leq \frac{1}{2}$.

5.1. Compactness: Higher Regularity

Let us focus now on the proof of Theorems 2 and 5. To show compactness, we follow the classical argument introduced in [28] and lately extended in [14], with some refinements. More precisely, the following.

Proof of Theorem 2. Observe that the space $\dot{H}^{s,q,\gamma}(\mathbb{R}^d)$ is reflexive. Then, it suffices to show that every given sequence u_j converging weakly to 0 in $H^{s,q,\gamma}_{rad}(\mathbb{R}^d)$, converges strongly in $L^p(\mathbb{R}^d)$, that is $\|u_j\|_{L^p(\mathbb{R}^d)} \to 0$. Given $\varepsilon > 0$, we split $\mathbb{R}^d$ in three parts, and thus:

$$\|u_j\|^p_{L^p(\mathbb{R}^d)} = \int_{|x|>R} |u_j(x)|^p dx + \int_{|x|<R^{-1}} |u_j(x)|^p dx + \int_{R \leq |x| \leq R^{-1}} |u_j(x)|^p dx, \tag{45}$$

where $R = R(\varepsilon)$ will be chosen later. Assume now that conditions (18) are satisfied. We have, arguing as in the proof of (37),

$$\int_{|x|>R} |u_j(x)|^p dx \leq C R^{d-p(\frac{d}{2}-s)} \leq \frac{\varepsilon}{3}, \tag{46}$$

for $R \geq R_1(\varepsilon)$, given that $p > \frac{2d}{d-2s}$. We have also, by using the inequality (6),

$$\begin{aligned}\int_{|x|<R^{-1}} |u_j(x)|^p dx &= \int_{|x|<R^{-1}} \frac{|u_j(x)|^q}{|x|^\gamma} |u_j(x)|^{p-q} |x|^\gamma dx \\ &\leq C \int_{|x|<R^{-1}} \frac{|u_j(x)|^q}{|x|^\gamma} |x|^{\gamma-\sigma(p-q)} dx \\ \leq CR^{\sigma(p-q)-\gamma} \int_{\mathbb{R}^d} \frac{|u_j(x)|^q}{|x|^\gamma} dx &\leq CR^{\sigma(p-q)-\gamma} < \frac{\varepsilon}{3},\end{aligned} \tag{47}$$

for $R \geq R_2(\varepsilon)$ and $\gamma > \sigma(p-q)$ which is fulfilled for $p < p^*_{s,\gamma}$ once $q > \frac{2d-2\gamma}{d-2s}$. Finally, by choosing $R = \max\{R_1(\varepsilon), R_2(\varepsilon)\}$, we observe that according to the Hölder continuity property (11) of the proposition, we have

$$\begin{aligned}\int_{R\leq|x|\leq R^{-1}} |u_j(x+y) - u_j(x)|^p dx &\leq \int_{R\leq|x|\leq R^{-1}} |u_j(x+y) - u_j(x)|^p dx \\ &\leq \|u_n\|^p_{\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)} \int_{R\leq|x|\leq R^{-1}} |y|^{p\alpha} dx,\end{aligned}$$

with $y \in \mathbb{R}^d$ and

$$\alpha = \frac{2qs - q}{2qs + 2 - q}.$$

By Proposition 1, the sequence u_j, $j \in \mathbb{N}$ admits a subsequence u_{j_k}, which converges almost everywhere to 0 on the compact set

$$\left\{x \in \mathbb{R}^d \mid R \leqslant |x| \leqslant R^{-1}\right\}.$$

By taking $j \in \mathbb{N}$ to be large enough, one obtains

$$\int_{R\leq|x|\leq R^{-1}} |u_j(x)|^p dx < \frac{\varepsilon}{3}. \tag{48}$$

Thus, by (46) and (47) and the above inequality, we work out $\|u_j\|_{L^p(\mathbb{R}^d)} \to 0$ for $p^*_s < p < p^*_{s,\gamma}$, as $j \to \infty$. The case depicted in (17) can be handled in a similar way as in [14], with the following difference that we argue as in the proof of (34) and exploit the bound,

$$\int_{|x|<R^{-1}} |u_j(x)|^p dx \leq CR^{p(\frac{1}{2}-\frac{s}{d})-1}, \tag{49}$$

if one uses again (45). The proof is now completed. □

5.2. Compactness: Unified Approach

In this section, inspired by [20], we present a method to show compactness with the main scope of treating both the cases of functions with low and high regularity in a unified manner. Let us consider now the following.

Proof of Theorem 4. We select $\varphi \in \mathcal{S}(\mathbb{R}^d)$. By the fractional Leibniz rule (12) and the Sobolev embedding, we obtain

$$\begin{aligned}\left\|(-\Delta)^{\frac{s}{2}}\varphi u\right\|_{L^2(\mathbb{R}^d)} &\lesssim \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2(\mathbb{R}^d)} \|\varphi\|_{L^\infty(\mathbb{R}^d)} + \left\|(-\Delta)^{\frac{s}{2}}\varphi\right\|_{L^{\frac{2d}{d+2s}}(\mathbb{R}^d)} \|u\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \\ &\lesssim \|\varphi u\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim \|u\|_{\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)},\end{aligned}$$

where the last inequality is provided by Theorem 3. For all $\tilde{R} > 0$, we pick a smooth $\varphi(x)$ such that $\varphi(x) = 1$ in $\mathcal{B}_{\tilde{R}}(0)$ and $\varphi(x) = 0$ in $\mathcal{B}^c_{2\tilde{R}}(0)$. Let us set φu_j, with u_n, $n \in \mathbb{N}$, being a bounded sequence in $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$. Furthermore, one has that φu_j is bounded also in $H^s(\mathbb{R}^d)$

because of the continuous embedding $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d)$ which is a consequence of Theorem 3. In fact, if $q < p^*_{s,\gamma} < p < p^*_s$ with $q < \frac{2d-2\gamma}{d-2s}$ as in (25), one can see that

$$p^*_{s,\gamma} := q + \frac{((2s-1)q+2)\gamma}{2s(d-1)-(2s-1)\gamma} > 2,$$

while the case (44), with $p > p^*_s$ is straightforward. This bears to the fact that φu_j converges weakly to some w in $L^2(\mathbb{R}^d)$ with support still in $B_{2\tilde{R}}(0)$. Notice that we have also $\widehat{w} \in L^\infty(\mathbb{R}^d)$. By application of Plancharel's identity, we achieve

$$\|\varphi u_j - w\|_{L^2(\mathbb{R}^d)} \leq \|\widehat{\varphi u_j} - \widehat{w}\|_{L^2(\mathcal{B}_R(0))} + \|\widehat{\varphi u_j} - \widehat{w}\|_{L^2(\mathcal{B}^c_R(0))} \tag{50}$$

for any $R > 0$. Then

$$\|\widehat{\varphi u_j} - \widehat{w}\|_{L^2(\mathcal{B}^c_R(0))} \leq \frac{1}{R^s}\|\varphi u_j - w\|_{H^s(\mathbb{R}^d)}, \tag{51}$$

which means that the quantity $\widehat{\varphi u_j}(\xi) - \widehat{w}(\xi)$ is uniformly small if $|\xi|$ is sufficiently large. In addition, if one observes that

$$\lim_{n\to\infty}\langle \varphi u_j - w, e^{ix\cdot\xi}\rangle_{L^2(\mathbb{R}^2)} = \lim_{n\to\infty}(\widehat{\varphi u_j} - \widehat{w}) = 0,$$

by the definition of the Fourier transform and of the weak convergence in $L^2(\mathbb{R}^d)$, we have $\widehat{\varphi u_j}(\xi)$ tends to $\widehat{w}(\xi)$ almost everywhere as $j \to \infty$. By (50) and (51) and Hölder's inequality, we have

$$\left\|\varphi u_j - v\right\|_{L^1(\mathcal{B}_{\tilde{R}}(0))} \lesssim \left\|\varphi u_j - w\right\|_{L^2(\mathbb{R}^d)} \lesssim R^{\frac{d}{2}}\left\|\widehat{\varphi u_j} - \widehat{w}\right\|_{L^\infty(\mathbb{R}^d)} + \frac{1}{R^s}\left\|\varphi u_j - w\right\|_{H^s(\mathbb{R}^d)} \tag{52}$$

for a suitable $R > 0$. Additionally, by an application of the Young–Hausdorff inequality (14) and again Hölder's inequality, we see that

$$\left\|\widehat{\varphi u_j} - \widehat{w}\right\|_{L^\infty(\mathbb{R}^d)} \lesssim \left\|\varphi u_j - v\right\|_{L^1(\mathcal{B}_{2\tilde{R}}(0))} \lesssim \tilde{R}^{\frac{d}{2}}\left\|\varphi u_j - v\right\|_{L^2(\mathcal{B}_{2\tilde{R}}(0))} \lesssim \tilde{R}^{\frac{d}{2}}\left\|\varphi u_j - v\right\|_{H^s(\mathbb{R}^d)}. \tag{53}$$

The bounds (52) and (53) allow us to acquire the uniform estimate

$$\begin{aligned} &\left\|\varphi u_j - v\right\|_{L^1(\mathcal{B}_{\tilde{R}}(0))} \\ &\lesssim (R\tilde{R})^{\frac{d}{2}}\left\|\varphi u_j - v\right\|_{H^s(\mathbb{R}^d)} + \frac{1}{R^s}\left\|\varphi u_j - w\right\|_{H^s(\mathbb{R}^d)} \lesssim \left\|\varphi u_j - w\right\|_{\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)}. \end{aligned} \tag{54}$$

By a use of Lebesgue's dominated convergence theorem, we have that u_j converges to u in the $L^1(B_{\tilde{R}}(0))$ and thus almost everywhere, once $j \to \infty$. This shows that $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ is compactly embedded in $L^1_{loc}(\mathbb{R}^d)$. To deal with the general case, we shall use a continuity argument in conjunction with a perturbation argument. Namely, if $0 < s \leq \frac{1}{2}$, $q < p^*_{s,\gamma} < p < p^*_s$ with $q < \frac{2d-2\gamma}{d-2s}$ enjoying (25), we note that the constraint (42) is fulfilled with the strict inequality. We pick a $\gamma(\varepsilon) = \gamma + \varepsilon$ with $\varepsilon > 0$ that gives rise to a new set of parameters $(p, q, \gamma(\varepsilon), p^*_{s,\gamma(\varepsilon)}, \beta(\varepsilon), r(\varepsilon))$. We have that

$$\lim_{\varepsilon\to 0}(p^*_{s,\gamma(\varepsilon)}, \beta(\varepsilon), r(\varepsilon)) = (p^*_{s,\gamma}, \beta, r).$$

By (43), one can readily see that $p^*_{s,\gamma(\varepsilon)}$ approaches $p^*_{s,\gamma}$ since it is a decreasing function of ε. Moreover, by (41), we obtain

$$\beta(\varepsilon) = \frac{d - p(d-2s)}{p}\left(1 + \frac{d(p-q)}{\gamma(\varepsilon)p - d(p-q)}\right)$$

and that $\beta(\varepsilon) \nearrow \beta < 0$, $r(\varepsilon) \searrow r$ as $\varepsilon \to 0$. In conclusion, we can choose ε to be suitably small while still ensuring $q < \frac{2d-2\gamma}{d-2s}$, $p > p^*_{s,\gamma}$, and one can proceed as for (40) and deduce by the Hölder inequality the following

$$\begin{aligned}\int_{\mathcal{B}^c_R(0)} |u|^p dx &\leq \left(\int_{\mathcal{B}^c_R(0)} |u(x)|^r |x|^{\gamma(\varepsilon)\frac{r(\varepsilon)r-p}{p-q}} dx\right)^{\frac{p-q}{r(\varepsilon)-q}} \left(\int_{\mathcal{B}^c_R(0)} \frac{|u(x)|^q}{|x|^{\gamma+\varepsilon}} dx\right)^{\frac{r(\varepsilon)-p}{r(\varepsilon)-q}} \\ &\lesssim \frac{1}{R^{\varepsilon\frac{r(\varepsilon)-p}{r(\varepsilon)-q}}} \left(\int_{\mathcal{B}^c_R(0)} |u(x)|^r |x|^{-r(\varepsilon)r\beta(\varepsilon)} dx\right)^{\frac{p-q}{r(\varepsilon)-q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx\right)^{\frac{r(\varepsilon)-p}{r(\varepsilon)-q}} \\ &\lesssim \frac{1}{R^{\varepsilon\frac{r(\varepsilon)-p}{r(\varepsilon)-q}}} \|u\|^{\frac{p-q}{r(\varepsilon)-q}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx\right)^{\frac{r(\varepsilon)-p}{r(\varepsilon)-q}}.\end{aligned} \tag{55}$$

Let $s > \frac{1}{2}$. Selecting again $\gamma(\varepsilon) = \gamma + \varepsilon$, we can see that if ε is small enough so that $q < \frac{2d-2\gamma(\varepsilon)}{d-2s}$, the inequality

$$\frac{\gamma(\varepsilon)}{p-q} < \sigma(\varepsilon) = \frac{2s(d-1) - \gamma(\varepsilon)(2s-1)}{(2s-1)q+2}, \tag{56}$$

is still valid. We obtain then, similarly as for (35),

$$\begin{aligned}\int_{\mathcal{B}^c_R(0)} |u(x)|^p dx &\leq \sup_{|x|>R} \left(|u(x)||x|^{\frac{\gamma(\varepsilon)}{p-q}}\right)^{p-q} \int_{\mathcal{B}^c_R(0)} \frac{|u(x)|^q}{|x|^{\gamma(\varepsilon)}} dx \\ &\lesssim \frac{1}{R^\varepsilon} \|u\|^{\frac{2(p-q)}{(2s-1)q+2}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx\right)^{\frac{(1-\eta)(p-q)}{q}} \int_{\mathcal{B}^c_R(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \\ &\lesssim \frac{1}{R^\varepsilon} \|u\|^{\frac{2(p-q)}{(2s-1)q+2}}_{\dot{H}^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx\right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2}+1},\end{aligned} \tag{57}$$

where in the second inequality, we use that $\sigma(\varepsilon) \nearrow \sigma$ for $\varepsilon \to 0$, with $\sigma(\varepsilon)$ as in (56) and $|x|^{\sigma(\varepsilon)} \lesssim |x|^\sigma$ for $|x| \geq R > 1$. In the case (44), $p > p^*_s$, we recall instead that we have, by (10),

$$\int_{\mathcal{B}^c_R(0)} |u(x)|^p dx \lesssim \frac{1}{R^{p\left(\frac{d}{2}-s\right)-d}} \|u\|^p_{\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)}, \tag{58}$$

for $0 < s \leq \frac{1}{2}$ and, by (37), one can write the similar inequality

$$\int_{B^c_R(0)} |u(x)|^p dx \lesssim CR^{d-p\left(\frac{d}{2}-s\right)} \|u\|^p_{\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)}, \tag{59}$$

when $s > \frac{1}{2}$. The previous (55), (56), (58), and (59) give that

$$\lim_{R\to\infty} \sup_{j\in\mathbb{N}} \|u_j\|_{L^p(\mathcal{B}^c_R(0))} \to 0,$$

with $p \geq 1$. The proof of the theorem follows by interpolation with the case $p = 1$ and by the above embedding $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^1_{loc}(\mathbb{R}^d)$. $\square$

We conclude the section with the following.

Proof of Theorem 5. To show (27) if (28) is satisfied, we shall estimate again the L^p norm of $u \in \dot{H}^{s,q,\gamma}_{rad}$ in $\mathcal{B}_R(0)$ and in $\mathcal{B}^c_R(0)$. The bound in $B_R(0)$, when $p < \frac{2d}{d-2s}$, is the same as in (34). As far as the bound in $\mathcal{B}^c_R(0)$ is concerned, we have

$$\int_{\mathcal{B}_R^c(0)} |u(x)|^p dx \leq \sup_{|x|>R} \left(|u(x)||x|^{-\frac{\gamma'}{p-q}} \right)^{p-q} \int_{\mathcal{B}_R^c(0)} |x|^{\gamma'} |u(x)|^q dx$$
$$\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} |x|^{\gamma} |u(x)|^q dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2}} \int_{\mathcal{B}_R^c(0)} |x|^{\gamma'} |u(x)|^q dx \tag{60}$$
$$\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} |x|^{\gamma} |u(x)|^q dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2}+1},$$

where in the second line of the above inequality, we utilize $|x|^{-\frac{\gamma'}{p-q}} \lesssim |x|^{\frac{\gamma'}{p-q}} \lesssim |x|^{\sigma'}$, for $|x| \gtrsim 1$ and (6), once

$$\frac{\gamma'}{p-q} \leq \sigma' = \frac{2s(d-1) + (2s-1)\gamma'}{(2s-1)q+2} < \sigma, \tag{61}$$

where we took into account that $\gamma' < \gamma$ and $|x|^{\gamma'} \lesssim |x|^{\gamma}$ for $|x| \geq R > 1$. We observe also that (61) is satisfied for $p^*_{s,\gamma} < q < p$ and $q < \frac{2d+2\gamma}{d-2s}$, with $p^*_{s,\gamma}$ defined as in (19). In the frame of (29), we have again (37) in $B^c_R(0)$, when $p > \frac{2d}{d-2s}$. To estimate in $\mathcal{B}_R(0)$, with $p^*_s < p < p^*_{s,\gamma} < q$ and $q > \frac{2d+2\gamma}{d-2s}$, we catch that, by the Hölder inequality,

$$\int_{\mathcal{B}_R(0)} |u(x)|^p dx \lesssim R^{d(q-p)} \int_{\mathcal{B}_R(0)} |u(x)|^q dx \lesssim \int_{\mathcal{B}_R(0)} |x|^{d(q-p)} |u(x)|^q dx$$
$$\lesssim \int_{\mathcal{B}_R(0)} |x|^{\gamma} |u(x)|^q dx \lesssim \|u\|^p_{\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)}, \tag{62}$$

by the bound $|x|^{d(q-p)} \lesssim |x|^{\gamma}$, for $|x| \lesssim 1$, if $\gamma \leq d(q-p)$. As far as compactness is concerned, we choose $\varepsilon > 0$, then again we take

$$\|u_j\|^p_{L^p(\mathbb{R}^d)} = \int_{|x|>R} |u_j(x)|^p dx + \int_{|x|<R^{-1}} |u_j(x)|^p dx + \int_{R \leq |x| \leq R^{-1}} |u_j(x)|^p dx, \tag{63}$$

where $R = R(\varepsilon)$ is selected analogously as in the proof of Theorem 2. In the regime (28), we estimate the second and the third integrals on the right-hand side of the above inequality as in (49) and (48), respectively. For the first one, we achieve

$$\int_{|x|>R} |u_j(x)|^p dx = \int_{|x|>R} \frac{|u_j(x)|^{p-q}}{|x|^{\gamma'}} |u_j(x)|^q |x|^{\gamma'} dx$$
$$\leq C \int_{|x|>R} |u_j(x)|^q |x|^{\gamma'} |x|^{\gamma' - \sigma'(p-q)} dx \leq C R^{\gamma' - \sigma'(p-q)} \int_{\mathbb{R}^d} \frac{|u_j(x)|^q}{|x|^{\gamma}} dx \tag{64}$$
$$\leq C R^{\gamma' - \sigma'(p-q)} < \frac{\varepsilon}{3},$$

from (61) if one follows the steps used to prove (60). If one considers now (29), we control the first and the third integrals on the right-hand side of (63) as in (46) and (48), respectively. For the second, we obtain

$$\int_{|x|<R^{-1}} |u(x)|^p dx \lesssim R^{-d(q-p)+\gamma} \int_{|x|<R^{-1}} |x|^{\gamma} |u(x)|^q dx < \frac{\varepsilon}{3}, \tag{65}$$

for $\gamma < d(q-p)$, as we did in (62). The proof is thus accomplished. □

Remark 6. *To demonstrate the compactness of the embedding (27), one can employ the approach illustrated in Theorem 4, considering the estimates provided in (60) and (62). This fact allows us to treat the full range* $0 < s < \frac{d}{2}$ *in a unified fashion, avoiding also the use of the equicontinuity property stated in Proposition 5, which seems to work for* $s > \frac{1}{2}$ *only (see the proof of Theorem 4.1 in [14]. See also [19] for a better understanding of the role played by the equicontinuity in the compact embeddings for fractional spaces).*

In order to have a self-contained treatise, we need to prove the following.

Proposition 10. *Let* $d \geq 1$, $s > 0$, *and* $1 \leq q < \infty$, $-d(q-1) < \gamma < d$, *Then the space* $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ *is complete.*

Proof. Assume that $\gamma > 0$ and consider the Cauchy sequence $u_j \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$, $j \in \mathbb{N}$. Then $(-\Delta)^{\frac{s}{2}} u_j$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$, and thus there exists $f \in L^2(\mathbb{R}^d)$ such that the sequence $(-\Delta)^{\frac{s}{2}} u_j$ converges strongly as $j \to \infty$, to f in $L^2(\mathbb{R}^d)$. On the other hand, we have, for every $R > 0$,

$$\int_{\mathcal{B}_R(0)} |u(x)|^q dx \lesssim R^\gamma \int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \tag{66}$$

which gives

$$\lim_{l,j\to\infty} \int_{\mathcal{B}_R(0)} |u_j(x) - u_l(x)|^q = 0.$$

There exists thus a measurable function $u : \mathbb{R}^d \to \mathbb{R}$ such that u_j converges, as $j \to \infty$, to u in $L^q_{\text{loc}}(\mathbb{R}^d)$. By Fatou's lemma, we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} \frac{|u_j(x) - u(x)|^q}{|x|^\gamma} dx \leq \lim_{j\to\infty} \liminf_{l\to\infty} \int_{\mathbb{R}^d} \frac{|u_j(x) - u_l(x)|^q}{|x|^\gamma} dx = 0. \tag{67}$$

We observe that by (66) we can get also

$$\lim_{j\to\infty} \sup_{R>0} \frac{1}{R^\gamma} \int_{\mathcal{B}_R(0)} |u_j(x) - u(x)|^q \lesssim \lim_{j\to\infty} \int_{\mathbb{R}^d} \frac{|u_j(x) - u(x)|^q}{|x|^\gamma} dx = 0, \tag{68}$$

since (67). Furthermore, by the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} |(u_j(x) - u(x))\varphi| dx \\
&\leq \int_{\mathcal{B}_R(0)} |(u_j(x) - u(x))\varphi(x)| dx + \int_{\mathcal{B}^c_R(0)} |x|^{-\gamma q} |(u_j(x) - u(x))| |x|^{\gamma q} |\varphi(x)| dx \\
&\lesssim \sup_{0\leq k\leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(\mathcal{B}_R(0))} \|u_j - u\|_{L^q(\mathcal{B}_R(0))} \\
&\quad + \sup_{0\leq k\leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(\mathcal{B}^c_R(0))} \left(\int_{\mathcal{B}^c_R(0)} |x|^{-\gamma} |(u_j(x) - u(x))|^q dx \right)^{\frac{1}{q}} \\
&\lesssim \sup_{0\leq k\leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(\mathbb{R}^d)} \left(\|u_j - u\|_{L^q(\mathcal{B}_R(0))} + \left(\int_{\mathbb{R}^d} |x|^{-\gamma} |(u_j(x) - u(x))|^q dx \right)^{\frac{1}{q}} \right),
\end{aligned} \tag{69}$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $N \geq q\gamma$. A use again of (67) in combination with (68) guarantees

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} |(u_n(x) - u(x))\varphi| dx = 0.$$

For this reason, $u_n(x) - u(x)$ if $n \to \infty$ converges to 0 as tempered distributions on $\mathbb{R}^d$. Therefore, $\widehat{(-\Delta)^{\frac{s}{2}}} u_j$ converges to $\widehat{(-\Delta)^{\frac{s}{2}}} u$ as distributions on $\mathbb{R}^d$. This fact and the above consideration on the convergence of $(-\Delta)^{\frac{s}{2}} u_j$ in $L^2(\mathbb{R}^d)$ imply that $(-\Delta)^{\frac{s}{2}} u = f$. Let now $\gamma < 0$, $d \geq 2$, and select, as the above, a Cauchy sequence $u_j \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$, $j \in \mathbb{N}$, converging strongly as $j \to \infty$ to f in $L^2(\mathbb{R}^d)$. One sees that for $R > 0$ and $q \leq \frac{2d}{d-2s}$, by the Sobolev embedding,

$$\int_{\mathcal{B}_R(0)} |u_j(x) - u_l(x)|^q dx \lesssim R^{1-q\left(\frac{1}{2}-\frac{s}{d}\right)} \|u_j(x) - u_l(x)\|^p_{\dot{H}^s(\mathbb{R}^d)}$$

and for $q > \frac{2d}{d-2s}$,

$$\int_{\mathcal{B}_R(0)} |u_j(x) - u_l(x)|^q dx \lesssim \int_{\mathbb{R}^d} |x|^\gamma \left|u_j(x) - u_l(x)\right|^q dx,$$

which enhance to

$$\lim_{l,j\to\infty} \int_{\mathcal{B}_R(0)} \left|u_j(x) - u_l(x)\right|^q = 0.$$

Then we can find a measurable function $u : \mathbb{R}^d \to \mathbb{R}$ such that u_j converges, as $j \to \infty$, to u in $L^q_{\text{loc}}(\mathbb{R}^d)$. Fatou's lemma shows that

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} |x|^\gamma \left|u_j(x) - u(x)\right|^q dx \leq \lim_{j\to\infty} \liminf_{l\to\infty} \int_{\mathbb{R}^d} |x|^\gamma \left|u_j(x) - u_l(x)\right|^q dx = 0. \tag{70}$$

As the above,

$$\begin{aligned} &\int_{\mathbb{R}^d} |(u_j(x) - u(x))\varphi| dx \\ &\leq \int_{\mathcal{B}_R(0)} |(u_j(x) - u(x))\varphi(x)| dx + \int_{\mathcal{B}^c_R(0)} |x|^{-\gamma q} |(u_j(x) - u(x))| |x|^{\gamma q} |\varphi(x)| dx \\ &\lesssim \sup_{0\leq k\leq N} \left\| |x|^k |\varphi(x)\right\|_{L^\infty(\mathcal{B}_R(0))} \|u_j - u\|_{L^q(\mathcal{B}_R(0))} \\ &+ \sup_{0\leq k\leq N} \left\| |x|^k \varphi(x)\right\|_{L^\infty(\mathcal{B}^c_R(0))} \left(\int_{\mathcal{B}^c_R(0)} |x|^\gamma |(u_j(x) - u(x))|^q dx\right)^{\frac{1}{q}} \\ &\lesssim \sup_{0\leq k\leq N} \left\| |x|^k \varphi(x)\right\|_{L^\infty(\mathbb{R}^d)} \left(\|u_j - u\|_{L^q(\mathcal{B}_R(0))} + \left(\int_{\mathbb{R}^d} |x|^\gamma |(u_j(x) - u(x))|^q dx\right)^{\frac{1}{q}}\right), \end{aligned} \tag{71}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $N \geq q\gamma$. The inequality above and a further application of (70) infer

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} |(u_j(x) - u(x))\varphi| dx = 0.$$

The remaining part of the proof is the same as the one carried out above for the case $\gamma > 0$. Therefore, we omit it. □

6. Gagliardo–Nirenberg Inequalities

This section is addressed to present the proof of the Gagliardo–Nirenberg-type inequalities (30) and (31).

Proof of Theorem 6. We shall treat only the case $\gamma > 0$ because the proof for $\gamma < 0$ can be carried out in a similar manner, with some minor changes. Let us consider the scaling $u_\chi(x) = \chi^{\frac{d}{p}} u(\chi x)$ such that $\|u_\chi\|_{L^p(\mathbb{R}^d)} = \|u\|_{L^p(\mathbb{R}^d)}$. The embedding leads to

$$\|u_\chi\|^2_{L^p(\mathbb{R}^d)} \leq C \left\|(-\Delta)^{\frac{s}{2}} u_\chi\right\|^2_{L^2(\mathbb{R}^d)} + C \left(\int_{\mathbb{R}^d} \frac{|u_\chi(x)|^q}{|x|^\gamma} dx\right)^{\frac{2}{q}},$$

which implies the following

$$
\begin{aligned}
\|u\|^2_{L^p(\mathbb{R}^d)} \leq & C\chi^{\frac{2d}{p}-d+2s}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2_{L^2(\mathbb{R}^d)} + C\chi^{\frac{2d}{p}-\frac{2(d-\gamma)}{q}}\left(\int_{\mathbb{R}^d}\frac{|u(x)|^q}{|x|^\gamma}dx\right)^{\frac{2}{q}} \\
:= & C\chi^{\frac{2d}{p}-d+2s}A + C\chi^{\frac{2d}{p}-\frac{2(d-\gamma)}{q}}B. \qquad (72)
\end{aligned}
$$

By optimizing the sum on the left-hand side of the above inequality (72), one obtains that the minimum of the above sum is attained at

$$
\widetilde{\chi} = \left(A^{-1}B\frac{2p(d-\gamma)-2dq}{q(2d-p(d-2s))}\right)^{\frac{q}{2d-2\gamma-q(d-2s)}} = C(p,q,d,\gamma,s)\left(A^{-1}B\right)^{\frac{q}{2d-2\gamma-q(d-2s)}}, \qquad (73)
$$

with $C = C(p,q,d,\gamma,s) > 0$. By plugging the previous (73) into (72), we arrive at

$$
\begin{aligned}
\|u\|^2_{L^p(\mathbb{R}^d)} \leq & \ C\chi^{\frac{2d}{p}-d+2s}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2_{L^2(\mathbb{R}^d)} + C\chi^{\frac{2d}{p}-\frac{2(d-\gamma)}{q}}\left(\int_{\mathbb{R}^d}\frac{|u(x)|^q}{|x|^\gamma}dx\right)^{\frac{2}{q}} \\
\leq & \ CA^{\frac{2d-dp+2ps}{q(d-2s)-(2d-2\gamma)}+1}B^{\frac{q(2d-p(d-2s))}{p(2d-2\gamma-q(d-2s))}} + CA^{\frac{(2d-2\gamma)p-2dq}{p(2d-2\gamma-q(d-2s))}}B^{\frac{2dq-(2d-2\gamma)p}{p(2d-2\gamma-q(d-2s))}+1} \qquad (74) \\
\leq & \ CA^{\frac{(2d-2\gamma)p-2dq}{p(2d-2\gamma-q(d-2s))}}B^{\frac{q(2d-p(d-2s))}{p(2d-2\gamma-q(d-2s))}},
\end{aligned}
$$

which gives (30) with $p \neq p^*_{s,\gamma}$ and $p \neq p^*_s$, where $p^*_{s,\gamma}$, p^*_s, γ, q as in (17) and (18) or (28) and (29), with $p^*_{s,\gamma}$, p^*_s defined as in (19). □

We are in a position now to give the following.

Proof of Corollary 1. The proof is a direct consequence of the scaling invariant inequality

$$
\int_{\mathbb{R}^d}(|x|^{(\alpha-d)} * |u|^p)|u|^p dx \leq C(d,p,\alpha)\|u\|^{2p}_{L^{\frac{2pd}{d+\alpha}}(\mathbb{R}^d)}, \qquad (75)
$$

arising from (13) in Proposition 8 and of (30) for $p^*_{s,\gamma} \leq \frac{2pd}{d+\alpha} \leq p^*_s$. □

7. Minimization Problems

In this section, we go over the proofs of the theorems connected to the minimization problems (7) and (2).

Proof of Theorem 7. The fact that $m > 0$ follows by Theorem 6. We will prove now that there is a function $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ such that $W^{p,q,s,\gamma}_1(u) = m$ with $W^{p,q,s,\gamma}_1(u)$ as in (32). For this proposal, pick up a minimizing sequence $u_j \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$, $j \in \mathbb{N}$ converging weakly to $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ such that

$$
\int_{\mathbb{R}^d} u_j^p(x)dx = 1, \quad \int_{\mathbb{R}^d}\frac{u_j^q(x)}{|x|^\gamma}dx = 1, \quad W^{p,q,s,\gamma}_1(u_j) \to m,
$$

for $j \to \infty$. We may assume also $u_j \geq 0$ because of the bound

$$
\begin{aligned}
\left\|(-\Delta)^{\frac{s}{2}}|u|\right\|^2_{L^2(\mathbb{R}^d)} = & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{||u(x)|-|u(y)||^2}{|x-y|^{d+2s}}dxdy \\
\lesssim & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dxdy = \left\|(-\Delta)^{\frac{s}{2}}u\right\|^2_{L^2(\mathbb{R}^d)}. \qquad (76)
\end{aligned}
$$

By Proposition 6, we have

$$
\left\|(-\Delta)^{s/2}u\right\| \leq m, \quad \int_{\mathbb{R}^d}\frac{u^q(x)}{|x|^\gamma}dx \leq 1, \quad \int_{\mathbb{R}^d}u^p(x)dx \leq 1.
$$

By the compact embedding $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^d)$ of Theorems 2 and 5, we have that $u_j \to u$ almost everywhere and

$$\lim_{j\to\infty}\int_{\mathbb{R}^d} u_j^p(x)dx = \int_{\mathbb{R}^d} u^p(x)dx = 1.$$

This will imply $W_1^{p,q,s,\gamma}(u) \leq m$. Nevertheless, by the definition of m, we arrive at $W_1^{p,q,s,\gamma}(u) = m$. Then, $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ is the required minimizer, and the proof is complete. □

Proof of Corollary 2. We know that $d > 0$ by Corollary 1. Choose, as the above, a non-negative minimizing sequence $u_j \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$, $j \in \mathbb{N}$ converging weakly to $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}(|x|^{(\alpha-d)} * |u_j|^p)|u_j|^p = 1, \quad \int_{\mathbb{R}^d}\frac{u_j^q(x)}{|x|^\gamma}dx = 1, \quad W_2^{p,q,s,\alpha,\gamma}(u_j) \to m,$$

with $W_2^{p,q,s,\alpha,\gamma}(u)$ as in (33), for $j \to \infty$. Proposition 6 and inequality (75) bring

$$\left\|(-\Delta)^{s/2}u\right\| \leq m, \quad \int_{\mathbb{R}^d}\frac{u^q(x)}{|x|^\gamma}dx \leq 1.$$

The compact embedding $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^{\frac{2pd}{d+\alpha}}(\mathbb{R}^d)$ of Theorems 2 and 5 guarantees that $u_j \to u$ almost everywhere, with $u_j, u \in L^{\frac{2pd}{d+\alpha}}(\mathbb{R}^d)$. Then, by (15) in Lemma 1, we obtain

$$\lim_{j\to\infty}\int_{\mathbb{R}^d}(|x|^{(\alpha-d)} * |u_j|^p)|u_j|^p = \int_{\mathbb{R}^d}(|x|^{(\alpha-d)} * |u|^p)|u|^p = 1.$$

This gives $W_2^{p,q,s,\alpha,\gamma}(u) \leq m$. We conclude, as the above, that $W_2^{p,q,s,\alpha,\gamma}(u) = m$. Then, we find a minimizer function $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$. The proof is completed. □

We obtain the following.

Corollary 3. *Let* $\frac{1}{2} < s < 1$, $p^*_{s,\gamma} - 2 < \tilde{p} < \frac{4s}{d-2s}$, *with* $p^*_{s,\gamma}$, $q = \tilde{q} + 2$ *and* γ *as in Theorem 6. Then, there exists a positive function* $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$, *a solution to* (2) *with* $f(x,u)$ *as in* (3) *such that*

$$W_1^{p,q,s,\gamma}(u) = \min_{v\in\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)} W_1^{p,q,s,\gamma}(v).$$

We obtain also the following.

Corollary 4. *Let* $\frac{1}{2} < s < 1$, $p^*_{s,\alpha,\gamma} < p < \frac{d+\alpha}{d-2s}$, *with* $p^*_{s,\alpha,\gamma}$, $q = \tilde{q} + 2$, α *and* γ *as in Corollary 1. Then, there exists a positive function* $u \in \dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ *solution to* (2) *with* $f(x,u)$ *as in* (4) *such that*

$$W_2^{p,q,s,\alpha,\gamma}(u) = \min_{v\in\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)} W_2^{p,q,s,\alpha,\gamma}(v).$$

Remark 7. *In Corollary 3, we improve the result in* [15]. *To be more precise, we extend the lower bound of the domain of admissibility for p from* $\frac{2}{d-2s}$ *to* $\frac{4s\gamma}{2s(d-1)-(2s-1)\gamma}$. *We generalize it then to the case* $q > 1$ *and* $\gamma \neq 1$. *Corollary 4 is instead new in the literature.*

Remark 8. *We emphasize that the existence of positive minimizer solutions for* (2) *are pivotal in the study of the dynamics of certain nonlinear evolution equations. To have full insight into the argument and its association with stability and scattering analysis, we cite, for instance* [30–35], *along with the references provided therein.*

8. Conclusions

We extend the outcomes obtained in [14–16] by broadening the range of parameters p, q, γ, and s associated with the embedding of $\dot{H}^{s,q,\gamma}_{rad}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$. Additionally, we enhance Gagliardo–Nirenberg-type inequalities, incorporating symmetry akin to (2), thus generalizing them to a nonlocal framework. This extension sheds light on extremals within the corresponding minimization problems. Notably, our work includes a compact embedding result that improves the findings in [14] by extending them to $s \leq \frac{1}{2}$. While Strauss's radial inequality suggests the existence of a continuous representative, validating it only for $s > \frac{1}{2}$, we emphasize that this constraint does not pose a significant limitation. This restriction is primarily structural, indicating a lack of pointwise representations for functions within $H^s(\mathbb{R}^d)$ with small s. Furthermore, aligning with the classical Sobolev embedding theorem stating that $H^s(\mathbb{R}^d)$ embeds into $C^{0,s-d/2}(\mathbb{R}^d)$ (as seen in [18,19]), we overcome this obstacle by leveraging a set of radial inequalities tailored for addressing the case of $s \leq \frac{1}{2}$, coupled with a continuity argument. This enables us to treat the complete range $0 < s < \frac{d}{2}$ comprehensively, without relying on the equicontinuity property stated in Proposition 5, which works for $s > \frac{1}{2}$ only.

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