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The Equivalence of Two Modes of Order Convergence

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Abstract: It is well known that if a poset satisfies Property A and its dual form, then the o-convergence and o_2-convergence in the poset are equivalent. In this paper, we supply an example to illustrate that a poset in which the o-convergence and o_2-convergence are equivalent may not satisfy Property A or its dual form, and carry out some further investigations on the equivalence of the o-convergence and o_2-convergence. By introducing the concept of the local Frink ideals (the dually local Frink ideals) and establishing the correspondence between ID-pairs and nets in a poset, we prove that the o-convergence and o_2-convergence of nets in a poset are equivalent if and only if the poset is ID-doubly continuous. This result gives a complete solution to the problem of E.S. Wolk in two modes of order convergence, which states under what conditions for a poset the o-convergence and o_2-convergence in the poset are equivalent.

Keywords: order convergence; local Frink ideal (dually local Frink ideal); ID-doubly continuous poset

MSC: 06A06; 06B10

1. Introduction

Let P be a poset and $(x_i)_{i\in I}$ a net on an up-directed set I with value in the poset P. The concept of order convergence of nets in a poset P was introduced by Birkhoff [1], Mcshane [2], Frink [3], Rennie [4] and Ward [5]. It is worth noting that the authors may have attached different meanings to the order convergence. Following the formulation of Wolk [6], we correspond to the following two modes of order convergence:

Definition 1 ([1–3]). *A net $(x_i)_{i\in I}$ in a poset P is said to* o-converge *to an element $x \in P$ (in symbol $(x_i)_{i\in I} \xrightarrow{o} x$) if there exist subsets M and N of P such that*

(A0) M is up-directed and N is down-directed;
(B0) $\sup M = x = \inf N$;
(C0) For every $m \in M$ and $n \in N$, $m \leqslant x_i \leqslant n$ holds eventually, i.e., there is $i_0 \in I$ such that $m \leqslant x_i \leqslant n$ for all $i \geqq i_0$.

Definition 2 ([4–6]). *A net $(x_i)_{i\in I}$ in a poset P is said to* o_2-converge *to an element $x \in P$ (in symbol $(x_i)_{i\in I} \xrightarrow{o_2} x$) if there exist subsets M and N of P such that*

(A2) $\sup M = x = \inf N$;
(B2) For every $m \in M$ and $n \in N$, $m \leqslant x_i \leqslant n$ holds eventually.

A research topic concerning the o-convergence and o_2-convergence, which are closely related to our work, is from the topological aspect. The o-convergence in a poset P may not be topological, i.e., there does not exist a topology τ on the poset P such that the o-convergent class and the convergent class with respect to the topology τ are equivalent. In [7], based on the introduction of Condition(*) and the double continuity for posets,

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Zhou and Zhao proved that, for a double continuous poset P with Condition(*), the o-convergence in the poset P is topological. As a further result, Condition ($\triangle$), a weaker condition than Condition(*), and the $\mathcal{O}$-doubly continuous posets were defined in [8]. It was shown that, for a poset P with Condition ($\triangle$), the o-convergence in the poset P is topological if and only if the poset P is $\mathcal{O}$-doubly continuous. Following the ideal in [8], Sun and Li [9] studied the B-topology on posets and found that the o-convergence in a poset P is topological if and only if the poset P is S^*-doubly continuous, which demonstrates the equivalence between the o-convergence being topological and the S^*-double continuity of a poset. Moreover, the ideal-o-convergence, a generalized form of o-convergence established via ideals, was defined in posets by Georgiou et al. [10,11]. Also, the authors obtained that the ideal-o-convergence in a poset P is topological if and only if the poset P is S^*-doubly continuous. This generalized the previous results on the o-convergence.

On the other hand, the o_2-convergence is also not topological generally. To characterize these posets so that the o_2-convergence is topological, Zhao and Li [12] studied the notions of α-double continuous posets and α^*-double continuous posets. Under some additional conditions, the o_2-convergence in these posets is topological. Ulteriorly, Li and Zou [13] proposed the concept of O_2-doubly continuous posets and showed that the o_2-convergence in a poset P is topological if and only if the poset P is O_2-doubly continuous, meaning that they gave a sufficient and necessary condition for the o_2-convergence to be topological. Further, Georgiou et al. [14] extended the o_2-convergence to be the ideal-o_2-convergence via ideals, and showed that the O_2-double continuity can equivalently characterize such a convergence to be topological.

From the order-theoretical aspect, by the definitions, one can readily verify that the o-convergence implies the o_2-convergence, i.e., if a net $(x_i)_{i\in I}$ in a poset P o-converges to an element $x \in P$, then it o_2-converges to x. However, the converse implication is not true. This fact can be demonstrated by the example in [6]. Hence, in [6], Wolk posed the following fundamental problem:

Problem 1. *Under what conditions for a poset P do the o-convergence and o_2-convergence in P agree?*

A well-known result on this problem is that the o-convergence and o_2-convergence in a lattice are equivalent. Then, Wolk [6] obtained a result on the characterization of posets for the associated o-inf convergence (a counterpart of o-convergence) and o_2-inf convergence (a counterpart of o_2-convergence) being equivalent, which provides an approximate solution to the fundamental problem, using the concepts of Frink ideals and dual Frank ideals [15].

Motivated by these results toward the problem mentioned above, in this paper, we continue to make some further investigations on the o-convergence and o_2-convergence, hoping to clarify the order-theoretical condition of a poset P, which is sufficient and necessary for the o-convergence and o_2-convergence to be equivalent.

To this end, in Section 2, following the Frink ideal (the dual Frink ideal), the concepts of local Frink ideals (dually local Frink ideals) and ID-pairs in posets are further proposed, and then the relationship between ID-pairs and nets is presented. Section 3 is devoted to the order-theoretical characterization of the local Frink ideal (the dually local Frink ideal) generated by a general set. Using this characterization, we prove that the ID-double continuity is the precise feature for those posets for which the two modes of order convergence are equivalent.

For the unexplained notions and concepts, one can refer to [6,16,17].

2. Local Frink Ideal (Dually Local Frink Ideal) in Posets

We appoint some conventional notations to be used in the sequel. Let X be a set. We take $F \sqsubseteq X$ to mean that F is a finite subset of the set X, including the empty set $\emptyset$. Given a poset P and $K, L \subseteq P$. The notations K^u and L^l are used to denote the set of all upper bounds of K and the set of all lower bounds of L, respectively, i.e., $K^u = \{y \in P : (\forall p \in K)\ y \geqslant p\}$

and $L^l = \{z \in P : (\forall p \in L)\ z \leqslant p\}$. Particularly, if the sets K and L are all reduced to be a singleton $\{y\}$, then the notations $\uparrow y$ and $\downarrow y$ are reserved to denote the sets $\{y\}^u$ and $\{y\}^l$, respectively.

Since the Frink ideal (the dual Frink ideal) in posets plays a fundamental role in the discussion of this section, we first review its definition.

Definition 3 ([15]). *Let P be a poset.*

(1) *A subset K of the poset P is called a* Frink ideal *if, for every $F \sqsubseteq K$, we have $(F^u)^l \subseteq K$. Furthermore, a Frink ideal K is said to be* normal *if $(K^u)^l = K$.*
(2) *A subset L of the poset P is called a* dual Frink ideal *if, for every $S \sqsubseteq L$, we have $(S^l)^u \subseteq L$. Furthermore, a dual Frink ideal L is said to be* normal *if $(L^l)^u = L$.*

Based on the Frink ideal (the dual Frink ideal), we further define the local Frink ideal (the dually local Frink ideal) in posets.

Definition 4. *Let P be a poset and $K, L \subseteq P$.*

(1) *The subset K is called a* local Frink ideal in L *if, for every $F \sqsubseteq K$ and every $S \sqsubseteq L$, we have $(F^u \bigcap S^l)^l \subseteq K$.*
(2) *The subset L is called a* dually local Frink ideal in K *if, for every $F \sqsubseteq K$ and every $S \sqsubseteq L$, we have $(F^u \bigcap S^l)^u \subseteq L$.*

Example 1. *Let $\mathbb{R}$ be the set of all real numbers, in its usual order, and let $a \in \mathbb{R}$. If we take $K = (-\infty, a]$ and $L = [a, +\infty)$, then, by Definition 4, the interval K is a local Frink ideal in L and the interval L is a dually local Frink ideal in K.*

Given a poset P and $K, L \subseteq P$. We simply denote by $\mathfrak{L}(L)$ the family of all local Frink ideals in L and, by $\mathfrak{D}(K)$, the family of all dually local Frink ideals in K.

Remark 1. *Let P be a poset and $K, L \subseteq P$. Then,*

(1) *From the logic viewpoint, it is reasonable to stipulate that $\emptyset^u = \emptyset^l = P$. Thus, for every $L \subseteq P$ and every $K \in \mathfrak{L}(L)$, we have $\perp \in K$ if the poset P has the least element $\perp$. Dually, for every $K \subseteq P$ and every $L \in \mathfrak{D}(K)$, we have $\top \in L$ if the greatest element $\top$ exists in the poset P.*
(2) *If $K \in \mathfrak{L}(L)$, then $K \in \mathfrak{L}(L_0)$ for every $L_0 \subseteq L$. And, dually, if $L \in \mathfrak{D}(K)$, then $L \in \mathfrak{D}(K_0)$ for every $K_0 \subseteq K$.*
(3) *The subset K is a Frink ideal if and only if $K \in \mathfrak{L}(\emptyset)$. And, dually, the subset L is a dual Frink ideal if and only if $L \in \mathfrak{D}(\emptyset)$.*

Proposition 1. *Let P be a poset and $K, L \subseteq P$.*

(1) *If $K \in \mathfrak{L}(L)$, then the subset K is a Frink ideal.*
(2) *If $L \in \mathfrak{D}(K)$, then the subset L is a dual Frink ideal.*

Proof. (1): Suppose that $K \in \mathfrak{L}(L)$. Then, we have $(F^u \bigcap S^l)^l \subseteq K$ for every $F \sqsubseteq K$ and $S \sqsubseteq L$. This implies that $(F^u)^l \subseteq (F^u \bigcap S^l)^l \subseteq K$. Thus, we conclude that $(F^u)^l \subseteq K$ for every $F \sqsubseteq K$. This shows that the subset K is a Frink ideal.

(2): The proof is similar to that of (1). □

However, the converse implications of Proposition 1 may not be true. This fact can be clarified in Example 7.

Definition 5. *Let P be a poset. A pair (K, L) consisting of subsets K and L of P is called an* ID-pair *in P if $K \in \mathfrak{L}(L)$ and $L \in \mathfrak{D}(K)$. Moreover, an ID-pair (K, L) in P is said to be* nontrivial *if one of the following conditions is exactly satisfied:*

(1) $|P| = 1$, *where* $|P|$ *denotes the cardinal of the poset* P;
(2) $|P| \geq 2$ *and* $(K, L) \neq (P, P)$.

Example 2. *Let* $P = \{a, b\} \bigcup \{\bot, \top\}$, *with the partial order* $\leqslant$ *defined by*

- $\bot \leqslant a \leqslant \top$;
- $\bot \leqslant b \leqslant \top$.

Take $K = \{\bot\}$ *and* $L = \{\top\}$. *Then, it is easy to see from Definitions 4 and 5 that the pair* $(K, L) = (\{\bot\}, \{\top\})$ *is a nontrivial ID-pair.*

Proposition 2. *Let* (K, L) *be an ID-pair in a poset* P. *Then, the ID-pair* (K, L) *is nontrivial if and only if* $F^u \bigcap S^l \neq \emptyset$ *for every* $F \sqsubseteq K$ *and every* $S \sqsubseteq L$.

Proof. ($\Rightarrow$): Let (K, L) be a nontrivial ID-pair in a poset P. We consider the following cases:

(i) $|P| = 1$, i.e., the poset $P = \{p\}$ contains only one element p.
It is easy to check that $F^u \bigcap S^l = \{p\} \neq \emptyset$ for every $F \sqsubseteq K$ and every $S \sqsubseteq L$.

(ii) $|P| \geq 2$.
Suppose that $(F_0)^u \bigcap (S_0)^l = \emptyset$ for some $F_0 \sqsubseteq K$ and $S_0 \sqsubseteq L$. Then, we have $[(F_0)^u \bigcap (S_0)^l]^l = P \subseteq K$ and $[(F_0)^u \bigcap (S_0)^l]^u = P \subseteq L$ since (K, L) is an ID-pair in the poset P. This implies that $(K, L) = (P, P)$, which is a contradiction to the assumption that the ID-pair (K, L) is nontrivial. Hence, we have that $F^u \bigcap S^l \neq \emptyset$ for every $F \sqsubseteq K$ and every $S \sqsubseteq L$.

By (i) and (ii), we conclude that $F^u \bigcap S^l \neq \emptyset$ for every $F \sqsubseteq K$ and every $S \sqsubseteq L$.

($\Leftarrow$): Suppose that (K, L) is an ID-pair such that $F^u \bigcap S^l \neq \emptyset$ for every $F \sqsubseteq K$ and every $S \sqsubseteq L$. If $(K, L) \neq (P, P)$, then the ID-pair (K, L) is nontrivial by Definition 5. If $(K, L) = (P, P)$, i.e., $K = L = P$, then, by the assumption, we have $\{p\}^u \bigcap \{q\}^l = \uparrow p \bigcap \downarrow q \neq \emptyset$ and $\{q\}^u \bigcap \{p\}^l = \uparrow q \bigcap \downarrow p \neq \emptyset$ for all $p, q \in P$. It follows that $p = q$ for all $p, q \in P$. Hence, we conclude that $|P| = 1$. This shows, by Definition 5, that the ID-pair (K, L) is nontrivial. □

In fact, given a poset P and a Frink ideal K (resp. a dual Frink ideal L) of the poset P, we can select a subset L (resp. a subset K) of P such that the pair (K, L) is a nontrivial ID-pair.

Theorem 1. *Let* P *be a poset.*

(1) *If* K *is a Frink ideal of the poset* P, *then the pair* (K, L) *is a nontrivial ID-pair for some subset* L *of the poset* P;
(2) *If* L *is a dual Frink ideal of the poset* P, *then the pair* (K, L) *is a nontrivial ID-pair for some subset* K *of the poset* P.

Proof. (1): Suppose that K is a Frink ideal of P. Set $L = \bigcup\{(F^u)^u : F \sqsubseteq K\}$. Now, we process to show that the pair (K, L) is an ID-pair. Let $F_0 \sqsubseteq K$ and $S_0 \sqsubseteq L$. We consider the following two cases:

(i) $S_0 = \emptyset$.
Since K is a Frink ideal, by the definition of L, we have

$$[(F_0)^u \cap (S_0)^l]^l = [(F_0)^u \cap P]^l = [(F_0)^u]^l \subseteq K,$$

and

$$[(F_0)^u \cap (S_0)^l]^u = [(F_0)^u \cap P]^u = [(F_0)^u]^u \subseteq L.$$

(ii) $S_0 = \{s_1, s_2, \ldots, s_m\} \neq \emptyset$.

By the definition of L, there exists $F_i \sqsubseteq K$ such that $s_i \in [(F_i)^u]^u$ for every $i \in \{1, 2, \ldots, m\}$. This means that $(F_i)^u \subseteq \downarrow s_i$ for every $i \in \{1, 2, \ldots, m\}$. Thus, we have $(F_1 \cup F_2 \cup \cdots \cup F_m)^u \subseteq (S_0)^l$, which implies that

$$\begin{aligned}((F_0)^u \cap (S_0)^l)^l \subseteq& [(F_0 \cup F_1 \cup F_2 \cup \cdots \cup F_m)^u \cap (S_0)^l]^l \\ =& [(F_0 \cup F_1 \cup F_2 \cup \cdots \cup F_m)^u]^l \\ \subseteq& K,\end{aligned}$$

and

$$\begin{aligned}((F_0)^u \cap (S_0)^l)^u \subseteq& [(F_0 \cup F_1 \cup F_2 \cup \cdots \cup F_m)^u \cap (S_0)^l]^u \\ =& [(F_0 \cup F_1 \cup F_2 \cup \cdots \cup F_m)^u]^u \\ \subseteq& L.\end{aligned}$$

The combination of (i) and (ii) shows that the pair (K, L) is an ID-pair in P. Finally, we prove that the ID-pair (K, L) is nontrivial. Assume that $(K, L) = (P, P)$. Let $x, y \in L = P$. Then, by the definition of L, there exists $F_y \sqsubseteq P$ such that $y \in [(F_y)^u]^u$, which implies that $(F_y)^u \subseteq \downarrow y$. Since $(\{x\} \cup F_y)^u \subseteq (F_y)^u \subseteq \downarrow y$, we have $x \in \downarrow y$, i.e., $x \leqslant y$. Similarly, we can prove that $y \leqslant x$. This means that $x = y$, and thus we have $|P| = 1$. By Definition 5, it follows that the ID-pair (K, L) is nontrivial.

(2): By a similar verification to that of (1). □

Example 3. *Let P be a chain, i.e., for all $x, y \in P$, either $x \leqslant y$ or $y \leqslant x$. For every $x \in P$, by Definition 4 we have that the set $\downarrow x$ is a Frink ideal. Obviously, by Definitions 4 and 5, the set $\uparrow x$ can be selected such that the pair $(\downarrow x, \uparrow x)$ is a nontrivial ID-pair in P.*

Given a poset P and a net $(x_i)_{i \in I}$ in the poset P, an element $p \in P$ is called an *eventually lower bound* of the net $(x_i)_{i \in I}$ provided that there exists $i_0 \in I$ such that $x_i \geqslant p$ for all $i \geqq i_0$. An *eventually upper bound* of the net $(x_i)_{i \in I}$ is defined dually. Following the notations of Wolk [6], we also take the symbols P_x and Q_x to mean the set of all eventually lower bounds of the net $(x_i)_{i \in I}$ and the set of all eventually upper bounds of the net $(x_i)_{i \in I}$, respectively. If we denote $E_x(i_0) = \{x_i \in P : i \geqq i_0\}$, then $P_x = \bigcup\{[E_x(i)]^l : i \in I\}$ and $Q_x = \bigcup\{[E_x(i)]^u : i \in I\}$. For a set X, the symbol $Y \subset X$ means that Y is a proper subset of the set X, i.e., $Y \subseteq X$ and $Y \neq X$. In the following, we always take $\geq_o$ to represent the ordinary order on $\mathbb{N}$, the set of all positive integers.

Now, we can establish a correspondence between the nets and the ID-pairs:

Theorem 2. *Let P be a poset. Then, a pair (K, L) in P is a nontrivial ID-pair if and only if there exists a net $(x_i)_{i \in I}$ in P such that $P_x = K$ and $Q_x = L$.*

Proof. ($\Leftarrow$): Let (K, L) be a pair of subsets of the poset P. Suppose also that $(x_i)_{i \in I}$ is a net in the P such that $P_x = K$ and $Q_x = L$. For every $F \sqsubseteq P_x = K$ and every $S \sqsubseteq Q_x = L$, we consider the following cases:

(i) $F = S = \emptyset$.
Since $F = \emptyset$ and $S = \emptyset$, we have that $E_x(i) \subseteq F^u \bigcap S^l = P \neq \emptyset$ for all $i \in I$. This implies that $(F^u \bigcap S^l)^l \subseteq [E_x(i)]^l$ and $(F^u \bigcap S^l)^u \subseteq [E_x(i)]^u$ for all $i \in I$. Hence, $(F^u \bigcap S^l)^l \subseteq P_x$ and $(F^u \bigcap S^l)^u \subseteq Q_x$.

(ii) $F = \emptyset \sqsubseteq P_x$ and $S = \{s_1, s_2, \ldots, s_n\} \sqsubseteq Q_x$.
Since $S = \{s_1, s_2, \ldots, s_n\} \sqsubseteq Q_x$, for every $1 \leq_o t \leq_o n$, there exists $i_t \in I$ such that $E_x(i_t) \subseteq \downarrow s_t$. Take $i_0 \in I$ such that $i_0 \geqq i_t$ for all $1 \leq_o t \leq_o n$. Then, we have $E_x(i_0) \subseteq \bigcap\{E_x(i_t) : 1 \leq_o t \leq_o n\} \subseteq \bigcap\{\downarrow s_t : 1 \leq_o t \leq_o n\} = S^l$, which implies that $E_x(i_0) \subseteq F^u \bigcap S^l = S^l \neq \emptyset$, $(F^u \bigcap S^l)^l \subseteq [E_x(i_0)]^l$ and $(F^u \bigcap S^l)^u \subseteq [E_x(i_0)]^u$. It follows that $(F^u \bigcap S^l)^l \subseteq P_x$ and $(F^u \bigcap S^l)^u \subseteq Q_x$.

(iii) $F = \{e_1, e_2, \ldots, e_m\} \sqsubseteq P_x$ and $S = \emptyset \sqsubseteq Q_x$.

By a similar verification to that of (ii), we can also prove that $F^u \bigcap S^l \neq \emptyset$, $(F^u \bigcap S^l)^l \subseteq P_x$ and $(F^u \bigcap S^l)^u \subseteq Q_x$.

(iv) $F = \{e_1, e_2, \dots, e_m\} \sqsubseteq P_x$ and $S = \{s_1, s_2, \dots, s_n\} \sqsubseteq Q_x$.
Since $F = \{e_1, e_2, \dots, e_m\} \sqsubseteq P_x$ and $S = \{s_1, s_2, \dots, s_n\} \sqsubseteq Q_x$, there exist $i_r, i_t \in I$ such that $E_x(i_r) \subseteq \uparrow e_r$ and $E_x(i_t) \subseteq \downarrow s_t$ for all $1 \leq_o r \leq_o m$ and $1 \leq_o t \leq_o n$. Take $i_0 \in I$ such that $i_0 \geqq i_r, i_t$ for all $1 \leq_o r \leq_o m$ and $1 \leq_o t \leq_o n$. Then, we have $E_x(i_0) \subseteq \bigcap\{\uparrow e_r : 1 \leq_o r \leq_o m\} \bigcap \bigcap\{\downarrow s_t : 1 \leq_o t \leq_o n\} = F^u \bigcap S^l$, which implies that $F^u \bigcap S^l \neq \emptyset$, $(F^u \bigcap S^l)^l \subseteq [E_x(i_0)]^l$ and $(F^u \bigcap S^l)^u \subseteq [E_x(i_0)]^u$. Thus, $(F^u \bigcap S^l)^l \subseteq P_x$ and $(F^u \bigcap S^l)^u \subseteq Q_x$.

By (i)–(iv), Definition 4 and Proposition 2, we conclude that the pair $(P_x, Q_x) = (K, L)$ is a nontrivial ID-pair in the poset P.

($\Rightarrow$): Assume that the pair (K, L) is a nontrivial ID-pair in the poset P. We take the following cases into consideration:

(v) Either the set K or the set L is infinite.
Without loss of generality, we can assume that the set K is infinite. As the ID-pair (K, L) is nontrivial, we have that $F^u \bigcap S^l \neq \emptyset$ for every $F \sqsubseteq K$ and every $S \sqsubseteq L$ by Proposition 2. Let κ_F^S be the cardinal, linearly ordered by $\geq_F^S$, of the set $F^u \bigcap S^l$, and $a_F^S : \kappa_F^S \to F^u \bigcap S^l$ be a one-to-one function from κ_F^S onto $F^u \bigcap S^l$ for every $F \sqsubseteq K$ and every $S \sqsubseteq L$. Put $I = \{(F, S, \lambda) : F \sqsubseteq K, S \sqsubseteq L, \lambda \in \kappa_F^S\}$. For any $(F_1, S_1, \lambda_1), (F_2, S_2, \lambda_2) \in I$, we define $(F_2, S_2, \lambda_2) \geqq (F_1, S_1, \lambda_1)$ if and only if one of the following conditions is satisfied:

(1) $F_1 = F_2$, $S_1 = S_2$ and $\lambda_2 \geq_{F_1}^{S_1} \lambda_1$;
(2) $F_1 \subset F_2$ and $S_1 \subseteq S_2$.

Now, one can readily check that the ordered set I is up-directed. Let the net $(x_i)_{i \in I}$ in the poset P be defined by $x_{(F,S,\lambda)} = a_F^S(\lambda)$ for every $(F, S, \lambda) \in I$. Next, we proceed to prove that $P_x = K$ and $Q_x = L$. Let $p \in P_x$. Then, there exists $(F_1, S_1, \lambda_1) \in I$ such that $p \in [E_x((F_1, S_1, \lambda_1))]^l$. Take $S_2 = S_1$ and $F_2 \sqsubseteq K$ with $F_1 \subset F_2$. Then, we have $[(F_2)^u \bigcap (S_2)^l]^l \subseteq K$ since the pair (K, L) is a nontrivial ID-pair. According to the definition of I, it follows that $(F_2, S_2, \lambda_2) \geqq (F_1, S_1, \lambda_1)$ for every $\lambda_2 \in \kappa_{F_2}^{S_2}$, which implies that $x_{(F_2,S_2,\lambda_2)} = a_{F_2}^{S_2}(\lambda_2) \in E_x((F_1, S_1, \lambda_1))$ for every $\lambda_2 \in \kappa_{F_2}^{S_2}$. Hence, we conclude that $(F_2)^u \bigcap (S_2)^l \subseteq E_x((F_1, S_1, \lambda_1))$. This shows that $p \in [E_x((F_1, S_1, \lambda_1))]^l \subseteq [(F_2)^u \bigcap (S_2)^l]^l \subseteq K$. Thus, $P_x \subseteq K$. Conversely, let $q \in K$. Set $F_0 = \{q\} \sqsubseteq K$ and $S_0 = \emptyset \sqsubseteq L$. Then, by the definition of I, it is easy to see that $(F_0, S_0, \lambda_0) \in I$ for all $\lambda_0 \in \kappa_{F_0}^{S_0}$. For every $(F, S, \lambda) \in I$ with $(F, S, \lambda) \geqq (F_0, S_0, \lambda_0)$, by the definition of I, we have $F_0 \subseteq F$ and $S_0 \subseteq S$, which implies that $F^u \bigcap S^l \subseteq (F_0)^u \bigcap (S_0)^l = \uparrow q$. It follows that $x_{(F,S,\lambda)} = a_F^S(\lambda) \in F^u \bigcap S^l \subseteq \uparrow q$ for every $(F, S, \lambda) \geqq (F_0, S_0, \lambda_0)$. This means that $q \in P_x$. Hence, we conclude that $K \subseteq P_x$. This shows that $P_x = K$. It can be similarly proved that $Q_x = L$.

(vi) Both the sets K and L are finite.
Since the pair (K, L) is a nontrivial ID-pair in the poset P, it follows that $K^u \bigcap L^l \neq \emptyset$, $(K^u \bigcap L^l)^l \subseteq K$ and $(K^u \bigcap L^l)^u \subseteq L$. Let κ_K^L, well ordered by $\geq_K^L$, denote the cardinal of the set $K^u \bigcap L^l$, and $a_K^L : \kappa_K^L \to K^u \bigcap L^l$ be a one-to-one function from the cardinal κ_K^L onto the set $K^u \bigcap L^l$. Set $I = \{(n, \lambda) : n \in \mathbb{N}, \lambda \in \kappa_K^L\}$. For any $(n_1, \lambda_1), (n_2, \lambda_2) \in I$, we define $(n_2, \lambda_2) \geqq (n_1, \lambda_1)$ if and only if one of the following conditions is satisfied:

(3) $n_1 = n_2$ and $\lambda_2 \geq_K^L \lambda_1$;
(4) $n_1 \neq n_2$ and $n_2 \geq_o n_1$.

It can easily be checked that the ordered I is up-directed. Let $(x_i)_{i \in I}$ be the net in the poset P by defining $x_{(n,\lambda)} = a_K^L(\lambda) \in K^u \bigcap L^l$ for all $\lambda \in \kappa_K^L$. Now, it remains to show that $K = P_x$ and $L = Q_x$. Let $q \in K$. Then, we have $K^u \bigcap L^l \subseteq \uparrow q$. By the definition of the net $(x_i)_{i \in I}$, it follows that $x_{(n,\lambda)} = a_K^L(\lambda) \in K^u \bigcap L^l \subseteq \uparrow q$ for all $(n, \lambda) \in I$. This means that $q \in [E_x((n, \lambda))]^l$ for all $(n, \lambda) \in I$. Hence, we conclude that $q \in P_x$, which shows that $K \subseteq P_x$. Conversely, let $p \in P_x$. Then, there exists $(n_0, \lambda_0) \in I$ such that $p \in [E_x((n_0, \lambda_0))]^l$. Since $(n_0 + 1, \lambda) \geqq (n_0, \lambda_0)$, for all $\lambda \in \kappa_K^L$, it follows that $x_{(n_0+1,\lambda)} = a_K^L(\lambda) \in K^u \bigcap L^l$ for all $\lambda \in \kappa_K^L$. This implies that $K^u \bigcap L^l \subseteq E_x((n_0, \lambda_0))$. Hence, we have $p \in [E_x((n_0, \lambda_0))]^l \subseteq (K^u \bigcap L^l)^l \subseteq K$. This shows that $P_x \subseteq K$. Therefore, $P_x = K$. A similar verification can show that $Q_x = L$.

By (v) and (vi), we can conclude that there exists a net $(x_i)_{i \in I}$ in the poset P such that $P_x = K$ and $Q_x = L$. Thus, the proof is completed. □

Example 4. *Let $P = \{\top\} \bigcup \{a_1, a_2, \ldots, a_n, \ldots\}$ with the partial order $\leqslant$ defined by*

- $(\forall n)\ a_n \leqslant \top$.

Consider the net $(x_n)_{n \in \mathbb{N}}$ defined by

$$(\forall n \in \mathbb{N})\ x_n = a_n,$$

where the up-directed set $\mathbb{N}$ is the set of all positive integers in its usual order. By the definition of the net $(x_n)_{n \in \mathbb{N}}$, we have $P_x = \emptyset$ and $Q_x = \{\top\}$. On the other hand, it follows from Definition 4 and Definition 5 that the pair $(\emptyset, \{\top\})$ is a nontrivial ID-pair. This demonstrates Theorem 2 in the case.

The combination of Proposition 1 and Theorems 1 and 2 indicates that the eventually lower bounds P_x and eventually upper bounds Q_x of a net $(x_i)_{i \in I}$ are precisely a Frink ideal and a dual Frink ideal, respectively (see Corollary 1). However, they are not independent. Theorem 2 clarifies the correlation between the Frink ideal P_x and the dual Frink ideal Q_x from the point of view of order; that is, the Frink ideal P_x and the dual Frink ideal Q_x must be matched as a nontrivial ID-pair. Also, this is the initial motivation of introducing the local Frink ideal (the dually local Frink ideal) and ID-pair for posets in the sequel.

Corollary 1 ([6]). *Let P be a poset and $K, L \subseteq P$. Then,*

(1) The subset K is a Frink ideal if and only if $P_x = K$ for some net $(x_i)_{i \in I}$ in the poset P;
(2) The subset L is a dual Frink ideal if and only if $Q_y = L$ for some net $(y_j)_{j \in J}$ in the poset P.

3. ID-Doubly Continuous Posets

Given a poset P and $M, N \subseteq P$, let $\mathfrak{L}_M(N) = \{K \in \mathfrak{L}(N) : M \subseteq K\}$. Then, one can readily verify by Definition 4 that the intersection $\bigcap \mathfrak{L}_M(N)$ contains the set M and is again a local Frink ideal in the set N. This local Frink ideal is called the *local Frink ideal generated by the set M* and denoted by $IG_N(M)$. The *dually local Frink ideal generated by the set N* is defined dually, and denoted by $DG_M(N)$. Next, we clarify the structure of $IG_N(M)$ and $DG_M(N)$:

Proposition 3. *Let P be a poset and $M, N \subseteq P$. Then,*

(1) $IG_N(M) = \{p \in P : (\exists M_0 \sqsubseteq M)\ (\exists N_0 \sqsubseteq N)\ (M_0)^u \bigcap (N_0)^l \subseteq \uparrow p\}$;
(2) $DG_M(N) = \{q \in P : (\exists M_{00} \sqsubseteq M)\ (\exists N_{00} \sqsubseteq N)\ (M_{00})^u \bigcap (N_{00})^l \subseteq \downarrow q\}$.

Proof. (1): Denote the set $\{p \in P : (\exists M_0 \sqsubseteq M)\ (\exists N_0 \sqsubseteq N)\ (M_0)^u \bigcap (N_0)^l \subseteq \uparrow p\}$ by $\overline{M}_N$. Then, it is easy to see that $M \subseteq \overline{M}_N$. Now, we proceed to prove that $\overline{M}_N \in \mathfrak{L}(N)$. Let $F \sqsubseteq \overline{M}_N$ and $S \sqsubseteq N$. We should consider the following cases:

(i) $F = \emptyset$.
Since $F = \emptyset$, it follows that $(M_a)^u \bigcap S^l \subseteq F^u \bigcap S^l = S^l$ for all $M_a \sqsubseteq M$, which implies that $(F^u \bigcap S^l)^l \subseteq [(M_a)^u \bigcap S^l]^l$ for all $M_a \sqsubseteq M$. This means that $(M_a)^u \bigcap S^l \subseteq \uparrow p'$ for all $p' \in (F^u \bigcap S^l)^l$. Hence, we infer that $(F^u \bigcap S^l)^l \subseteq \overline{M}_L$.

(ii) $F = \{e_1, e_2, \ldots, e_m\} \neq \emptyset$.

It follows by the definition of $\overline{M}_N$ that, for every $1 \leq_o r \leq_o m$, there exist $M_r \sqsubseteq M$ and $N_r \sqsubseteq N$ such that $(M_r)^u \bigcap (N_r)^l \subseteq \uparrow e_r$. Take $M_F = \bigcup\{M_r : 1 \leq_o r \leq_o m\}$ and $N_F = \bigcup\{N_r : 1 \leq_o r \leq_o m\} \bigcup S$. Then, we have that $M_F \sqsubseteq M$, $N_F \sqsubseteq N$ and

$$\begin{aligned}(M_F)^u \bigcap (N_F)^l =& \bigcap\{(M_r)^u \bigcap (N_r)^l : 1 \leq_o r \leq_o m\} \bigcap S^l \\ \subseteq& \bigcap\{\uparrow e_r : 1 \leq_o r \leq_o m\} \bigcap S^l \\ =& F^u \bigcap S^l.\end{aligned}$$

This implies that $(F^u \bigcap S^l)^l \subseteq [(M_F)^u \bigcap (N_F)^l]^l$, which means that $(M_F)^u \bigcap (N_F)^l \subseteq \uparrow p'$ for all $p' \in (F^u \bigcap S^l)^l$. Thus, we conclude that $(F^u \bigcap S^l)^l \subseteq \overline{M}_N$ by the definition of $\overline{M}_N$.

According to (i), (ii) and Definition 4, we show that $\overline{M}_N \in \mathfrak{L}(N)$.

To complete the proof, it suffices to prove that $\overline{M}_N \subseteq K$ for every $K \in \mathfrak{L}(N)$ with $M \subseteq K$. Let $p \in \overline{M}_N$. Then, by the definition of $\overline{M}_N$, there exist $M_0 \sqsubseteq M$ and $N_0 \sqsubseteq N$ such that $(M_0)^u \bigcap (N_0)^l \subseteq \uparrow p$. This means that $p \in [(M_0)^u \bigcap (N_0)^l]^l$. Since $M \subseteq K$ and $K \in \mathfrak{L}(N)$, it follows that $p \in [(M_0)^u \bigcap (N_0)^l]^l \subseteq K$. So, we have that $\overline{M}_N \subseteq K$. Consequently, we infer that $IG_N(M) = \overline{M}_N = \{p \in P : (\exists M_0 \sqsubseteq M)\ (\exists N_0 \sqsubseteq N)\ (M_0)^u \bigcap (N_0)^l \subseteq \uparrow p\}$.

(2): The proof is similar to that of (1). □

Lemma 1. *Let P be a poset and $M, N \subseteq P$. Then, we have that $IG_N(M) \in \mathfrak{L}(DG_M(N))$ and $DG_M(N) \in \mathfrak{D}(IG_N(M))$, i.e., the pair $(IG_N(M), DG_M(N))$ is an ID-pair in the poset P.*

Proof. We only show that $IG_N(M) \in \mathfrak{L}(DG_M(N))$; the fact $DG_M(N) \in \mathfrak{D}(IG_N(M))$ can be similarly proved. Let $F \sqsubseteq IG_N(M)$ and $S \sqsubseteq DG_M(N)$. We consider the following cases:

(i) $F = \emptyset$ and $S = \emptyset$.
If the least element $\perp$ exists in the poset P, then we have that $\perp \in IG_N(M)$ by Remark 1. It follows that $(F^u \bigcap S^l)^l = \{\perp\} \subseteq IG_N(M)$. If the poset P has no least element, then $(F^u \bigcap S^l)^l = \emptyset \subseteq IG_N(M)$ by Remark 1 again. This shows that $(F^u \bigcap S^l)^l \subseteq IG_N(M)$.

(ii) $F = \{e_1, e_2, \ldots, e_m\} \neq \emptyset$ and $S = \emptyset$.
By Proposition 3, there exist $M_r \sqsubseteq M$ and $N_r \sqsubseteq N$ such that $(M_r)^u \bigcap (N_r)^l \subseteq \uparrow e_r$ for all $1 \leq_o r \leq_o m$. Take $M_0 = \bigcup\{M_r : 1 \leq_o r \leq_o m\}$ and $N_0 = \bigcup\{N_r : 1 \leq_o r \leq_o m\}$. Then, we have that $M_0 \sqsubseteq M$, $N_0 \sqsubseteq N$ and

$$\begin{aligned}(M_0)^u \bigcap (N_0)^l =& \bigcap\{(M_r)^u \bigcap (N_r)^l : 1 \leq_o r \leq_o m\} \\ \subseteq& \bigcap\{\uparrow e_r : 1 \leq_o r \leq_o m\} \\ =& F^u = F^u \bigcap S^l.\end{aligned}$$

It follows that $(F^u \bigcap S^l)^l \subseteq [(M_0)^u \bigcap (N_0)^l]^l$, which implies that $(M_0)^u \bigcap (N_0)^l \subseteq \uparrow p$ for all $p \in (F^u \bigcap S^l)^l$. Thus, by Proposition 3, we have that $p \in IG_N(M)$ for all $p \in (F^u \bigcap S^l)^l$. This means that $(F^u \bigcap S^l)^l \subseteq IG_N(M)$.

(iii) $F = \emptyset$ and $S = \{s_1, s_2, \ldots, s_n\} \neq \emptyset$.
Proceeding as in the proof of (ii), we can again have $(F^u \bigcap S^l)^l \subseteq IG_N(M)$.

(iv) $F = \{e_1, e_2, \ldots, e_m\} \neq \emptyset$ and $S = \{s_1, s_2, \ldots, s_n\} \neq \emptyset$.
By Proposition 3, there exist $M_r^F, M_t^S \sqsubseteq M$ and $N_r^F, N_t^S \sqsubseteq N$ such that $(M_r^F)^u \bigcap (N_r^F)^l \subseteq \uparrow e_r$ and $(M_t^S)^u \bigcap (N_t^S)^l \subseteq \downarrow s_t$ for all $1 \leq_o r \leq_o m$ and $1 \leq_o t \leq_o n$. Set $M_0 = \bigcup\{M_r^F : 1 \leq_o r \leq_o m\} \bigcup \bigcup\{M_t^S : 1 \leq_o t \leq_o n\}$ and $N_0 = \bigcup\{N_r^F : 1 \leq_o r \leq_o m\} \bigcup \bigcup\{N_t^S : 1 \leq_o t \leq_o n\}$. Then, we have that $M_0 \sqsubseteq M$, $N_0 \sqsubseteq N$ and

$$\begin{aligned}(M_0)^u \bigcap (N_0)^l =& \bigcap\{(M_r^F)^u \bigcap (N_r^F)^l : 1 \leq_o r \leq_o m\} \\ & \bigcap \bigcap\{(M_t^S)^u \bigcap (N_t^S)^l : 1 \leq_o t \leq_o n\} \\ \subseteq & \bigcap\{\uparrow e_r : 1 \leq_o r \leq_o m\} \bigcap \bigcap\{\downarrow s_t : 1 \leq_o t \leq_o n\} \\ =& F^u \bigcap S^l.\end{aligned}$$

This implies that $(F^u \bigcap S^l)^l \subseteq [(M_0)^u \bigcap (N_0)^l]^l$, which concludes that $(M_0)^u \bigcap (N_0)^l \subseteq \uparrow p$ for all $p \in (F^u \bigcap S^l)^l$. Hence, by Proposition 3, we have $(F^u \bigcap S^l)^l \subseteq IG_N(M)$.

According to (i)–(iv) and Definition 4, we infer that $IG_N(M) \in \mathfrak{L}(DG_M(N))$. □

Lemma 2. *Let P be a poset and $M, N \subseteq P$. If $\sup M = x = \inf N \in P$, then we have $\sup IG_N(M) = x = \inf DG_M(N)$.*

Proof. Let $\sup M = x = \inf N \in P$. Then, one can readily check, by Proposition 3, that $M \subseteq IG_N(M) \subseteq \downarrow x$ and $N \subseteq DG_M(N) \subseteq \uparrow x$. It follows that $\sup IG_N(M) = x = \inf DG_M(N)$. □

We turn to define the ID-double continuity for posets. Since the ID-double continuity has a close relationship to Property A, proposed by Wolk, we review Property A and its dual form for posets in the following:

Definition 6 ([6]). *A poset P has* Property A *if, for every non-normal Frink ideal K with $\sup K = x \in P$, there exists an up-directed subset $K_U \subseteq K$ such that $\sup K_U = x$. Dually, a poset P has* Property DA *if, for every non-normal dual Frink ideal L with $\inf L = y \in P$, there exists a down-directed subset $L_D \subseteq L$ such that $\inf L_D = y$.*

Definition 7. *A poset P is called an* ID-doubly continuous poset *if, for every ID-pair (K, L) in the poset P with $\sup K = x = \inf L \in P$, there exist an up-directed subset $K_U \subseteq K$ and a down-directed subset $L_D \subseteq L$ such that $\sup K_U = x = \inf L_D$.*

Example 5. *(1) Every finite poset is ID-doubly continuous;*

(2) Every lattice is ID-doubly continuous.

Suppose that P is a finite poset and (K, L) is an ID-pair with $\sup K = x = \inf L \in P$. Then, we have that $K, L \sqsubseteq P$ and $K^u \bigcap L^l = \{x\}$. Since the pair (K, L) is an ID-pair, by Definition 4 and Definition 5, it follows that $(K^u \bigcap L^l)^l = \downarrow x \subseteq K$ and $(K^u \bigcap L^l)^u = \uparrow x \subseteq L$, which implies that $x \in K$ and $x \in L$. This means that the singleton $\{x\}$ is an up-directed subset of K and also a down-directed subset of L such that $\sup\{x\} = x = \inf\{x\}$. So, by Definition 7, the finite poset P is ID-doubly continuous.

The fact that every lattice is ID-doubly continuous can also be readily checked by Definition 7.

Proposition 4. *Let P be a poset. If the poset P has Property A and Property DA, then it is an ID-doubly continuous poset.*

Proof. Let (K, L) be an ID-pair in the poset P with $\sup K = x = \inf L \in P$. Then, by Proposition 1, the set K is a Frink ideal. If $x \in K$, then we have that $\{x\}$ is an up-directed subset of K and $\sup\{x\} = x$. If $x \notin K$, then K is a non-normal Frink ideal since $x \in (K^u)^l = \downarrow x \neq K$. By Property A, it follows that there exists an up-directed subset $K_U \subseteq K$ such that $\sup K_U = x$. A similar verification can prove that there exists a down-directed subset $L_D \subseteq L$ such that $\inf L_D = x$. Hence, the poset P is ID-doubly continuous. □

In general, an ID-doubly continuous poset may not possess Property A and Property DA. For such an example, one can refer to Example 7 in Section 4.

Now, we arrive at the main result:

Theorem 3. *A poset P is ID-doubly continuous if and only if the o-convergence and o_2-convergence in the poset P are equivalent.*

Proof. ($\Rightarrow$): Suppose that a poset P is ID-doubly continuous. To prove the equivalence between the o-convergence and o_2-convergence, it suffices to show that, for every net $(x_i)_{i\in I}$ in the poset P, we have

$$(x_i)_{i\in I} \xrightarrow{o_2} x \in P \Rightarrow (x_i)_{i\in I} \xrightarrow{o} x.$$

Let $(x_i)_{i\in I} \xrightarrow{o_2} x$. Then, by Definition 2, there exist subsets $M, N \subseteq P$ such that $\sup M = x = \inf N$, and, for every $m \in M$ and every $n \in N$, $m \leqslant x_i \leqslant n$ holds eventually. This means that $M \subseteq P_x$ and $N \subseteq Q_x$, which implies that $IG_N(M) \subseteq P_x$ and $DG_M(N) \subseteq Q_x$ by Remark 1 and Theorem 2. According to Lemma 1 and 2, it follows that $(IG_N(M), DG_M(N))$ is an ID-pair with $\sup IG_N(M) = x = \inf DG_M(N)$. Since the poset P is ID-doubly continuous, we have that $\sup M_U = x = \inf N_D$ for some up-directed subset $M_U \subseteq IG_N(M) \subseteq P_x$ and some down-directed subset $N_D \subseteq DG_M(N) \subseteq Q_x$. This concludes $(x_i)_{i\in I} \xrightarrow{o} x$.

($\Leftarrow$): Assume that the o-convergence and o_2-convergence in a poset P are equivalent. Let (K, L) be an ID-pair in the poset P with $\sup K = x = \inf L \in P$. Since $x \in F^u \bigcap S^l \neq \emptyset$ for all $F \sqsubseteq K$ and $S \sqsubseteq L$, the pair (K, L) is a nontrivial ID-pair by Proposition 2. According to Theorem 2, there exists a net $(x_i)_{i\in I}$ in the poset P such that $K = P_x$ and $L = Q_x$. Thus, we have $(x_i)_{i\in I} \xrightarrow{o_2} x$. By the hypothesis, it follows that $(x_i)_{i\in I} \xrightarrow{o} x$. This means that $\sup K_U = x = \inf L_D$ for some up-directed subset $K_U \subseteq K = P_x$ and some down-directed subset $L_D \subseteq L = Q_x$. So, the poset P is an ID-doubly continuous poset. □

By Example 5 and Theorem 3, we immediately have the following:

Example 6. *(1) In every finite poset, the o-convergence and the o_2-convergence are equivalent;*
(2) In every lattice, the o-convergence and the o_2-convergence are equivalent.

By Proposition 4 and Theorem 3, or by Definition 2 and Theorem 2 and 5 in [6], we readily have the following:

Corollary 2. *If a poset P has Property A and Property DA, then the o-convergence and o_2-convergence in the poset P are equivalent.*

4. Example

In this section, we mainly give an example to clarify the following facts:

(1) A Frink ideal K of a poset P may not be a local Frink ideal in every nonempty subset L of P; Dually, a dual Frink ideal K need not be a dually local Frink ideal in every nonempty subset K of P.
(2) An ID-doubly continuous poset fails to satisfy Property A and Property DA.

Example 7. *Let $P = \{x\} \bigcup \{a_1, a_2, \ldots, a_n, \ldots\} \bigcup \{b_1, b_2, \ldots, b_n, \ldots\} \bigcup \{c_1, c_2, \ldots, c_n, \ldots\} \bigcup \{d_1, d_2, \ldots, d_n, \ldots\}$ (see Figure 1). Define the partial order $\leqslant$ on P by setting*

- $\downarrow x = \{x\} \bigcup \{a_1, a_2, \ldots, a_n, \ldots\} \bigcup \{b_1, b_2, \ldots, b_n, \ldots\} \bigcup \{c_1, c_2, \ldots, c_n, \ldots\}$;
- $(\forall n)\ \downarrow a_n = \{a_1, a_2, \ldots, a_n\}$;
- $(\forall n)\ \downarrow b_n = \{b_n\}$;
- $(\forall n)\ \downarrow c_n = \{c_n\} \bigcup \{a_1, a_2, \ldots, a_n\} \bigcup \{b_1, b_2, \ldots, b_n\}$;
- $(\forall n)\ \downarrow d_n = \{d_n\} \bigcup \{b_1, b_2, \ldots, b_n\}$.

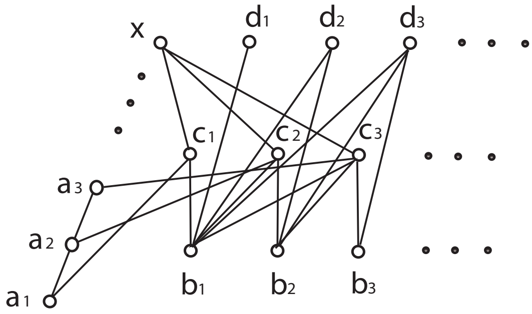

Figure 1. The diagram for the poset in Example 7.

Let $K = \{b_1, b_2, \ldots, b_n, \ldots\}$. Then, the set K is a non-normal Frink ideal by Definition 3 and the definition of the poset P. However, the poset P does not process Property A since we can easily see that $\sup K = x$, and $\sup K_U \neq x$ for every up-directed subset $K_U \subseteq K$. We next show that $K \notin \mathfrak{L}(L)$ for any nonempty subset L of the poset P by analyzing the following cases:

(i) $a_i \in L$ *(resp.* $b_i \in L$, $c_i \in L$, $d_i \in L$*) for some* $i \in \mathbb{N}$.
Take $j \in \mathbb{N}$ such that $j >_o i$. Then, we have $\{b_j\} \sqsubseteq K$, $\{a_i\} \sqsubseteq L$ (resp. $\{b_i\} \sqsubseteq L$, $\{c_i\} \sqsubseteq L$, $\{d_i\} \sqsubseteq L$) and $(\{b_j\}^u \bigcap \{a_i\}^l)^l = P \nsubseteq K$ (resp. $(\{b_j\}^u \bigcap \{b_i\}^l)^l = P \nsubseteq K$, $(\{b_j\}^u \bigcap \{c_i\}^l)^l = P \nsubseteq K$, $(\{b_j\}^u \bigcap \{d_i\}^l)^l = P \nsubseteq K$). This implies that $K \notin \mathfrak{L}(L)$ by Definition 4.

(ii) $x \in L$.
It is easy to see that $\{b_1, b_2\} \sqsubseteq K$, $\{x\} \sqsubseteq L$ and $(\{b_1, b_2\}^u \bigcap \{x\}^l)^l = \{a_1, a_2\} \bigcup \{b_1, b_2\} \nsubseteq K$. This implies that $K \notin \mathfrak{L}(L)$ by Definition 4.

The combination of (i) and (ii) shows that the set K is not a local Frink ideal in any nonempty subset L of the poset P.

Now, we are going to verify that P is an ID-doubly continuous poset. Let (K', L') be an ID-pair in the poset P with $\sup K' = p = \inf L'$. We consider the following cases:

(iii) $p = a_i$ *(resp.* $p = b_i, c_i, d_i$*) for some* $i \in \mathbb{N}$.
It is easy to see, by the definition of the poset P, that there exist $K_0 \sqsubseteq K'$ and $L_0 \sqsubseteq L'$ such that $\sup K_0 = \inf L_0 = p = a_i$. Since the pair (K', L') is an ID-pair, we have $[(K_0)^u \bigcap (L_0)^l]^l = \downarrow a_i \subseteq K'$ and $[(K_0)^u \bigcap (L_0)^l]^u = \uparrow a_i \subseteq L'$, i.e., $a_i \in K'$ and $a_i \in L'$. Take $K'_U = L'_D = \{a_i\}$. Then, the set K'_U is an up-directed subset of the set K', the set L'_D is a down-directed subset of the set L' and $\sup K'_U = a_i = \inf L'_D$.

(iv) $p = x$ *and* $x \in K'$.
Since $\inf L' = x$, one can readily check that $L' = \{x\}$. Take $K'_U = L'_D = \{x\}$. Then, we have that the set K'_U is an up-directed subset of the set K', the set L'_D is a down-directed subset of the set L' and $\sup K'_U = x = \inf L'_D$.

(v) $p = x$ *and* $a_i \in K'$ *for some* $i \in \mathbb{N}$.
Since $\inf L' = x$, it is easy to see that $L' = \{x\}$. If the set $K' \bigcap \{a_1, a_2, \ldots\}$ is infinite, then we have that the set $K'_U = K' \bigcap \{a_1, a_2, \ldots\}$ is an up-directed subset of the set K', the set $L'_D = \{x\}$ is a down-directed subset of the set L' and $\sup K'_U = x = \inf L'_D$.
If the set $K' \bigcap \{a_1, a_2, \ldots\}$ is finite, then we have that the set $K' \bigcap \{b_1, b_2, \ldots\}$ is also finite. Otherwise, suppose that the set $K' \bigcap \{b_1, b_2, \ldots\}$ is infinite. Then, there exists $\{b_{i_1}, b_{i_2}, \ldots\} \subseteq K'$. Since the pair (K', L') is an ID-pair in the poset P, we have that $a_{i_k} \in (\{b_{i_1}, b_{i_k}\}^u \bigcap \{x\}^l)^l$ for every $k \in \mathbb{N}$ with $k \geq_o 2$. This means that $\{a_{i_2}, a_{i_3}, \ldots\} \subseteq K' \bigcap \{a_1, a_2, \ldots\}$, contradicting the hypothesis that the set $K' \bigcap \{a_1, a_2, \ldots\}$ is finite. Let $\{a_{j_1}, a_{j_2}, \ldots, a_{j_m}\} = K' \bigcap \{a_1, a_2, \ldots\}$ and $\{b_{i_1}, b_{i_2}, \cdots_{i_n}\} = K' \bigcap \{b_1, b_2, \ldots\}$, and let $j_0 = \max\{j_1, j_2, \ldots, j_m\}$ and $i_0 = \max\{i_1, i_2, \ldots, i_n\}$. Since $\sup K' = x$, we also take the following cases into consideration:

(v1) $x \in K'$.
In this case, we can return the verification to Case (iv).

(v2) $c_{i^0} \in K'$ *for some* $i^0 \in \mathbb{N}$ *with* $i^0 <_o j_0$.

In this case, if we take $K_0 = \{a_{j_0}, c_{i^0}\}$ *and* $L_0 = \{x\}$, *then we have* $K_0 \sqsubseteq K'$ *and* $L_0 \sqsubseteq L'$ *with* $\sup K_0 = x = \inf L_0$. *By a similar verification to that of Case (iii), there exist an up-directed subset* K'_U *of the set* K' *and a down-directed subset* L'_D *of the set* L' *such that* $\sup K'_U = x = \inf L'_D$.

(v3) $c_{i^1} \in K'$ *for some* $i^1 \in \mathbb{N}$ *with* $i^1 <_o i_0$.
In this case, if we take $K_0 = \{b_{i_0}, c_{i^1}\}$ *and* $L_0 = \{x\}$, *then we have* $K_0 \sqsubseteq K'$ *and* $L_0 \sqsubseteq L'$ *with* $\sup K_0 = x = \inf L_0$. *By a similar verification to that of (iii), there exist an up-directed subset* K'_U *of the set* K' *and a down-directed subset* L'_D *of the set* L' *such that* $\sup K'_U = x = \inf L'_D$.

(v4) $c_{i^2}, c_{i^3} \in K'$ *for some* $i^2, i^3 \in \mathbb{N}$.
In this case, if we take $K_0 = \{c_{i^2}, c_{i^3}\}$ *and* $L_0 = \{x\}$, *then we have* $K_0 \sqsubseteq K'$ *and* $L_0 \sqsubseteq L'$ *with* $\sup K_0 = x = \inf L_0$. *By a similar verification to (iii), there exist an up-directed subset* K'_U *of the set* K' *and a down-directed subset* L'_D *of the set* L' *such that* $\sup K'_U = x = \inf L'_D$.

(vi) $p = x$ *and* $c_i \in K'$ *for some* $i \in \mathbb{N}$.
Since the pair (K', L') *is an ID-pair, we have* $a_i \in (\{c_i\}^u \bigcap \{x\}^l)^l \subseteq K'$. *So, we can return the verification to Case (v).*

(vii) $p = x$ *and* $b_i \in K'$ *for some* $i \in \mathbb{N}$.
We consider the following cases:

(vii1) $b_i, b_j \in K' \bigcap \{b_1, b_2, \dots\}$ *for some* $i, j \in \mathbb{N}$.
Since the pair (K', L') *is an ID-pair, we have* $a_i \in (\{b_i, b_j\}^u \bigcap \{x\}^l)^l \subseteq K'$. *So, we can return the verification to Case (v).*

(vii2) $\{b_i\} = K' \bigcap \{b_1, b_2, \dots\}$.
Since $\sup K' = x$, *there exists* $j \in \mathbb{N}$ *such that* $a_j \in K'$ *(resp.* $c_j \in K'$, $x \in K'$*). So, we can return the verification to Case (v) (resp. Case (vi), Case (iv)).*

By Definition 7 and the combination of Cases (iii)–(vii), we conclude that the poset P is an ID-doubly continuous poset.

5. Discussion

This paper introduced the notion of ID-pairs in posets. It was shown that the set of all eventually lower bounds and the set of all eventually upper bounds of a net in a given poset can be precisely paired to be an ID-pair. This result provides a potential approach for dealing with the general nets in posets, since some kinds of order convergent nets, such as the o-convergent nets and o_2-convergent nets, are uniquely determined by their eventually lower bounds sets and eventually upper bounds sets.

Furthermore, in order to characterize these posets in which the o-convergence and o_2-convergence are equivalent, the concept of ID-doubly continuous posets is proposed. It is proved that the equivalence of the o-convergence and o_2-convergence in a poset is equivalent to the ID-double continuity of the poset. This result provides a sufficient and necessary condition for the o-convergence and o_2-convergence to be equivalent.

However, it may be complicated to verify the ID-double continuity for some posets, such as the poset in Example 7. On the contrary, the lattices, a special kind of poset, can be easily proved to be ID-double continuous. This indicates that the ID-double continuity has some close relationships with some special kinds of posets. These relationships deserve further investigation.

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